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EDITORS

K. HUSIMI, *Osaka* M. KOBAYASI, *Kyoto*
M. KOTANI, *Tokyo* T. MUTO, *Tokyo*
S. SAKATA, *Nagoya* S. TOMONAGA, *Tokyo*
H. YUKAWA, *Columbia University*

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An Algebraic Theory of the Density Matrix, II

Toshiyuki NISHIYAMA

Department of Physics, Osaka University

(Received August 23, 1950)

§ 1. Introduction

In part I¹⁾ some theorems concerning the structure of symmetric operator have been applied. Our present objective consists in the verification of those theorems by studying the structure theory of the algebra in question, appearing in the expression for the density operator of the form

$$\rho(q^n, q^{n'}) = \frac{1}{n!} \varphi^*(q_1') \cdots \varphi^*(q_n') \varphi(q_n) \cdots \varphi(q_1).$$

The structure theory of the algebra corresponds to an extension mainly in the two directions of the former theories of P. Jordan²⁾ and H. Ostertag.³⁾ They dealt namely with electrons only, while our theory is applicable to Fermi particles with arbitrary intrinsic degrees of freedom. They took only the exchange operators into consideration, while ours is available to obtain representations of all the kinds of permutation operators. The structure theory obtained is analogous to the scheme of group algebra due to Weyl,⁴⁾ which is included in our theory as a matter of course.

A special attention is paid for the exchange operators, which represent exchanges of spins or of charges, or both of spin and charge for electrons or for nucleons. And also some remarks on the relations to the full linear group and spinor theory are pointed out.

If one wants to dispense with the group theory, he will find in the following an interesting way recognizing old things from a new point of view, although there are no fundamentally new results.

§ 2. Permutation operators.

The density operator $\rho(q^n, q^{n'})$ takes the following form. If we separate the intrinsic state functions,

$$\begin{aligned} \frac{1}{n!} \sum_{(\lambda)} \sum_{(r)} \sum_{(r')} \alpha_{r_1 \lambda_1}^* \alpha_{r_2 \lambda_2}^* \cdots \alpha_{r_n \lambda_n}^* \alpha_{r_n' \lambda_n} \cdots \alpha_{r_2' \lambda_2} \alpha_{r_1' \lambda_1} \phi_{r_1'}(q_1) \phi_{r_2'}(q_2) \\ \cdots \phi_{r_n'}(q_n) \overline{\phi_{r_1}(q_1')} \overline{\phi_{r_2}(q_2')} \cdots \overline{\phi_{r_n}(q_n')}, \end{aligned}$$

where any intrinsic state, is denoted by λ_i . One used to specialize the operator in this manner taking average over the intrinsic states, as quantum numbers concerning orbital states were separable from those concerning intrinsic states. (Russel-Saunders' case). Put

$$\begin{aligned} a_{r_1\lambda_1}^* a_{r_2\lambda_2}^* \cdots a_{r_n\lambda_n}^* &= C^*(r_1\lambda_1, r_2\lambda_2, \dots, r_n\lambda_n), \\ a_{r_1\lambda_1} a_{r_2\lambda_2} \cdots a_{r_n\lambda_n} &= C(r_1\lambda_1, r_2\lambda_2, \dots, r_n\lambda_n). \end{aligned} \quad (2.1)$$

We ask for the structure of the operator $\sum_{(\lambda)} C^* C$. The permutation of n orbital states are transformed into those of n intrinsic states. As we have already mentioned, the intrinsic state λ_1 is attached to the orbital state r_1 , etc. The intrinsic state of the state r_1 may be denoted by $\lambda(r_1)$ instead of λ_1 . A permutation of the orbital state is designated by

$$\begin{bmatrix} 1, & 2, & \dots, & n \\ r_1, & r_2, & \dots, & r_n \end{bmatrix} = P, \quad (2.2)$$

and a permutation of the intrinsic states is designated by

$$\begin{bmatrix} 1', & 2', & \dots, & n' \\ \lambda(r_1), & \lambda(r_2), & \dots, & \lambda(r_n) \end{bmatrix} = R. \quad (2.3)$$

If we rearrange the orbital states, the arguments of $\lambda(r)$, in their own natural order 1, 2, ..., n , the intrinsic states are accordingly rearranged in the order $\lambda(1), \lambda(2), \dots, \lambda(n)$:

$$\begin{bmatrix} 1', & 2', & \dots, & n' \\ \lambda(1), & \lambda(2), & \dots, & \lambda(n) \end{bmatrix} = P^{-1}R. \quad (2.4)$$

It means the arrangement $\lambda(1), \lambda(2), \dots, \lambda(n)$ is obtained from the natural arrangement by the operation

$$[\{1', 2', \dots, n'\} P^{-1}\} R].$$

We consider in the above formulation only such a special case that there are n different orbital and intrinsic states. For the other cases we have only to modify our description slightly.

The operator in question is expressed by

$$\begin{aligned} &\sum_{(\lambda)} C^*(r_1\lambda_1, r_2\lambda_2, \dots, r_n\lambda_n) C(r_1'\lambda_1', r_2'\lambda_2', \dots, r_n'\lambda_n') \\ &= \sum_{(\lambda)} \sum_R (-1)^{P+Q} C^*(1'1', 2'2', \dots, nn', PR) C(1'1', 2'2', \dots, nn', QR). \end{aligned} \quad (2.5)$$

Here PR and QR mean permutations with respect to intrinsic states, and $(-1)^{P+Q} = +1$ if P and Q are both odd or even permutations and -1 for the other cases. $\sum_{(\lambda)}$ means sum over all the allowable intrinsic states and R runs over the non-equivalent permutations. Henceforth we shall use the notation $C(; PR)$ instead of $C(1'1', 2'2', \dots, nn', PR)$.

Now our objective is the operator in the following form :

$$\begin{aligned} \sum_R C^*(; PR) C(; QR) &= \sum_R C^*(; R) C(; QP^{-1}R) \\ &= \sum_R C^*(; PQ^{-1}R) C(; R) = QP^{-1} = S. \end{aligned} \quad (2.6)$$

§ 3. Structure of symmetric algebras

It is a rather mathematical problem to study the structure theory of the operator. We shall start with a Kronecker product composed of the so-called elementary algebras. Every elementary algebra describes an orbital state and has the following 2^k primitive idempotents

$$\begin{aligned} e_1 &= (1 - N_1)(1 - N_2) \cdots (1 - N_k), \\ \begin{pmatrix} k \\ 1 \end{pmatrix} &\left\{ \begin{array}{l} e_2 = N_1(1 - N_2) \cdots (1 - N_k) \\ \vdots \\ e_i = N_1 \cdots N_{P_1-1}(1 - N_{P_1})N_{P_1+1} \cdots (1 - N_{P_f-1})N_{P_f}(1 - N_{P_f+1}) \cdots (1 - N_k) \\ \vdots \\ e_k = N_1N_2 \cdots N_k. \end{array} \right. \end{aligned} \quad (3.1)$$

Elementary algebra E_k describes an orbital energy state and has subalgebras whose principal idempotents $N(f)$ are given as follows :

$$\begin{aligned} N(0) &= (1 - N_1)(1 - N_2) \cdots (1 - N_k), \\ N(1) &= \sum_{P=1} (1 - N_1) \cdots (1 - N_{P-1})N_P(1 - N_{P+1}) \cdots (1 - N_k), \\ &\vdots \\ N(f) &= \sum_{P_1 < P_2 < \cdots < P_f} (1 - N_1) \cdots (1 - N_{P_1-1})N_{P_1}(1 - N_{P_1+1}) \cdots \\ &\quad (1 - N_{P_f-1})N_{P_f}(1 - N_{P_f+1}) \cdots (1 - N_k), \\ &\vdots \\ N(k) &= N_1N_2 \cdots N_k. \end{aligned} \quad (3.2)$$

Here $N(f)$ indicates that the state is f -fold occupied. These subalgebras are total matrix algebras whose quantities have the following form

$$C^*(f; P) C(f'; Q) \quad \text{degree} \begin{pmatrix} k \\ f \end{pmatrix}, \quad (3.3)$$

where f means the occupation number and symbol $(;')$ means set (λ') of intrinsic states in C which may be different from set (λ) in C^* . These quantities are briefly denoted by

$$e(f)_{\lambda\lambda'}. \quad (3.4)$$

It holds the associative law :

$$e(f)_{\lambda\rho} e(f)_{\rho'\lambda'} = e(f)_{\lambda\lambda'} \delta_{\rho\rho'}.$$

A quantity of the direct power of elementary algebras has the form

$$C^*(; ; P) C(; ; Q), \quad (3.5)$$

where the sets of the orbital and intrinsic states are denoted by symbols $(:)$ and $(;)$ respectively.

This direct power \mathbf{A} has the principal idempotent $U(f)$ corresponding to a prescribed orbital configuration (f) and this idempotent is given by the product of principal idempotents of the elementary algebras $\mathbf{F}_k^{(r)}$ as follows:

$$\begin{aligned} U(f) &= N_1(n_1)N_2(n_2)\cdots N_n(n_n), \\ (f)^* &= (n_1, n_2, \dots, n_n), \end{aligned} \quad (3.6)$$

and $U(f)$ is the direct sum of $\binom{k}{n_1}\binom{k}{n_2}\cdots\binom{k}{n_n}$ primitive idempotents of the algebra \mathbf{A} . The algebra $U(f)\mathbf{A}U(f)$ is generated by elements

$$C^*(;P)C(;Q), \quad (3.7)$$

with the same orbital states in both C^* and C . We can readily see that the algebra \mathbf{M}_f generated by such elements $C^*(;P)C(;Q)$ with the same orbital and intrinsic states in both C^* and C is contained in the algebra $U(f)\mathbf{A}U(f)$. \mathbf{M}_f is marked by a certain spin magnetic quantum number $M=(M_1, M_2, \dots, M_k)$, which means the first intrinsic state is M_1 -fold occupied, the second state is M_2 -fold occupied, and the last k -th state is $M_k=(n-\sum_{i=1}^{k-1}M_i)$ -fold occupied. Hereby we understand the permutation operators in question generate a subalgebra of \mathbf{M}_f and thus symmetric algebra of type π_f is determined. In the following we shall consider especially the structure of symmetric algebra of type unity, which has at most as many centums as the number of classes of the symmetric group π . By a suitable linear combination of quantities of the centrum of symmetric algebra in question, we can obtain the unity quantity of an ideal marked by a prescribed quantum number of multiplets designated by a partition of n : (A_1, A_2, \dots, A_k) , $\sum_{i=1}^k A_i = n$.

Let quantities of the centrum belonging to magnetic quantum number M be $Z_1, Z_2, \dots, Z_\alpha$. Then the unity quantity $e(A)$ is given as follows: (Appedix. II)

$$\begin{aligned} e(A) &= \sum_{\tau=1}^{\alpha} C_{\Lambda}(\gamma) Z_{\tau}, & C_{\Lambda}(\gamma) &= \frac{m}{h} x_{\Lambda}(\gamma), \\ e(A)e(A') &= e(A)\delta_{\Lambda\Lambda'}, \end{aligned} \quad (3.8)$$

h : number of the quantities of symmetric algebra,

$x_{\Lambda}(\gamma)$: character, m : multiplicity.

The unity quantities thus obtained make this symmetric algebra (\mathcal{S}) break up into ideals as follows:

$$(\mathcal{S}) = (\mathcal{S})e(A_1) + (\mathcal{S})e(A_2) + \cdots + (\mathcal{S})e(A_{\alpha}), \quad (3.9a)$$

$$\begin{aligned} (\mathcal{S})e(A_p) &= e(A_p)(\mathcal{S}) = \mathbf{I}(A_p), \\ &\text{(the } A_p\text{-th ideal).} \end{aligned} \quad (3.9b)$$

This algebraic operation composing the unity quantity of "elements of the centrum"

is equivalent to that well-known reduction of the Kronecker power ("die spezielle symmetrische Transformation \sum_0 " of Weyl) furnished by the transformation induced on the tensors of rank n under the influence of k -dimensional rotation c_k . We understand that every quantity of the representation module of a simple ideal marked by a certain term of spin multiplets is an eigenfunction belonging to the intrinsic quantum number.

Now we ask for the other kinds of the orbital configurations of type π_f . Symmetric algebras, contra and unity quantities of ideals of type π_f are uniquely derived from the above mentioned similar quantities of type unity. If all the quantities of symmetric algebra composing the centrum of type unity are replaced by the corresponding quantities of symmetric algebra of type π_f , we get the centrum of type π_f . To be explicit, let two elements of symmetric group be R , S and one of the quantities of the centrum of type unity be

$$R \equiv S \pmod{\pi_f},$$

say

$$Z = b_R \mathbf{R} + b_S \mathbf{S} + b_T \mathbf{T} \quad (3.10a)$$

and then we get

$$Z_f = (b_R + b_S) \mathbf{R}_f + b_T \mathbf{T}_f \quad (3.10b)$$

as a quantity of the centrum of type π_f , where the correspondence between \mathbf{R} , \mathbf{S} and \mathbf{R}_f , \mathbf{S}_f is uniquely determined. Let the unity quantity of an ideal be

$$e = C_1 Z_1 + C_2 Z_2, \quad (2.11a)$$

$$Z_1 = b_R \mathbf{R} + b_S \mathbf{S}, \quad (2.11b)$$

$$Z_2 = b_U \mathbf{U} + b_V \mathbf{V}, \quad (2.11c)$$

$$R \equiv S \equiv U \pmod{\pi_f}.$$

Then we get

$$e_f = (c_1 b_R + c_1 b_S + c_2 b_U) \mathbf{R}_f + c_2 b_V \mathbf{V}_f, \quad (3.12)$$

$$e_f \cdot e_f = e_f$$

as an unity quantity belonging to one of the intrinsic quantum numbers. One can generally define the traces of the regular representation of symmetric algebra by the coefficients of the operators. For example:

$$\text{trace } \mathbf{R} = h c_1 b_R \quad (h: \text{order of } (\mathbf{S})). \quad (3.13)$$

The theorem that the trace of \mathbf{P}_f is developed by the traces of quantities of symmetric algebra of type unity is readily obtained from (3.12), (3.13). For example:

$$\begin{aligned} \text{trace}_f \mathbf{R}_f &= h' (c_1 b_R + c_1 b_S + c_2 b_U) \\ &= \frac{1}{g_f} (\text{trace } \mathbf{R} + \text{trace } \mathbf{S} + \text{trace } \mathbf{U}), \end{aligned} \quad (3.14)$$

where

$$g_f = \frac{h}{h'}, \quad \text{the order of the subgroup } \pi_f.$$

§ 4. The case of many electrons

We shall consider a special case that the number of intrinsic states is two, namely $k=2$. The elementary algebra has four primitive idempotents:

$$\begin{aligned} e_1 &= a_+^* a_+ a_-^* a_- = N_+ N_-, \\ e_2 &= a_+^* a_+ a_- a_-^* = N_+ (1 - N_-), \\ e_3 &= a_+ a_+^* a_-^* a_- = (1 - N_+) N_-, \\ e_4 &= a_+ a_+^* a_- a_-^* = (1 - N_+) (1 - N_-). \end{aligned} \quad (4.1)$$

The symmetric algebra of type unity belonging to a prescribed magnetic quantum number M_z has as many simple ideals as the number of different total spin quantum numbers S , which satisfy the following inequality

$$\frac{n}{2} \geq S \geq |M_z|, \quad M_z = \frac{n}{2} - \mu, \quad (4.2)$$

where μ is the number of minus spin in the configuration of intrinsic states under consideration. An arbitrary permutation operator belonging to a given magnetic quantum number takes the form

$$P_{M_z} = \sum_R C^*(; ; R) C(; ; PR), \quad (4.3)$$

where the set of intrinsic states $(;)$ belongs to magnetic quantum number M_z . Especially the exchange operator exchanging two orbital states r and s is expressed by

$$\begin{aligned} ((rs))_{M_z} &= \sum_R C^*(; ; R) C(; ; (rs)R) \\ &= \sum_{\lambda=\pm} \sum_{\lambda'=\pm} e_{\lambda\lambda'}^{(r)} e_{\lambda'\lambda}^{(s)} U(1, 1, \dots, 1), \end{aligned} \quad (4.4)$$

where $e_{\lambda\lambda'}$ is a quantity of the subalgebra of elementary algebra $N(1) \mathbf{F}_2 N(1)$ and $U(1, 1, \dots, 1) = N_1(1) N_2(1) \dots N_n(1)$. The non-trivial part of this operator is denoted by X_{rs} , namely

$$X_{rs} = \sum_{\lambda=\pm} \sum_{\lambda'=\pm} e_{\lambda\lambda'}^{(r)} e_{\lambda'\lambda}^{(s)} \quad (4.5)$$

and this operator takes the following alternative form

$$X_{rs} = \frac{1}{2} (1_r 1_s + \sigma_r \cdot \sigma_s), \quad (4.6)$$

if one replaces the algebra $N(1) \mathbf{F}_2 N(1)$ with degree two by the quaternion algebra as follows:

$$\begin{aligned} 1 &= e_{11} + e_{22} \\ \sigma^{(1)} &= e_{12} + e_{21}, \quad \sigma^{(2)} = -ie_{12} + ie_{21}, \quad \sigma^{(3)} = e_{11} - e_{22}, \\ \sigma &= \sigma^{(1)} \mathbf{i} + \sigma^{(2)} \mathbf{j} + \sigma^{(3)} \mathbf{k}. \end{aligned} \quad (4.7)$$

The most important character of symmetric algebra (\mathcal{S}) is the commutativity with the direct power of two dimensional full linear groups which are represented by

$$\mathbf{L}_2 = \sum_{i,k}^{1 \rightarrow 2} L_{ik}^{(2)} e_{ik}, \quad (4.8)$$

where e_{ik} is one of the bases of elementary algebra \mathbf{F}_2 . Any element of the algebra defined by the direct sum of symmetric algebras (\mathcal{S}_m) belonging to allowable magnetic quantum numbers:

$$(\mathcal{S}_0) = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} (\mathcal{S}_m), \quad (4.9)$$

which is commutative with the direct power of \mathbf{L}_2 denoted by $[\mathbf{L}_2]^n$:

$$(\mathcal{S}_0)[\mathbf{L}_2]^n = [\mathbf{L}_2]^n (\mathcal{S}_0). \quad (5.10)$$

Hereby we are led to the well-known theorem that the state belonging to total spin S has $(2S+1)f$ degeneracy and this state appears in every state of magnetic quantum number M_z only once, whose absolute value is not larger than S , because $[\mathbf{L}_2]^n$ must have f equivalent irreducible representations of degree $2S+1$, if the representation of (\mathcal{S}) contains $2S+1$ equivalent irreducible parts of degree f . This is the well-known reciprocal relation.

§ 5. The case of four intrinsic degrees of freedom

We shall consider \mathbf{A}_4 , a direct power of \mathbf{F}_4 's, which is defined in Section 3. We ask for the structure of symmetric algebra of type unity, which is the subalgebra of \mathbf{B}_4 defined by

$$\begin{aligned} \mathbf{B}_4 &= U(1, 1, \dots, 1) \mathbf{A}_4 U(1, 1, \dots, 1) \\ &= U(1, 1, \dots, 1) \mathbf{F}_4^{(1)} \times \mathbf{F}_4^{(2)} \times \dots \times \mathbf{F}_4^{(n)} U(1, 1, \dots, 1), \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} U(1, 1, \dots, 1) &= N_1(1) N_2(1) \dots N_n(1), \\ N_r(1) &= N_{r1}(1 - N_{r2})(1 - N_{r3})(1 - N_{r4}) \\ &\quad + N_{r2}(1 - N_{r1})(1 - N_{r3})(1 - N_{r4}) \\ &\quad + N_{r3}(1 - N_{r1})(1 - N_{r2})(1 - N_{r4}) \\ &\quad + N_{r4}(1 - N_{r1})(1 - N_{r2})(1 - N_{r3}). \end{aligned} \quad (5.2)$$

The quantities of \mathbf{B}_4 are given by

$$C^*(:; PR) C(:; QR),$$

and the quantities of symmetric algebra of type unity belonging to a prescribed magnetic quantum number M take the form

$$P_M = \sum_R C^*(; ; R) C(; ; PR).$$

Here symbol $(;)$ means a set of intrinsic states which belong to magnetic quantum number $M = (M_1, M_2, M_3, M_4)$ representing a partition of $N = M_1 + M_2 + M_3 + M_4$. Every intrinsic state is designated by double suffices $(-, -)$, $(-, +)$, $(+, -)$, $(+, +)$ by which we imply that the state of the particle is proton with $(-)$ spin, proton with $(+)$ spin, neutron with $(-)$ spin and neutron with $(+)$ spin, respectively. Other interpretations might be possible. For instance, we may understand elementary algebra describing one proton with four degrees of freedom in the sense of Bhabha. In the following, however, we shall content with the phenomenological treatment of nucleon intrinsic states. Now all symmetric algebras are semisimple and break up into simple algebras marked by Wigner's super multiplets (S, T, Y) .⁵⁾ The numbers S, T, Y are related to a partition $N = A_1 + A_2 + A_3 + A_4$ in the following manner:

$$S = \frac{1}{2} (A_4 + A_3 - A_2 - A_1), \quad (5.3a)$$

$$T = \frac{1}{2} (A_4 - A_3 + A_2 - A_1), \quad (5.3b)$$

$$Y = \frac{1}{2} (A_4 - A_3 - A_2 + A_1), \quad (5.3c)$$

$$A_4 \geq A_3 \geq A_2 \geq A_1,$$

and

$$A_4 \geq M_4, \quad A_4 + A_3 \geq M_4 + M_3, \quad (5.4)$$

$$A_4 + A_3 + A_2 \geq M_4 + M_3 + M_2 \quad \text{or} \quad A_1 \leq M_1.$$

These inequalities are obtained from the branching rule, whose proof in Weyl's book⁴⁾ remains valid in our scheme, too. Especially we ask for the expression for the exchange operator $((rs))$, which exchanges the orbital states r and s resulting in the following form

$$\begin{aligned} ((rs)) &= \sum_R C^*(; ; R) C(; ; (rs)R) \\ &= \sum_{\lambda=1}^4 \sum_{\lambda'=1}^4 e_{\lambda\lambda'}^{(r)} e_{\lambda'\lambda}^{(s)} N_1(1) \cdots N_n(1) \\ &= \sum_{\lambda=1}^4 \sum_{\lambda'=1}^4 e_{\lambda\lambda'}^{(r)} e_{\lambda'\lambda}^{(s)} U(1, 1, \dots, 1), \end{aligned} \quad (5.5)$$

where $e_{\lambda\lambda'}^{(r)}$ is a quantity of $N_r(1) F_4^{(r)} N_r(1)$ (total matrix algebra). The non-trivial part of the operator is denoted by X_{rs} , namely

$$X_{rs} = \sum_{\lambda=1}^4 \sum_{\lambda'=1}^4 e_{\lambda\lambda'}^{(r)} e_{\lambda'\lambda}^{(s)}; \quad (5.6)$$

If one remembers that algebra F_4 is isomorphic to Sedenion algebra (Appendix VI) with 16 elements $\zeta_1, \zeta_2, \dots, \zeta_{16}$, X_{rs} takes the alternative form

$$X_{rs} = \sum_{k=1}^{16} \xi_k^{(r)} \xi_k^{(s)} \quad (5.7)$$

whose analogy to Dirac's exchange operator is to be stressed.

We introduce further the following operators,

$$Z_{rs,1} = \sum_{i,k=1,2} \ell_{ik}^{(r)} \ell_{ki}^{(s)} + \sum_{i,k=3,4} \ell_{ik}^{(r)} \ell_{ki}^{(s)}, \quad (5.8a)$$

$$Z_{rs,2} = \sum_{i,k=1,3} \ell_{ik}^{(r)} \ell_{ki}^{(s)} + \sum_{i,k=2,4} \ell_{ik}^{(r)} \ell_{ki}^{(s)}, \quad (5.8b)$$

$$Z_{rs,3} = \sum_{i,k=1,4} \ell_{ik}^{(r)} \ell_{ki}^{(s)} + \sum_{i,k=2,3} \ell_{ik}^{(r)} \ell_{ki}^{(s)}, \quad (5.8c)$$

which are commutative with X_{rs} . Hereby the relationship to the vector model of spin and isotopic spin is established by assuming that $G_4 = N(1)F_4N(1)$ is a direct product of two simple algebra of degree two as follows:

$$G_4 = F_2 \times F_2'. \quad (5.9)$$

We assume further the idempotents of F_2 are e_{11} (— spin) and e_{22} (+ spin) and those of F_2' are e_{11}' (proton) and e_{22}' (neutron), then Z_1 , Z_2 and Z_3 take the well-known form

$$Z_{rs,1} = \frac{1}{4} (1_r 1_s + \sigma_r \cdot \sigma_s) (1_r' 1_s' + \tau_r^{(3)} \tau_s^{(3)}): \text{Bartlett}, \quad (5.10a)$$

$$Z_{rs,2} = \frac{1}{4} (1_r' 1_s' + \tau_r \cdot \tau_s) (1_r 1_s + \sigma_r^{(3)} \sigma_s^{(3)}): \text{Heisenberg}, \quad (5.10b)$$

$$Z_{rs,3} = \frac{1}{4} (1_r 1_s + \sigma_r \cdot \sigma_s) (1_r' 1_s' + \tau_r \cdot \tau_s): \text{Majorana}, \quad (5.10c)$$

where σ and τ are three dimensional vector operators, whose components are

$$1 = e_{11} + e_{22}, \quad \sigma^{(1)} = e_{12} + e_{21}, \quad \sigma^{(2)} = -i e_{12} + i e_{21}, \quad \sigma^{(3)} = e_{11} - e_{22}, \quad (5.11a)$$

and

$$1' = e_{11}' + e_{22}', \quad \tau^{(1)} = e_{12}' + e_{21}', \quad \tau^{(2)} = -i e_{12}' + i e_{21}', \quad \tau^{(3)} = e_{11}' - e_{22}', \quad (5.11b)$$

respectively. By the definition, we get

$$X_{rs} = Z_{rs,1} + Z_{rs,2} + Z_{rs,3}. \quad (5.12)$$

We meet in the theory of nuclear force with the operator

$$Y_{rs} = C_0 + C_1 Z_{rs,1} + C_2 Z_{rs,2} + C_3 Z_{rs,3}. \quad (5.13)$$

The coefficients C_0 , C_1 , C_2 , C_3 are given by Breit and Feenberg and Kemmer from the saturation property of nuclear force.⁽⁶⁾ Y_{rs} is also commutative with X_{rs} .

§ 6. Summary

The algebraic structure of permutation operators has been studied. In particular, the properties of exchange operators concerning ordinary spin and isotopic spin are examined from our standpoint. Appendix contains some examples of

various kinds of the representations of creation and annihilation operators, and also those of symmetric algebra of order twenty-four followed by some remarks on the orthogonality relations, an infinitesimal transformation concerning the exchange operator and Sedenion algebra.

The author would like to express his gratitude to Professor K. Husimi for the stimulating lectures and valuable discussions.

Appendix I. The representations of $\varphi^*(q)$ and $\varphi(q)$

According to the theory of second quantization, the representation of creation operator $\varphi^*(q)$ (annihilation operator $\varphi(q)$) is defined in n -subspace (n -Teilraum).⁷⁾ If one writes the density operator in the form,

$$\rho(q^n, q^{n'}) = \frac{1}{n!} \varphi^*(q_1') \cdots \varphi^*(q_n') \varphi(q_n) \cdots \varphi(q_1),$$

the representation of $\varphi^*(q)$ is given by the relation,

$$\varphi^*(q) \Phi^*(q_1, \dots, q_n) = \sqrt{n+1} \Phi^*(q, q_1, \dots, q_n),$$

$$\Phi^*(q_1, \dots, q_n) = \frac{1}{\sqrt{n!}} \varphi^*(q_1) \cdots \varphi^*(q_n),$$

and the representation of $\varphi(q)$ is obtained from the commutation relation as follows:

$$\begin{aligned} & \varphi(q) \Phi^*(q_1, \dots, q_n) \\ &= \sum_{i=1}^n (\pm 1)^{i-1} \delta(q - q_i) \Phi^*(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) / \sqrt{n} + (\pm 1)^n \Phi^*(q_1, \dots, q_n) \varphi(q), \end{aligned}$$

where $+1$ for Bose statistics and -1 for Fermi statistics.

Multiplying the operator of Bose statistics

$$u = \frac{1}{n!} \left(\sum_{n=1}^{\infty} e_{n-1, n}^{(1)} \right) \times \cdots \times \left(\sum_{n=1}^{\infty} e_{n-1, n}^{(n)} \right)$$

from the right side, and taking the trace in n -subspace by the restriction operator used in Section 1 of part I,

$$\begin{aligned} U(n_1, \dots, n_n) &= e_{n_1, n_1}^{(1)} \cdots e_{n_n, n_n}^{(n)}, \\ n_1 + \cdots + n_n &= n, \end{aligned}$$

one gets the usual formulae⁸⁾ as follows

$$\text{trace}_{(n+1)} \varphi^*(q) \Phi^*(q_1, \dots, q_n) u = \sqrt{n+1} \overline{\Psi(q, q_1, \dots, q_n)},$$

$$\text{trace}_{(n-1)} \varphi(q) \Phi^*(q_1, \dots, q_n) u = \sum_{i=1}^n \delta(q - q_i) \overline{\Psi(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)} / \sqrt{n},$$

where

$$\text{trace}_{(n)} \Phi^*(q_1, \dots, q_n) u = \overline{\Psi(q_1, \dots, q_n)} = \sum_{(n_i)} \frac{1}{\sqrt{n! n_1! \cdots n_n!}} \det^{(+)} \overline{\psi_r(q_k)},$$

and

$$\text{trace}_{(n)} \Phi^*(q_1, \dots, q_n) \Phi(q) u = 0.$$

Here the creation and annihilation operators create and annihilate just one particle as their names suggests us. The alternative conventional definition is given by the relations,

$$\begin{aligned} \Phi^*(q) \Phi(q_1, \dots, q_n) &= \sum_{i=1}^n (\pm 1)^{i-1} \delta(q - q_i) \Phi(n-1) / \sqrt{n} + (\pm 1)^n \Phi(n) \Phi^*(q), \\ \Phi(q) \Phi(q_1, \dots, q_n) &= \sqrt{n+1} \Phi(q, q_1, \dots, q_n). \end{aligned}$$

Appendix II. Centra, ideals, regular representations and the reciprocal algebra

According to the interpretation of Part I, the density matrix is obtained by taking the trace of the following operator:

$$\sum_{(f)} U_f \rho(q^n, q^{n'}) U_f = \frac{1}{n!} \sum_{(f)} \sum_{\substack{P \\ \text{mod. } \pi_f}} \sum_Q (-1)^{P+Q} g_f(QP^{-1})_f \Psi_f(Q) \overline{\Psi_f(P)},$$

where f indicates an orbital configuration, P_f is a permutation operator of type π_f and g_f is the order of the subgroup π_f .

Now our problem is reduced to obtain the trace of P_f , quantities of symmetric algebra (S_f). In the following we consider an example obtaining centra and ideals of symmetric algebra belonging to an orbital configuration $f^* = (1, 1, 1, 1)$ and magnetic quantum number $M = (1, 1, 1, 1)^{(9)}$.

Hence the direct product $A_4^4 = F_4^{(1)} \times F_4^{(2)} \times F_4^{(3)} \times F_4^{(4)}$ is to be restricted to the algebra $B_4^4 = U(1, 1, 1, 1) A_4^4 U(1, 1, 1, 1)$. Any element P_M is given by,

$$P_M = \sum_R C^*(; R) C(; PR) \quad (\text{II.1})$$

$$= \sum_R C^*(; R) C(; RP), \quad (\text{II.2})$$

where P and Q mean, contrary to Section 2, permutations of the order of the particles. In the last expression the notations R and RP do not operate on $(;)$, suffices of intrinsic states, but on $(:)$, those of orbital states.

The centrum has five independent quantities corresponding to the classes of the symmetric group with four unary cycles, one binary cycle, two binary cycles, one trinary cycle and one quaternary cycle respectively.

$$Z_{(i)(j)(k)(l)} = E_M,$$

$$Z_{(ij)(k)(l)} = \sum_P P_M \quad \text{summed over the second class,}$$

$$Z_{(ij)(kl)} = \sum_P P_M \quad \text{summed over the third class,}$$

$$Z_{(ijk)(l)} = \sum_P P_M \quad \text{summed over the fourth class,}$$

$$Z_{(ijkl)} = \sum_P P_M \quad \text{summed over the fifth class.}$$

From the multiplication table of these elements, one gets readily five ideals belonging to the five multiplets, $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$, $\{1, 1, 1, 1\}$. The unity quantities of the ideals are as follows:

$$e\{4\} = \frac{1}{24} (Z_{(i)(j)(k)(l)} + Z_{(ij)(k)(l)} + Z_{(ij)(kl)} + Z_{(ijk)(l)} + Z_{(ijkl)}),$$

$$e\{3, 1\} = \frac{1}{8} (3Z_{(i)(j)(k)(l)} + Z_{(ij)(k)(l)} - Z_{(ij)(kl)} - Z_{(ijk)(l)}),$$

$$e\{2, 2\} = \frac{1}{12} (2Z_{(i)(j)(k)(l)} + 2Z_{(ij)(kl)} - Z_{(ijk)(l)}),$$

$$e\{2, 1, 1\} = \frac{1}{8} (3Z_{(i)(j)(k)(l)} - Z_{(ij)(k)(l)} - Z_{(ij)(kl)} + Z_{(ijk)(l)}),$$

$$e\{1, 1, 1, 1\} = \frac{1}{24} (Z_{(i)(j)(k)(l)} - Z_{(ij)(k)(l)} + Z_{(ijk)(l)} - Z_{(ijkl)}).$$

Taking the trace with respect to $\{3, 1\}$, one gets

$$\text{trace } Z_{(i)(j)(k)(l)} = \text{trace } E = x(E)f = 3^2,$$

$$\frac{1}{6} \text{trace } Z_{(ij)(k)(l)} = \text{trace } ((12)) = x(12)f = 3,$$

$$\frac{1}{3} \text{trace } Z_{(ij)(kl)} = \text{trace } ((12))((34)) = x\{(12)(34)\}f = -3, \quad (\text{II} \cdot 3)$$

$$\frac{1}{8} \text{trace } Z_{(ijk)(l)} = \text{trace } ((123))((4)) = x\{(123)(4)\}f = 0,$$

$$\frac{1}{6} \text{trace } Z_{(ijkl)} = \text{trace } ((1234)) = x\{(1234)\}f = -3,$$

where x means the character and f is the multiplicity of the multiplets $\{3, 1\}$, in magnetic quantum number M . In our case which gives the normal representation, f must be equal to $x(E)$. The degree of the simple ideal and propriety of ideal suffix can be reexamined by the formula of simple characteristic. For the other orbital configurations, say $f^* = (2, 2)$, the centrum is easily obtained noticing that $P \equiv PS$ if S is in the subgroup $\pi_{(2)}$. Then there are only six independent quantities of the algebra instead of twenty-four quantities and only three quantities of the centrum instead of five quantities. If some quantities in the expression of the unity quantities of the ideals are identified, as above mentioned, the three unity quantities of the ideals are obtained,

$$e_{(2,2),1} = \frac{1}{6} (Z_{(2,2),1} + Z_{(2,2),2} + Z_{(2,2),3}),$$

$$e_{(2,2),2} = \frac{1}{6} (2Z_{(2,2),1} - Z_{(2,2),2} + 2Z_{(2,2),3}), \quad (\text{II} \cdot 4)$$

$$e_{(2,2),3} = \frac{1}{2} (Z_{(2,2),1} - Z_{(2,2),3}).$$

If one wants to compare these quantities of the centrum with those of the centrum of type unity, one will see the connection between them in the following relations

$$\begin{aligned} Z_{(i)(j)(k)(l)} &\rightarrow Z'_{(i)(j)(k)(l)} = Z_{(2,2),1}, \\ Z_{(ij)(k)(l)} &\rightarrow Z'_{(ij)(k)(l)} = Z_{(2,2),2} + 2Z_{(2,2),1}, \\ Z_{(ij)(kl)} &\rightarrow Z'_{(ij)(kl)} = Z_{(2,2),1} + Z_{(2,2),3}, \\ Z_{(ijk)(l)} &\rightarrow Z'_{(ijk)(l)} = 2Z_{(2,2),2}, \\ Z_{(ijkl)} &\rightarrow Z'_{(ijkl)} = Z_{(2,2),2} + Z_{(2,2),3}. \end{aligned} \quad (\text{II} \cdot 5)$$

From (II.3), (II.4) and (II.5) one finds

$$\text{trace}_{(2,2),3} ((13))((24))_{(2,2)} = \frac{2}{4} \{ \text{trace}_{\{3,1\}}((12))((34)) + \text{trace}_{\{3,1\}}((1234)) \} = -3.$$

To obtain the ideals, it will be better to start with the three quantities of the centrum directly by means of the following multiplication table.

$Z_{(2,2),2}$	$4Z_{(2,2),1} + 2Z_{(2,2),2} + 4Z_{(2,2),3}$	
$Z_{(2,2),3}$	$Z_{(2,2),2}$	$Z_{(2,2),1}$
	$Z_{(2,2),2}$	$Z_{(2,2),3}$

Table 1

The regular representation of symmetric algebra S is obtained by regarding the linear set of the quantities of symmetric algebra, on which the representation is considered, as the linear set over the algebra V , a subfield of (S) , generated by the idempotents of the centrum. Let three orthogonal linear sets, which are equivalent to the three ideals of (S) , be

$$\begin{aligned} L_1 &: S_1^{e_1}, \\ L_1 &: S_1^{e_2}, S_2^{e_2}, \quad V = (e_1, e_2, e_3), \\ L_3 &: S_1^{e_3}, S_2^{e_3}, S_3^{e_3}. \end{aligned}$$

Then one gets one dimensional, two dimensional and three dimensional irreducible representations as linear transformations on L_1 , L_2 and L_3 respectively.

Next we add some remarks on the so-called reciprocal algebra. One can regard (S) as a linear transformation on a linear set of order six over a field K , in other words, as a subfield of B . Any one of the elements of S corresponds to a transformation

$$R \rightarrow R^Pr = RP.$$

Another type of the transformation is as follows:

$$R \rightarrow R^{P\iota} = PR.$$

These transformations are summed up to a subfield (S^{-1}) of \mathbf{B} , which is reciprocal to (S) and has the same centrum as (S) .¹⁰⁾ Explicitly

$$(S) : \sum_R C^*(:R) C(:RP) \text{ etc.},$$

$$(S^{-1}) : \sum_R C^*(:R) C(:PR) \text{ etc.}$$

Appendix III. The orthogonality of characters

Any element of symmetric algebra takes the following form in the Λ -th simple ideal.

$$P(\Lambda) = \sum_{i,k}^{f_\Lambda} P_{ik}(\Lambda) e_{ik}(\Lambda), \quad e_{ij}(\Lambda) e_{kl}(\Lambda') = e_{ik}(\Lambda) \delta_{jl} \delta_{\Lambda\Lambda'}.$$

Characters are defined by

$$x_\Lambda(P) = \text{trace } P(\Lambda) = \sum_{i=1}^{f_\Lambda} P_{ii}(\Lambda),$$

where f_Λ is the multiplicity of the multiplets $\{\Lambda\}$ in the algebra \mathbf{B} . The completeness of characters can be secured, if set

$$\sum_{\Lambda} n(a) x_\Lambda(a) x_{\Lambda'}(a) = h' \delta_{\Lambda\Lambda'},$$

$$\sum_{\Lambda} x_\Lambda(a) x_\Lambda(a') = \frac{h'}{n(a')} \delta_{aa'},$$

where h' means the order of the algebra (S) (the number of the quantities) and $n(a)$ is the number of quantities contained in the a -th quantity of the centrum. Λ runs over as many allowable multiplets quantum numbers as number of quantities of the centrum. These follow from the fact that for any quantity of the Λ -th simple ideal, the operator

$$\sum_P P^{-1}(\Lambda) e_{ik}(\Lambda) P(\Lambda)$$

is a scalar multiple of the unity quantity $e(\Lambda)$.

Appendix IV

One will readily find the following relationship for the traces with respect to \mathbf{B}^n by the definition of \mathbf{R} and $[L_2]^n$

$$\text{trace } \mathbf{R}[L_2]^n = \sum_i x_s(\mathbf{R}) x_s([L_2]^n) = \sigma_1^{\alpha_1} \dots \sigma_n^{\alpha_n},$$

$$\sigma_k = \sum_{i=1}^2 L_{ii}^{(2)k} = \sum_{i=1}^2 \epsilon_i^k, \quad \sum_{k=1}^n k a_k = n,$$

where ϵ_i is an eigen value of $L_{ii}^{(2)}$ and R belongs to a class denoted by the partition (a_1, \dots, a_n) of n .¹¹⁾

From the first orthogonality relation of the traces, $x_s([L_2]^n)$ is solved by

$$x_s([L_2]^n) = \sum_{(a)} \frac{x_s(a)}{a_1! \dots a_n!} \left(\frac{\sigma_1}{1}\right)^{a_1} \dots \left(\frac{\sigma_n}{n}\right)^{a_n}.$$

This is the so-called simple characteristic taking the same expression as partition function $f_s^{(n),1}$.

Appendix V. An infinitesimal transformation

In the theory of the Ising model one meets with the operator

$$D = \sum_{n=1}^N X_{nn+1} = \frac{1}{2} \sum_{n=1}^N (1_n 1_{n+1} + \sigma_n \cdot \sigma_{n+1}),$$

a sum of Dirac's exchange operators. We separate out the operator $1_n 1_{n+1} + \sigma_n^{(3)} \sigma_{n+1}^{(3)}$ from $\sigma_n^{(1)} \sigma_{n+1}^{(1)} + \sigma_n^{(2)} \sigma_{n+1}^{(2)}$. The former might be treated by means of the canonical transformation. Here we introduce a two dimensional vector operator and further anticommutative operators analogous to the spinor analysis as follows:

$$\mathbf{q}_n = (\sigma_n^{(1)}, \sigma_n^{(2)}),$$

$$\vec{\mathcal{E}}_n = \sigma_1^{(3)} \dots \sigma_{n-1}^{(3)} \mathbf{q}_n; \quad \vec{\mathcal{E}}_n \vec{\mathcal{E}}_m + \vec{\mathcal{E}}_m \vec{\mathcal{E}}_n = 2\delta_{nm} \mathbf{1}_m.$$

Using the normal coordinates the operator D' takes following form

$$D' = \sum_{n=1}^N \mathbf{q}_n \cdot \mathbf{q}_{n+1} = i \sum_{k=-N/2}^{N/2} \omega_k \mathbf{Q}_k^{(1)} \cdot \mathbf{Q}_{-k}^{(2)} = \sum_{k=-N/2}^{N/2} \omega_k R_k,$$

where

$$\omega_k = \cos \frac{2\pi k}{N},$$

$$\mathbf{Q}_k = \frac{1}{\sqrt{2N}} \sum_{n=1}^N \vec{\mathcal{E}}_n \exp\left(\frac{2\pi k}{N} ni\right),$$

and

$$R_k \tilde{R}_k R_k = R_k, \quad R_k R_k = \tilde{R}_k \tilde{R}_k = 0.$$

Let the real and imaginary parts of R_k be X_k and Y_k respectively, then one finds.¹²⁾

$$X_k^2 = X_k, \quad Y_k^2 = Y_k.$$

Further, $\exp(-\theta X_k)$ means a rotation of the imaginary angle $\frac{2\theta}{i}$ about X -axis.

Appendix VI. Total matrix algebra of degree four

The isomorphism of a total matrix algebra of degree four to the algebra generated by the four anticommutative quantities is as follows.

$$\beta_i \beta_k + \beta_k \beta_i = 2\delta_{ik} \quad (i, k=1, 2, 3, 4),$$

$$\begin{aligned} e_{11} &= (1 + \beta_4)(1 - i\beta_1\beta_2), & e_{22} &= (1 + \beta_4)(1 + i\beta_1\beta_2), \\ e_{13} &= (1 - \beta_4)(1 - i\beta_1\beta_2), & e_{44} &= (1 - \beta_4)(1 + i\beta_1\beta_2), \\ e_{12} &= -(1 + \beta_4)(\beta_1 + i\beta_2)\beta_3, & e_{21} &= (1 + \beta_4)(\beta_1 - i\beta_2)\beta_3, \\ e_{13} &= -i(1 + \beta_4)(1 - i\beta_1\beta_2)\beta_3, & e_{31} &= i(1 - \beta_4)(1 - i\beta_1\beta_2)\beta_3, \\ e_{14} &= -i(1 + \beta_4)(\beta_1 + i\beta_2), & e_{41} &= i(1 - \beta_4)(\beta_1 - i\beta_2), \\ e_{23} &= -i(1 - \beta_4)(\beta_1 - i\beta_2), & e_{32} &= i(1 + \beta_4)(\beta_1 + i\beta_2), \\ e_{24} &= i(1 + \beta_4)(1 + i\beta_1\beta_2)\beta_3, & e_{42} &= -i(1 - \beta_4)(1 + i\beta_1\beta_2)\beta_3, \\ e_{34} &= -(1 - \beta_4)(\beta_1 + i\beta_2)\beta_3, & e_{43} &= (1 - \beta_4)(\beta_1 - i\beta_2)\beta_3. \end{aligned}$$

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On the Theory of Cooperative Phenomena*

Tomoyasu TANAKA, Hiiooshi KATSUMORI and Soichiro TOSHIMA

Department of Physics, Faculty of Science, Kyushu University

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We obtain the low temperature expansions of partition functions for several crystal types by direct counting of complexions and then discuss the typical cooperative phenomena.

(1) Spontaneous magnetizations of the three dimensional Ising ferromagnets (in case of the simple cubic, the body centered cubic and the face centered cubic) are given.

(2) Using the grand partition function for the face centered cubic lattice, the properties of the regular solution are discussed. We get there a condition for the appearance of an upper consolute temperature or a lower consolute temperature, taking account of the effects on free volume of each molecule due to the species of the nearest neighbor molecules.

(3) Considering the regular solution as a mixture of holes and molecules, we are led to the properties of liquid.

(4) Using the grand partition function for the triangular lattice, we discuss the adsorption phenomenon as the localized monolayer.

§ 1. Introduction

In recent years the ingenious developments in the mathematical theory of cooperative phenomena have been made by several authors.¹⁾ But unfortunately no rigorous theory has hitherto succeeded in solving the three dimensional model corresponding to the real crystal. And it seems difficult to get a rigorous theory applicable to all types of the real crystals, such as the body centered cubic and the face centered cubic.

For the present stage, we will try to expand the partition functions for several crystal types by direct counting of the complexions. This method of counting contains no newer part at all, and indeed it was used by Bloch²⁾ to obtain the partition functions of ferromagnetic lattices. Lately we heard G. S. Rushbrooke³⁾ had independently treated of the theory of regular solution along the same line as we had. His counting is, however, only for the simple cubic lattice, so it may be permitted for us to publish the results by our elementary countings.

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§ 2. Method of direct counting of complexions

Since the theory of the Ising model of a ferromagnet is analogously extended to the theories of the other cooperative phenomena, for example, the theory of regular solution, the hole theory of liquid and the theory of localized monolayers of the gas adsorption, we shall begin by constructing the partition functions of the Ising models for the several crystal types at low temperatures.

We shall deal with an assembly of N spins, each of magnetic moment m . Each spin is assumed capable of interacting only with its nearest neighbors and of having either the same or the opposite direction along an external magnetic field H . Then as an interaction energy between two neighboring spins we shall denote $-J$ when they are parallel, and J when antiparallel.

Writing

$$\alpha = e^{-C} = e^{-mH/kT} \quad \text{and} \quad \beta = e^{-K} = e^{-J/2kT}, \quad (1)$$

these quantities are small at low temperatures, so we may expand the partition function as a power series in them. By direct counting of complexions, we may obtain a partition function for N spins

$$\begin{aligned} f^N &= \sum_{(N_{\uparrow} + N_{\downarrow} = N)} g(N_{\uparrow}, N_{\downarrow}, N_{\uparrow\downarrow}) (\alpha^{-1})^{N_{\uparrow}} (\alpha)^{N_{\downarrow}} (\beta^{-1})^{N_{\uparrow\downarrow}} \\ &= (\alpha^{-1})^N (\beta^{-1})^{\frac{zN}{2}} \sum_{N_{\uparrow}, N_{\uparrow\downarrow}} g(N; N_{\uparrow}, N_{\uparrow\downarrow}) \alpha^{2N_{\uparrow}} \beta^{2N_{\uparrow\downarrow}} \end{aligned} \quad (2)$$

where N_{\uparrow} = number of spins parallel to H , N_{\downarrow} = number of spins antiparallel to H and $N_{\uparrow\downarrow}$ = number of pairs of neighbors, one of which is parallel and the other is antiparallel to H . For example in case of the face centered cubic lattice, we have

$$\begin{aligned} f^N &= (\alpha^{-1}\beta^{-6})^N \left\{ 1 + N\alpha^2\beta^{24} + \alpha^4 \left[6N\beta^{44} + \left(\frac{1}{2} N(N-1) - 6N \right) \beta^{48} \right] \right. \\ &\quad + \alpha^6 \left[8N\beta^{60} + 42N\beta^{64} + (6N^2 - 120N) \beta^{68} + \frac{N}{6} (N^2 - 39N + 422) \beta^{72} \right] \\ &\quad \left. + \alpha^8 [2N\beta^{72} + 24N\beta^{76} + 123N\beta^{80} + (8N^2 + 126N) \beta^{84} + \dots] + \dots \right\} \end{aligned}$$

which leads to an expression for f by taking N -th root.

Such countings for several crystal types give the following results:

the 2-dimensional triangular lattice—*

$$f = \alpha^{-1}\beta^{-3} [1 + \alpha^2\beta^{12} + \alpha^4(3\beta^{20} - 3\beta^{24}) + \alpha^6(2\beta^{24} + 9\beta^{28} - 27\beta^{32} + 16\beta^{36})$$

* There is a rigorous theory in this case¹⁾ as well as in case of the square lattice, but we shall add it here, because it may be applied to adsorption phenomenon on localized monolayers which will be discussed in the next section, and it may be compared with another types of crystal.

$$\begin{aligned}
 & + \alpha^8 (3\beta^{28} + 12\beta^{32} + 7\beta^{36} - 165\beta^{40} + 249\beta^{44} - 106\beta^{48}) \\
 & + \alpha^{10} (6\beta^{32} + 21\beta^{36} + 21\beta^{40} - 171\beta^{44} - 654\beta^{48} + \dots) \\
 & + \alpha^{12} (14\beta^{36} + 42\beta^{40} + 15\beta^{44} - 258\beta^{48} + \dots) \\
 & + \alpha^{14} (\beta^{36} + 30\beta^{40} + 105\beta^{44} + 38\beta^{48} \dots) \\
 & + \alpha^{16} (6\beta^{40} + 60\beta^{44} + 216\beta^{48} + \dots) + \alpha^{18} (27\beta^{44} + 78\beta^{48} + \dots) \\
 & + \alpha^{20} (3\beta^{44} + 86\beta^{48} + \dots) + \alpha^{22} (24\beta^{48} + \dots) \\
 & + \alpha^{24} (2\beta^{48} + \dots) + \dots], \tag{3}
 \end{aligned}$$

the simple cubic lattice—*

$$\begin{aligned}
 f = & \alpha^{-1} \beta^{-3} [1 + \alpha^2 \beta^{12} + \alpha^4 (3\beta^{20} - 3\beta^{24}) + \alpha^6 (15\beta^{28} - 33\beta^{32} + 18\beta^{36}) \\
 & + \alpha^8 (3\beta^{32} + 83\beta^{36} - 309\beta^{40} + 360\beta^{44} - 137\beta^{48}) \\
 & + \alpha^{10} (48\beta^{40} + 429\beta^{44} - 2676\beta^{48} + 5055\beta^{52} - 4041\beta^{56} + \dots) \\
 & + \alpha^{12} (18\beta^{44} + 496\beta^{48} + 1632\beta^{52} - 21366\beta^{56} + \dots) \\
 & + \alpha^{14} (8\beta^{48} + 378\beta^{52} + 3906\beta^{56} + \dots) \\
 & + \alpha^{16} (\beta^{48} + 306\beta^{56} + \dots) \\
 & + \alpha^{18} (24\beta^{56} + \dots) + \dots], \tag{4}
 \end{aligned}$$

the body centered cubic lattice—

$$\begin{aligned}
 f = & \alpha^{-1} \beta^{-4} [1 + \alpha^2 \beta^{16} + \alpha^4 (4\beta^{28} - 4\beta^{32}) + \alpha^6 (28\beta^{40} - 60\beta^{44} + 32\beta^{48}) \\
 & + \alpha^8 (12\beta^{48} + 148\beta^{52} - 762\beta^{56} + 868\beta^{60} - 266\beta^{64}) \\
 & + \alpha^{10} (12\beta^{56} + 216\beta^{60} + 1274\beta^{64} + \dots) \\
 & + \alpha^{12} (3\beta^{64} + \dots) + \dots], \tag{5}
 \end{aligned}$$

the face centered cubic lattice—

$$\begin{aligned}
 f = & \alpha^{-1} \beta^{-6} [1 + \alpha^2 \beta^{24} + \alpha^4 (6\beta^{44} - 6\beta^{48}) \\
 & + \alpha^6 (8\beta^{60} + 42\beta^{64} - 114\beta^{68} + 64\beta^{72}) \\
 & + \alpha^8 (2\beta^{72} + 24\beta^{76} + 123\beta^{80} + 134\beta^{84} + \dots) + \dots]. \tag{6}
 \end{aligned}$$

§ 3. Application to typical cooperative phenomena

(i) Ferromagnetism

If we use the partition function obtained in the preceding section, the magnetic moment of the Ising ferromagnet consisting of N spins is readily given by

* G. S. Rushbrooke and A. J. Wakefield⁽³⁾ have calculated to β^{56} and our independent calculation to β^{52} exactly coincides.

$$M = Nm \frac{\partial \log f}{\partial C} = -M_{\infty} a \frac{\partial \log f}{\partial a}. \quad (M_{\infty} = Nm) \quad (7)$$

Then inserting the expression of f into (7) and making a tend to 1 (namely $H \rightarrow 0$), we have the "spontaneous magnetization":

the simple cubic lattice—

$$\begin{aligned} \frac{M}{M_{\infty}} = & 1 - 2\beta^{12} - 12\beta^{20} + 14\beta^{24} - 90\beta^{28} + 192\beta^{32} - 792\beta^{36} + 2148\beta^{40} - 7716\beta^{44} \\ & + 23262\beta^{48} - 79512\beta^{52} + 252054\beta^{56} - \dots, \end{aligned} \quad (8)$$

the body centered cubic lattice—

$$\begin{aligned} \frac{M}{M_{\infty}} = & 1 - 2\beta^{16} - 16\beta^{28} + 18\beta^{32} - 168\beta^{40} + 384\beta^{44} - 314\beta^{48} - 1184\beta^{52} + 6264\beta^{56} \\ & - 9744\beta^{60} - 10174\beta^{64} + \dots, \end{aligned} \quad (9)$$

the face centered cubic lattice—

$$\begin{aligned} \frac{M}{M_{\infty}} = & 1 - 2\beta^{24} - 24\beta^{44} + 26\beta^{48} - 48\beta^{60} - 252\beta^{64} + 720\beta^{68} - 438\beta^{72} \\ & - 192\beta^{76} - 984\beta^{80} - 1008\beta^{84} - \dots. \end{aligned} \quad (10)$$

From (8), (9) and (10), we may plot the curves of spontaneous magnetization versus temperatures. Probably these curves are invalid near the Curie temperatures, since the low temperature partition functions will diverge near there. In spite of these circumstances it will be possible to expect that we may roughly determine the Curie point for each crystal type, by careful numerical examination of the above power series. These discussions will be done in detail and besides with an alternative method of combining with the high temperature partition functions by some of us in the separate paper.

(ii) *Regular Solution*

The partition function for the whole assembly consisting of N_A molecules A and N_B molecules B , treated as a single system, can be written in the form

$$f^N = \sum_{N_{AB}} g(N_A, N_B, N_{AB}) e^{-N_{AB} w_{AB}/kT} \{ \phi_A v_A e^{(\lambda_A + kT)/kT} \}^{N_A} \{ \phi_B v_B e^{(\lambda_B + kT)/kT} \}^{N_B} \quad (11)$$

where N_{AB} is the number of AB pairs of neighbors and $g(N_A, N_B, N_{AB})$ is the number of distinguishable arrangements of the N_A molecules A and N_B molecules B with specified value of N_{AB} and the definitions of w_{AB} , λ_A , ϕ_A , v_A etc. are usual ones.* Corresponding to (11) for the partition function for given N_A , N_B , the grand partition function for given λ_A , λ_B , ($\lambda = e^{\mu/kT}$ and μ is the corresponding chemical potential) is

* See, Fowler and Guggenheim; *Statistical Thermodynamics*, pp. 351-352 and p. 362.

$$\begin{aligned} \Xi = \sum_{N_A, N_B} \sum_{N_{AB}} g(N_A, N_B, N_{AB}) e^{-N_{AB} v_{AB}/kT} (\phi_A v_A e^{[x_A + kT]/kT})^{N_A} \\ \times (\phi_B v_B e^{[x_B + kT]/kT})^{N_B} \lambda_A^{N_A} \lambda_B^{N_B}. \end{aligned} \quad (12)$$

Writing for brevity

$$\xi_A = \lambda_A \phi_A v_A e^{[x_A + kT]/kT}, \quad \xi_B = \lambda_B \phi_B v_B e^{[x_B + kT]/kT} \quad \text{and} \quad \eta_{AB} = e^{-v_{AB}/kT}, \quad (13)$$

(12) becomes

$$\Xi = \sum_{N_A, N_B} \sum_{N_{AB}} g(N_A, N_B, N_{AB}) \eta_{AB}^{N_{AB}} \xi_A^{N_A} \xi_B^{N_B}. \quad (14)$$

Since the term in Ξ is proportional to the probability of the configuration of specified N_A , N_B and N_{AB} , the average values of N_A and N_B are given by

$$\bar{N}_A = \xi_A \frac{\partial \log \Xi}{\partial \xi_A} \quad \text{and} \quad \bar{N}_B = \xi_B \frac{\partial \log \Xi}{\partial \xi_B}. \quad (15)$$

Therefore, if we write for small $\xi = \xi_A/\xi_B$

$$\Xi = \sum_N \xi_B^N \sum_{N_A, N_{AB}} g(N; N_A, N_{AB}) \eta_{AB}^{N_{AB}} (\xi_A/\xi_B)^{N_A} = \sum_N \xi_B^N \Lambda(\xi)^N = \frac{1}{1 - \xi_B \Lambda(\xi)} \quad (16)$$

and for small $\xi^{-1} = \xi_B/\xi_A$

$$\Xi = \sum_N \xi_A^N \sum_{N_B, N_{AB}} g(N; N_B, N_{AB}) \eta_{AB}^{N_{AB}} (\xi_B/\xi_A)^{N_B} = \sum_N \xi_A^N \Lambda(\xi^{-1})^N = \frac{1}{1 - \xi_A \Lambda(\xi^{-1})}, \quad (16')$$

the mole-fractions are given by

$$x_A = \frac{\bar{N}}{\bar{N}_A + \bar{N}_B} = \xi \frac{\partial \log \Lambda(\xi)}{\partial \xi} \quad \text{and} \quad x_B = 1 - x_A \quad (\text{for small } \xi) \quad (17)$$

or

$$x_B = \frac{\bar{N}}{\bar{N}_A + \bar{N}_B} = \xi^{-1} \frac{\partial \log \Lambda(\xi^{-1})}{\partial (\xi^{-1})} \quad \text{and} \quad x_A = 1 - x_B \quad (\text{for small } \xi^{-1}). \quad (17')$$

Now we will deal with the face centered cubic lattice ($z=12$). Noting the relation between the present notations and those of ferromagnetism

$$a\beta^{z/2} = a\beta^6 \rightarrow \xi_A, \quad a^{-1}\beta^{z/2} = a^{-1}\beta^6 \rightarrow \xi_B \quad \text{and} \quad \beta^2 \rightarrow \eta_{AB} (= \eta) \quad (18)$$

and comparing (16) and (16') with (2) and (6), we are led to

$$\begin{aligned} \Lambda(\xi) = 1 + \xi \eta^{12} + \xi^2 (6\eta^{22} - 6\eta^{24}) + \xi^3 (8\eta^{30} + 42\eta^{32} - 114\eta^{34} + 64\eta^{36}) \\ + \xi^4 (2\eta^{36} + 24\eta^{38} + 123\eta^{40} + 134\eta^{42} + \dots) + \dots \end{aligned} \quad (19)$$

and

$$A(\xi^{-1}) = 1 + \xi^{-1}\eta^{12} + \xi^{-2}(6\eta^{22} - 6\eta^{24}) + \xi^{-3}(8\eta^{30} + 42\eta^{32} - 114\eta^{34} + 64\eta^{36}) \\ + \xi^{-4}(2\eta^{36} + 24\eta^{38} + 123\eta^{40} + 134\eta^{42} + \dots) + \dots \quad (19')$$

Then, substituting this into (17) and (17'), we obtain the mole-fraction x_A in the form

$$x_A(\xi) = \xi \frac{\partial \log A(\xi)}{\partial \xi} \\ = \frac{1}{A(\xi)} \{ \xi\eta^{12} + 2\xi^2(6\eta^{22} - 6\eta^{24}) + 3\xi^3(8\eta^{30} + 42\eta^{32} - 114\eta^{34} + 64\eta^{36}) \\ + 4\xi^4(2\eta^{36} + 24\eta^{38} + 123\eta^{40} + 134\eta^{42} + \dots) + \dots \} \\ \text{for } \xi < 1 \text{ (small concentration of } A \text{ molecules)} \quad (20)$$

and

$$x_A(\xi^{-1}) = 1 - x_B(\xi^{-1}) = 1 - \xi^{-1} \frac{\partial \log A(\xi^{-1})}{\partial (\xi^{-1})} \\ = 1 - \frac{1}{A(\xi^{-1})} \{ \xi^{-1}\eta^{12} + 2\xi^{-2}(6\eta^{22} - 6\eta^{24}) + 3\xi^{-3}(8\eta^{30} + 42\eta^{32} - 114\eta^{34} + 64\eta^{36}) \\ + 4\xi^{-4}(2\eta^{36} + 24\eta^{38} + 123\eta^{40} + 134\eta^{42} + \dots) + \dots \} \\ \text{for } \xi > 1 \text{ (small concentration of } B \text{ molecules).} \quad (20')$$

In order to examine the behavior in the vicinity of $\xi=1$, we must take the sum

$$\xi_A^N A(\xi^{-1}) + \xi_B^N A(\xi) \quad (21)$$

instead of an individual term. Whereby the mole-fraction x_A becomes

$$\theta(\xi) = \frac{x_A(\xi) + \xi^N (A(\xi^{-1})/A(\xi))^N (1 - x_B(\xi^{-1}))}{1 + \xi^N (A(\xi^{-1})/A(\xi))^N} \quad (22)$$

Since for a thermodynamical assembly N is very large, $\xi^N=0$ when $\xi < 1$ and $\xi^{-N}=0$ when $\xi > 1$. Therefore we obtain

$$\lim_{\xi \rightarrow 1-0} \theta(\xi) = \lim_{\xi \rightarrow 1-0} x_A(\xi) = \theta(1-0) < \frac{1}{2} \quad (23)$$

and

$$\lim_{\xi \rightarrow 1+0} \theta(\xi) = \lim_{\xi \rightarrow 1+0} (1 - x_B(\xi^{-1})) = \theta(1+0) > \frac{1}{2}, \quad (24)$$

where the inequalities are given by numerical calculation of (20) and (20') for sufficiently small η . On the other hand, if we use the relation $[x_A(\xi)]_{\xi=1} = [x_B(\xi^{-1})]_{\xi=1}$ which is obvious from (20) and (20'), (22) leads to

$$[\theta(\xi)]_{\xi=1} = \theta(1) = \frac{1}{2}. \quad (25)$$

This fact means mathematically that $\xi=1$ is the singular point of the expression (22) and three values $\theta(1-0)$, $\theta(1)$ and $\theta(1+0)$ correspond at this point, in other words, $\theta(\xi)$ has the triple roots at $\xi=1$. The physical meaning of this fact is that at low temperatures we have phase separation into two phases, of concentrations $\theta(1-0)$ and $\theta(1+0)$ and that the phase-boundary is given by

$$[\theta(\xi)]_{\xi=1\pm 0} = (\text{power series in } \eta). \quad (26)$$

Remembering N is large and using (15), (16) and (16'), we get the results³⁾

$$\xi_A = \xi / \Lambda(\xi) \quad \text{and} \quad \xi_B = 1 / \Lambda(\xi) \quad \text{for small } \xi \quad (27)$$

or

$$\xi_A = 1 / \Lambda(\xi^{-1}) \quad \text{and} \quad \xi_B = \xi^{-1} / \Lambda(\xi^{-1}) \quad \text{for small } \xi^{-1}.$$

Assuming a perfect gas phase in equilibrium with the solution, that is, assuming the relations

$$p_A / p_A^0 = \xi_A \quad \text{and} \quad p_B / p_B^0 = \xi_B \quad (28)$$

where p_A and p_B denote the partial vapor pressures and suffix o means that of pure vapor, thus we can show the partial vapor pressure curve for a small value of η (< 1) in Fig. 1. Moreover, phase boundary curve shown in Fig. 2 is obtained by (26). Here T_C corresponds to the divergence point of $[\Lambda(\xi)]_{\xi=1} = [\Lambda(\xi^{-1})]_{\xi=1}$. Comparison between (26) and (10) shows that this phase boundary curve of the regular solution is essentially equivalent to the spontaneous magnetization curve of the ferromagnet.

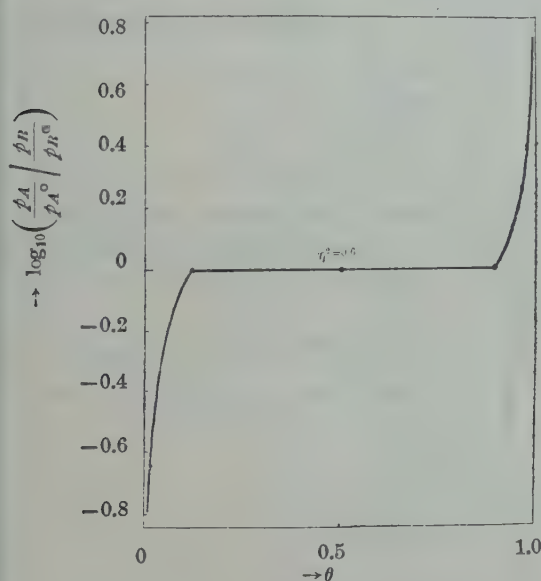


Fig 1

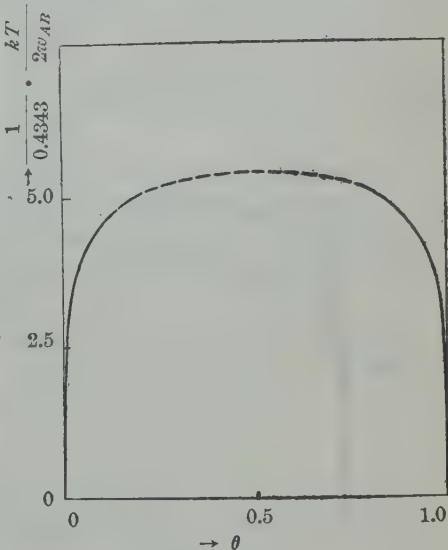


Fig. 2

A remark on a lower consolute temperature should be finally added. If we take into account the effects on the free volume of each molecule due to the species of nearest neighbors, and use the approximation of the geometric mean

of free volumes proposed by S. Ono,⁴⁾ e.g. $v_A^z \rightarrow v_{AA}^z v_{AB}^{z-x}$ when x of z nearest neighbors around a molecule A are molecules A and $z-x$ are molecules B , where v_{AB} is the free volume of a molecule A with z nearest neighbor molecules B and v_{BA} , v_{AA} and v_{BB} are given by the similar definitions,* we get the same grand partition function as before except that we must substitute

$$\eta_{AB}^z = e^{-z v_{AB}/kT} \rightarrow \zeta_{AB} = \left(\frac{v_{AB} v_{BA}}{v_{AA} v_{BB}} \right) \eta_{AB}^z = e^{-\left(z v_{AB}/kT + \log \frac{v_{AA} v_{BB}}{v_{AB} v_{BA}} \right)}. \quad (29)$$

Therefore the necessary and sufficient condition for occurrence of phase separation is

$$z v_{AB}'/kT + \log \left(\frac{v_{AA} v_{BB}}{v_{AB} v_{BA}} \right) > z v_{AB}/kT_C. \quad (30)$$

In other words,

$$\text{for } w_{AB}' > 0, \quad z v_{AB}/kT_C > \log \frac{v_{AA} v_{BB}}{v_{AB} v_{BA}} \quad (31)$$

and

$$\text{for } w_{AB}' < 0, \quad z v_{AB}/kT_C > \log \frac{v_{AA} v_{BB}}{v_{AB} v_{BA}}.$$

The second condition of (31) suggests the appearance of a lower consolute temperature. The phase boundary curves for several positive and negative values of w_{AB}' are shown in Fig. 3 and Fig. 4 respectively.

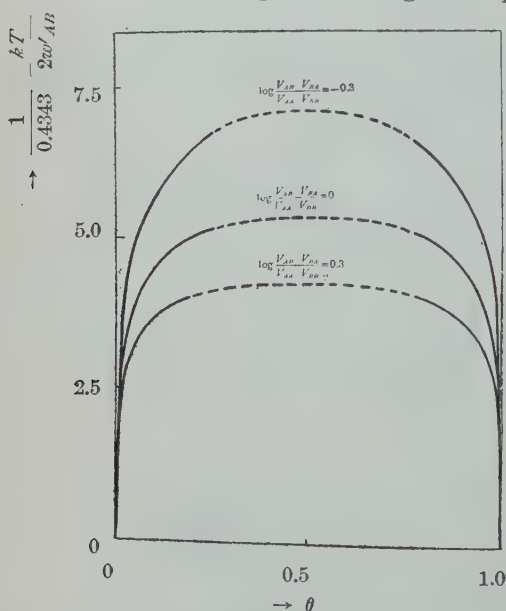


Fig. 3

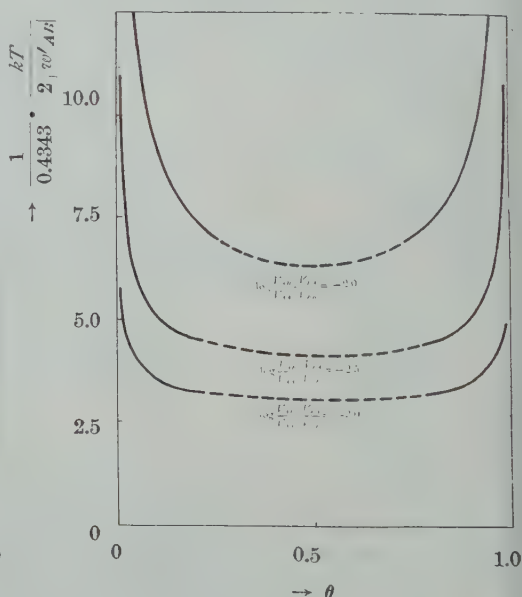


Fig. 4

* In this case $v_{AB} = v_{BA}$ is clear, but on purpose we distinguish v_{AB} from v_{BA} , since in the hole theory of liquid which will be discussed later we must distinguish between them.

(iii) *Hole Theory of Liquid*

If we take a mixture of holes and molecules for a model of liquid,⁴⁾ we can follow the steps of the above argument on the regular solution. In fact, substituting

$$\left. \begin{array}{l} v_{AA} \rightarrow \omega \\ v_{AB} \rightarrow \tau \\ \xi_A \rightarrow \xi \end{array} \right\}, \left. \begin{array}{l} v_{BB} \rightarrow 1 \\ v_{BA} \rightarrow 1 \\ \xi_B \rightarrow 1 \end{array} \right\} \quad \text{and} \quad \eta_{AB}^e \rightarrow \eta^{e/2}(\tau/\omega) = e^{-\{zw/2kT + \log(\omega/\tau)\}} \quad (32)$$

we can obtain the grand partition functions for small ξ (gas phase) and for large ξ (liquid phase). It is obvious that $\xi=1$ corresponds to the condensation point. And if we write the densities of gas phase and liquid phase by ρ_G and ρ_L respectively, the relation between $x_L = \rho_L/(\rho_G + \rho_L)$ and kT/w can be obtained. This relation is essentially equivalent to the spontaneous magnetization. In addition we may get the equation of state as the power series in ξ (gas phase) and in ξ^{-1} (liquid phase).

(iv) *Gas Adsorption Phenomenon as Localized Monolayers*

In order to treat of the adsorption phenomenon of gas molecules on the solid surface, we shall consider the lattice points on the solid surface as the triangular net. Thus from (3) we may obtain the grand partition functions for small ξ (small number of adsorbed molecules) and for large ξ (large number of adsorbed molecules) substituting

$$\left. \begin{array}{l} a\beta^{z/2} = a\beta^3 \rightarrow \xi = e^{(\mu + W + w)/kT} \\ a^{-1}\beta^{z/2} = a^{-1}\beta^3 \rightarrow 1 \end{array} \right\} \quad \text{and} \quad \beta^4 \rightarrow \eta = e^{-w/kT}, \quad (33)$$

where μ is the chemical potential of gas molecules and w is a potential energy between two neighboring molecules and W is a potential energy between a gas molecule and solid surface. These grand partition functions are very similar to those of the hole theory of liquid without considering free volumes;

$$\begin{aligned} A(\xi) = & 1 + \xi\eta^3 + \xi^2(3\eta^5 - 3\eta^6) + \xi^3(2\eta^6 + 9\eta^7 - 27\eta^8 + 16\eta^9) \\ & + \xi^4(3\eta^7 + 12\eta^8 + 7\eta^9 - 165\eta^{10} + 249\eta^{11} - 106^{12}) \\ & + \xi^5(6\eta^8 + 21\eta^9 + 21\eta^{10} - 171\eta^{11} - 654\eta^{12} + \dots) \\ & + \xi^6(14\eta^9 + 42\eta^{10} + 15\eta^{11} - 258\eta^{12} + \dots) \\ & + \xi^7(\eta^9 + 30\eta^{10} + 105\eta^{11} + 38\eta^{12} + \dots) \\ & + \xi^8(6\eta^{10} + 60\eta^{11} + 216\eta^{12} + \dots) + \xi^9(27\eta^{11} + 78\eta^{12} + \dots) \\ & + \xi^{10}(3\eta^{11} + 86\eta^{12} + \dots) + \xi^{11}(24\eta^{12} + \dots) \\ & + \xi^{12}(2\eta^{12} + \dots) + \dots, \end{aligned} \quad (34)$$

$$A(\xi^{-1}) = (\text{the same expression as the above one but } \xi \rightarrow \xi^{-1}). \quad (34')$$

By the similar discussion as the theory of regular solution, we have the relation between the fraction of the sites that are occupied and the vapor pressures of regular monolayers. Thus we may plot the curves of the adsorption isotherm for several values of η (Fig. 5).

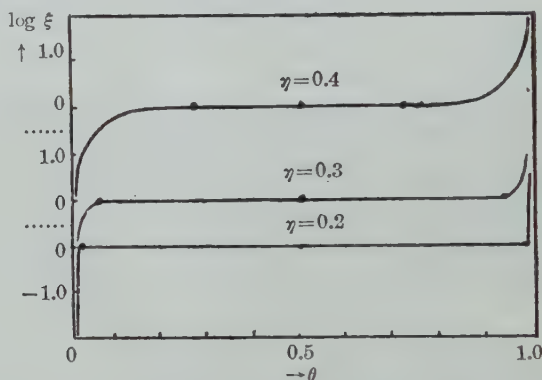


Fig. 5

§ 4. Concluding remarks

The numbers of various complexions, $g(N_A, N_B, N_{AB})$'s, were counted by means of an elementary method for two and three dimensional lattices. The results were used to construct the partition functions for the regular cooperative assemblies. For two dimensional lattices many authors have given exact solutions in the case $\xi=1$. But for three dimensional lattices and even for two dimensional lattices when ξ is not equal to unity, we have still to resort to approximate methods, and the result obtained in this article may be used for the checking of the validity of the approximations.

In conclusion, this work is indebted to the Ministry of Education for the research grant.

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Interaction of μ Meson with Matter, I.

—Nuclear Excitation by Electromagnetic Interaction—

Toshinosuke MUTO and Makoto TANIFUJI

Institute of Science and Technology, University of Tokyo

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§ 1. Introduction

Since the π - and μ -mesons were recently discovered experimentally in the cosmic ray observations and later produced artificially in the laboratory by the impact of the charged particle of the extremely high energies upon the various kinds of targets, the reliable informations about these new particles have been accumulated rather rapidly, thus enabling us to make the detailed theoretical discussions of the interaction of these particles with matter. In fact, the entire range of the characteristic processes, in which the high energy μ meson passes through the matter, losing its energy mainly through the ionization loss, and then is captured in the bound states around the nucleus, finally subjecting to the spontaneous decay or being captured by the nucleus through the charge exchange reactions, has been elucidated to a considerable extent by the thorough theoretical consideration of Fermi, Weisskopf, Teller,¹⁾ Wheeler, Tiomno²⁾ and others.³⁾ Although the main features of the processes due to the interaction of μ meson with matter may be considered to be clarified by the mentioned authors, the closer examinations reveal that some processes accessible on the basis of the present-day theory still remain unattacked so far.

According to Chang's experiments⁴⁾ on cosmic ray μ meson stopped in the plate absorbers within the cloud chamber, there appear the soft γ -rays which are reasonably attributed to the processes of Bohr orbit transitions⁵⁾ of μ -meson and the nuclear transitions from the excited states left after the charge exchange reaction.⁶⁾ The other possible type of γ -ray emission, however, may be considered to be realized from the excited nucleus by the electromagnetic interaction of μ meson with the nucleus. In order to decide whether the mentioned process is responsible for a part of Chang's γ -rays or not, it is required to work out theoretically the cross section of the mentioned process and particularly its magnitude dependent upon the energy of μ -meson. The complications of such investigation, however, come from the intricacy of nucleonic dynamics. In order to make any progress at all, it will be necessary to use some of the nuclear model proposed so far for avoiding the well-known mathematical difficulties involved. For this purpose we have taken, for the following calculation, the liquid drop model of heavy atomic nuclei, which have been used successfully so

far in making clear a number of physical properties of nuclear phenomena except for the case of the extremely high energy reactions of atomic nuclei, in which case the rather individual motion of each nucleon may be considered to play an important role as shown by Serber.⁷⁾ On the other hand, it will be worthwhile working the characteristic behaviour of the liquid drop model in such a nuclear reaction in view of extending the field of validity by an application of the liquid drop model to some other nuclear phenomena than the hitherto discussed one.⁸⁾

§ 2. Quantized liquid drop model of atomic nuclei⁹⁾

As is well-known, the nuclear liquid drop is capable of making a rotation as a whole, the surface vibration and the volume vibration, of which the surface vibration may reasonably be considered to be responsible for the soft γ -ray emission observed in Chang's observations from the view point of the energy considerations.

According to Flügge,¹⁰⁾ the Hamiltonian of the surface vibration of a spherical liquid drop may be written as

$$H = \frac{g}{2} \int |\Delta\Phi|^2 d\tau + \frac{\gamma}{2} \int \left[\frac{1}{\sin^2\theta} \left(\frac{\partial\zeta}{\partial\varphi} \right)^2 + \left(\frac{\partial\zeta}{\partial\theta} \right)^2 - 2\zeta^2 \right] dS, \quad (1)$$

where g denotes the mass density of nucleus, γ its surface tension, Φ the velocity potential, ζ the surface displacement, $d\tau$ the volume element and dS the surface element.

The expansion of the velocity potential and the surface displacement into the spherical harmonics gives

$$\left. \begin{aligned} \Phi &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2} \{ \beta_{l,m} e^{im\varphi} + \beta_{l,m}^* e^{-im\varphi} \} P_l^m(\cos\theta) r^l, \\ \zeta &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2} \{ a_{l,m} e^{im\varphi} + a_{l,m}^* e^{-im\varphi} \} P_l^m(\cos\theta). \end{aligned} \right\} \quad (2)$$

From the boundary condition the following relation is derived:

$$\frac{d}{dt} a_{l,m} = l \beta_{l,m} R_0^{l-1}, \quad (3)$$

where R_0 represents the nuclear radius of the spherical shape without the surface vibration, being proportional to $A^{1/3}$. (A =mass number).

Putting (2) into (1), we get

$$\begin{aligned} H = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\gamma}{2} \cdot \frac{4\pi}{2l+1} \cdot \frac{(l+m)!}{(l-m)!} (l-1)(l+2) \left\{ (a_{l,m} a_{l,m}^* + a_{l,m}^* a_{l,m}) + \right. \\ \left. + \frac{1}{\omega_l^2} (\dot{a}_{l,m} \dot{a}_{l,m}^* + \dot{a}_{l,m}^* \dot{a}_{l,m}) \right\}, \end{aligned} \quad (4)$$

where

$$\omega_l = \{ (\rho_m R_0^3 / \gamma) / l(l-1)(l-2) \}^{1/2} \quad (5)$$

is the characteristic frequency associated with the l -st normal mode of vibration and ρ_m denotes the particle density.

Making use of the relation

$$\frac{d}{dt} a_{l,m} = i \omega_l a_{l,m}, \quad (6)$$

which is derived from the equation of motion, and further the abbreviations

$$\left. \begin{aligned} a_{l,m} &= \frac{l}{\omega_l} \left(\frac{2l+1}{4\pi} \cdot \frac{(l-m)!}{(l+m)!} \right)^{1/2} \left(\frac{2\pi}{3} \frac{\hbar \omega_l}{LAM} \right)^{1/2} a_{l,m} = A(l, m)^{-1/2} a_{l,m}, \\ \beta_{l,m} &= \frac{i}{R_0^{l-1}} \left(\frac{2l+1}{4\pi} \cdot \frac{(l-m)!}{(l+m)!} \right)^{1/2} \left(\frac{2\pi}{3} \frac{\hbar \omega_l}{LAM} \right)^{1/2} a_{l,m}, \end{aligned} \right\} \quad (7)$$

(M =nucleon's mass).

Hamiltonian (4) may be readily simplified into the following form

$$H = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2} \hbar \omega_l (a_{l,m} a_{l,m}^* + a_{l,m}^* a_{l,m}). \quad (8)$$

In order to quantize the surface vibration of the mentioned model along the ordinary manner, we shall hereafter introduce the quantum condition

$$[a_{l,m}, a_{l',m'}^*] = \delta_{l,l'} \delta_{m,m'}, \quad (9)$$

the other commutators being zero.

On account of (9), the eigenvalue of (8) may easily be evaluated to become

$$E = \sum_{l=0}^{\infty} \sum_{m=-l}^l (n_{l,m} + 1/2) \hbar \omega_l, \quad (10)$$

where $n_{l,m}$ represents the integral positive number including zero value. In such a representation, $a_{l,m}$ and $a_{l,m}^*$ are well-known to represent the emission and absorption operators corresponding to the normal mode of vibration specified by l and m respectively.

§ 3. Electromagnetic interaction of the nuclear liquid drop model with μ -meson

According to the Klein-Møller's procedure⁽¹¹⁾ in the correspondence-quantum-electrodynamics, the electromagnetic field associated with the transition of μ meson may be described, as usual, by the four vector potential.

$$\left. \begin{aligned} \mathbf{A} &= -\frac{4\pi\hbar^2}{c} \frac{\mathbf{i}_{pp'}}{(p-p')^2 - \left(\frac{E-E'}{c}\right)^2}, & \mathbf{i}_{pp'} &= e c (\phi_{p'}^* \boldsymbol{\alpha} \phi_p), \\ A_0 &= 4\pi\hbar^2 \frac{\sigma_{pp'}}{(p-p')^2 - \left(\frac{E-E'}{c}\right)^2}, & \sigma_{pp'} &= e (\phi_{p'}^* \phi_p), \end{aligned} \right\} \quad (11)$$

where p, E and p', E' represent the momenta and energies of μ -meson before and after the scattering process respectively, $\boldsymbol{\alpha}$ Dirac's spin vector, and ϕ_p and $\phi_{p'}$ the spinor wave functions associated with the p and p' respectively.

Next, the interaction of the liquid drop model with the electromagnetic

field (11) induced by the scattering of μ -meson may be written as

$$H = \int d\tau H' = \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} H' d\varphi \quad (12)$$

$$= H_s + H_{v,1} + H_{v,2} + H_{v,3},$$

where H' expresses the interaction energy density,

$$H' = -\frac{i}{c} \mathbf{A} \cdot \rho \mathbf{v} + \rho A_0 = \frac{\rho}{c} 4\pi e \hbar^2 \frac{1}{(\mathbf{p} - \mathbf{p}')^2 - \left(\frac{E - E'}{c}\right)^2} \{ (\psi_p^* \alpha \psi_p) \nabla \Phi + c (\psi_p^* \psi_p) \}, \quad (13)$$

ρ and $i = \rho \mathbf{v} = -\rho \nabla \Phi$ being the charge and current density within the nucleus respectively.

The substitution of (2) into (12) and (13) gives

$$H_s = \sum_{l=0}^{\infty} \sum_{m=-l}^l K_l' \theta_l^m (\cos \theta') \{ e^{im\varphi'} a_{l,m} + e^{-im\varphi'} a_{l,m}^* \} (u_p^* u_p), \quad (14)$$

$$H_{v,1} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ -K_{l-1}' \sqrt{(l-m)(l+m-1)} \theta_{l-1}^{m+1} e^{i(m+1)\varphi'} a_{l,m} \right. \\ \left. + K_{l-1}' \sqrt{(l+m)(l+m-1)} \theta_{l-1}^{m-1} e^{-i(m-1)\varphi'} a_{l,m}^* \right\} (u_p^* (a_x - i a_y) u_p), \quad (15)$$

$$H_{v,2} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ +K_{l-1}' \sqrt{(l+m)(l+m-1)} \theta_{l-1}^{m-1} e^{i(m-1)\varphi'} a_{l,m} \right. \\ \left. - K_{l-1}' \sqrt{(l-m)(l-m-1)} \theta_{l-1}^{m+1} e^{-i(m+1)\varphi'} a_{l,m}^* \right\} (u_p^* (a_x + i a_y) u_p), \quad (16)$$

$$H_{v,3} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left\{ K_{l-1}' \sqrt{(l-m)(l+m)} \theta_{l-1}^m (\cos \theta') [e^{im\varphi'} a_{l,m} + e^{-im\varphi'} a_{l,m}^*] \right\} (u_p^* a_z u_p), \quad (17)$$

where

$$K_l' = l \sqrt{\frac{2l-1}{2l+1}} \frac{\bar{q}}{\varepsilon} K_{l-1}', \quad (18.1)$$

$$K_{l-1}' = i\pi \sqrt{\frac{8\pi(2l+1)}{3} \frac{\hbar \omega_l}{l(2l-1)AM}} R_0^3 \bar{q}^{-3/2} J_{l+1/2}(\bar{q}), \quad (18.2)$$

$$\theta_l^m (\cos \theta') = \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m (\cos \theta'), \quad (18.3)$$

$$\mathbf{q} = \mathbf{p} - \mathbf{p}', \quad \bar{q} = \frac{|\mathbf{q}| R_0}{\hbar}, \quad \varepsilon = E - E', \quad \bar{\varepsilon} = \frac{\varepsilon R_0}{\hbar c}, \quad (18.4)$$

$$\psi_p = u_p e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r}}, \quad \psi_{p'} = u_{p'} e^{\frac{i}{\hbar} \mathbf{p}' \cdot \mathbf{r}}. \quad (18.5)$$

Since (14), (15), (16) and (17) are seen to involve linearly the operators $a_{l,m}$ and $a_{l,m}^*$, the electromagnetic interaction (12) may be considered to induce the nuclear excitation accompanied by the inelastic scattering of μ -meson.

§ 4. Cross section

The scattering cross section of μ -meson accompanied by the nuclear excitation may be readily evaluated by the ordinary method in the perturbation theory, in which case the square of the scattering matrix element is to be averaged over the spin of the incident meson, and summed over both spins of the scattered meson and the quantum number m .

Thus, we have, denoting the final state density by ρ_E ,

$$dW = \frac{2\pi}{\hbar} \sum_m \sum_{\substack{\text{final} \\ \text{spin}}} \frac{1}{2} \sum_{\substack{\text{initial} \\ \text{spin}}} |(i|H|f)|^2 \rho_E \quad (19)$$

for the differential probability for the mentioned scattering.

Expressing by $d\sigma_l$ the differential cross section of the μ -meson scattering accompanied by the excitation of the l -th normal mode of surface vibration, we have, from (19),

$$d\sigma_l = 6\pi \left(\frac{e^2 Z}{c} \right)^2 \left(\frac{R_0}{Mc\hbar} \right) \left(\frac{1}{A} \right) \frac{\bar{\epsilon}_l E E' \bar{p}'}{\bar{p}} \frac{2l+1}{l(2l-1)} \frac{[J_{l+1/2}(\bar{q})]^2}{\bar{q}^3 (\bar{q}^2 - \bar{\epsilon}_l^2)^2} K(l, \bar{q}), \quad (20)$$

$$K(l, \bar{q}) = \frac{1}{2} \{ K^{\uparrow\uparrow}(l, \bar{q}) + K^{\uparrow\downarrow}(l, \bar{q}) + K^{\downarrow\uparrow}(l, \bar{q}) + K^{\downarrow\downarrow}(l, \bar{q}) \}, \quad (20')$$

where $K^{\uparrow\uparrow}$ and $K^{\downarrow\downarrow}$ correspond to the scattering without the spin change, while $K^{\uparrow\downarrow}$ and $K^{\downarrow\uparrow}$ the one with the spin change. Further, we obtain

$$\begin{aligned} K^{\uparrow\uparrow}(l, \bar{q}) + K^{\downarrow\downarrow}(l, \bar{q}) &= \frac{2}{(\bar{\mu}^2 + \bar{p}^2)(\bar{\mu}'^2 + \bar{p}'^2)} \left\{ \sum_m \left\{ \sqrt{\frac{2l-1}{2l+1}} (l-m) \theta_l^m(\cos \theta') \bar{\mu} \bar{q} \right. \right. \\ &\quad \left. \left. - \sqrt{l^2 - m^2} \theta_{l-1}^m(\cos \theta') (\bar{\mu} + \bar{\mu}') \bar{p} \right\}^2 + l^2 \left(\frac{2l-1}{2} \right) \frac{\bar{q}^2}{\bar{\epsilon}^2} (\bar{\mu} \bar{q}' + \bar{p}^2 - \bar{p} \bar{q} \cos \theta')^2 \right. \\ &\quad \left. + 2l \sqrt{\frac{2l-1}{2l+1}} \left(\frac{\bar{q}}{\bar{\epsilon}} \right) \sum_m \theta_l^m(\cos \theta') \{ \bar{\mu} \bar{\mu}' + \bar{p}^2 - \bar{p} \bar{q} \cos \theta' \} \left\{ \sqrt{\frac{2l-1}{2l+1}} (l-m) \theta_l^m(\cos \theta') \bar{\mu} \bar{q} \right. \right. \\ &\quad \left. \left. - \sqrt{l^2 - m^2} \theta_{l-1}^m(\cos \theta') (\bar{\mu} + \bar{\mu}') \bar{p} \right\} \right\}, \quad (21) \end{aligned}$$

and

$$\begin{aligned} K^{\uparrow\downarrow}(l, \bar{q}) + K^{\downarrow\uparrow}(l, \bar{q}) &= \sum_m \left\{ \left[\sqrt{\frac{2l-1}{2l+1}} \sqrt{(l+m)(l-m-1)} \theta_{l-1}^{m-1}(\cos \theta') \bar{\mu} \bar{q} \right. \right. \\ &\quad \left. \left. - \sqrt{(l+m)(l+m-1)} \theta_{l-1}^{m-1}(\cos \theta') (\bar{\mu} - \bar{\mu}') \bar{p} \right]^2 \right. \\ &\quad \left. + \left[\sqrt{\frac{2l-1}{2l+1}} \sqrt{(l-m)(l+m-1)} \theta_l^{m+1}(\cos \theta') \bar{\mu} \bar{q} \right. \right. \\ &\quad \left. \left. - \sqrt{(l-m)(l-m-1)} \theta_{l-1}^{m+1}(\cos \theta') (\bar{\mu} - \bar{\mu}') \bar{p} \right]^2 \right\} \\ &\quad + l^2 \left(\frac{2l-1}{2} \right) \frac{\bar{q}^2}{\bar{\epsilon}^2} (\bar{p} \bar{q} \sin \theta')^2 \end{aligned}$$

$$\begin{aligned}
& + 2l \sqrt{\frac{2l-1}{2l+1}} \left(\frac{\bar{q}}{\bar{\epsilon}} \right) \sum \bar{p} \bar{q} \sin \theta' \theta_l^m(\cos \theta') \times \\
& \times \left\{ \sqrt{\frac{2l-1}{2l+1}} \left[\sqrt{(l+m)(l-m-1)} \theta_{l-1}^{m-1}(\cos \theta') \right. \right. \\
& \quad \left. \left. + \sqrt{(l-m)(l+m-1)} \theta_l^{m+1}(\cos \theta') \right] \bar{\mu} \bar{q} \right. \\
& \left. - \left[\sqrt{(l+m)(l+m-1)} \theta_{l-1}^{m-1}(\cos \theta') \right. \right. \\
& \quad \left. \left. + \sqrt{(l-m)(l-m-1)} \theta_{l-1}^{m+1}(\cos \theta') \right] (\bar{\mu} - \bar{\mu}') \bar{p} \right\}, \quad (22)
\end{aligned}$$

where

$$\bar{q} = (\bar{p}^2 + \bar{p}'^2 - 2\bar{p}\bar{p}' \cos \theta_{p'})^{1/2}, \quad \bar{q} \cos \theta' = \bar{p} - \bar{p}' \cos \theta_{p'}, \quad (23)$$

$$\mu = M'c + \frac{E}{c}, \quad \mu' = M'c + \frac{E'}{c}, \quad \epsilon_l = \hbar \omega_l, \quad (24)$$

$$\bar{\mu} = \frac{M'R_0c}{\hbar} + \bar{E}, \quad \bar{\mu}' = \frac{M'R_0c}{\hbar} + \bar{E}', \quad \bar{\epsilon}_l = \frac{\bar{\epsilon}_l R_0}{\hbar c}, \quad (24')$$

$$\bar{p} = \frac{pR_0}{\hbar}, \quad \bar{E} = \frac{ER_0}{\hbar c}, \quad E = c\sqrt{p^2 + M'^2c^2}, \quad (25)$$

$$\bar{p}' = \frac{p'R_0}{\hbar}, \quad \bar{E}' = \frac{E'R_0}{\hbar c}, \quad E' = c\sqrt{p'^2 + M'^2c^2}, \quad (25')$$

$$A = \text{mass number}, \quad Z = \text{nuclear charge}, \quad (26)$$

$$M = \text{nuclear mass}, \quad M' = \text{meson mass} = 210m_e. \quad (26')$$

Integrating $d\sigma_l$ with respect to the direction of the scattered μ -meson, we get for the total cross section,

$$\sigma_l = 12\pi^2 \left(\frac{e^2 Z}{Mc^2} \right)^2 \left(\frac{McR_0}{\hbar} \right) \left(\frac{1}{A} \right) \bar{\epsilon}_l \cdot \frac{\bar{E}\bar{E}'}{\bar{p}^2} \cdot \frac{2l+1}{l(2l-1)} \int_{\bar{p}-\bar{p}'}^{\bar{p}+\bar{p}'} \frac{[J_{l+1/2}(\bar{q})]^2}{\bar{q}^3 (\bar{q}^2 - \bar{\epsilon}_l^2)^2} K(l, \bar{q}) d\bar{q}. \quad (27)$$

The integral in (27) can not be expressed by simple analytical functions but its evaluation is forced to be carried out by the numerical method in which the characteristic properties of the scattering nucleus should be actually specified. We, therefore, shall work out the mentioned cross section for the case of ${}_{82}\text{Pb}^{207}$ nucleus, since the liquid drop model is well-known to be safely valid for the heavier nuclei ($A \gg 50$).

Inserting the value of the surface tension γ , derived from the mass formula of ${}_{82}\text{Pb}^{207}$, the energies of the excited states belonging to $l=2$ and $l=3$ become

$$\epsilon_2 = 1.09 \text{ Mev} \quad (l=2),$$

$$\text{and} \quad \epsilon_3 = 2.12 \text{ " } \quad (l=3), \quad (28)$$

respectively. Using (20) and (27), we have evaluated numerically the angular distribution of the scattered μ meson and the energy dependences of the total cross section, which results are shown graphically in Fig. 1 and 2.

§ 4. Discussion of the results

As seen in Fig. 1, for the inelastic scattering of μ -meson accompanied by the $l=2$ mode of surface vibration of the liquid drop model, there appears the maximum intensity in the direction of about 10° with respect to the incident beam and the scattering in the larger angles decreases rather slowly, thus the appreciable amount of intensity being still remained in the large angle scattering. Further, the forward scattering is seen to become rather predominant with the increase of the μ -meson energy. With the increase of the energy loss of μ -meson, the backward scattering associated with the excitation of the $l=3$ mode of surface vibration is found to become of appreciable amount in contrast to that of the former case. These states of affairs in the angular distribution of μ -mesons are clearly seen to be in sharp contrast to those of the inelastic scattering associated with both atom-excitation and -ionization process, which are evaluated by putting the μ -meson mass in place of the electron mass in the ordinary scattering formula valid for the electron.¹²⁾

Fig. 1
Angular distribution of the scattered μ mesons
(scatterer= Pb^{207})

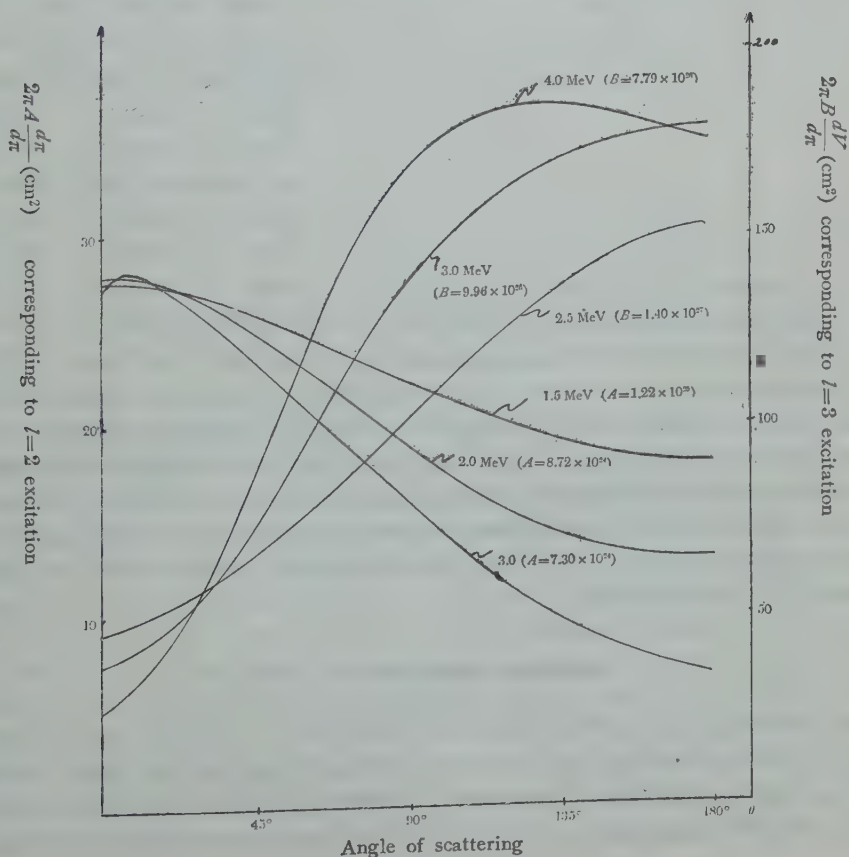
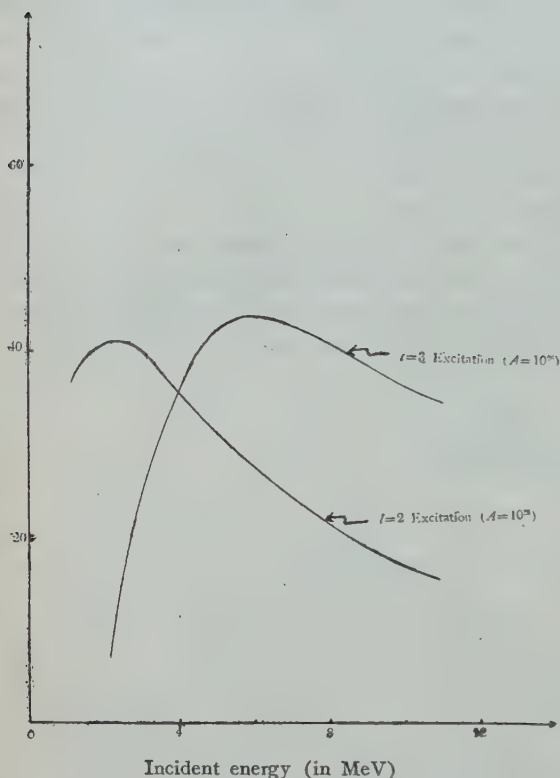


Fig. 2
Total cross section of the inelastic
scattering of the μ mesons
(scatterer= Pb^{207})



meson impact, the cross sections are shown, for the incident energies of 1 MeV from 7 MeV, to become of approximately $10^{-18}\text{cm}^2 \sim 10^{-19}\text{cm}^2$ and $10^{-20}\text{cm}^2 \sim 10^{-21}\text{cm}^2$ respectively, which values are found to be extremely large compared with the cross section of the nuclear excitation by μ -meson impact under consideration. We, therefore, may conclude that the nuclear excitation due to the electromagnetic interaction of μ -meson becomes of less importance in comparison with the ionization process for the relatively low energy region of μ -mesons, particularly taking into account the relatively low accuracy available in the cosmic ray measurements.

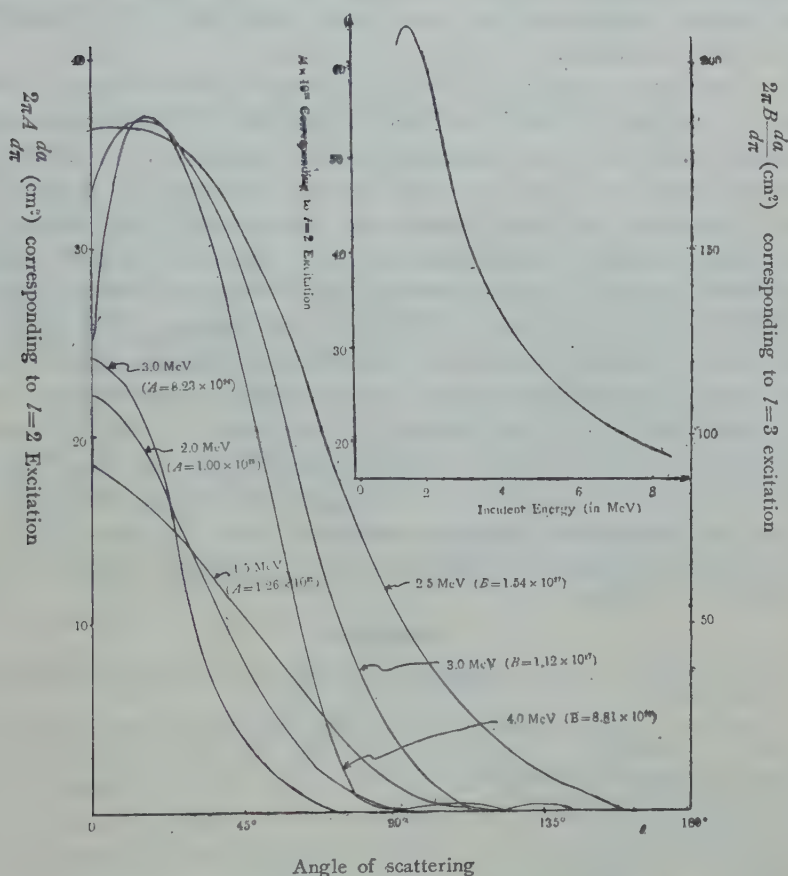
§ 5. Inelastic scattering of protons¹³⁾

The differential and total cross sections (20) and (27) of the inelastic scattering accompanied by the nuclear excitation will be effectively valid for protons by the replacement of μ -meson mass M' by the proton mass since both μ -meson and proton are known to be described by the spinor wave functions and,

The magnitudes of the total cross sections and their energy-dependences are described in Fig. 2. The main features of the total cross section dependent upon the energy of incident μ -meson are that the predominant maximum makes its appearance at some energy a little higher than the threshold one and, after reaching the maximum, the cross section decreases rather monotonously with the incident energies. The magnitude of the cross sections for several MeV of incident energies are found to be of 10^{-24}cm^2 or 10^{-25}cm^2 according as the $l=2$ mode of surface vibration or its $l=3$ one is excited. In this connection, it should be remarked that, for the case of the excitation and ionization of a valency electron of the neutral atom by the μ -

Fig. 3

Angular distribution of the scattered protons and its total cross-section
(scatterer= Pb^{207})



further, the mutual interactions with the nucleus here considered are restricted merely to be of the electromagnetic one. We, therefore, have worked numerically the angular distribution and the total cross section of the inelastically scattered proton by $_{82}\text{Pb}^{207}$, which results are shown graphically in Fig. 3. On account of the different range of the permissible behaviours of the angular distribution of the scattered protons are clearly seen to be quite different from those of the scattered μ -mesons. In fact, the remarkable features of the former may be considered, as seen in Fig. 3, to lie in the predominant forward scattering and, further, the appearance of the second maximum of scattered intensity in a direction of relatively large angle in addition to the sharp maximum in the small angle direction in the case of the excitation of $l=3$ mode of vibration for the incident energies of 3.0 MEV and 4.0 MEV. It should be remarked that the second

maximum in the scattered intensity seems to increase in magnitude with the increase of incident energies of proton. The total cross sections for the proton scattering are roughly 10 times smaller than those for the μ -meson scattering and their energy dependences are seen to be quite similar qualitatively to those of the μ -meson scattering.

Now, we have already a vast experimental observations of the inelastic scattering of protons by atomic nucleus, a large part of which may be considered to be mainly caused by the proper nucleonic interaction, providing the useful informations about the quantum states in the nucleus. In such experiments, therefore, it seems to be hardly possible to discriminate the inelastic scattering due to the electromagnetic interaction from that due to the proper nucleonic one, which situation makes it quite difficult to make a reasonable comparison of the present computations with the experimental observations. The most favorable condition of experiments to observe the scattering process under consideration may be supposed, from the theoretical side, to prepare a proton beam of suitable energies sufficiently below the Coulomb barrier, which, in addition, are required not to lie in the resonance region of the nuclear levels. In view of the above situations, therefore, the detailed analysis of the inelastic scattering of protons by the nucleus shall be reserved in later occasion.

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Note on the Eigenvalue Problem in the Quantum Field Theory

Kazuhiko NISHIJIMA

Department of Physics, University of City Osaka

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In the quantum mechanics of atoms and molecules, many eigenvalue problems are solved in good agreement with experiments. While, in the quantum mechanics of wave fields, the problem is so complicated because of its infinite degrees of freedom that we had to abandon the hope of obtaining the solution. The recent vivid development in the quantum electrodynamics had brought brilliant informations on such problems as scattering, production and capture. These calculations are essentially based on the S matrix formalism¹⁾²⁾ and shows the usefulness of the concept of S matrix, in which it is assumed that the interaction is switched on adiabatically in the remote past and switched off again adiabatically in the remote future. And as its result, the kinetic energy and momentum of the total system is conserved. But in some respect, the use of S matrix in the eigenvalue problem seems to be inadequate, since the truly conserved quantity should be the total energy including the potential (or interaction) energy, but not the mere kinetic energy. This fact seems to show us that the total energy is not an adiabatic invariant, and we should not use the concept of "remote past or future" in the eigenvalue problem.

In this short note, we discuss how to formulate the problem, and how the qualitative nature of the eigenvalues will be. And in this problem, it is pointed out that we should always carefully doubt the properties of operators if they are surely hermitian, surely unitary.

§ 1. Formulation

The eigenvalue problem of the wave fields can be formulated in a covariant way as the natural generalization of the problem in the ordinary quantum mechanics. Such a formulation is desirable in order to clarify the Lorentz invariant character of the eigenvalues. From now on we always employ the interaction representation for this purpose.

The notations used are as follows:

$$\begin{aligned}x^\mu &= (x, y, z, ct), & x_\mu &= (x, y, z, -ct), \\dX &= dx^1 dx^2 dx^3 dx^4, \\dF_\mu &= (dx^2 dx^3 dx^4, dx^1 dx^3 dx^4, dx^1 dx^2 dx^4, dx^1 dx^2 dx^3).\end{aligned}$$

The distinction between cogradient and contragradient suffices enables us to use only real variables and not to pay attention to the operators whether they are hermitian or anti-hermitian. As for other vectors, the same distinction should be understood.

(a) The energy momentum operator

The total Lagrangian density of interacting fields is separated into two parts:

$$L = L_f + L_i,$$

where L_f and L_i refer to the free and interaction parts respectively. Then the canonical energy momentum tensor of free fields is given by

$$T_{\mu}^{\nu} = \sum_A \left[\frac{\partial U_{(A)}}{\partial x^{\mu}} \cdot \frac{\partial L_f}{\partial \left(\frac{\partial U_{(A)}}{\partial x^{\nu}} \right)} + \frac{\partial U_{(A)}^*}{\partial x^{\mu}} \cdot \frac{\partial L_f}{\partial \left(\frac{\partial U_{(A)}^*}{\partial x^{\nu}} \right)} \right] \\ + \sum_B \frac{\partial V_{(B)}}{\partial x^{\mu}} \cdot \frac{\partial L_f}{\partial \left(\frac{\partial V_{(B)}}{\partial x^{\nu}} \right)} - \delta_{\mu}^{\nu} L_f,$$

where U, U^* and V represent the charged and neutral fields wave functions respectively, and we have the following divergence equation:

$$\partial T_{\mu}^{\nu} / \partial x^{\nu} = 0.$$

As the immediate consequence of the above equation, the energy momentum operator \mathfrak{P}^0 of free fields defined by the following equation does not depend on the choice of the space-like surface C .

$$\mathfrak{P}_{\mu}^0 = \int^C T_{\mu}^{\nu} dF_{\nu}. \quad (1)$$

Since \mathfrak{P}^0 is the displacement operator³⁾ in the interaction representation, we have for arbitrary physical quantity $S(X)$ or $U[C]$

$$[\mathfrak{P}_{\mu}^0, S(X)] = \frac{\hbar}{i} \frac{\partial}{\partial x^{\mu}} S(X), \\ [\mathfrak{P}_{\mu}^0, U[C]] = \frac{\hbar}{i} \int^C dF_{\mu} \frac{\delta}{\delta C_X} U[C], \quad (2)$$

where it must be noticed that the operator $U[C]$ should be completely determined by the field quantities in the neighbourhood of the surface C .

Now the total energy momentum operator \mathfrak{P}_{μ} is given by

$$\mathfrak{P}_{\mu} = \mathfrak{P}_{\mu}^0 + \int^C H(X) dF_{\mu}, \quad (3)$$

where $H(X)$ is the interaction Hamiltonian density. According to the conservation law,⁴⁾ it satisfies the following equation:

$$\frac{d\mathfrak{P}_{\mu}}{dC_X} = \left[\frac{\delta}{\delta C_X} + \frac{i}{\hbar} H(X), \mathfrak{P}_{\mu} \right] = 0. \quad (4)$$

Solving (4), we have

$$\mathfrak{P}_{\mu}[C] = T[C, C_0] \mathfrak{P}_{\mu}[C_0] T[C, C_0]^{-1}, \quad (5)$$

where $T[C, C_0]$ is the generalized transformation function defined by

$$\left\{ \frac{\delta}{\delta C_X} + \frac{i}{\hbar} H(X) \right\} T[C, C_0] = 0, \quad T[C_0, C_0] = 1.$$

(b) Stationary state

The stationary state in the ordinary quantum mechanics is defined as the eigenstate of the Hamiltonian: $(H-E)\phi=0$.

As its natural generalization, we define "the stationary state" relativistically as the eigenstate of the energy momentum operator \mathfrak{P} , i.e.

$$(\mathfrak{P}-p)\mathcal{T}_p[C]=0. \quad (6)$$

This is a simultaneous eigenvalue equation and is compatible with each other because of the commutation relation:

$$[\mathfrak{P}_\mu, \mathfrak{P}_\nu] = \int^C dF_\mu \int^C dF'_\nu \left[H(X) + \frac{\hbar}{i} \frac{\partial}{\partial C_X}, H(X') + \frac{\hbar}{i} \frac{\partial}{\partial C_{X'}} \right] = 0,$$

since the Poisson bracket in the integrand vanishes so long as the integrability condition is satisfied. Moreover (6) is compatible with the generalized Schrödinger equation by (4). Our problem is to obtain the eigenvalue p .

In order to solve this problem, we assume two postulates:

Postulate (I). *Though the state vector space \mathfrak{S} , the set of all state vectors, is a continuously infinite dimensional space, we can apply the theory of Hilbert space.*

Postulate (II). *\mathfrak{P} is an hermite operator, and its eigenvectors form a complete system in the space \mathfrak{S} .*

Since the operator $\mathfrak{P}[C]$ changes only by an unitary transformation for the deformation of the space-like surface C , its spectrum of eigenvalues does not depend on the surface C .

§ 2. The case of free fields

In this section, we will solve the eigenvalue problem of free fields, which, though at first sight seems to be trivial, is very instructive for the further purpose.

When there are several fields *without* interaction, the state vector space \mathfrak{S} is given by the direct product of individual ones:

$$\mathfrak{S} = \mathfrak{S}^1 \times \mathfrak{S}^2 \times \dots \times \mathfrak{S}^n,$$

and the total energy momentum operator is the sum of individual ones:

$$\mathfrak{P}^0 = \mathfrak{P}^{(1)} + \mathfrak{P}^{(2)} + \dots + \mathfrak{P}^{(n)}.$$

The operator $\mathfrak{P}^{(s)}$, the energy momentum of the s -th field, must be written in the strict sense of the word, as follows:

$$\mathfrak{P}^{(s)} = \mathbf{I}^{(1)} \times \mathbf{I}^{(2)} \times \dots \times \mathfrak{P}^s \times \dots \times \mathbf{I}^{(n)},$$

where $\mathbf{I}^{(s)}$ and \mathfrak{P}^s are the unit and energy momentum operators in the space \mathfrak{S}^s , respectively. The eigenvalue problem of the free fields is characterized by the following equation:

$$(\mathfrak{P}^0 - p)\chi = 0,$$

which is separated into eigenvalue equations of individual fields:

$$(\mathfrak{P}^s - \mathbf{p}^s)\chi^s = 0,$$

with

$$\mathbf{p} = \sum \mathbf{p}^s, \quad \chi = \chi^1 \chi^2 \cdots \chi^n, \quad \chi^s \in \mathfrak{S}^s. \quad (7)$$

To discuss the eigenvalue problem of an isolated field, we will illustrate by the electromagnetic and electron-positron fields. We begin with the definition of the vacuum state. According to Schwinger³⁾, the vacuum χ_0^s of the isolated s -th field is such a state for which the eigenvalue of any time-like component of the energy momentum four vector, is an absolute minimum. And as was shown by him, the vacuum expectation value of the symmetrical energy momentum tensor is of the form:

$$\langle \theta_\mu^\nu \rangle_0 = K \delta_\mu^\nu.$$

and therefore

$$\langle \mathfrak{P}_\mu \rangle_0 = \int^c \langle T_\mu^\nu \rangle_0 dF_\nu = \int^c \langle \theta_\mu^\nu \rangle_0 dF_\nu = \int^c K dF_\mu.$$

In the case of the electromagnetic field, we fortunately have $K=0$, and it yields

$$\mathfrak{P}^0 \chi_0 = 0.$$

In the case of the electron-positron field, however, the c -number K does not vanish and, indeed, is divergent.

There can be no objection, however, to altering the definition of the energy momentum tensor by the addition of a suitable multiple of δ_μ^ν , which is so chosen that the vacuum expectation value of θ_μ^ν is zero, as stated by Schwinger. The same procedure is applicable to the other kinds of fields.

And the altering does not change the content of this paper, since only commutation relations are utilized.

With this understanding, we always have

$$\mathfrak{P}^0 \chi_0^s = 0. \quad (8)$$

Then the total vacuum is given by $\chi_0 = \chi_0^1 \chi_0^2 \cdots \chi_0^n$.

Since the vacuum is the minimum energy state, the eigenvalues of any time-like component of \mathfrak{P}^0 are never negative, i.e., $p^4 \geq 0$. And since the vacuum state is unique, a state with the eigenvalue $p^4 = 0$ must be the vacuum, for $p^4 = 0$ requires $\mathbf{p} = 0$.

After defining the vacuum state, we will proceed to the eigenvalue problem in the s -th field. This field is considered to be an ensemble of particles with the definite rest mass, spin and charge, and the wave function to be the operator creating and annihilating the particles. So that the Fourier component of the wave function represents the operator of creation or annihilation of a particle with definite energy-momentum \mathbf{p} , and charge state, spin orientation expressed by a symbol λ .

Now such a creation operator as mentioned above be $b^*(p_\lambda)$, then it satisfies the following commutation relation:

$$[\mathfrak{P}^s, b^*(p_\lambda)] = p_\lambda b^*(p_\lambda), \quad (p_\lambda^2 + m^2 = 0).$$

Let χ^s be a state defined by

$$\chi^s = b^*(p_\lambda) b^*(p'_{\lambda'}) b^*(p''_{\lambda''}) \cdots \chi_0^s,$$

then by the above commutation relation and the definition of the vacuum, we get

$$\begin{aligned} \mathfrak{P}^s \chi^s &= \mathfrak{P}^s (b^*(p_\lambda) b^*(p'_{\lambda'}) \cdots \chi_0^s) \\ &= [\mathfrak{P}^s, b^*(p_\lambda) b^*(p'_{\lambda'}) \cdots] \chi_0^s + b^*(p_\lambda) b^*(p'_{\lambda'}) \cdots (\mathfrak{P}^s \chi_0^s) \\ &= (\sum p_\lambda) (b^*(p_\lambda) b^*(p'_{\lambda'}) \cdots \chi_0^s) = (\sum p_\lambda) \chi^s. \end{aligned}$$

Thus the eigenvalue problem of the s -th field is completely solved, and the eigenvalues and eigenvectors are

$$p^s = \sum p_\lambda, \quad \chi^s = b^*(p_\lambda) b^*(p'_{\lambda'}) \cdots \chi_0^s.$$

The state vectors $\{\chi^s\}$ will form a complete system in the space \mathfrak{S}^s for the various choice of p_λ 's.

As for the total free fields, the construction of the solution from the above results and (7) is a trivial work.

§ 3. The case of interacting fields

In order to discuss the eigenvalue problem of interacting fields, we construct a transformation operator $U[C]$, which is not unitary but useful for our case.

The operator $U[C]$ is defined by

$$U[C] = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \int^C dX_1 \int^{C(X_1)} dX_2 \cdots \int^{C(X_{n-1})} dX_n H(X_1) H(X_2) \cdots H(X_n), \quad (9)$$

where the volume integral $\int^C dX$ should be performed in Tati-Tomonaga's sense⁹⁾, but not from $-\infty$ to C . That is, expressing the integrand by Fourier integral

$$G(X) = \int G(k) e^{ikx} (dk),$$

we define the volume integral by

$$\int^C dX \cdot G(X) = \int^C dF_\mu \int (dk) G(k) \frac{k^\mu}{k^2} e^{ikx}, \quad (k^2 = k^\mu k_\mu)$$

The operator $U[C]$ satisfies the following functional differential equation as easily be shown

$$\left\{ \frac{\partial}{\partial C_x} + \frac{i}{\hbar} H(X) \right\} U[C] = 0. \quad (10)$$

Combining the equations (2) and (10), one obtains at once

$$[\mathfrak{P}_\mu^0, U[C]] = - \left(\int^c dF_\mu H(X) \right) U[C]. \quad (11)$$

This equation is considered to be the relativistic generalization of Møller's equation⁶⁾:

$$\mathcal{W}W - W\mathcal{W} = V\mathcal{W}, \quad (12)$$

where W , V and \mathcal{W} are the kinetic, potential energies and wave matrix, and correspond to \mathfrak{P}_μ^0 , $\int^c dF_\mu H(X)$ and $U[C]$ respectively. The equation (11) is satisfied by various choice of the operator $U[C]$, and in fact, it is possible to construct a solution which satisfies the outgoing wave condition from $T[C, C_0]$ by a suitable averaging with respect to the surface C_0 ⁷⁾.

Thus a complete correspondence to Møller's S matrix theory is possible, but we will not discuss it here since they are out of our question. Here it must be remembered that $U[C]$ is completely determined by the neighbourhood of the surface C , for the operators which does not satisfy this condition (11) is no more valid. For instance

$$[\mathfrak{P}_\mu^0, T[C, C_0]] = - \left(\int^c dF_\mu H(X) \right) T[C, C_0] + T[C, C_0] \left(\int^{C_0} dF_\mu H(X) \right).$$

By the formula (11), one easily obtains

$$\begin{aligned} \mathfrak{P}_\mu[C]U[C] &= \mathfrak{P}_\mu^0 U[C] + \left(\int^c dF_\mu H(X) \right) U[C] \\ &= [\mathfrak{P}_\mu^0, U[C]] + U[C] \mathfrak{P}_\mu^0 + \left(\int^c dF_\mu H(X) \right) U[C] \\ &= - \left(\int^c dF_\mu H(X) \right) U[C] + U[C] \mathfrak{P}_\mu^0 + \left(\int^c dF_\mu H(X) \right) U[C] \\ &= U[C] \mathfrak{P}_\mu^0. \end{aligned} \quad (13)$$

It is interesting to compare (13) with the following relation:

$$\mathfrak{P}_\mu[C] T[C, C_0] = T[C, C_0] \mathfrak{P}_\mu[C_0].$$

Now it must be noticed that the operator $U[C]$ has not its inverse $U[C]^{-1}$ which is defined by

$$U[C]^{-1}U[C] = U[C]U[C]^{-1} = 1.$$

If there were such operator $U[C]$, then by the following identity derived from (13)

$$U[C]^{-1} \mathfrak{P}_\mu[C] U[C] = \mathfrak{P}_\mu^0,$$

we have the complete information on the spectrum of \mathfrak{P}_μ , that it is identical with the case of free fields. But as will be demonstrated later, \mathfrak{P}_μ has eigenvalues that \mathfrak{P}_μ^0 has not, so the existence of $U[C]^{-1}$ is fictitious. As the eigenvalue

problem of the free fields is completely solved, we will utilize its solution, the eigenvalue \mathbf{p} and the corresponding eigenvector $\chi_{\mathbf{p}}$, i.e.

$$(\mathfrak{P}^0 - \mathbf{p})\chi_{\mathbf{p}} = 0, \quad (14)$$

especially we know $p^4 \geq 0$. Now put

$$\mathcal{P}_{\mathbf{p}}[C] = U[C]\chi_{\mathbf{p}}, \quad (15)$$

then from (13) and (14), we get

$$(\mathfrak{P}[C] - \mathbf{p})\mathcal{P}_{\mathbf{p}}[C] = 0. \quad (16)$$

Therefore, so long as $\mathcal{P}_{\mathbf{p}}[C]$ does not vanish, $\mathcal{P}_{\mathbf{p}}[C]$ is the eigenvector of \mathfrak{P} corresponding to the eigenvalue \mathbf{p} .

The equation (16) shows that the energy momentum operator $\mathfrak{P}[C]$ of interacting fields includes the eigenvalues identical with the free fields.

The eigenvector of the hermite operator $\mathfrak{P}[C]$ are orthogonal to each other, so if there were such a state vector as

$$U[C]\chi = 0, \quad (17)$$

then from the completeness of the set $\{\chi_{\mathbf{p}}\}$, χ is expanded as

$$\chi = \sum a(\mathbf{p})\chi_{\mathbf{p}},$$

and

$$U[C]\chi = \sum a(\mathbf{p})\mathcal{P}_{\mathbf{p}}[C],$$

therefore, there must be eigenvectors $\chi_{\mathbf{p}}$ satisfying

$$U[C]\chi_{\mathbf{p}} = 0, \quad (18)$$

because of the orthogonality of \mathcal{P} 's.

Let the set of state vectors satisfying (17) be \mathfrak{R}_2 , i.e.

$$\mathfrak{R}_2 = \{\chi \mid U[C]\chi = 0, \chi \in \mathfrak{S}\},$$

then χ 's satisfying (18) are the bases of the space \mathfrak{R}_2 . The space \mathfrak{S} is decomposed into two subspaces \mathfrak{R}_1 and \mathfrak{R}_2 :

$$\mathfrak{S} = \mathfrak{R}_1 + \mathfrak{R}_2, \quad \mathfrak{R}_1 \perp \mathfrak{R}_2. \quad (19)$$

It seems to us that $\mathfrak{S} = \mathfrak{R}_1$ and \mathfrak{R}_2 is a null set, but it is not proved yet. It is clear that $\mathfrak{S} = \mathfrak{R}_1$ for the free fields, since $U[C] = 1$ in this case. Thus we get the eigenvectors \mathcal{P} 's, by the transformation $U[C]$ on χ 's belonging to the space \mathfrak{R}_1 . Now the set of linear combinations of \mathcal{P} 's obtained in (15) be \mathfrak{W}_1 , i.e.

$$\mathfrak{W}_1 = \{\mathcal{P} \mid \mathcal{P} = U[C]\chi, \chi \in \mathfrak{S} \text{ or } \mathfrak{R}_1\}. \quad (20)$$

If the interaction is absent, \mathfrak{W}_1 coincides with \mathfrak{S} , but not in the present case. So we will decompose \mathfrak{S} into two subspaces \mathfrak{W}_1 and \mathfrak{W}_2 analogous to (19):

$$\mathfrak{S} = \mathfrak{W}_1 + \mathfrak{W}_2, \quad \mathfrak{W}_1 \perp \mathfrak{W}_2. \quad (21)$$

As will be shown later, \mathfrak{B}_2 is an infinite dimensional space so long as the interaction is present.

With the help of these subspaces, $U(C)$ and \mathfrak{P}_μ are expressed as follows:

$$\begin{array}{cc}
 (\mathfrak{R}_1) & (\mathfrak{R}_2) \\
 U(C) = \begin{array}{|c|c|} \hline \text{diagonal lines} & \circ \\ \hline \circ & \circ \\ \hline \end{array} & , \quad \mathfrak{P} = \begin{array}{|c|c|} \hline \text{diagonal line with } \times & \circ \\ \hline \circ & \text{diagonal lines} \\ \hline \end{array} \\
 (\mathfrak{B}_2) & (\mathfrak{B}_1) \quad (\mathfrak{B}_2)
 \end{array} \quad (22)$$

We have obtained the eigenvalues of \mathfrak{P} belonging to the subspace \mathfrak{B}_1 , but the part to \mathfrak{B}_2 is not diagonalized yet.

For $\Psi \in \mathfrak{B}_2$, $\chi \in \mathfrak{S}$, we have

$$(\Psi, U(C)\chi) = 0.$$

Therefore immediately we get

$$(U^\dagger(C)\Psi, \chi) = 0, \quad (23)$$

where $U^\dagger(C)$ is the adjoint operator of $U(C)$. Since χ is arbitrarily chosen, we get from (23)

$$U^\dagger(C)\Psi = 0, \quad \text{for } \Psi \in \mathfrak{B}_2.$$

If we define A and M by

$$A = U^\dagger(C)U(C), \quad M = U(C)U^\dagger(C),$$

then A and M are the unnormalized projection operators of the subspaces \mathfrak{R}_1 and \mathfrak{B}_1 respectively and intimately related to the normalization of the operator $U(C)$. They satisfy

$$(A, \mathfrak{P}^\circ) = (M, \mathfrak{P}) = 0.$$

Next we will try to prove the existence of the subspace \mathfrak{B}_2 , or the non-existence of the operator $U(C)^{-1}$.

Consider the vacuum state χ^0 defined in § 3,

$$\mathfrak{P}^0 \chi^0 = 0. \quad (25)$$

Then it is clear that

$$\mathfrak{P} \chi^0 = \left(\mathfrak{P}^0 + \int^c dF H(X) \right) \chi^0 = \left(\int^c dF H(X) \right) \chi^0 \neq 0, \quad (26)$$

since $H(X)$ involves creation operators.

If \mathfrak{B}_2 were a null set, then $\{\Psi_P C\}$ form a complete system of \mathfrak{S} , and χ_0 can be expanded in Ψ 's:

$$\chi_0 = \sum a^P[C] \Psi_P[C]. \quad (27)$$

For a Hamiltonian density $H(X)$ which does not involve the normal dependent part, we have in general

$$\langle H(X) \rangle_0 = (\chi_0, H(X) \chi_0) = 0. \quad (28)$$

The proof is restricted only to these cases.

Combining (25) and (28), we get

$$(\chi_0, \mathfrak{P} \chi_0) = 0. \quad (29)$$

Inserting (27) into (29), we have

$$\sum p |a_p[C] \Psi_p[C]|^2 = 0. \quad (30)$$

But since $p^4 \geq 0$ for \mathfrak{B}_1 , it is necessary to assume $a_p[C] = 0$ for $p \neq 0$. So we know

$$\chi_0 = a_0[C] \Psi_0[C],$$

and therefore

$$\mathfrak{P} \chi_0 = 0,$$

which contradicts to (26). This result means that \mathfrak{B}_2 is not a null set, and (30) requires the existence of the eigenvalues $p^4 < 0$.

So let an eigenvalue with $p^4 < 0$ be p , then from the Lorentz invariance of the whole theory, L_p is also an eigenvalue, where L is an arbitrary Lorentz transformation.

For instance consider the following transformation :

$$p^{4'} = \frac{p^4 - \beta p}{\sqrt{1 - \beta^2}}, \quad p' = \frac{p - \beta p^4}{\sqrt{1 - \beta^2}},$$

with β/p and $(\beta p) > 0$. Then we readily see

$$0 > p^4 > p^{4'}.$$

Continuing such processes, we know that *there are infinitely many states with negative energy which do not belong to \mathfrak{B}_1 .*

This fact shows that the subspace \mathfrak{B}_2 has really an infinite dimension. In the above discussion, we have utilized the Lorentz degeneracy of the eigenvalues. In order to obtain the energy levels of bound states, however, we should solve the eigenvalue problem of the rest mass of a given system rather than the energy eigenvalue. In short we must solve

$$(\mathfrak{P}^4 - p^4) \Psi = 0,$$

with the additional condition

$$\mathfrak{P} \Psi = 0,$$

which enables us to get rid of the Lorentz degeneracy, and corresponds to the separation of the freedom of the centre of mass.

§ 4. Canonical transformation

In the previous section, we have discussed the general features of the eigenvalue problem, but when we proceed to actual problems, a concrete method is required. The most important and familiar one is the method of canonical transformations.

So we give here a theorem useful for practical purposes.

Theorem: If a unitary operator $S[C]$ satisfies the following two conditions:

$$(1) \quad S[C]^{-1} \left(\frac{\partial}{\partial C_X} + \frac{i}{\hbar} H(X) \right) S[C] = \frac{\partial}{\partial C_X} + \frac{i}{\hbar} \tilde{H}(X),$$

$$(2) \quad [\mathfrak{P}_\mu^0, S[C]] = \frac{\hbar}{i} \int^c dF_\mu \frac{\partial}{\partial C_X} S[C],$$

then the following formula is obtained:

$$S[C]^{-1} \left(\mathfrak{P}_\mu^0 + \int^c H(X) dF_\mu \right) S[C] = \mathfrak{P}_\mu^0 + \int^c \tilde{H}(X) dF_\mu.$$

(proof)

$$\begin{aligned} S[C]^{-1} \left(\mathfrak{P}_\mu^0 + \int^c H(X) dF_\mu \right) S[C] &= \mathfrak{P}_\mu^0 + S[C]^{-1} [\mathfrak{P}_\mu^0, S[C]] \\ &\quad + \int^c S[C]^{-1} H(X) S[C] dF_\mu \\ &= \mathfrak{P}_\mu^0 + \int^c dF_\mu \left(\frac{\hbar}{i} S[C]^{-1} \frac{\partial S[C]}{\partial C_X} + S[C]^{-1} H(X) S[C] \right) \\ &= \mathfrak{P}_\mu^0 + \int^c dF_\mu \left\{ \frac{\hbar}{i} S[C]^{-1} \left(\frac{\partial}{\partial C_X} \cdot S[C] - S[C] \cdot \frac{\partial}{\partial C_X} \right) + S[C]^{-1} H(X) S[C] \right\} \\ &= \mathfrak{P}_\mu^0 + \int^c dF_\mu \left\{ S[C]^{-1} \left(\frac{\hbar}{i} \frac{\partial}{\partial C_X} + H(X) \right) S[C] - \frac{\hbar}{i} \frac{\partial}{\partial C_X} \right\} \\ &= \mathfrak{P}_\mu^0 + \int^c dF_\mu \left\{ \frac{\hbar}{i} \frac{\partial}{\partial C_X} + \tilde{H}(X) - \frac{\hbar}{i} \frac{\partial}{\partial C_X} \right\} = \mathfrak{P}_\mu^0 + \int^c dF_\mu \tilde{H}(X). \quad \text{Q. E. D.} \end{aligned}$$

An example of such a unitary transformation⁵⁾ is given by

$$S_1[C] = \exp \left(\frac{\hbar}{i} \int^c H(X) dX \right). \quad (31)$$

We do not suffer from the pole $1/k^2$ in Tati-Tomonaga's integral in this case, since the first order Hamiltonian causes only virtual processes, so that $\int^c H(X) dX$ is hermitian and $S_1[C]$ is unitary.

Though at first sight it seems possible to eliminate the interaction Hamiltonian by successive canonical transformations, it is indeed impossible. For if it were possible, \mathfrak{P} will always has the same eigenvalues with \mathfrak{P}^0 , in contrast to the

result obtained in the previous section.

For instance, the ordinary canonical transformation⁵⁾ applied after the previous canonical transformation $S_1[C]$ is

$$S_2[C] = \exp \left\{ -\frac{\hbar^2}{2} \int^{G(X)} dX' \int^{G(X')} dX'' [H(X'), H(X'')] \right\}. \quad (32)$$

The application of this transformation to unbound states would be valid, but no more valid to bound states, for the integral contains the pole $1/k^2$ and the transformed Hamiltonian can cause real processes.

What we must notice here is the word "bound state" which should be understood as a state belonging to a discrete level of the rest mass. Therefore in the strict sense of the word, $S_2[C]$ is not unitary and the unitarity is destroyed in the bound states.

In general careless use of an incorrect canonical transformation gives up the bound states. The reason why we couldn't obtain the bound states by the use of the operator $U[C]$ is thus clarified.

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Note added in proof In the former part of section 4, we have concluded that the energy momentum four vector includes identical eigenvalues with the free case, but in order to obtain the correct level shift due to self energy an attention should be paid to separate the total energy momentum. The correct separation is

$$P_\mu = \left(P_\mu^0 + \int dF_\mu H_{self} \right) + \int (H_{int} - H_{self}) dF_\mu.$$

Thus $H_{int} - H_{self}$ must be employed in the integrand of the operator $U(C)$ and free particles have renormalized masses instead of bare ones. The author is indebted to Mr. G. Takeda on this point.

Effects of Nuclear Motion on the Fine and the Hyperfine Structure of Hydrogen, I

Takehiko ISHIDZU

*Department of Dynamics, 1st Faculty of Engineering,
Tokyo University*

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The effects of nuclear motion on the energy levels of the hydrogen atom are studied by means of the relativistic 16-component wave equation. The former works of Breit and Brown and of Breit and Meyerott are rearranged at first from a somewhat different point of view. Next, it is shown that the original 16-component wave equation is reducible to a two-component radial equation in special case that the total angular momentum \mathbf{F} vanishes, and the latter equation is solved by expansion in power series in $\beta (= m/M)$. In general cases in which \mathbf{F} is not zero, only expansion of the equation in power series in a^2 ($a = e^2/c\hbar$) enables us to solve the equation successively. This procedure is essentially equivalent to that of Bechert and Meixner, but the process of approximation seems to become clearer in our treatment and some refinement in results is obtained. The result for the correction due to nuclear motion on the fine structure is shown to be in close agreement with those of former works. The correction factor for the hyperfine structure splittings is given by $(1+\beta)^{-3}$ as usual for the two extreme hyperfine components $j=l+\frac{1}{2}$, $F=l+1$ and $j=l-\frac{1}{2}$, $F=l-1$, but is shown to be given by a new factor $(1+\beta)^{-2}$ for the other two components $j=l\pm\frac{1}{2}$, $F=l$. Experimental proof of this factor would be very difficult, however, because the term responsible for this correction factor is absent in the energy expression for s terms, so that this appears only for the components of non-vanishing l terms. Finally some considerations are given about the validity of our results, with special reference to an iterated term of Breit's interaction V .

§ 1. Introduction

The effect of nuclear mass motion on the energy levels of hydrogen atoms was first studied by Darwin¹⁾ as a prequantum-mechanical problem and later by Bechert and Meixner²⁾ and also by Lowen³⁾ by quantum mechanical method. Recently after discovery of the Lamb shift⁴⁾, and that of the discrepancies of experimental value of the hyperfine separation of the ground state of hydrogen with theory⁵⁾, more accurate determination of the fine and the hyperfine structure has become of great importance. In connection with this, Breit and his collaborators⁶⁾⁻⁹⁾ have re-examined the problem and carried out similar calculations. Though it has been proved that the Lamb shift as well as the discrepancies of the hyperfine separation can be essentially explained both from the new quantum-electrodynamical point of view, by taking interactions with radiation field into consideration, it is still important for strict proof of the theory to give an answer for the two-particle problem without such modification as accurately as possible. With this purpose we have made also similar calculations to those mentioned above

and obtained some refinement in the former results. This paper gives the processes and the results of our calculations, clarifying their relation to other works especially with regard to the process of approximation. Before entering our work, however, we shall summarize the results hitherto obtained.

Darwin¹⁾ derived in his paper in 1920 an expression for the retarded potential between two charged particles accurate to order $v_1 v_2 / c^2$ relative to the Coulomb potential, where v_1 and v_2 are the velocities of the two particles, and applied it to calculation of the hydrogen spectrum by Sommerfeld's old quantum theory. His result gave, as a correction for the non-relativistic Rydberg energy, the Bohr's simple reduced mass correction factor $(1 + \beta)^{-1}$, where $\beta = m/M$, m and M are, respectively, the masses of an electron and a proton, and besides, though he could not give, of course, the correct fine structure of the terms, an additional energy expressed by

$$-\frac{\beta a^4}{8n^4} \cdot \frac{1}{(1 + \beta)^3} \quad (1)$$

in units of mc^2 , where a is the fine structure constant and n is the principal quantum number. This additional energy is of the same order as that of the hyperfine structure splittings as seen below, but depends only on the principal quantum number. This energy shall be called the "Darwin's term" hereafter. Later in 1935 Bechert and Meixner²⁾ calculated the effect on the fine structure and found that it was also representable by the reduced mass correction factor $(1 + \beta)^{-1}$ with an additional energy equivalent to the Darwin's term, though they neglected terms of relative order β^2 and of higher orders. Their calculation was based on the reduction of the relativistic 16-component equation for a proton and an electron to a 4-component wave equation, including Breit's interaction Y which is a quantum mechanical equivalent to the Darwin's expression for the retarded potential (see below). Lowen³⁾ started also from the same equation, but he applied the so-called "large component" approximation only to proton states, reducing the equation to 8-component in place of 4-component equation. His calculation was not, however, carried out far enough to deduce any conclusion about the fine structure and the hyperfine structure of hydrogen.

The calculations of Breit and his collaborators developed since 1947 also lie in the same line as that of the Lowen's. The effect on the fine structure was calculated by Breit and Brown⁶⁾ by method of taking the expectation value of that part of the interaction Y which is independent of nuclear spin, after reducing the equation to 8-component equation. They discussed the validity of their approximation from various view-points in detail, though their result obtained to the first power in β agreed entirely with those of Bechert and Meixner and of Darwin. The effect on the hyperfine structure, on the other hand, was calculated by Breit and Meyerott^{7,8)} by taking, this time, the remaining part of Y which depends on the nuclear spin and was found to be represented by a correc-

tion factor $(1+\beta)^{-3}$ or, on account of the validity of approximation, by $1-3\beta$. Comparison of the theoretical value of the hyperfine separation of the ground state of hydrogen with experiment has been actually done by taking account of this correction factor and experimental values of the fine structure constant α obtained from it has been discussed by Bethe and others¹⁰⁾¹¹⁾. Recently Breit, Brown and Arfken⁹⁾ have improved the former work and discussed a higher order effect appearing except for s terms.

In the next section of this paper the calculations of Breit and Brown⁹⁾ and of Breit and Meyerott⁷⁾ are rearranged at first from a somewhat different point of view and shown to be equivalent to a second-order perturbation calculation. In §3 the 16-component equation including the interaction V is reduced to a dimensionless form for future reference. The eigenfunctions of the angular part corresponding to a given total angular momentum F , M_F are constructed and the matrix elements of the angular part of various operators are obtained also for future use. Actual elimination of the angular part from the equation is carried out in §4 for special case $F=0$ and each of two resulting radial equations with four components, each corresponding to the $p_{1/2}$ or $s_{1/2}$ state respectively, is shown to be reducible to a new equation with two components. The latter may be solved easily by expansion in power series in β for both cases; the first order perturbation energy is obtained up to the order α^4 , being in accord with former results. In §5 the 16-component equation is solved for general values of F successively by expansion in power series in α^2 . The equation in the zeroth approximation reduces to the Schroedinger equation with reduced mass $m/(1+\beta)$ and energies of order α^4 are obtained in the next approximation. This procedure of approximation is essentially equivalent to Breit's general reduction¹²⁾²⁾ of 16-component to 4-component equation and hence our results also agree with that of Bechert and Meixner or of Breit and others, though these are restricted in the first power of β . The process of approximation, however, seems to have become more obvious in our treatment, and besides, effects of order β^2 and of higher order neglected in Bechert and Meixner have been evaluated rigorously, as well as the perturbation energy between different hyperfine components. The result for the fine structure gives the simple reduced mass correction factor $(1+\beta)^{-1}$ rigorously, i.e. with no approximation regarding the entrance of β ; that for the hyperfine structure gives also the correction factor $(1+\beta)^{-3}$ in agreement with Breit and Meyerott for the two hyperfine components $j=l+\frac{1}{2}$, $F=l+1$ and $j=l-\frac{1}{2}$, $F=l-1$ of a given term, but a different factor $(1+\beta)^{-2}$ for the other two components $j=l\pm\frac{1}{2}$, $F=l$. This latter factor, however, comes out of a term which disappears in the energy expression for s terms ($l=0$), so the hyperfine separation of the ground state of hydrogen, or generally, of every s state is not affected by this asymmetric factor, being given for the component $j=\frac{1}{2}$, $F=0$ also by the factor $(1+\beta)^{-3}$ or $1-3\beta$. The term in question which is proportional to β^2 has been neglected in Bechert and Meixner. This effect corresponds to the one called "special mass

effect" by Breit and others⁹). For s terms, in fact, special treatment is necessary in the calculation by expansion in α^2 and this modification of theory is shown in the last subsection of § 5, actually giving the symmetric factor $(1+\beta)^{-3}$ for every component of s terms.

In the last section the validity of approximation in our calculation is discussed with special reference to iteration energy of Breit's interaction Y which has been omitted throughout in our calculation. The effect of the asymmetric factor on the hyperfine separation of terms with non-vanishing l -value is mentioned briefly.

In Appendix the hyperfine structure formula for hydrogenlike atoms is derived by regarding the nucleus as a fixed dipole for a Dirac electron. Our derivation seems to be more general than that of Breit and others^{13,14}), especially because it requires no modification for s terms. In the first subsection (A.1) a few preparations for this are made.

Notations

c =velocity of light,

\hbar =Planck's constant divided by 2π ,

e =electronic charge,

m =electronic mass,

M =mass of proton,

$\beta=m/M=1/\mu$, $z=1/(1+\beta)$,

$\alpha=e^2/c\hbar$ =fine structure constant,

$a_0=\hbar^2/me^2$ =Bohr radius,

$\alpha_e, \beta_e; \alpha_M, \beta_M$ =the Dirac's four matrices for electron and proton respectively,

$p_0=-(\hbar/ic)(\partial/\partial t)-V/c$,

$E=Mc^2+mc^2\varepsilon$ =total energy of the system,

$V=-e^2/r=mc^2\alpha^2v$ =the Coulomb potential between proton and electron,

$Y=(e^2/r)^{1/2}\{(\alpha_e\alpha_M)+(\alpha_e\mathbf{r})(\alpha_M\mathbf{r})/r^2\}$ =Breit's interaction,

$Y'=(e^2/r)S=mc^2\alpha^2uS$,

$S=\frac{1}{2}\{(\sigma_e\sigma_M)+(\sigma_e\mathbf{r})(\sigma_M\mathbf{r})/r^2\}$,

$u=-v=1/\rho$, $r=a_0\rho$,

$\mathbf{r}=a_0\rho$ =position vector of electron relative to proton,

$\mathbf{p}_e, \mathbf{p}_M$ =momenta of electron and proton respectively,

$\mathbf{p}=-imc\alpha\nabla$ =momentum of the relative motion,

∇ =gradient vector in atomic units,

$s\hbar=\frac{1}{2}\sigma_e\hbar$, $I\hbar=\frac{1}{2}\sigma_M\hbar$ =spin vectors of electron and proton respectively,

$l\hbar, l, m_l$ =orbital angular momentum of the relative motion and the quantum numbers of its magnitude and z -component,

$\mathbf{j}=\mathbf{l}+\mathbf{s}$; and j, m_j =the quantum numbers of magnitude and z -component of \mathbf{j} ,

$\mathbf{F}=\mathbf{j}+\mathbf{I}$; and F, M_F =the quantum numbers of magnitude and z -component of \mathbf{F} ,

$k=j+\frac{1}{2}$; $x=\pm k$ for $l=j\pm\frac{1}{2}$ respectively,

n =principal quantum number,
 $O(a^2) \cdot A$ =the part of order a^2 of A ,
 $\langle A \rangle$ =expectation value of A .

§ 2. 8-Component approximation of Breit and others*

2.1. Wave equation of the system

The wave equation of the system of a proton and an electron including Breit's interaction Y may be written as

$$\{p - c_0 + \beta_M Mc^2 + \beta_e mc^2 - c(\alpha_M p_M) - c(\alpha_e p_e) + Y\} \Psi = 0, \quad (2)$$

where Ψ is the 16-component wave function of the system of which "large" components for both particles precede its small ones for convenience, contrary to customary convention. The operator $c p_0$ is equivalent for definite total energy E to

$$c p_0 = E - V,$$

where V is the Coulomb potential between the two particles. The interaction Y is given by

$$Y = (e^2/2r) \{ (\alpha_e \alpha_M) + (\alpha_e \mathbf{r})(\alpha_M \mathbf{r})/r^2 \}. \quad (2a)$$

Assuming that the total momentum vanishes: $p_e + p_M = 0$, one has

$$p_e = -p_M = p,$$

and Eq.(2) may be then replaced by

$$\{-c p_0 + \beta_M Mc^2 + \beta_e mc^2 + c(\alpha_M p) - c(\alpha_e p) + Y\} \Psi = 0, \quad (2')$$

which depends on the relative coordinates alone. Since total angular momentum $\mathbf{F} = \mathbf{j} + \mathbf{I}$ commutes with every operator appearing in (2'), eigenfunctions of the system are characterized by the quantum numbers F and M_F defined by

$$\mathbf{F}^2 \Psi = F(F+1) \Psi, \quad F_z \Psi = M_F \Psi.$$

2.2. 8-Component approximation as a second-order perturbation theory

The calculations of Breit and Brown⁶⁾ and of Breit and Meyerott⁷⁾ are shown to be expressible in the form of a second-order perturbation calculation as in the following. We shall take as a zeroth approximation to Eq.(2') the Hamiltonian

$$H^{(0)} = \beta_M Mc^2 + H_0,$$

where H_0 is the Dirac's Hamiltonian for a fixed nucleus

$$H_0 = V + \beta_e mc^2 - c(\alpha_e p), \quad (3)$$

and as a perturbation to $H^{(0)}$ the remaining part of Eq.(2')

$$H^{(1)} = c(\alpha_M p) + Y.$$

Then we have the zeroth-order eigenfunctions

$$\Psi^{(0)} = (\Phi, 0) \quad \text{or} \quad \Psi^{(0)} = (0, \Phi)$$

with the aid of any one of the eigenfunctions of H_0 , Φ :

$$H_0 \Phi = E_0 \Phi,$$

and the eigenvalues

$$E^{(0)} = Mc^2 + E_0 \quad \text{or} \quad E^{(0)} = -Mc^2 + E_0,$$

corresponding to positive- or negative-energy states of the proton respectively; Φ represents a 8-component function here.

The first-order perturbation energy $E^{(1)}$ always vanishes since $H^{(1)}$ contains α_M linearly which interchanges the large and the small components of $\Psi^{(0)}$. The second order energy $E^{(2)}$ is given by

$$E^{(2)} = \sum_f |H_{if}^{(1)}|^2 / (E_i^{(0)} - E_f^{(0)}), \quad (4)$$

where suffices i, f refer to initial and final states respectively. Assuming that an initial state belongs to positive-, and final states belong to negative-energy states of proton respectively and neglecting the difference $E_{0i} - E_{0f}$ compared to $2Mc^2$, i.e. putting $E_i^{(0)} - E_f^{(0)} = 2Mc^2 + E_{0i} - E_{0f} \doteq 2Mc^2$, one has in place of (4)

$$E^{(2)} = \langle (H^{(1)})^2 \rangle_i / 2Mc^2, \quad (4')$$

where the symbol $\langle \rangle$ means the expectation value. Since

$$(H^{(1)})^2 = c^2 (\alpha_M \mathbf{p})^2 + c (\alpha_M \mathbf{p}) Y + c Y (\alpha_M \mathbf{p}) + Y^2, \quad (4a)$$

it is easily verified that the Breit and Brown's and the Breit and Meyerott's calculations are equivalent to evaluation of the expectation value of (4a) except the last term which means an iteration of the interaction Y . In fact, one has

$$(\alpha_M \mathbf{p})^2 = \mathbf{p}^2,$$

and linearizing $c (\alpha_M \mathbf{p}) Y + c Y (\alpha_M \mathbf{p})$ in σ_M with the aid of the relation:

$(\alpha \mathbf{a}_M)(\alpha_M \mathbf{b}) = (\mathbf{a} \mathbf{b}) + i(\sigma_M [\mathbf{a} \times \mathbf{b}])$, \mathbf{a} and \mathbf{b} being any vectors commuting with σ_M , and besides, using some other relations similar to Eqs.(25) below, it follows

$$c (\alpha_M \mathbf{p}) Y + c Y (\alpha_M \mathbf{p}) = (ce^2/r) \{ (\alpha_e \mathbf{p}) + r^{-2} (\alpha_e \mathbf{r}) (\mathbf{r} \mathbf{p}) \} \\ + (ce^2 \hbar / r^3) (\sigma_M [\alpha_e \times \mathbf{r}]). \quad (4b)$$

The expectation value of $c^2 \mathbf{p}^2$ and of the part of (4b) independent of nuclear spin σ_M gives, being divided by $2Mc^2$, up to the energy of order $mc^2 \alpha^4$

$$E_a^{(2)} = (m^2 c^4 - E_0^2) / 2Mc^2, \quad (5)$$

as Breit and Brown have shown. $E_a^{(2)}$ proves to be equal, in the first power of β , to the simple reduced mass correction to the Rydberg and the fine structure energies plus the Darwin's term (1) (see § 4.3, especially Eqs. (16) and (17a)).

The spin-dependent part of (4b) is identical with the expression for the

hyperfine splitting energy due to a fixed idealized proton having one nuclear magneton $\mu_K = e\hbar/2Mc$, as is shown in Eq.(A7') in Appendix. Breit and Meyerott have found the correction factor $(1+\beta)^{-3}$ by evaluating corrections due to the nuclear motion to the integral $\int_0^\infty f g dr$, essential for the hyperfine structure theory.

§ 3. Eigenfunctions of the system and the matrix elements of angular operators

3.1. The equation in terms of 4-component wave functions

Decomposition of the 16-component Ψ into four 4-component wave functions by

$$\Psi = (\varphi, \chi, \bar{\varphi}, \bar{\chi}) \quad (6)$$

yields from Eq.(2')

$$\left. \begin{aligned} (Mc^2 + mc^2 + V - E)\varphi - c(\sigma_e \mathbf{p})\chi + c(\sigma_M \mathbf{p})\bar{\varphi} + Y'\bar{\chi} &= 0, \\ -c(\sigma_e \mathbf{p})\varphi + (Mc^2 - mc^2 + V - E)\chi + Y'\bar{\varphi} + c(\sigma_M \mathbf{p})\bar{\chi} &= 0, \\ c(\sigma_M \mathbf{p})\varphi + Y'\chi + (-Mc^2 + mc^2 + V - E)\bar{\varphi} - c(\sigma_e \mathbf{p})\bar{\chi} &= 0, \\ Y'\varphi + c(\sigma_M \mathbf{p})\chi - c(\sigma_e \mathbf{p})\bar{\varphi} + (-Mc^2 - mc^2 + V - E)\bar{\chi} &= 0, \end{aligned} \right\} \quad (7)$$

where

$$Y' = (e^2/2r) \{ (\sigma_e \sigma_M) + (\sigma_e \mathbf{r})(\sigma_M \mathbf{r})/r^2 \}, \quad (2a')$$

and φ, χ correspond to positive, $\bar{\varphi}, \bar{\chi}$ to negative-energy states of proton on the one hand, while, on the other hand, $\varphi, \bar{\varphi}$ correspond to positive-, $\chi, \bar{\chi}$ to negative-energy states of electron, each having four components respectively.

To facilitate the expansion in power series in β or α^2 we shall write the whole equations of (7) with dimensionless quantities, dividing these by mc^2 and introducing the atomic units for length by $\mathbf{r} = a_0 \rho$. Further replacing χ by $i\chi$, $\bar{\varphi}$ by $i\bar{\varphi}$ and $\bar{\chi}$ by $-\bar{\chi}$ we obtain

$$\left. \begin{aligned} (1 + \alpha^2 v - \epsilon)\varphi - \alpha(\sigma_e \nabla)\chi + \alpha(\sigma_M \nabla)\bar{\varphi} - \alpha^2 u S \bar{\chi} &= 0, \\ \alpha(\sigma_e \nabla)\varphi + (-1 + \alpha^2 v - \epsilon)\chi + \alpha^2 u S \bar{\varphi} + \alpha(\sigma_M \nabla)\bar{\chi} &= 0, \\ -\alpha(\sigma_M \nabla)\varphi + \alpha^2 u S \chi + (-2\mu + 1 + \alpha^2 v - \epsilon)\bar{\varphi} - \alpha(\sigma_e \nabla)\bar{\chi} &= 0, \\ -\alpha^2 u S \varphi - \alpha(\sigma_M \nabla)\chi + \alpha(\sigma_e \nabla)\bar{\varphi} + (-2\mu - 1 + \alpha^2 v - \epsilon)\bar{\chi} &= 0, \end{aligned} \right\} \quad (7')$$

denoting $(E - Mc^2)/mc^2 = \epsilon$, $M/m = \mu (= 1/\beta)$, $V/mc^2 = \alpha^2 v$, $Y'/mc^2 = \alpha^2 u S$, $\mathbf{p}/mc = -i\alpha \nabla$, where $-v = u = 1/\rho$, $S = \frac{1}{2} \{ (\sigma_e \sigma_M) + (\sigma_e \mathbf{r})(\sigma_M \mathbf{r})/\rho^2 \}$ and ∇ is the gradient operator with respect to ρ .

3.2. FM_F -eigenfunctions⁽²⁾¹⁵⁾

The two-component l_j, m_j -eigenfunctions for a single electron can be constructed by the ordinary normalized spherical harmonics $Y(l, m_l)$'s for two allowed l -values $l^\pm = j \pm \frac{1}{2}$ for a given j respectively as follows:

$$Y(l, \pm j, m_j) = \begin{pmatrix} a(l^\pm) Y(l^\pm, m_j - \frac{1}{2}) \\ b(l^\pm) Y(l^\pm, m_j + \frac{1}{2}) \end{pmatrix},$$

where

$$\left. \begin{aligned} a(l^\pm) &= \sqrt{(l^\pm \mp m_j + \frac{1}{2}) / (2l^\pm + 1)}, \\ b(l^\pm) &= \mp \sqrt{(l^\pm \pm m_j + \frac{1}{2}) / (2l^\pm + 1)}, \end{aligned} \right\}$$

and the two components correspond to the states $\sigma_{ez} = \pm 1$ respectively. The $Y(l, \pm j, m_j)$ is also normalized since $\{a(l^\pm)\}^2 + \{b(l^\pm)\}^2 = 1$. Quite similarly, four-component normalized $ljFM_F$ -functions for the system of a proton and an electron can be constructed by the ljm_j -eigenfunctions just obtained above as follows:

$$Y(l, j^\pm, F, M_F) = \begin{pmatrix} a(j^\pm) Y(l, j^\pm, M_F - \frac{1}{2}) \\ b(j^\pm) Y(l, j^\pm, M_F + \frac{1}{2}) \end{pmatrix}, \quad (8)$$

where the first two components refer to the state $\sigma_{M_z} = +1$, the other two refer to the state $\sigma_{M_z} = -1$, and $a(j)$ and $b(j)$ are obtained by replacing l by j and m_j by M_F in $a(l)$ and $b(l)$ respectively. The j 's can take two values $j^\pm = F \pm \frac{1}{2}$ for a given F and for each of these the l 's can take also two values $l^\pm = j \pm \frac{1}{2}$ respectively. We shall denote these four values of l by $l^{++} = j^+ + \frac{1}{2} = F + 1$, $l^{-+} = j^+ - \frac{1}{2} = F$, $l^{+-} = j^- + \frac{1}{2} = F$ and $l^{--} = j^- - \frac{1}{2} = F - 1$, and also the four-component $Y(l, j^\pm, F, M_F)$'s corresponding to these l -values by Y^{++} , Y^{-+} , Y^{+-} and Y^{--} for simplicity; that is

$$Y^{\pm\pm} = Y(l^{\pm\pm}, j^\pm, F, M_F), \quad Y^{\pm-} = Y(l^{\pm-}, j^\pm, F, M_F). \quad (8')$$

Finally 16-component FM_F -eigenfunctions Ψ for the system of a proton and an electron may be obtained by taking each component of (6) as a product of a radial function with one of these four-component FM_F -eigenfunctions; that is, with any Y of (8') $\varphi = \varphi(\rho) Y(l, j, F, M_F)$, and so on for other components.

3.3. Matrix elements of angular operators in $ljFM_F$ -scheme

The angular part of the matrix elements of any scalar product operator such as appearing in Eqs.(7') can easily be found in the above defined $ljFM_F$ -scheme by the method developed in Chap. 3 of the Condon-Shortley's book¹⁶⁾. We shall give here the elements of such operators for future use.

First one has the non-vanishing elements of the operator $(\sigma_e \nabla)$ between the states of same j but different l -values:

$$(l, \pm j | (\sigma_e \nabla) | l', \pm j) = (d/d\rho) + (1 + \kappa^\mp) / \rho = d_\mp, \quad (9a)$$

where $\kappa^\pm = \pm(j + \frac{1}{2})$ and the integration of radial parts is not written explicitly. (This is the same as in the following.) For those of the operator $(\sigma_M \nabla)$, using the symbols just defined above, one finds between the states with l -values differ-

ing by one from each other

$$\left. \begin{aligned} (l^{\pm}, \pm j, \pm F | (\sigma_M \nabla) | l^{\mp}, \mp j, \mp F) &= -f d_{\mp\pm}, \\ (l^{\pm}, \mp j, \mp F | (\sigma_M \nabla) | l^{\mp}, \mp j, \mp F) &= f d_{\mp\pm}, \\ (l^{\pm}, \pm j, \pm F | (\sigma_M \nabla) | l^{\pm}, \mp j, \mp F) &= -2\sqrt{-}f d_{\mp\pm}, \\ (l^{\pm}, \pm j, \pm F | (\sigma_M \nabla) | l^{\mp}, \mp j, \mp F) &= -2\sqrt{-}f d_{\mp\pm}, \end{aligned} \right\} \quad (9b)$$

where e.g. $d_{\pm\pm} = (d/d\rho) + (1 + x^{\pm\pm})/\rho$ with $x^{\pm\pm} = \pm k^{\pm} = \pm(j^{\pm} + \frac{1}{2})$ and $f = 1/(2F + 1)$, $\sqrt{-} = \sqrt{F(F+1)}$. Similarly one obtains the matrix elements of $(\sigma_e \sigma_M)$ and $(\sigma_e \rho)(\sigma_M \rho)/\rho^2$ appearing in Y' ; the former do not vanish only between the states with same l , while the latter do not vanish between the states with same l and with l differing by two from each other. They are given by

$$\left. \begin{aligned} (l^{\pm}, \pm j, \pm F | (\sigma_e \sigma_M) | l^{\pm}, \pm j, \pm F) &= 1, \\ (l^{\pm}, \pm j, \pm F | (\sigma_e \sigma_M) | l^{\mp}, \mp j, \mp F) &= -f(2F+3), \\ (l^{\pm}, \mp j, \mp F | (\sigma_e \sigma_M) | l^{\pm}, \mp j, \mp F) &= -f(2F-1), \\ (l^{\pm}, \mp j, \mp F | (\sigma_e \sigma_M) | l^{\mp}, \mp j, \mp F) &= -4\sqrt{-}f, \end{aligned} \right\} \quad (9c)$$

and

$$\left. \begin{aligned} (l^{\pm}, \pm j, \pm F | (\sigma_e \rho)(\sigma_M \rho)/\rho^2 | l^{\pm}, \pm j, \pm F) &= -f, \\ (l^{\pm}, \mp j, \mp F | (\sigma_e \rho)(\sigma_M \rho)/\rho^2 | l^{\pm}, \mp j, \mp F) &= f, \\ (l^{\pm}, \mp j, \mp F | (\sigma_e \rho)(\sigma_M \rho)/\rho^2 | l^{\mp}, \mp j, \mp F) &= -2\sqrt{-}f. \end{aligned} \right\} \quad (9d)$$

Further for the operators useful in § 5 one has in a similar fashion

$$(l, \pm j | (\sigma_e l) | l, \pm j) = -(1 + x^{\pm}), \quad (10a)$$

and

$$\left. \begin{aligned} (l^{\pm}, \pm j, \pm F | (\sigma_M l) | l^{\pm}, \pm j, \pm F) &= -(F+2), \\ (l^{\pm}, \mp j, \mp F | (\sigma_M l) | l^{\pm}, \mp j, \mp F) &= -fF(2F+3), \\ (l^{\pm}, \mp j, \mp F | (\sigma_M l) | l^{\pm}, \mp j, \mp F) &= f(F+1)(2F-1), \\ (l^{\pm}, \mp j, \mp F | (\sigma_M l) | l^{\mp}, \mp j, \mp F) &= F-1, \\ (l^{\pm}, \mp j, \mp F | (\sigma_M l) | l^{\mp}, \mp j, \mp F) &= 2\sqrt{-}f. \end{aligned} \right\} \quad (10b)$$

3.4. Correct eigenfunctions for the total system

It follows from the non-vanishing matrix elements given by (9) that the correct eigenfunctions of our equation (2') or (7') should be given by either one of the following two linear combinations:

$$\Psi_1 = \Psi_{++} + \Psi_{--}, \quad \Psi_2 = \Psi_{-+} + \Psi_{+-}, \quad (11)$$

where

$$\Psi_{\pm+} = \begin{pmatrix} \varphi_{\pm+}(\rho) Y^{\pm+} \\ \chi_{\pm+}(\rho) Y^{\mp+} \\ \bar{\varphi}_{\pm+}(\rho) Y^{\mp+} \\ \bar{\chi}_{\pm+}(\rho) Y^{\pm+} \end{pmatrix}, \quad \Psi_{\pm-} = \begin{pmatrix} \varphi_{\pm-}(\rho) Y^{\pm-} \\ \chi_{\pm-}(\rho) Y^{\mp-} \\ \bar{\varphi}_{\pm-}(\rho) Y^{\mp-} \\ \bar{\chi}_{\pm-}(\rho) Y^{\pm-} \end{pmatrix}, \quad (11a)$$

and the radial functions $\varphi_{++}(\rho)$, $\chi_{++}(\rho)$ etc. should be determined by Eqs.(7') hereafter. On substituting (9a), (9b), (9c), and (9d) in Eqs.(7') for Ψ_1 or Ψ_2 , elimination of the angular functions $Y^{\pm+}$ etc. from the equations may be carried out straightforwardly, leaving us a set of simultaneous equations for eight radial functions in each case. Nevertheless, the set of Eqs.(7') will be left unreduced for later use, except in special case $F=0$ which will be treated in the next section.

§ 4. Solution by expansion in power series in $\beta(=m/M)$ for special case $F=0$

4.1. Reduction to two-component equations

Since in special case $F=0$ only allowed value of j is $j^{\pm}=\frac{1}{2}$, Ψ_1 and Ψ_2 defined by (11) reduce simply to Ψ_{++} and Ψ_{--} which correspond to $p_{1/2}$ and $s_{1/2}$ states respectively. These shall be denoted by Ψ_+ and Ψ_- for simplicity. After eliminating the angular parts from (7') for Ψ_{\pm} by the use of Eqs.(9) and replacing every double suffix $\pm\pm$ simply by \pm , one has

$$\left. \begin{aligned} (1 + a^2 v - \epsilon_{\pm}) \varphi_{\pm}(\rho) - a d_{\mp}(\chi_{\pm}(\rho) + \bar{\varphi}_{\pm}(\rho)) - a^2 u S_{\pm} \bar{\chi}_{\pm}(\rho) &= 0, \\ a d_{\pm}(\varphi_{\pm}(\rho) - \bar{\chi}_{\pm}(\rho)) + (-1 + a^2 v - \epsilon_{\pm}) \chi_{\pm}(\rho) + a^2 u S_{\mp} \bar{\varphi}_{\pm}(\rho) &= 0, \\ a d_{\pm}(\varphi_{\pm}(\rho) - \bar{\chi}_{\pm}(\rho)) + a^2 u S_{\mp} \chi_{\pm}(\rho) + (-2\mu + 1 + a^2 v - \epsilon_{\pm}) \bar{\varphi}_{\pm}(\rho) &= 0, \\ -a^2 u S_{\pm} \varphi_{\pm}(\rho) + a d_{\mp}(\chi_{\pm}(\rho) + \bar{\varphi}_{\pm}(\rho)) + (-2\mu - 1 + a^2 v - \epsilon_{\pm}) \bar{\chi}_{\pm}(\rho) &= 0, \end{aligned} \right\} \quad (12)$$

where S_{\pm} denotes the matrix elements of the operator S given by (9c) and (9d):

$$(\iota^{\pm}, {}^+j, {}^+F | S | \iota^{\pm}, {}^+j, {}^+F) = S_{\pm}.$$

Each set of the simultaneous Eqs.(12) is readily shown to be reducible to a pair of new simultaneous equations for two functions ϕ_{\pm} and X_{\pm} introduced by

$$\phi_{\pm} = \varphi_{\pm} - \bar{\chi}_{\pm}, \quad X_{\pm} = \chi_{\pm} + \bar{\varphi}_{\pm}$$

respectively, owing to special condition $(\sigma_n \nabla) = -(\sigma_e \nabla)$ for $F=0$. Further introducing two-component functions by $\psi_{\pm} = (\phi_{\pm}, X_{\pm})$ one has then

$$H_{\pm} \psi_{\pm} = \epsilon_{\pm} \psi_{\pm} \quad (12')$$

with

$$H_{\pm} = H_{\pm}^{(0)} + \beta H'_{\pm}, \quad (12a)$$

$$H_{\pm}^{(0)} = \begin{pmatrix} 1 + a^2 v & -a d_{\mp} \\ a d_{\pm} & -1 + a^2 v \end{pmatrix}, \quad (12b)$$

$$2H_{\pm}' = \begin{pmatrix} 1 - (\epsilon_{\pm} - a^2 v)^2 + a^4 u^2 S_{\pm}^2 & -2a(\epsilon_{\pm} - a^2 v + a^2 u S_{\pm}) d_{\mp} \\ 2a(\epsilon_{\pm} - a^2 v + a^2 u S_{\mp}) d_{\pm} & 1 - (\epsilon_{\pm} - a^2 v)^2 + a^4 u^2 S_{\mp}^2 \end{pmatrix}. \quad (12c)$$

Thus far no approximation has been introduced or no term has been omitted from Eqs.(7'). Eqs.(12') may be solved now by expansion in power series in β according to usual perturbation theory as will be shown in the following.

4.2. Solution by expansion in power series in β

Substitution of the expansions

$$\epsilon_{\pm} = \sum_{\nu=0}^{\infty} \beta^{\nu} \epsilon_{\pm}^{(\nu)}, \quad \psi_{\pm} = \sum_{\nu=0}^{\infty} \beta^{\nu} \psi_{\pm}^{(\nu)} \quad \text{and} \quad H_{\pm} = \sum_{\nu=0}^{\infty} \beta^{\nu} H_{\pm}^{(\nu)}$$

in (12') yields first in the zeroth approximation

$$H_{\pm}^{(0)} \psi_{\pm}^{(0)} = \epsilon_{\pm}^{(0)} \psi_{\pm}^{(0)}, \quad (13)$$

which expresses, as properly expected, the radial equation of Dirac for $x = \pm 1$ respectively. Hence from the Gordon-Darwin's solutions one may put, say,

$$\psi_{\pm n}^{(0)}(\rho) = (g_{\pm n}(\rho), f_{\pm n}(\rho)) \quad (13a)$$

and

$$\epsilon_{\pm n}^{(0)} = \epsilon_n^{(0)} = [1 + \{a/(n - k + \gamma)\}^2]^{-1/2} \quad (13b)$$

with $k=1$, where $\gamma = \sqrt{k^2 - a^2}$ and n indicates the principal quantum number. H_{\pm}' becomes from (13) for the unperturbed orthogonal systems $\psi_{\pm}^{(0)}$

$$H_{\pm}' = (1 - \epsilon_{\pm}^2)/2 + (1 + \epsilon^{(0)}) (\epsilon_{\pm} - a^2 v) + a^4 v^2/2 - 2 \begin{pmatrix} \epsilon_{\pm} - a^2 v & 0 \\ 0 & 0 \end{pmatrix} + H_{\pm}'',$$

and substitution of (13b) for $\epsilon_{\pm}^{(0)}$ yields

$$H_{\pm n}^{(1)} = (1 - \epsilon_n^{(0)2})/2 + (1 + \epsilon^{(0)}) (\epsilon_n^{(0)} - a^2 v) + a^4 v^2/2 - 2 \begin{pmatrix} \epsilon_n^{(0)} - a^2 v & 0 \\ 0 & 0 \end{pmatrix} + H_{\pm}'' \quad (14)$$

with

$$H_{\pm}'' = -a^2 u \begin{pmatrix} S_{\pm} (1 - \epsilon^{(0)} + a^2 v - a^2 u S_{\pm}/2) & 0 \\ 0 & S_{\mp} (-1 - \epsilon^{(0)} + a^2 v - a^2 u S_{\mp}/2) \end{pmatrix}, \quad (14a)$$

and

$$H_{\pm n}^{(2)} = -\epsilon_n^{(0)} \epsilon_{\pm n}^{(1)} + (1 + \epsilon^{(0)}) \epsilon_{\pm n}^{(1)} - 2 \begin{pmatrix} \epsilon_{\pm n}^{(1)} & 0 \\ 0 & 0 \end{pmatrix},$$

and so on.

4.3. Eigenvalues of order β

The first-order perturbation energy $\epsilon_{\pm,n}^{(1)}$ is easily calculated by taking the expectation value of (14); that is

$$\epsilon_{\pm,n}^{(1)} = \langle H_{\pm,n}^{(1)} \rangle_n.$$

We shall obtain this up to the order a^4 in units of $mc^2\beta$ by expanding it further in power series in a^2 . Taking into account that the small component f_n is of order a compared with g_n and by means of the exact relation¹⁷⁾:

$$\int_0^\infty \rho^2 d\rho \cdot g_n^2(\rho) = (1 + \epsilon_n^{(0)})/2 \quad (15a)$$

with normalization $\int_0^\infty \rho^2 d\rho \cdot \{g_n^2(\rho) + f_n^2(\rho)\} = 1$, one has first for energy of order βa^2

$$O(a^2) \cdot \epsilon_{\pm,n}^{(1)} = O(a^2) \cdot (1 - \epsilon_n^{(0)2})/2 = a^2/2n^2 \quad (16)$$

in agreement with Eq.(5) in § 2.2, showing, from the expansion

$$\epsilon_n^{(0)} = 1 - \frac{a^2}{2n^2} - \frac{a^4}{2n^3} \left(\frac{1}{k} - \frac{3}{4n} \right) - \dots, \quad (13b')$$

the simple reduced mass correction to the Rydberg energy $-a^2/2n^2$, though only in the first power of β .

The energy of order βa^4 becomes from (14) and (14a), noting $-v=u=1/\rho$ and (13b') for $k=1$,

$$\begin{aligned} O(a^4) \cdot \epsilon_{\pm,n}^{(1)} &= O(a^4) \cdot (1 - \epsilon_n^{(0)2})/2 - (a^4/2n^2) \cdot (1 + S_{\pm}) \cdot O(a^0) \cdot \langle \rho^{-1} \rangle \\ &\quad + (a^4/2) (1 + 2S_{\pm}) \cdot O(a^0) \cdot \langle \rho^{-2} \rangle + 2a^2 (1 + S_{\mp}) \\ &\quad \times O(a^2) \cdot \int_0^\infty \rho d\rho \cdot f_{\pm,n}^2 + (a^4 S_{\pm}^2/2) \cdot O(a^0) \cdot \langle \rho^{-2} \rangle. \end{aligned} \quad (17)$$

We have by similar calculation to that of Bechert¹⁷⁾

$$O(a^2) \cdot \int_0^\infty \rho d\rho \cdot f_{\pm,n}^2(\rho) = (a^2/2n^3) (1 - 1/2n) \quad \text{for } k=1 \quad (15b)$$

and besides, known relations for nuclear charge Ze

$$O(a^0) \cdot \langle \rho^{-1} \rangle = Z/n^2, \quad (15c)$$

$$O(a^0) \cdot \langle \rho^{-2} \rangle = Z^2/n^3 (l + \frac{1}{2}). \quad (15d)$$

Further noting $S_+ = 0$, $S_- = -2$ from (9c) and (9d) Eq.(17) yields finally for the two hyperfine levels respectively

$$\left. \begin{aligned} O(a^4) \cdot \epsilon_{+,n}^{(1)} &= O(a^4) \cdot (1 - \epsilon_n^{(0)2})/2 - (2/3n^3)a^4 & \text{for } np_{1/2}, F=0, \\ O(a^4) \cdot \epsilon_{-,n}^{(1)} &= O(a^4) \cdot (1 - \epsilon_n^{(0)2})/2 - (2/n^3)a^4 & \text{for } ns_{1/2}, F=0. \end{aligned} \right\} \quad (17')$$

Here the last term in Eq.(17) proportional to S_{\pm}^2 arising from an iteration of the interaction Y is omitted as has been done throughout in similar calculations, inclusive of two-electron problem¹⁸⁾. We shall give a brief account for reasons why it should be done so. Actually it is shown soon below that reserving this term gives an uncorrect result for the hyperfine splittings.

• The first term in Eq.(17') gives from (13b')

$$O(a^4) \cdot \frac{1 - \epsilon_n^{(0)2}}{2} = \frac{a^4}{2n^3} \left(\frac{1}{k} - \frac{3}{4n} \right) - \frac{a^4}{8n^4}, \quad (17a)$$

showing that the correction to the fine structure can be expressed, in the first power of β , also by the simple reduced mass correction factor with the Darwin's additional term. This result corresponds exactly to Eq.(5) in §2.2.

The latter parts of (17') correspond to the hyperfine structure splittings of the two levels due to an idealized proton having one nuclear magneton, as is readily shown from the general formulas (A17) in Appendix. If we reserve here the term proportional to S_{\pm}^2 , it would give, in addition to (17'), $+(4/n^3)\beta a^4$ for $ns_{1/2}$, while zero for $np_{1/2}$ in obvious contradiction with experiment. Since the hyperfine splittings themselves are of order βa^4 in units of mc^2 , so the correction to them as well as the higher order correction in β to the fine structure, should be obtained only by evaluating the second order energy $\epsilon_{\pm,n}^{(2)}$; this would need a very laborious work and was not treated in this paper.

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Theory of the Temperature Effect of Electronic Energy Bands in Crystal, II

Toshinosuke MUTO and Seichi ÔYAMA

*Institute of Science and Technology, University of Tokyo and
Faculty of Science, Seizyô University*

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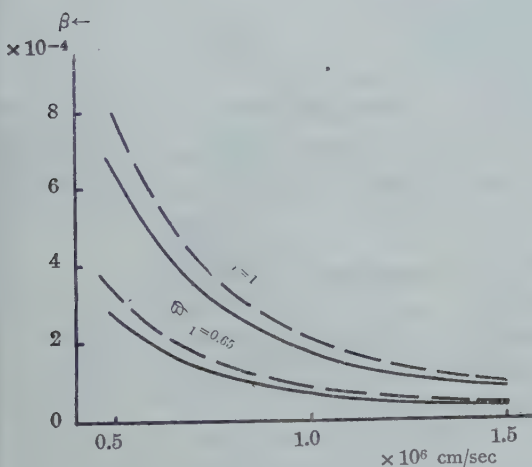


Fig. 1. Temperature coefficients β of Si versus the various values of its sound velocity c which is not measured directly.

--- $C_1^2 = 104 \text{ ev}^2$
— when $C_1^2 = 842 \text{ ev}^2$

According to our theoretical result for the temperature-shift of the energy gap in semiconductor and insulator derived in Part I¹⁾, the temperature coefficient β in the formula is observed to depend sensibly upon the sound velocities in the mentioned crystals, and the mobilities and effective masses of both electrons and positive holes at the ends of the energy bands, the reliable estimates of which quantities are now hardly possible unfortunately in view of the situation that the measured values are at variance with each other at present. So, for Si, Ge and Diamond, we have worked numerically α and β

in our formula for the temperature effect of the energy gap, corresponding to a number of possible values of the mentioned physical quantities involved, which results shall be reported here.

As for Si, the temperature coefficient β dependent upon the sound velocity is shown graphically in Fig. 1, since we have unfortunately no direct measurement of the sound velocity in Si at present.

For Ge ($M = 121.5 \times 10^{-24} \text{ gr}$, $a = 2.823 \times 10^{-8} \text{ cm}$), we get easily, after putting the numerical values of the various physical quantities into the equation (20) of Part I,

$$E_G = E_0 - 0.548 \times 10^2 \frac{\epsilon_1 C_1^2}{c} \left[1 + \frac{\epsilon_2 C_2^2}{\epsilon_1 C_1^2} - 2.52 \times 10^{-8} \epsilon_1 c \left(1 - \frac{\epsilon_2^2 C_2^2}{\epsilon_1^2 C_1^2} \right) \right]$$

$$- 2.09 \times 10^5 \frac{\epsilon_1 C_1^2}{\epsilon_1 c^2} \left(1 + \frac{\epsilon_2 C_2^2}{\epsilon_1 C_1^2} \right) T \equiv E_0 - \alpha - \beta T \text{ ev.}$$

$$(\theta \ll T; C_i \text{ in ev, } c \text{ in cm/sec}).$$

Further, it follows, for the sound velocity c , the mean kinetic energies C_1^2 and C_2^2 ,

$$\begin{aligned} c &= 0.445 \times 10^6 \text{ cm/sec from Debye temperature}^{(2)}, \\ &= 0.5357 \times 10^6 \text{ cm/sec from Shockley-Bardeen's letter}^{(3)}, \\ C_1^2 &= 24.0 \text{ (ev)}^2 \text{ from Seitz's computation of the mobilities}^{(2)}, \\ &= 11.2 \text{ (ev)}^2 \text{ from the measured mobility}^{(4)}, \\ &= 4.15 \text{ (ev)}^2 \text{ from the measurement of J. R. Haynes (refer to Shockley-Bardeen's letter}^{(3)} \text{ mentioned above)}, \\ &= 5.56 \text{ (ev)}^2 \text{ from Pearson's measurement}^{(5)}, \\ &= 3.24 \text{ (ev)}^2 \text{ from Dunlap's measurement}^{(6)}, \\ C_2^2/C_1^2 &= 2.06 \text{ } (\epsilon_1 = \epsilon_2 = 1) \text{ from Shockley-Bardeen's letter}^{(3)}, \\ &= 1.5 \text{ } (\epsilon_1 = \epsilon_2 = 1) \text{ Pearson's and Dunlap's observations.}^{(5)(6)} \end{aligned}$$

Similarly for Diamond ($M = 20.1 \times 10^{-24}$ gr, $a = 1.79 \times 10^{-8}$ cm, $c = 1.75 \times 10^6$ cm/sec), we have; inserting the required values into the general formula of Part I⁽¹⁾,

$$\begin{aligned} E_G &= E_0 - 2.1 \times 10^2 \frac{\epsilon_1 C_1^2}{c} \left[1 + \frac{\epsilon_2 C_2^2}{\epsilon_1 C_1^2} - 1.6 \times 10^{-8} \epsilon_1 c \left(1 - \frac{\epsilon_2 C_2^2}{\epsilon_1 C_1^2} \right) \right] \\ &\quad - 5.1 \times 10^5 \frac{\epsilon_1 C_1^2}{c^2} \left(1 + \frac{\epsilon_2 C_2^2}{\epsilon_1 C_1^2} \right) T \equiv E_0 - a - \beta T \text{ ev,} \\ &(\theta \ll T; C_i \text{ in ev, } c \text{ in cm/sec}). \end{aligned}$$

Further, it follows, for C_1^2 and C_2^2 ,

$$\begin{aligned} C_1^2 &= 936 \text{ (ev)}^2 \text{ from Seitz's computation of the mobilities}^{(2)}, \\ &= 1460 \text{ (ev)}^2 \text{ from the measurement of Bell Telephone Lab}^{(7)}, \\ &= 162 \text{ (ev)}^2 \text{ from the Hall effect measurement of Klick-Maurer}^{(8)}, \\ C_2^2/C_1^2 &\leq 4.5 \text{ } (\epsilon_1 = \epsilon_2 = 1) \text{ from the mobilities } \mu_1 = 900 \text{ and } \mu_2 \geq 200, \\ &\text{in which we shall tentatively assume the extreme values of } \mu_2 = 200 \text{ and the corresponding } C_2^2/C_1^2 = 4.5 \text{ in the actual evaluation of } a \text{ and } \beta \text{ for Diamond.} \\ C_1^2 &= 37.4 \text{ (ev)}^2 \text{ and } C_2^2/C_1^2 = 0.813 \text{ } (\epsilon_1 = \epsilon_2 = 1) \text{ from the recent measurement of Pearlstein and Sutton.}^{(9)} \end{aligned}$$

Table 1. Ge

$c \times 10^{-6}$ cm/sec	Seitz		Bell Lab		Haynes		Pearson		Dunlap	
	$a \times 10^4$	$\beta \times 10^4$	$a \times 10^4$	$\beta \times 10^4$	$a \times 10^4$	$\beta \times 10^4$	$a \times 10^4$	$\beta \times 10^4$	$a \times 10^4$	$\beta \times 10^4$
0.40	100	0.96	47	0.45	18	0.17	19	0.18	11	0.11
45	90	76	42	36	16	13	17	15	10	8
50	80	62	38	29	14	11	15	12	9	7
55	74	51	34	24	13	9	14	10	8	6
60	67	43	31	20	12	7	13	8	7	5
1.00	40	16	19	7	7	3	7	3	4	2

$$\Delta E = E_0 - a - \beta T \text{ ev.}$$

$$m_1^* = \epsilon_1 m, \quad m_2^* = \epsilon_2 m \quad (\epsilon_1 = \epsilon_2 = 1).$$

Table 2. Diamond

Seitz $a \times 10^4$ $\beta \times 10^4$		Bell Lab. $a \times 10^4$ $\beta \times 10^4$		Klick-Maurer $a \times 10^4$ $\beta \times 10^4$		Pearlstein et al $a \times 10^4$ $\beta \times 10^4$	
6300	8.7	9800	14	1100	1.5	83	0.12

$\Delta E = E_0 - \alpha - \beta T$ ev.

$m_1^* = \epsilon_1 m, \quad m_2^* = \epsilon_2 m \quad (\epsilon_1 = \epsilon_2 = 1).$

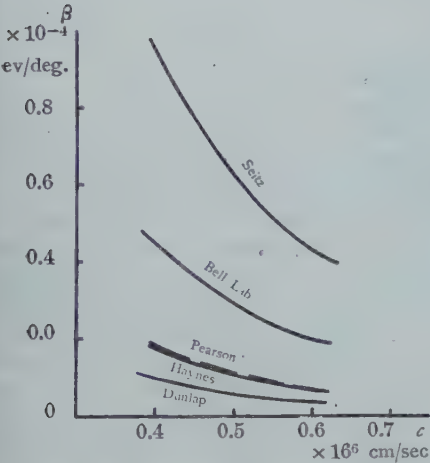


Fig. 2. Temperature coefficients β of Ge versus the various values of its sound velocity c which is not measured directly.

The numerical values of α and β for Ge and Diamond are shown, corresponding to each value of the mentioned physical quantities involved, in the accompanying tables and graphs.

Taking into consideration the unfavourable conditions which the accurate knowledges of the various kinds of physical quantities characteristic for the mentioned crystals are still lacking as far as we know, the quantitative comparison of our results with those of the available measurements seems to be premature but the order of magnitude of β is found to be in fair agreement with the observed result of Si ($\beta = 3 \times 10^{-4}$ ev/degree).¹⁰⁾ As for Ge, the observed value ($\beta = 1.1 \times 10^{-4}$ ev/degree)¹¹⁾ seems to be rather large in comparison with those of our calculation, the difference being probably due to the effect of the thermal expansion. The above mentioned observed values for Si and Ge were obtained through the detailed analysis of the resistivities and the Hall constants at relatively high temperature range ($T \gtrsim 500^\circ K$). From the photoelectric data at lower temperature range ($90^\circ \sim 290^\circ K$ for Si and $80^\circ \sim 300^\circ K$ for Ge), we get $\beta = 4.7 \sim 5 \times 10^{-4}$ ev/degree for Si¹²⁾ and $\beta = 4.6 \times 10^{-4}$ ev/degree for Ge¹³⁾ respectively. The examination of our theoretical formula valid at high and low temperatures shows that there exists really the suitable low temperature region in which the temperature coefficient β has relatively large values compared with the high temperature ones. Such theoretical situation seems to be responsible for a part of the mentioned, disagreements between the high and low temperature values of β , although the other possible causes for such disagreements may be considered to exist in addition to the above mentioned ones, as pointed out by Johnson and Fan¹¹⁾. As for Diamond, we unfortunately have no measurement of β as far as we know, and the discussion of β for such case shall be reserved for later occasion.

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On the Covariant Formalism of the Quantum Theory of Fields, II

Ryôyû UTIYAMA

Osaka University

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In the preceding paper of the same author,¹⁾ he has investigated only those cases where the field equations have no functional dependence among themselves, or more exactly, the action integral admits no groups of transformations depending on arbitrary functions. If we consider, however, any field interacting with the electromagnetic field, it will not be possible to make use of the method stated in (I) without any modification, because the canonical conjugate quantity to the scalar potential vanishes identically, and the equations of the electromagnetic field are not mutually independent.

In order to remove this formal difficulty, Fermi has proposed to modify the Lagrange function of the electromagnetic field by adding an adequate term $-1/2 \cdot (\partial A^\mu / \partial x^\mu)^2$ to it. An alternative method was suggested by Rosenfeld.²⁾ He has formulated the electrodynamics by using the elegant theory of invariant variation. At first sight his formalism appears to be covariant under Lorentz transformation, but it is actually difficult to consent to his assertion of the covariance of his theory. Accordingly the present author has tried to reformulate the same problem, following Rosenfeld's line of reasoning and at the same time preserving the required invariance.

In the quantum theory of fields, every field quantity is described with reference to any one Lorentz frame (let us call this x -system for simplicity), and the field equations give how the field quantities change with the lapse of time x^0 (in Heisenberg picture). But the identification of the time as the observation parameter with the time coordinate of the x -system, in general, destroys the apparent covariance of the field equations and

the commutation relations ($[C.R.]$). Therefore it is necessary to distinguish between these two time concepts. The observation parameter τ , if set equal to a constant, will describe a space-like hyperplane σ (or in general a space-like hypersurface) in x -system. In order to show the Lorentz covariance of the whole theory, we must prove it for any deformation of σ . Hence it is convenient to introduce, from the outset, an arbitrary system of curvilinear coordinate, as was done in (I).

In § 1, we shall present the outline of the classical electrodynamics from the standpoint stated above. In § 2 the quantization will be performed by the Heisenberg-Pauli's method by introducing a new quantity \mathcal{G}^0 which is canonically conjugate to the scalar potential. Further the character of the time derivative of the scalar potential will be investigated which was assumed to be an arbitrary c -number function in Rosenfeld's theory. In § 3 we shall show the covariance of our theory under both coordinate and gauge transformations. The field equations thus obtained differ from the usual ones. In particular, the so-called Lorentz condition, in our case, is not a condition imposed on the state vector, but a q -number relation. In §§ 4 and 5, the theory will be transformed into the interaction representation, where the Tomonaga-Schwinger equation emerges quite naturally, by virtue of our employment of the curved coordinate system.

Finally in the last two paragraphs the equivalence of our theory with the ordinary one will be proved and some remarks are given about the application of our method to the vector meson field.

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§ 1. Classical theory (Lagrange formalism)

Let us consider the electronic field interacting with the electromagnetic field. The Lagrangian density of the total system is, as usual, given by

$$\begin{aligned} L &= L_m + L_f - V, \\ L_m &= i\psi^\dagger \gamma^k \frac{\partial \psi}{\partial x^k} + i\alpha \psi^\dagger \psi, \quad L_f = -\frac{1}{4} f_{kl} f^{kl}, \\ V &= -j^k A_k, \quad j^k = e\psi^\dagger \gamma^k \psi, \end{aligned} \quad (1.1)^3$$

where A_1, A_2, A_3 and A_0 represent respectively the vector potential \mathbf{A} and minus the scalar potential $-\Phi$. In this paper we shall make use of the same notation as in (I).

Now all the expressions in (1.1) are expressed with reference to an arbitrarily chosen Lorentz frame (x -system). If we represent these expressions in regard to an arbitrary system of curvilinear coordinates (ξ -system), they will be written as follows ;

$$\begin{aligned} \mathfrak{L} &= D \cdot L = \mathfrak{L}_m + \mathfrak{L}_f - \mathfrak{B}, \\ \mathfrak{L}_m &= iD \cdot \{ \psi^\dagger \gamma^\mu(\xi) \frac{\partial \psi}{\partial \xi^\mu} + \alpha \psi^\dagger \psi \}, \\ \mathfrak{L}_f &= -\frac{1}{4} D \cdot f_{\mu\nu} f^{\mu\nu}, \\ f_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}, \\ \mathfrak{B} &= -D \cdot j^\mu A_\mu, \quad j^\mu = e\psi^\dagger \gamma^\mu(\xi) \psi, \end{aligned} \quad (1.1)'$$

where

$$\gamma^\mu(\xi) = h^\mu_k(\xi) \gamma^k, \quad A_\mu = h^\mu_k(\xi) A^k,$$

and ∇_μ represents the covariant derivative with regard to ξ^μ , whereas $A_{\nu,\mu}$ merely denotes $\partial A_\nu / \partial \xi^\mu$. Here it should be noticed that from the definition mentioned above, A_μ is a covariant vector, whereas ψ and ψ^\dagger are world scalars under general transformations of ξ -system (we shall call these ξ -transformations).

The field equations of the system considered are

$$\nabla_\nu f^{\mu\nu} = j^\mu \quad (1.2)$$

and

$$\gamma^\mu \left(\frac{\partial}{\partial \xi^\mu} - ie A_\mu \right) \psi + \alpha \psi = 0 \quad (1.3)$$

$$\left(\frac{\partial}{\partial \xi^\mu} + ie A_\mu \right) \psi^\dagger \gamma^\mu - \alpha \psi^\dagger = 0. \quad (1.3)^\dagger$$

3) The natural units are used, i.e. $\hbar = c = 1$.

It is easily seen that these equations are covariant under ξ -transformations.

Now the action integral $I = \int_{\Omega} \mathfrak{L}(d\xi)^4$ is invariant under the gauge transformation generated by the following infinitesimal one,

$$\begin{aligned} A_{\mu} &\rightarrow A'_{\mu} = A_{\mu} + \delta_G A_{\mu}, & \delta_G A_{\mu} &= \frac{\partial \lambda}{\partial \xi^{\mu}}, \\ \phi &\rightarrow \phi' = \phi + \delta_G \phi, & \delta_G \phi &= ie\lambda\phi, \\ \phi^{\dagger} &\rightarrow \phi'^{\dagger} = \phi^{\dagger} + \delta_G \phi^{\dagger}, & \delta_G \phi^{\dagger} &= -ie\lambda\phi^{\dagger}. \end{aligned} \quad (1.4)$$

Hence following the line of reasoning in § 2 (I), we obtain the following identities;

$$ie\{[\mathfrak{L}]_{\psi} \cdot \phi - [\mathfrak{L}]_{\psi^{\dagger}} \cdot \phi^{\dagger}\} - \frac{\partial}{\partial \xi^{\mu}} [\mathfrak{L}]_{A_{\mu}} \equiv 0, \quad (1.5)$$

and

$$\frac{\partial}{\partial \xi^{\nu}} \left\{ (ie \frac{\partial \mathfrak{L}}{\partial \phi_{,\nu}} \phi + [\mathfrak{L}]_{A_{\nu}}) \lambda + \frac{\partial \mathfrak{L}}{\partial A_{\mu,\nu}} \lambda_{,\mu} \right\} \equiv 0. \quad (1.6)$$

Taking notice of the fact that λ and its derivatives can be taken quite arbitrarily, the following identities are derived from (1.6)

$$\frac{\partial}{\partial \xi^{\mu}} (D \cdot j^{\mu}) - \frac{\partial}{\partial \xi^{\mu}} [\mathfrak{L}]_{A_{\mu}} \equiv 0, \quad (1.7)a$$

$$[\mathfrak{L}]_{A_{\mu}} - D j^{\mu} + \frac{\partial}{\partial \xi^{\nu}} \frac{\partial \mathfrak{L}}{\partial A_{\mu,\nu}} \equiv 0, \quad (1.7)b$$

$$\frac{\partial \mathfrak{L}}{\partial A_{\mu,\nu}} \equiv - \frac{\partial \mathfrak{L}}{\partial A_{\nu,\mu}}. \quad (1.7)c$$

The identity (1.7)b can be written as follows;

$$\frac{\partial \mathfrak{L}}{\partial A_{\mu}} = -ie \frac{\partial \mathfrak{L}}{\partial \phi_{,\mu}} \phi \equiv D \cdot j^{\mu}. \quad (1.7)'b$$

From this expression we know that the Lagrangian density must contain A_{μ} only in the combination $\partial \phi / \partial \xi_{\mu} - ie A_{\mu} \phi$. Furthermore, from (1.7)c, it is seen that $\partial A_{\mu} / \partial \xi^{\nu}$ must be contained in \mathfrak{L} in the combination $f_{\mu\nu}$. Making use of the equations of the electromagnetic field, we obtain from (1.7)a the equation of continuity;

$$\frac{\partial}{\partial \xi^{\mu}} (D \cdot j^{\mu}) = D \cdot \nabla_{\mu} j^{\mu} = 0. \quad (1.7)'a$$

Finally because of (1.7)c the canonical conjugate to the scalar potential vanishes identically. This fact interferes with the construction of the canonical formalism as already mentioned in the introduction.

Now if the field equations are used, the identity (1.6) gives us a constant of motion expressed as follows,

$$\begin{aligned}
 G &= \int_{\sigma} \left\{ \mathfrak{E}^{\mu} \frac{\partial \lambda}{\partial \xi^{\mu}} - D \cdot j^0 \lambda \right\} d\vec{\xi} \\
 &= \int_{\sigma} \left\{ \mathfrak{E}^0 \frac{\partial \lambda}{\partial \xi^0} - \left(\frac{\partial \mathfrak{E}^{\mu}}{\partial \xi^{\mu}} + D \cdot j^0 \right) \lambda \right\} d\vec{\xi}, \quad (1.8)
 \end{aligned}$$

where

$$\mathfrak{E}^{\mu} = \frac{\partial \mathfrak{L}}{\partial A_{\mu,0}} = D \cdot f^{\mu 0}, \quad (1.9)$$

and the integration should be performed over a space-like hypersurface σ which is defined by $\xi^0 = \text{constant}$.

In §3 we shall know that G is nothing but the infinitesimal generating operator for the gauge transformation (1.4). But in the present case we know from (1.9) and the field equations that G is actually equal to zero.

Apart from these characters of G , it is easily seen that G is invariant under both ξ - and Lorentz-transformations.

§ 2. Quantum theory

Let us introduce the canonical conjugate quantities as usual;

$$\begin{aligned}
 i\psi^* &= \frac{\partial \mathfrak{L}}{\partial \psi_0} = iD \cdot \phi_1 \dot{\psi}^0(\xi), \\
 \mathfrak{E}^{\mu} &= \frac{\partial \mathfrak{L}}{\partial A_{\mu,0}} = D \cdot f^{\mu 0}, \quad \mathfrak{E}^0 = \frac{\partial \mathfrak{L}}{\partial A_{0,0}} = 0. \quad (2.1)
 \end{aligned}$$

The last expression of (2.1) prohibits us from imposing the usual commutation relation $[[C, R]]$ for \mathfrak{E}^0 . Thus, following the reasoning of Rosenfeld let us introduce a new quantity \mathfrak{E}^0 which is canonically conjugate to $A_{0,0}$, and does not vanish identically. Then the $[C, R]$'s are written, as usual;

$$\begin{aligned}
 \{\phi_a^*(\vec{\xi}, \xi^0), \phi_b(\vec{\xi}', \xi^0)\} &= \delta_{ab} \delta(\vec{\xi} - \vec{\xi}'), \\
 [\mathfrak{E}^{\mu}(\vec{\xi}, \xi), A_{\nu}(\vec{\xi}', \xi^0)] &= -i\delta_{\nu}^{\mu} \delta(\vec{\xi} - \vec{\xi}'). \quad (2.2)
 \end{aligned}$$

All the other combinations are set to be commutative (or anticommutative).

In order to obtain the Hamiltonian density, the time derivative of any field quantity must be expressed in terms of canonical variables and their derivatives with regard to space-coordinates. But in the present case $\partial A_{0,0}/\partial \xi^0$ is indeterminate as we see from (2.1), i.e.,

$$A_{0,0} = A(\xi) = \text{indeterminate}.$$

Hence the Hamiltonian of the total system contains an indeterminate term and is given by

$$\bar{\mathfrak{H}} = \int_{\sigma} \mathfrak{H} d\vec{\xi},$$

where

$$\begin{aligned}
 \mathfrak{H} &= \mathfrak{H}_0 + \mathfrak{B} + \mathfrak{E}^\dagger A, \\
 \mathfrak{H}_0 &= -\frac{\gamma_{\bar{\mu}\bar{\nu}}}{2D \cdot \gamma^{\dot{0}\dot{0}}} \mathfrak{E}^{\bar{\mu}} \mathfrak{E}^{\bar{\nu}} + \mathfrak{E}^{\bar{\mu}} A_{\dot{0}, \bar{\mu}} + \frac{f_{\bar{\mu}\bar{\nu}}}{2\gamma^{\dot{0}\dot{0}}} \left((\gamma^{\dot{0}\bar{\nu}} \mathfrak{E}^{\bar{\mu}} - \gamma^{\dot{0}\bar{\mu}} \mathfrak{E}^{\bar{\nu}}) \right. \\
 &\quad \left. + \frac{1}{8} D \cdot f_{\bar{\mu}\bar{\nu}} f_{\bar{\rho}\bar{\sigma}} [(\gamma^{\bar{\nu}\bar{\sigma}} \gamma^{\bar{\mu}\bar{\rho}} - \gamma^{\bar{\mu}\bar{\sigma}} \gamma^{\bar{\nu}\bar{\rho}}) + \right. \\
 &\quad \left. + \frac{1}{\gamma^{\dot{0}\dot{0}}} (\gamma^{\bar{\mu}\bar{\sigma}} \gamma^{\bar{\nu}\bar{\rho}} \gamma^{\bar{\rho}\dot{0}} - \gamma^{\bar{\mu}\bar{\rho}} \gamma^{\bar{\nu}\bar{\sigma}} \gamma^{\bar{\sigma}\dot{0}} + \gamma^{\bar{\nu}\bar{\rho}} \gamma^{\bar{\mu}\bar{\sigma}} \gamma^{\bar{\sigma}\dot{0}} - \gamma^{\bar{\nu}\bar{\sigma}} \gamma^{\bar{\mu}\bar{\rho}} \gamma^{\bar{\rho}\dot{0}}) \right] \\
 &\quad \left. - i D \cdot \phi^\dagger \gamma^{\bar{\mu}} (\xi) \frac{\partial \psi}{\partial \xi^{\bar{\mu}}} - i x \psi^\dagger \psi \cdot D. \right. \quad (2.3)^{4)}
 \end{aligned}$$

Now Rosenfeld has assumed A to be a c -number function. But we shall propose, for the present, that A is an undeterminate linear combination of canonical variables $\mathfrak{E}^{\bar{\mu}}$ and $A_{\bar{\nu}}$. The restriction of linearity seems to be necessary in order to retain the linearity of the equations of the electromagnetic field. The precise determination of the form of A , will be given later.

The field equations are given by

$$\frac{d\Phi}{d\xi^{\dot{0}}} = i[\mathfrak{H}, \Phi]$$

where Φ is any field quantity. Especially, the equations of the electromagnetic field run as follows:

$$\frac{\partial A_{\bar{\mu}}}{\partial \xi^{\dot{0}}} = -\frac{1}{D \cdot \gamma^{\dot{0}\dot{0}}} \gamma_{\bar{\mu}\bar{\nu}} \mathfrak{E}^{\bar{\nu}} + A_{\dot{0}, \bar{\mu}} + \frac{\gamma^{\dot{0}\bar{\nu}}}{\gamma^{\dot{0}\dot{0}}} f_{\bar{\mu}\bar{\nu}} + i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') [A(\xi'), A_{\bar{\mu}}] d\xi^{\bar{\nu}}, \quad (2.4)a$$

$$\frac{\partial A^{\dot{0}}}{\partial \xi^{\dot{0}}} = A + i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') [A(\xi'), A^{\dot{0}}] d\xi^{\bar{\nu}}, \quad (2.4)b$$

$$\frac{\partial \mathfrak{E}^{\bar{\mu}}}{\partial \xi^{\dot{0}}} = -\frac{\partial}{\partial \xi^{\bar{\nu}}} (D \cdot f^{\bar{\mu}\bar{\nu}}) + i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') [A(\xi'), \mathfrak{E}^{\bar{\mu}}] d\xi^{\bar{\nu}} + D \cdot f^{\bar{\mu}}, \quad (2.4)c$$

$$\frac{\partial \mathfrak{E}^{\dot{0}}}{\partial \xi^{\dot{0}}} = \frac{\partial \mathfrak{E}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} + i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') [A(\xi'), \mathfrak{E}^{\dot{0}}(\xi)] d\xi^{\bar{\nu}} + D \cdot f^{\dot{0}}. \quad (2.4)d$$

From (2.4)c (2.4)d, we obtain

$$\begin{aligned}
 \frac{\partial^2 \mathfrak{E}^{\dot{0}}}{\partial \xi^{\dot{0}2}} &= i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') \left[A(\xi'), \frac{\partial \mathfrak{E}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} \right] d\xi^{\bar{\nu}} + i \int_{\sigma} \frac{\partial \mathfrak{E}^{\dot{0}}(\xi')}{\partial \xi^{\dot{0}}} [A(\xi'), \mathfrak{E}^{\dot{0}}] d\xi^{\bar{\nu}} \\
 &\quad + i \int_{\sigma} \mathfrak{E}^{\dot{0}}(\xi') \frac{\partial}{\partial \xi^{\dot{0}}} [A(\xi'), \mathfrak{E}^{\dot{0}}] d\xi^{\bar{\nu}}, \quad (2.5)
 \end{aligned}$$

4) As to the orders of non-commutative q -numbers, we assume, in this paper, that any suitable symmetrization procedure has been performed.

by virtue of the equation of continuity (1.7)'a which holds also in quantum theory. Further from (2.4)a, (2.4)c and (2.4)d, it results in

$$D\{\nabla_\nu f^{\mu\nu} - j^\mu\} = \left(\text{terms including } \mathfrak{E}^0 \text{ and } \frac{d\mathfrak{E}^0}{d\hat{\xi}^0}\right). \quad (2.6)$$

Therefore, in order to make our canonical equations (2.4) agree with the Maxwell's ones (1.2), it is necessary to impose the following conditions on the state vector Ψ_0 (in Heisenberg representation);

$$\mathfrak{E}^0(\hat{\xi})\Psi_0=0, \quad \frac{d\mathfrak{E}^0(\hat{\xi})}{d\hat{\xi}^0}\Psi_0=0, \quad \text{on the surface } \sigma. \quad (2.7)$$

Now, it is easily seen that the conditions (2.7) are not only necessary but also sufficient, from the fact that the time derivative of any order of \mathfrak{E}^0 can be expressed in terms of the linear combination of \mathfrak{E}^0 and $d\mathfrak{E}^0/d\hat{\xi}^0$ by virtue of eq. (2.5) and the linear character of Λ stated above. Therefore (2.7) holds at every instant and consequently it holds over the entire space-time;

$$(\nabla_\nu f^{\mu\nu} - j^\mu)\Psi_0=0. \quad (2.8)$$

This equation implies that our canonical equations (2.4)a, c, d are practically covariant.

Any transformation of the system of $\hat{\xi}$ -coordinates defined by-

$$\begin{aligned} \hat{\xi}^\mu &\rightarrow \hat{\xi}'^\mu = \hat{\xi}^\mu + \partial\hat{\xi}^\mu(\hat{\xi}), \\ A_\mu &\rightarrow A'_\mu(\hat{\xi}') = A_\mu(\hat{\xi}) + \partial A_\mu, \quad \partial A^\mu = -\frac{\partial\partial\hat{\xi}^\nu}{\partial\hat{\xi}^\mu} A_\nu, \\ \phi(\hat{\xi}) &\rightarrow \phi'(\hat{\xi}') = \phi(\hat{\xi}), \quad \partial\phi=0, \end{aligned} \quad (2.9)$$

can be expressed by a unitary transformation;

$$\Phi(\hat{\xi}) \rightarrow \Phi'(\hat{\xi}') = T\Phi(\hat{\xi})T^{-1}, \quad (2.10)$$

where Φ is any field quantity and T is a unitary operator which is given as follows in case of the infinitesimal transformation (2.9);

$$\begin{aligned} T &= 1 + iK, \\ K &= \int_\sigma (\mathfrak{E}^\mu \partial A_\mu - \mathfrak{T}^\mu_0 \partial\hat{\xi}^\mu) d\hat{\xi}, \\ \mathfrak{T}^\mu_0 &= \mathfrak{E}^\mu A_{\rho, \bar{\mu}} + i\psi^* \phi_{, \bar{\mu}}, \quad \mathfrak{T}^\mu_0 = \mathfrak{S}. \end{aligned} \quad (2.11)$$

From (2.10) and (2.11) it holds:

$$\delta^* \Phi = i[K, \Phi], \quad (2.12)$$

where

$$\delta^* \Phi = \Phi'(\hat{\xi}') - \Phi(\hat{\xi}) = \delta\Phi - \frac{d\Phi}{d\hat{\xi}^\mu} \partial\hat{\xi}^\mu.$$

Now, according to the definition of $\delta^*\Phi$, we know the following equality:

$$\frac{d}{d\xi^\mu}\delta^*\Phi = \delta^*\frac{d\Phi}{d\xi^\mu}.$$

Substituting from the field equations and the eq. (2.12), into the above equation, we obtain

$$\frac{d}{d\xi^0}\delta^*\Phi - \delta^*\frac{d\Phi}{d\xi^0} = i\left[\left(\frac{\partial K}{\partial\xi^0} + i[\tilde{\mathfrak{G}}, K] - \delta^*\tilde{\mathfrak{G}}\right), \Phi\right] = 0, \quad (2.13)^b$$

or at least,

$$\left[\left(\frac{dK}{d\xi^0} - \delta^*\tilde{\mathfrak{G}}\right), \Phi\right]\Psi_0 = 0, \quad (2.13)'$$

provided that the state vector Ψ_0 satisfies the condition (2.7). If (2.13) holds for every field quantity Φ then we can conclude that the expression (.....) has the following form;

$$M \equiv \frac{dK}{d\xi^0} - \delta^*\tilde{\mathfrak{G}} = c\text{-number}, \quad (2.14)$$

and we can easily show the covariance of all the equations under the transformation (2.9). On the contrary, if all the equations are covariant, then M must be a c -number.

In fact, (2.13)' holds for every field quantity Φ except for A_0 as we can see from (2.8). But if we assume Λ to be a c -number function, then it holds;

$$i[M, A_0] = \left\{ \frac{d}{d\xi^0}\delta^*A_0 - \delta^*\Lambda \right\}.$$

Therefore M must have the following form

$$M = \int_{\sigma} \mathfrak{G}^0 \left(\frac{d}{d\xi^0}\delta^*A_0 - \delta^*\Lambda \right) d\vec{\xi} + \int_{\sigma} M' d\vec{\xi},$$

where M' is commutative with A_0 and hence does not contain \mathfrak{G}^0 . On the other hand we know that the following equation holds for any field quantity Φ except for A_0 ;

$$[M, \Phi]\Psi_0 = 0.$$

Hence M' must be a c -number. Therefore M gets the following form;

$$M = \int_{\sigma} \mathfrak{G}^0 \left(\frac{d}{d\xi^0}\delta^*A_0 - \delta^*\Lambda \right) d\vec{\xi} + (c\text{-number}).$$

This expression contradicts with the requirement (2.14). Thus the assumption that Λ is a c -number interferes with the covariance of the field equations.

5) The notation δ^* represents the variation due to the explicit dependence on the quantities k_μ^k , k_μ^k etc.

Our next task is to determine the form of A so as to retain the covariance of the field equations.

The eq. (2.4)b can be generalized into the following tensor equations;

$$\frac{\partial A_\mu}{\partial \xi^\nu} = A_{\mu,\nu} \quad (2.15)$$

the (0, 0) component of which is just our eq. (2.4)b. Now let us, for the moment, deal with those cases where the ξ -system is a Lorentz frame and the transformation of it is restricted to the Lorentz transformations. Now (2.15) is reducible under the Lorentz transformation in the sense of the group representation, i.e. (2.15) can be separated into three irreducible parts. The first one is a symmetric tensor of the 2nd rank with the vanishing trace. The second is an antisymmetric tensor and the third is a scalar.

Our eq. (2.4)b is contained in the first and the third part. Therefore let us investigate these two cases separately.

i) *The case of the first irreducible part*

In this case we have to interpret eq. (2.4)b as it means

$$\frac{\partial A^\mu}{\partial \xi^\mu} = A^\mu{}_{,\mu} = 0 \quad (2.16)$$

i.e.

$$\frac{\partial A_0}{\partial \xi^0} = \frac{\partial A^\mu}{\partial \xi^\mu} \equiv A.$$

If we adopt again the curvilinear system, (2.16) must be replaced by

$$\nabla_\mu A^\mu = 0, \quad (2.16)'$$

i.e.

$$A \equiv -\frac{\gamma^{\dot{0}\dot{\rho}}}{\gamma^{\dot{0}\dot{0}}} A_{\dot{0},\dot{\rho}} - \frac{\gamma^{\dot{0}\dot{\rho}}}{\gamma^{\dot{0}\dot{0}}} f^{\dot{\rho}\dot{0}} - \frac{1}{\gamma^{\dot{0}\dot{0}}} \frac{\partial \gamma^{\rho\mu}}{\partial \xi^\mu} A_\rho - \frac{\gamma^{\dot{\mu}\dot{\nu}}}{\gamma^{\dot{0}\dot{0}}} A_{\nu,\dot{\mu}} - \frac{1}{\gamma^{\dot{0}\dot{0}}} \Gamma^{\mu}_{\dot{\mu}\dot{\nu}} A^\nu. \quad (2.16)''$$

Substituting from (2.1) into the second term of the right hand side of (2.16)'', and further from (2.16)'' into the Hamiltonian, then we obtain, instead of (2.4), the following eq.;

$$\frac{\partial A_{\dot{\mu}}}{\partial \xi^{\dot{0}}} = -\frac{1}{D\gamma^{\dot{0}\dot{0}}} \gamma^{\dot{\mu}\dot{\nu}} \mathfrak{E}^{\dot{\nu}} + A_{\dot{0},\dot{\mu}} + \frac{\gamma^{\dot{0}\dot{\nu}}}{\gamma^{\dot{0}\dot{0}}} f^{\dot{\mu}\dot{\nu}} - \frac{\gamma^{\dot{0}\dot{\nu}}}{D\gamma^{\dot{0}\dot{0}}} \mathfrak{E}^{\dot{0}}$$

or

$$\mathfrak{E}^{\dot{\mu}} = D \cdot f^{\dot{\mu}\dot{0}} + \frac{\gamma^{\dot{0}\dot{\mu}}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{E}^{\dot{0}}. \quad (2.17)$$

This equation contradicts to the original definition (2.1) of $\mathfrak{E}^{\dot{\mu}}$. However, the additional term containing $\mathfrak{E}^{\dot{0}}$ does not give rise to any change in eq. (2.17), by virtue of the commutativity of $\mathfrak{E}^{\dot{0}}$ and $A_{\dot{\mu}}$, on replacing the second term of (2.16)'' by

$$-\frac{\gamma^{\dot{0}\dot{\rho}}}{D\gamma^{\dot{0}\dot{0}}} \left\{ \mathfrak{E}^{\dot{\rho}} - \frac{\gamma^{\dot{0}\dot{\rho}}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{E}^{\dot{0}} \right\},$$

and further $\partial A_{\dot{\mu}}/\partial \xi^{\dot{0}}$ in the Hamiltonian by the righthand side of (2.17) instead of (2.1). The new Hamiltonian density, thus obtained, has the following form;

$$\mathfrak{H}' = \mathfrak{H}_0 + \mathfrak{B} - \frac{1 - \gamma^{\dot{0}\dot{0}} \gamma^{\dot{0}\dot{0}}}{2D \cdot (\gamma^{\dot{0}\dot{0}})^2} (\mathfrak{E}^{\dot{0}})^2 + \mathfrak{E}^{\dot{0}} A = \mathfrak{H}_0 + \mathfrak{B} + \mathfrak{E}^{\dot{0}} A', \quad (2.18)$$

where \mathfrak{G}_0 and \mathfrak{B} are given in (2.3), and A' is defined by

$$A' \equiv -\frac{1}{\gamma^{\dot{0}\dot{0}}} \left\{ \gamma^{\dot{0}\dot{p}} \bar{A}_{\dot{0}, \dot{p}} + \frac{\gamma^{\dot{0}\dot{p}}}{D} \left(\mathfrak{G}_{\dot{p}} - \frac{\gamma^{\dot{p}\dot{0}}}{2\gamma^{\dot{0}\dot{0}}} \mathfrak{G}_{\dot{0}} \right) + \frac{\partial \gamma^{\rho\mu}}{\partial \xi^\mu} A_\rho + \gamma^{\bar{\mu}\nu} A_{\nu, \bar{\mu}} + I_{\mu\nu}^\mu A^\nu \right\}. \quad (2.16)'''$$

In this way we can obtain new field eqs. without self-contradiction, which run on rearrangement, thus :—
The first set are

$$\mathfrak{G}_{\bar{\mu}} = D f^{\bar{\mu}\dot{0}} + \frac{\gamma^{\dot{0}\bar{\mu}}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{G}_{\dot{0}}. \quad (2.17)$$

If we introduce the new quantity E by

$$\mathfrak{G}_{\dot{0}} = -D \gamma^{\dot{0}\dot{0}} E, \quad (2.19)$$

(2.17) and (2.19) can be brought into a single compact form;

$$\mathfrak{G}^\mu = D \cdot (f^{\mu\dot{0}} - \gamma^{\mu\dot{0}} E). \quad (2.17)a$$

The remaining sets of the field equations instead of (2.4)b, c, d, are

$$F_{\mu} A^\mu = 0, \quad (2.17)b$$

and

$$F_{\nu} f^{\mu\nu} - \gamma^{\mu\nu} \frac{\partial E}{\partial \xi^\nu} = j^\mu. \quad (2.17)c$$

Substituting from (2.17)b into (2.17)c, we can rewrite (2.17)c into

$$\square A_\mu + \frac{\partial E}{\partial \xi^\mu} = -j_\mu, \quad (2.17)'c$$

since our underlying space-time is flat. Further from this equation, we deduce

$$\square E = 0 \quad (2.17)''c$$

provided that the equation of continuity and the eq. (2.17)b are taken into account. This is nothing but the eq. (2.5). The condition (2.7) can, in the present case, be replaced by

$$\mathfrak{G}_{\dot{0}} \cdot \Psi_0 = 0, \quad \mathfrak{F} \cdot \Psi_0 = 0 \quad (2.20)$$

where

$$\mathfrak{F} = \frac{\partial \mathfrak{G}_{\bar{\mu}}}{\partial \xi^\mu} + D \cdot j_{\dot{0}},$$

if we make use of the field equation.

ii) The case of the third irreducible part

In this case we have to take (2.4)b as meaning

$$\frac{\partial A^\mu}{\partial \xi^\mu} = (\text{a world scalar}) = \mathcal{S}. \quad (2.21)$$

The discussion of this case will be given in the later § 6.

In concluding this somewhat lengthy paragraph, we may summarize our problems in view to be investigated in the next paragraph. These consist in proving the following facts:

- i) E is a world scalar.
- ii) The condition (2.20) is covariant under ξ -transformation.
- iii) The two conditions (2.20) are mutually compatible.
- iv) The covariance of $[C.R.]$'s.
- v) The gauge-invariance of the whole system of equations and conditions.

§ 3. Coordinate- and gauge-transformation

We shall begin with the discussion of the ξ -transformation (2.9). The infinitesimal generating operator of (2.9) has been already given by (2.11).

Making use of the field equations (2.17)a, b, c, we can see the transformation character of \mathfrak{G}^ρ which is given by

$$\begin{aligned}\delta^*\mathfrak{G}^\rho &= i[K, \mathfrak{G}^\rho] = \delta\mathfrak{G}^\rho - \frac{\partial\mathfrak{G}^\rho}{\partial\hat{\xi}^\mu} \delta\hat{\xi}^\mu \\ &= \frac{\partial\delta\hat{\xi}^\rho}{\partial\hat{\xi}^\mu} \mathfrak{G}^\mu + \frac{\partial\delta\hat{\xi}^0}{\partial\hat{\xi}^\mu} \mathfrak{G}^{\rho\mu} + \frac{1}{D} \cdot \delta D \cdot \mathfrak{G}^\rho - \frac{\partial\mathfrak{G}^\rho}{\partial\hat{\xi}^\mu} \cdot \delta\hat{\xi}^\mu,\end{aligned}\quad (3.1)$$

where

$$\mathfrak{G}^{\rho\mu} = D \cdot \{ f^{\rho\mu} - \gamma^{\rho\mu} E \}, \quad \mathfrak{G}^\rho \equiv \mathfrak{G}^{\rho 0}.$$

This shows that $\mathfrak{G}^\rho (= \mathfrak{G}^{\rho 0})$ is the $(\rho, 0)$ component of a tensor density $\mathfrak{G}^{\rho\mu}$. Hence E must be a world scalar. Thus the first problem in §2 has been proved. From this statement we can easily understand the covariance of the equations of field (2.17)a, b, c. Therefore we obtain the following result from the reasoning of §2;

$$M \equiv \frac{dK}{d\hat{\xi}^0} - \delta^*\mathfrak{G}' = c\text{-number}.$$

In fact we can show that M is equal to zero by somewhat laborious calculation. Putting ρ equal to 0 in (3.1), we get

$$\delta^*\mathfrak{G}^0 = \left(2 \frac{\partial\delta\hat{\xi}^0}{\partial\hat{\xi}^\mu} \frac{\gamma^0_\mu}{\gamma^0_0} + \frac{1}{2} \gamma^{\mu\nu} \delta\gamma_{\mu\nu} \right) \mathfrak{G}^0 - \frac{\partial\mathfrak{G}^0}{\partial\hat{\xi}^\mu} \delta\hat{\xi}^\mu,$$

hence it holds

$$\delta^*\mathfrak{G}^0 \cdot \Psi_0 = 0, \quad (3.2)$$

if we make use of the condition (2.20).

In the same way we can also show

$$\delta^*\mathfrak{G}' \cdot \Psi_0 = 0. \quad (3.2)'$$

Thus the 2nd problem has been solved. The proof of the covariance of $[C.R]'$'s is trivial, i.e.

$$\begin{aligned}[A'(\vec{\xi}), B'(\vec{\eta})]_{\text{on}\sigma'} &= T[A(\vec{\xi}), B(\vec{\eta})]_{\text{on}\sigma} T^{-1} \\ &= [A(\vec{\xi}), B(\vec{\eta})]_{\text{on}\sigma} (= c\text{-number}),\end{aligned}$$

where σ' is a hypersurface defined by the same numerical value of $\hat{\xi}^0$ as that of $\hat{\xi}^0$ defining the original surface σ . Thus we have given the proof of covariance of the whole system of formulae under any transformation of $\hat{\xi}$ -system.

In the second place let us consider the gauge transformation (1.4), where λ was an arbitrary c -number and scalar function. But in the present case, λ must satisfy the following wave equation

$$\square\lambda \equiv F_\mu F^\mu \lambda = 0, \quad (3.3)$$

because of the gauge invariance of the eq. (2.17)b.

Now the gauge transformation (1.4) is expressed by a unitary transformation

$$\Phi \rightarrow \Phi' = T_G \Phi T_G^{-1} = \Phi + \delta_G \Phi, \quad (3.4)$$

where T_G is given by the following expression in case of the infinitesimal transformation;

$$T_G = 1 + iG, \quad G = \int_{\sigma} \left\{ \mathfrak{E}^0 \frac{\partial \lambda}{\partial \xi^0} - \mathfrak{F} \cdot \lambda \right\} d\vec{\xi}. \quad (3.5)$$

From (3.4) we see at once the gauge covariance of $[C.R.]$'s.

As the gauge transformation is commutative with the $\hat{\xi}$ -transformation, the following equality holds for any field quantity Φ ;

$$\delta_G \delta^* \Phi = \delta^* \delta_G \Phi. \quad (3.6)$$

Substituting from (3.4) and (2.12) into (3.6), we obtain

$$\delta^* G = 0. \quad (3.7)$$

If we restrict the $\hat{\xi}$ -transformation to the special one

$$\delta \xi^0 = \epsilon = \text{constant parameter},$$

$$\delta \xi^{\mu} = 0, \quad (3.8)$$

then (3.7) becomes

$$\frac{dG}{d\xi^0} = 0. \quad (3.7)'$$

Eq. (3.7) means the invariance of G under any $\hat{\xi}$ -transformation while (3.7)' means that G is a constant of motion. Further, if the $\hat{\xi}$ -transformation is restricted to another special one:

$$\delta \xi^0 = \begin{cases} < 0 & \text{near one particular point } \vec{\xi}_p \text{ on } \sigma, \\ = 0 & \text{at all the other points} \end{cases} \quad (3.9)$$

and

$$\frac{\partial \delta \xi^0}{\partial \xi^0} = 0 \quad \text{near the surface } \sigma,$$

$$\text{we obtain from (3.7)} \quad \frac{\partial G[\sigma]}{\partial \sigma_p} = 0 \quad (3.7)''$$

where $G[\sigma]$ is considered to be a functional of σ .

From (3.7) and (3.7)', it can be proved that the equation (2.12) and the field equations are gauge-invariant,

Now (3.7)' may, on the ground of (3.3), be computed to the form

$$\frac{dG}{d\xi^0} = \int_{\sigma} \left\{ \mathfrak{A} \frac{\partial \lambda}{\partial \xi^0} + \mathfrak{B} \lambda \right\} d\vec{\xi} = 0.$$

It follows

$$\mathfrak{A} = 0, \quad \text{and} \quad \mathfrak{B} = 0, \quad (3.10)$$

where

$$\begin{aligned}\mathfrak{A} &= \frac{d\mathfrak{G}^{\dot{0}}}{d\xi^{\dot{0}}} - \mathfrak{F} + 2 \frac{\partial}{\partial \xi^{\dot{\mu}}} \left(\frac{\gamma^{\dot{0}\dot{\mu}}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{G}^{\dot{0}} \right) - \frac{\mathfrak{G}^{\dot{0}}}{\gamma^{\dot{0}\dot{0}}} \left(\frac{\partial \gamma^{\dot{0}\dot{\nu}}}{\partial \xi^{\dot{\nu}}} + \frac{1}{2} \gamma^{\nu\sigma} \gamma_{\nu\sigma, \rho} \gamma^{\rho\dot{0}} \right), \\ \mathfrak{B} &= -\frac{d\mathfrak{F}}{d\xi^{\dot{0}}} - \frac{\partial^2}{\partial \xi^{\dot{\mu}} \partial \xi^{\dot{\nu}}} \left(\frac{\gamma^{\dot{\mu}\dot{\nu}}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{G}^{\dot{0}} \right) + \frac{\partial}{\partial \xi^{\dot{\mu}}} \left\{ \left(\frac{\partial \gamma^{\dot{\mu}\dot{\nu}}}{\partial \xi^{\dot{\nu}}} + \frac{1}{2} \gamma^{\nu\sigma} \gamma_{\nu\sigma, \rho} \gamma^{\rho\dot{\mu}} \right) \frac{\mathfrak{G}^{\dot{0}}}{\gamma^{\dot{0}\dot{0}}} \right\}.\end{aligned}\quad (3.11)$$

From (3.10) and (3.11) we can also show the compatibility of the condition (2.20) with the eqs. of field. In quite the same way it can also be proved from the eq. (3.7) that (2.20) is covariant under ξ -transformation.

The third problem mentioned in § 2, i.e. the compatibility between the two conditions (2.20) is easily proved because of the commutability of $\mathfrak{G}^{\dot{0}}$ and \mathfrak{F} . Further from this character of commutability, we know also the gauge invariance of the condition (2.20).

Thus we have proved the gauge invariance of all the formula.

We have, so far, expressed all the formulae in reference to the ξ -system. Now let us rewrite all the expressions with regard to the original Lorentz frame in order to show the apparent covariance under the Lorentz transformation. The field equations are

$$\begin{aligned}\gamma^k \left(\frac{\partial}{\partial x^k} - ieA_k \right) \psi + \kappa \psi &= 0, \\ \frac{\partial f^{kl}}{\partial x^l} - \frac{\partial E}{\partial x^k} &= j^k, \quad \square E = 0, \\ \frac{\partial A^k}{\partial x^k} &= 0, \quad E^k = f^{k0} - g^{k0} E.\end{aligned}\quad (3.12)$$

And the $[C.R.]$'s are;

$$\begin{aligned}\int_{\sigma} [\{ f^{kl}(x') - g^{kl} E(x') \}, A_m(x)] d\sigma'_l &= -i\delta_m^k, \\ \int_{\sigma} \{ (\psi^\dagger(x') \gamma^k)_\alpha, \psi_\beta(x) \} d\sigma'_k &= \delta_{\alpha\beta}.\end{aligned}\quad (3.13)$$

The condition (2.20) is expressed by

$$E \cdot \Psi_0 = 0, \quad n^k(x) \frac{\partial E}{\partial x^k} \cdot \Psi_0 = 0, \quad (3.14)$$

where n^k is the k -th component of a unit normal of the surface σ at the point (x) and is defined by

$$n^k(x) = -\frac{\dot{\gamma}^{\dot{0}k}}{\sqrt{-\dot{\gamma}^{\dot{0}\dot{0}}}}.$$

Finally, the generating operator G of the gauge transformation can be expressed by

$$G = \int_{\sigma} \left\{ (f^{kl} - g^{kl} E) \frac{\partial \lambda}{\partial x^k} - j^l \lambda \right\} d\sigma_l. \quad (3.15)$$

If we make use of the eqs (3.12), the G can be transformed into the the following form;

$$G = \int_{\sigma} \left\{ \frac{\partial E}{\partial x^k} \lambda - E \frac{\partial \lambda}{\partial x^k} \right\} d\sigma^k. \quad (3.15)'$$

From this expression, we can easily verify the eq. (3.7)'';

$$\frac{\delta G[\sigma]}{\delta \sigma_p} = \frac{\partial}{\partial x^k} \left\{ \frac{\partial E}{\partial x^k} \lambda - E \frac{\partial \lambda}{\partial x^k} \right\} = 0.$$

§ 4. Interaction representation

Let us consider the following unitary transformation;

$$\mathbf{A} \rightarrow A = U \mathbf{A} U^{-1} \quad (4.1)$$

where \mathbf{A} is any field quantity in Heisenberg representation and A is the same quantity in the interaction representation (operators in Heisenberg representation will be denoted by bold face letters). U is a unitary operator whose transformation character under the ξ -transformation is defined by

$$i\delta^* U = - \int_{\sigma} (\mathfrak{B} \delta \xi^0) d\xi^0 \cdot U, \quad (4.2)$$

$$\mathfrak{B} = DU V U^{-1}.$$

Then from (4.1) we obtain the transformation rule of any field quantity

$$\delta^* A = i[K_0, A], \quad (4.3)$$

if we make use of the definition (4.2) and the eq. (2.12). Here K_0 is the infinitesimal generating operator of the free fields, that is, K_0 does not contain the interaction term. If we restrict the ξ -transformation to the special one (3.8), we obtain from (4.3)

$$\frac{dA}{d\xi^0} = i[\bar{\mathfrak{S}}_f, A], \quad (4.4)$$

where

$$\bar{\mathfrak{S}}_f = \bar{\mathfrak{S}}_0 + \mathfrak{G}^0 A',$$

i.e. the eq. (4.4) is nothing but the equation of the free field. In this case, (4.2) is replaced by

$$i \frac{dU(\xi^0)}{d\xi^0} = \bar{\mathfrak{B}} U(\xi^0). \quad (4.5)$$

On the other hand, if the ξ -transformation is restricted to the second special case (3.9), then (4.2) becomes

$$i \frac{\delta U[\sigma]}{\delta \sigma_p} = V(p) \cdot U[\sigma], \quad (4.5)'$$

where U is considered to be a functional of σ rather than a function of ξ^0 .

The state vector $\Psi(\xi^0)$ in the interaction representation corresponding to (4.1), is defined as follows;

$$\begin{aligned}\Psi_0 \rightarrow \Psi(\xi^0) &= U(\xi^0) \cdot \Psi_0, \\ \text{or} \\ \Psi[\sigma] &= U[\sigma] \cdot \Psi_0.\end{aligned}\quad (4.6)$$

From (4.6) and the eqs. (4.5), (4.5)', we obtain the Tomonaga-Schwinger equation;

$$i \frac{d\Psi(\xi^0)}{d\xi^0} = \mathfrak{B} \cdot \Psi(\xi^0), \quad (4.7)$$

$$i \frac{\partial \Psi[\sigma]}{\partial \sigma_p} = V(P) \cdot \Psi[\sigma], \quad (4.7)'$$

or in general

$$i \partial^* \Psi = - \int_{\sigma} (\mathfrak{B} \cdot \partial \xi^0) d\vec{\xi} \cdot \Psi. \quad (4.7)''$$

The $[C.R]$'s and the condition (2.20) retain their original expressions also in this representation. Especially the latter is given by;

$$\mathfrak{E}^0(P) \cdot \Psi[\sigma] = 0, \quad \mathfrak{F}(P) \cdot \Psi[\sigma] = 0, \quad (4.8)$$

where the point P lies on the surface σ .

Next let us consider the gauge transformation, the generating operator of which keeps the same form as in case of the Heisenberg representation. Now let us separate G into two parts, one of which does not contain the interaction constant e while the other depends on e , i.e.

$$\begin{aligned}G &= G_0 + G_1, \\ G_0 &= \int_{\sigma} \left\{ \mathfrak{E}^0 \frac{\partial \lambda}{\partial \xi^0} - \frac{\partial \mathfrak{E}^p}{\partial \xi^p} \cdot \lambda \right\} d\vec{\xi}, \\ G_1 &= - \int_{\sigma} Df^0 \lambda d\vec{\xi}.\end{aligned}\quad (4.9)$$

The eq. (3.7) is, now, replaced by

$$\partial^* G - i \left[\int (\mathfrak{B} \partial \xi^0) d\vec{\xi}, G \right] = 0. \quad (4.10)$$

Making use of (4.3), this is written as follows;

$$\partial^* G + i[K_0, G] - i \left[\int (\mathfrak{B} \partial \xi^0) d\vec{\xi}, G \right] = 0. \quad (4.10)'$$

(4.10)' is an identity because it contains only canonical field quantities and their derivatives with regard to space coordinates. Hence there result the following identities from (4.10)' corresponding to each power of e ;

$$\delta^* G_0 = \delta^* G_0 + i[K_0, G_0] = 0, \quad (4.11)a$$

$$\delta^* G_1 - i \left[\int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi}, G_0 \right] = 0, \quad (4.11)b$$

$$\left[\int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi}, G_1 \right] = 0. \quad (4.11)c$$

Now from (4.10), the following relation is obtained ;

$$\begin{aligned} T_G (i\delta^* + \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi}) T_G^{-1} \cdot \Psi \\ = \{ i\delta^* + \delta^* G + \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi} + i[G, \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi}] \} \Psi \\ = (i\delta^* + \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi}) \Psi = 0, \end{aligned} \quad (4.12)$$

if we make use of the eq. (4.7)''.

Let us separate T_G into two factors in correspondence to (4.9), and put

$$\begin{aligned} T_G &= T_0 \cdot T_1 = T_1 \cdot T_0, \\ T_0 &= 1 + iG_0, \quad T_1 = 1 + iG_1, \end{aligned} \quad (4.13)$$

then, from (4.11)a it follows ;

$$T_0 \delta^* (T_0^{-1}) = 0.$$

Hence from (4.12) we obtain

$$\begin{aligned} T_0 \left\{ i\delta^* + \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi} \right\} T_0^{-1} \cdot (T_1^{-1} \cdot \Psi) \\ = \left\{ i\delta^* + \int_{\sigma} (\mathfrak{B}' \cdot \delta \dot{\xi}^0) d\vec{\xi} \right\} \Psi' = 0 \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} \mathfrak{B}' &= T_0 \mathfrak{B} T_0^{-1} = -Dj^{\mu} \left(A_{\mu} + \frac{\partial \lambda}{\partial \xi^{\mu}} \right), \\ \Psi' &= T_0^{-1} \Psi. \end{aligned} \quad (4.15)$$

Therefore, if we define the transformation character of the state vector Ψ and the field quantity Φ under the gauge transformation by the following equations ;

$$\begin{aligned} \Psi &\rightarrow \Psi' = T_1^{-1} \cdot \Psi = \Psi - iG_1 \Psi, \\ \Phi &\rightarrow \Phi' = T_0 \Phi T_0^{-1} = \Phi + i[G_0, \Phi], \end{aligned} \quad (4.16)$$

then the eq. (4.14) guarantees the gauge invariance of the eq. (4.7)''. Thus we have learned the two transformation character of Ψ ;

$$\delta^* \Psi = i \left\{ \int_{\sigma} (\mathfrak{B} \delta \dot{\xi}^0) d\vec{\xi} \right\} \cdot \Psi$$

and

$$\delta_G \Psi = -iG_1 \cdot \Psi. \quad (4.17)^6$$

As to the compatibility of these two equations, we can easily prove it by making use of (4.11)b and (4.11)c.

The field equations (4.4), the condition (4.8) and $[C.R]$'s are invariant for the transformation (4.16).

Our next task lies in the verification of the compatibility between the condition (4.8) and the equation (4.7)". Eq. (4.10) can be written as follows,

$$\delta^* G - i \left[\int_{\sigma} (\mathfrak{B} \delta \xi^{\dot{0}}) d\vec{\xi}, G \right] = \int_{\sigma} \left\{ \mathfrak{W} \frac{\partial \lambda}{\partial \xi^{\dot{0}}} + \mathfrak{B}' \lambda \right\} d\vec{\xi},$$

by virtue of (3.3). From this equation we infer

$$\mathfrak{W} = 0, \quad \mathfrak{B}' = 0, \quad (4.18)$$

where

$$\begin{aligned} \mathfrak{W} = & \delta^* \mathfrak{E}^{\dot{0}} - \mathfrak{E}^{\dot{0}} \frac{\partial \delta \xi^{\dot{0}}}{\partial \xi^{\dot{0}}} + \frac{\partial}{\partial \xi^{\dot{\mu}}} (\mathfrak{E}^{\dot{0}} \delta \xi^{\dot{\mu}}) + \mathfrak{F} \delta \xi^{\dot{0}} \\ & - 2 \frac{\partial}{\partial \xi^{\dot{\mu}}} \left(\frac{\dot{\gamma}^{\dot{0}\dot{\mu}}}{\dot{\gamma}^{\dot{0}\dot{0}}} \mathfrak{E}^{\dot{0}} \delta \xi^{\dot{0}} \right) + \frac{1}{\dot{\gamma}^{\dot{0}\dot{0}}} \left(\dot{\gamma}^{\dot{0}\nu} + \frac{1}{2} \dot{\gamma}^{\nu\sigma} \dot{\gamma}_{\nu\sigma, \rho} \dot{\gamma}^{\rho\dot{0}} \right) \mathfrak{E}^{\dot{0}} \delta \xi^{\dot{0}} - i \left[\int_{\sigma} (\mathfrak{B} \delta \xi^{\dot{0}}) d\vec{\xi}, \mathfrak{E}^{\dot{0}} \right], \end{aligned}$$

6) We can give the alternative method to obtain the transformation character of U (or $\Psi(\tau)$) under the gauge transformation. From the eq. (4.5), the following integral equation is derived (we shall write t instead of $\xi^{\dot{0}}$, for simplicity);

$$U(t) = 1 - i \int_{-\infty}^t \bar{\mathfrak{B}}(t') U(t') dt'.$$

where we assume $U(-\infty) = 1$. From this equation we get;

$$\begin{aligned} \delta_G U(t) = & i \int_{t'=-\infty}^t \int_{t=\text{const}} D(t) \dot{\gamma}^{\dot{\mu}}(t') \frac{\partial \lambda(t')}{\partial \xi^{\dot{\mu}} t'} d\vec{\xi}' \cdot U(t') dt' - i \int_{-\infty}^t \bar{\mathfrak{B}}(t') \delta_G U(t') dt' \\ = & -iG_1(t) \cdot U(t) - i \int_{-\infty}^t \bar{\mathfrak{B}}(t') \{ \delta_G U(t') + iG_1(t') \cdot U(t') \} dt', \end{aligned}$$

by making use of the equation of continuity. If we put

$$\chi(t) = \delta_G U(t) + iG_1(t) \cdot U(t),$$

then the above equation appears as follow;

$$\chi(t) = -i \int_{-\infty}^t \bar{\mathfrak{B}}(t') \cdot \chi(t') dt'. \quad (a)$$

Since it holds $\chi(-\infty) = 0$, and

$$\frac{d\chi(t)}{dt} = -i\bar{\mathfrak{B}}(t), \quad \chi(t)$$

in consequence of (a), we confirm that identically $\chi(t) \equiv 0$,

so that

$$\delta_G U(t) = -iG_1(t) \cdot U(t),$$

or

$$\delta_G \Psi(\tau) = -iG_1(t) \cdot \Psi(\tau).$$

This is just the second equation of (4.17).

$$\begin{aligned}
 \mathfrak{B}' = & -\delta^* \mathfrak{F} + \frac{\partial}{\partial \xi^\mu} \left(\mathfrak{G}^{\dot{0}} \frac{\partial \delta \xi^\mu}{\partial \xi^{\dot{0}}} \right) - \frac{\partial}{\partial \xi^\nu} \left(\mathfrak{F} \delta \xi^\nu \right) \\
 & + \frac{\partial^2}{\partial \xi^\mu \partial \xi^\nu} \left(\frac{\gamma^{\mu\nu}}{\gamma^{\dot{0}\dot{0}}} \mathfrak{G}^{\dot{0}} \delta \xi^{\dot{0}} \right) - \frac{\partial}{\partial \xi^\mu} \left[\left(\gamma^{\mu\nu} + \frac{1}{2} \gamma^{\nu\sigma} \gamma_{\nu\sigma, \rho} \gamma^{\rho\mu} \right) \frac{\mathfrak{G}^{\dot{0}}}{\gamma^{\dot{0}\dot{0}}} \delta \xi^{\dot{0}} \right] \\
 & + i \left[\int (\mathfrak{B} \delta \xi^{\dot{0}}) d\vec{\xi}, \mathfrak{F} \right].
 \end{aligned} \tag{4.19}$$

If the ξ -transformation is restricted to the special case (3.9), we obtain the following relations, from (4.18) with (4.19) on integration over a surface σ ;

$$\begin{aligned}
 \frac{\delta \mathfrak{G}^{\dot{0}}[\sigma]}{\delta \sigma_p} &= \frac{1}{D_p} \mathfrak{F}(P) + \frac{1}{\gamma^{\dot{0}\dot{0}}} \left(\gamma^{\dot{0}\nu} + \frac{1}{2} \gamma^{\nu\sigma} \gamma_{\nu\sigma, \rho} \gamma^{\rho\dot{0}} \right) \frac{\mathfrak{G}^{\dot{0}}(P)}{D_p} \\
 &\quad - i[V(P), \mathfrak{G}^{\dot{0}}[\sigma]], \\
 \frac{\delta \mathfrak{F}[\sigma]}{\delta \sigma_p} &= -i[V(P), \mathfrak{F}[\sigma]],
 \end{aligned} \tag{4.20}$$

where

$$\mathfrak{G}^{\dot{0}}[\sigma] \equiv \int_\sigma \mathfrak{G}^{\dot{0}}(\vec{\xi}, \xi^{\dot{0}}) d\vec{\xi}, \quad \mathfrak{F}[\sigma] \equiv \int_\sigma \mathfrak{F}(\vec{\xi}, \xi^{\dot{0}}) d\vec{\xi}.$$

Now, from the condition (4.8), we obtain

$$\frac{\delta}{\delta \sigma_p} (\mathfrak{G}^{\dot{0}}[\sigma], \Psi[\sigma]) = \frac{\delta \mathfrak{G}^{\dot{0}}[\sigma]}{\delta \sigma_p} \cdot \Psi[\sigma] - i \mathfrak{G}^{\dot{0}}[\sigma] \cdot V(P) \cdot \Psi[\sigma].$$

If we substitute from the first equation of (4.20) into the above, and make use of the condition (4.8), then we arrive at

$$\frac{\delta}{\delta \sigma_p} (\mathfrak{G}^{\dot{0}}[\sigma] \cdot \Psi[\sigma]) = 0.$$

In quite the same way, we can prove

$$\frac{\delta}{\delta \sigma_p} (\mathfrak{F}[\sigma] \cdot \Psi[\sigma]) = 0.$$

Thus the condition (4.8) is compatible with the eq. (4.7)''.

From now on, we shall avail ourselves exclusively with an rectilinear orthogonal ξ -system, the transformation of which being restricted to the Lorentz group. This restriction, however, does not destroy the generality of our theory because we have already shown the covariance of our formalism under the ξ -transformation. In this simple case we can easily solve the equations of field, the solutions of which will be given below. In particular we will express any

field quantity Φ at an arbitrary world point (ξ) in terms of the canonical quantities on the surface σ . As to E , for example, we get

$$E(\xi) = \int_{\sigma(\xi^0 = \text{const})} \left\{ D(\xi - \xi') \frac{\partial \bar{\mathcal{G}}^\mu(\xi')}{\partial \xi^{\mu'}} + \frac{\partial D(\xi - \xi')}{\partial \xi^0} E(\xi') \right\} d\xi'. \quad (4.21)$$

Making (4.21) operate on $\Psi[\sigma]$, we obtain the following condition on account of (4.8);

$$\left\{ E(\xi) + \int_{\sigma} D(\xi - \xi') j^0(\xi') d\xi' \right\} \Psi[\sigma] = 0. \quad (4.22)$$

This is more general than the first of the conditions (4.8) to the effect that the point (ξ) may well lie outside of σ . On differentiation of (4.22) with respect to ξ^0 , we get the generalized second condition

$$\left\{ \frac{\partial \bar{\mathcal{G}}^\mu(\xi)}{\partial \xi^\mu} + \int_{\sigma} \frac{\partial D(\xi - \xi')}{\partial \xi^0} j^0(\xi') d\xi' \right\} \cdot \Psi[\sigma] = 0. \quad (4.22)'$$

By means of these solutions, we can also generalize the $[C.R.]$'s as follows;

$$[E(\xi), A_\mu(\xi')] = -i \frac{\partial D(\xi - \xi')}{\partial \xi^\mu}$$

$$[A_\mu(\xi), A_\nu(\xi')] = -i \gamma_{\mu\nu} D(\xi - \xi') + i \int_{\sigma'} \frac{\partial D(\xi - \eta)}{\partial \eta^\mu} \frac{\partial D(\eta - \xi')}{\partial \eta^\nu} (d\eta), \quad (4.23)$$

where σ' is a hyper plane on which ξ^0 is a constant. As for the electron field the $[C.R.]$'s remain the same form with that of the usual ones.

In concluding this paragraph we want to write down several expressions with reference to the original Lorentz frame.

In the first place the eq. (4.7) runs as follows

$$i \frac{d\Psi(\tau)}{d\tau} = \bar{\mathfrak{B}} \Psi(\tau),$$

where

$$\tau = h_k^0 x^k = -n_k x^k,$$

$$\bar{\mathfrak{B}} = -n^k \int j^l A_l d\sigma_k,$$

and the more general eq. (4.7)'' becomes,

$$i \partial^* \Psi = -\frac{1}{2} \epsilon_{kl} \int_{\sigma} j^m A_m (n^k x^l - n^l x^k) d\sigma \cdot \Psi$$

or

$$\lim_{\epsilon_{kl} \rightarrow \infty} i \frac{\Psi(\sigma') - \Psi(\sigma)}{\epsilon_{kl}} = -\frac{1}{2} \int_{\sigma} j^m A_m (n^k x^l - n^l x^k) d\sigma \Psi(\sigma),$$

where we have in mind the infinitesimal Lorentz transformation of the orthogonal rectilinear ξ -system, i.e.

$$\delta \xi^\mu = \epsilon^\mu_\nu \xi^\nu, \quad \delta \xi^0 = -n_k \epsilon^k_l \xi^l$$

with

$$\epsilon_{kl} = -\epsilon_{lk},$$

and the hyper plane σ' is that on which the new time coordinate τ' has the same value as that of τ on the plane σ .

The condition (4.22) now is expressed by

$$\left\{ E(x) + \int_\sigma D(x-x') j^k(x') d\sigma'_k \right\} \Psi(\tau) = 0, \quad (4.24)$$

and the $[C.R.]$'s are rewritten as follows;

$$[E(x), A_k(x')] = -i \frac{\partial D(x-x')}{\partial x^k},$$

$$[A_k(x), A_l(x')] = -i g_{kl} D(x-x') + i \int_{\sigma'} \int_{\sigma''} \frac{\partial D(x-x'')}{\partial x^{k''}} \frac{\partial D(x''-x')}{\partial x^{l''}} (dx'')^4. \quad (4.25)$$

Here it should be noticed that the condition of the flatness of σ and σ' in (4.24), (4.25) is not necessary, but we can rather adopt any space-like hyper-surface as easily seen from the nature of these equations.

§ 5. Elimination of the longitudinal component

In this paragraph we deal with the special case of an orthogonal rectilinear ξ -system.

In the first place, let us separate A_μ into two parts;

$$A_\mu = B_\mu + \frac{\partial A}{\partial \xi^\mu}, \quad (5.1)$$

where we assume

$$\frac{\partial B^\mu}{\partial \xi^\mu} = 0, \quad B^0 = B_0 = 0.$$

Then from (2.17)b, A must satisfy the following equation

$$\Delta A - \frac{\partial A_0}{\partial \xi^0} = 0. \quad (5.2)$$

Introducing the Green function $G(\vec{\xi})$ of the Poisson equation, (5.2) can be solved as follows;

$$A(\vec{\xi}, \xi^0) = \int_\sigma G(\vec{\xi} - \vec{\xi}') \frac{\partial A_\mu(\vec{\xi}', \xi^0)}{\partial \xi^\mu} d\vec{\xi}' \quad (5.3)$$

where

$$\Delta G(\vec{\xi}) = \delta(\vec{\xi}).$$

The $[C.R]$'s between A and other quantities are given by;

$$[\mathfrak{E}^{\bar{\mu}}(\xi), A(\xi')] = i \frac{\partial G(\vec{\xi} - \vec{\xi}')}{\partial \xi^{\bar{\mu}}},$$

$$[E(\xi), A(\xi')] = 0,$$

$$[A_{\bar{\mu}}, A] = 0, \quad [A(\xi), A(\xi')] = 0, \quad [B_{\bar{\mu}}, A] = 0, \quad (5.4)$$

and further

$$\left[\frac{\partial \mathfrak{E}^{\bar{\mu}}(\xi)}{\partial \xi^{\bar{\mu}}}, A(\xi') \right] = i \delta(\vec{\xi} - \vec{\xi}'). \quad (5.5)$$

Making D'Alembertian operate on both sides of (5.3), we get

$$\square A = -E \quad (5.6)$$

by virtue of the field equation. Hence from (5.1) we obtain the following equation;

$$\square B_{\bar{\mu}} = 0. \quad (5.7)$$

The $[C.R]$'s between $B_{\bar{\mu}}$ and other quantities are given by;

$$[\mathfrak{E}^{\bar{\mu}}, B_{\bar{\nu}}] = -i \delta_{\bar{\nu}}^{\bar{\mu}} \delta(\vec{\xi} - \vec{\xi}') + i \frac{\partial^2 G(\vec{\xi} - \vec{\xi}')}{\partial \xi^{\bar{\mu}} \partial \xi^{\bar{\nu}}},$$

$$[E, B_{\bar{\nu}}] = 0, \quad [A_{\bar{\mu}}, B_{\bar{\nu}}] = 0,$$

$$\left[\frac{\partial \mathfrak{E}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}}, B_{\bar{\nu}} \right] = 0. \quad (5.8)$$

Now we can give the more general $[C.R]$'s by using the field equations and $[C.R]$'s (5.4), (5.8). To do this let us introduce a new function Q , which is defined by;

$$\frac{\partial^2 Q(\xi)}{\partial \xi^{\bar{0}2}} = D(\xi), \quad (5.9)$$

and at $\xi^{\bar{0}} = 0$

$$Q(\xi) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial \xi^{\bar{0}}} \Big|_{\xi^{\bar{0}}=0} = G(\vec{\xi}).$$

In terms of this new function, the generalized $[C.R]$'s are given as follows;

$$[A(\xi), A(\xi')] = i Q(\xi - \xi') + i \int_{\sigma'}^{\sigma} \int D(\xi - \eta) D(\eta - \xi') (d\eta)^4,$$

$$[B_{\bar{\mu}}, A] = 0,$$

$$[B_{\bar{\mu}}(\xi), B_{\bar{\nu}}(\xi')] = -i\eta_{\bar{\mu}\bar{\nu}}D(\xi - \xi') + i\frac{\partial^2 Q(\xi - \xi')}{\partial \xi^{\bar{\mu}} \partial \xi^{\bar{\nu}}}. \quad (5.10)$$

If we put

$$A_0 = \frac{\partial A'}{\partial \xi^0},$$

then A' satisfies the following equation which is derived from the eq. (2.17)' (where j^{μ} should be put equal to zero),

$$\square A' = -E. \quad (5.11)$$

Therefore the $[C.R]$'s between A' and other quantities are

$$\begin{aligned} [A'(\xi), A'(\xi')] &= i \int_{\sigma'}^{\sigma} D(\xi - \eta) D(\eta - \xi') (d\eta)^4, \\ [A'(\xi), A'(\xi')] &= -iQ(\xi - \xi') + i \int_{\sigma'}^{\sigma} D(\xi - \eta) D(\eta - \xi') (d\eta)^4, \\ [A'(\xi), B_{\bar{\mu}}(\xi')] &= 0. \end{aligned} \quad (5.12)$$

Now let us try to eliminate the charge density j^0 from the second condition of (4.8) so as to obtain the same one as in case of the free field.

Consider the unitary transformation;

$$\Psi(\tau) \rightarrow \Psi'(\tau) = S(\tau) \cdot \Psi(\tau), \quad (5.13)$$

where $S(\tau)$ is a unitary operator defined by

$$S = e^{iR(\tau)} \quad R(\tau) = - \int_{\tau=\text{const}} j^0 A(\vec{\xi}, \tau) d\vec{\xi}.$$

From the $[C.R]$'s (5.4), (5.5), it results:

$$[R, j^0] = 0, \quad [R, E] = 0,$$

$$i \left[R, \frac{\partial \mathcal{G}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} \right] + j^0 = 0,$$

so that it holds

$$S \mathcal{H} S^{-1} = \frac{\partial \mathcal{G}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}}, \quad S E S^{-1} = E.$$

Thus the condition (4.8) is transformed into

$$E \Psi'(\tau) = 0, \quad \frac{\partial \mathcal{G}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} \Psi'(\tau) = 0, \quad (5.14)$$

as anticipated.

Corresponding to the transformation (5.13), the Tomonaga-Schwinger equation is now written as follows;

$$i \frac{d\Psi'(\tau)}{d\tau} = \bar{V}' \Psi'(\tau), \quad (5.15)$$

where

$$\bar{V}'(\tau) = i \frac{dS}{d\tau} S^{-1} + S \bar{V} S^{-1},$$

and this is expressed by

$$\begin{aligned} \bar{V}' = & - \int_{\tau=\text{const}} j^{\bar{\mu}} B_{\bar{\mu}} d\vec{\xi} + \frac{1}{2} \iint_{\tau=\text{const}} j^{\dot{0}}(\xi) G(\vec{\xi} - \vec{\xi}') j^{\dot{0}}(\xi') d\vec{\xi} d\vec{\xi}' \\ & + \iint_{\tau=\text{const}} j^{\dot{0}}(\xi) G(\vec{\xi} - \vec{\xi}') \frac{\partial \mathcal{G}^{\bar{\mu}}(\xi')}{\partial \xi^{\bar{\mu}'}} d\vec{\xi} d\vec{\xi}', \end{aligned} \quad (5.16)$$

by making use of the definition of $R(\tau)$ and the eq. (5.1). Substituting from (5.16) into (5.15), the last integral of (5.16) disappears because of the condition (5.14).

Next let us consider the gauge transformation. The transformation of $B_{\bar{\mu}}$, A and A' are given by

$$\begin{aligned} \delta_G B_{\bar{\mu}} &= i[G_0, B_{\bar{\mu}}] = 0, \\ \delta_G A &= \lambda, \quad \delta_G A' = \lambda, \end{aligned} \quad (5.17)$$

where the following $[C, R]$'s are used;

$$\left. \begin{aligned} [A_0(\xi), A'(\xi')] &= -iG(\vec{\xi} - \vec{\xi}'), \\ [E, A'] &= 0, \\ \left[\frac{\partial \mathcal{G}^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}}, A' \right] &= i\delta(\vec{\xi} - \vec{\xi}') \end{aligned} \right\} \quad \text{for } \xi^{\dot{0}} = \xi'^{\dot{0}}.$$

The state vector $\Psi'(\tau)$ is transformed in the following way:

$$\delta_G \Psi' = (\delta_G S \cdot S^{-1} - iSG_1 S^{-1}) \Psi',$$

The right-hand side of this equation turns out to vanish, i.e. Ψ' reveals itself to be gauge-invariant, if we take into account the eqs. (4.17) and (5.17). From this fact, we can see the gauge-invariance of eq. (5.15). In reference to the original Lorentz frame the eq. (5.1) runs—

$$A_k(x) = B_k + \frac{\partial A}{\partial x^k} + n_k n^i \frac{\partial}{\partial x^i} (A - A'), \quad (5.18)$$

where B_k is defined by

$$B_k = l_k^{\mu} B_{\mu},$$

and satisfies the following condition

$$\frac{\partial B^k}{\partial x^k} = 0, \quad n_k B^k = 0. \quad (5.19)$$

The last equation of (5.10) is replaced by

$$[B_k(x), B_l(x')] = -i(g_{kl} + n_k n_l) D(x - x') + i \left(\frac{\partial}{\partial x^k} + n_k \left(n^j \frac{\partial}{\partial x^j} \right) \right) \cdot \left(\frac{\partial}{\partial x^l} + n_l \left(n^h \frac{\partial}{\partial x^h} \right) \right) Q(x - x'), \quad (5.20)$$

and the definition of Q is now

$$\left(n^k \frac{\partial}{\partial x^k} \right)^2 Q = D$$

$$\lim_{(n\alpha) \rightarrow 0} Q = 0, \quad \lim_{(n\alpha) \rightarrow 0} \left(n^k \frac{\partial}{\partial x^k} \right) Q = - \frac{1}{\sqrt{x_k x^k}} \Big|_{(n\alpha) \rightarrow 0}. \quad (5.21)$$

Finally the eq. (5.15) is represented by

$$i \frac{d\Psi'(\tau)}{d\tau} = \bar{V}'' \Psi'(\tau),$$

$$\bar{V}''(\tau) = - \int_{\sigma} j^k B_k n^l d\sigma_l + \frac{1}{2} \iint_{\sigma} j^k(x) n^h \frac{\partial Q(x - x')}{\partial x^h} j^l(x') d\sigma_k d\sigma'_l. \quad (5.22)$$

All the expressions from (5.18) to (5.22) agree with those given by Schwinger.

§ 6. Proof of the equivalence of the present theory with the ordinary one (Fermi's theory)

In this paragraph we shall refer all quantities to a rectilinear coordinate system and also to the Heisenberg representation.

Let us recall the main features of the current theory due to Fermi.

The Lagrangian used in Fermi's theory is

$$L_F = -\frac{1}{4} f_{kl} f^{kl} - \frac{1}{2} \left(\frac{\partial A^k}{\partial x^k} \right)^2 + L_m - V,$$

where L_m and V are given in (1.1). The canonically conjugate quantity to the electromagnetic potential is accordingly defined by

$$E^k = f^{k0} - g^{k0} \left(\frac{\partial A^l}{\partial x^l} \right),$$

the Hamiltonian density:

$$H_F = H - \frac{1}{2} (E^0)^2,$$

where H is the Hamiltonian density of our own theory. The field equations are;

$$\square A_k = -j_k, \quad E^0 = -\frac{\partial A^k}{\partial x^k}, \quad \square E^0 = 0.$$

And finally the supplementary condition is

$$E^0 \Psi_0 = 0, \quad \frac{\partial E^0}{\partial x^0} \Psi_0 = 0,$$

or

$$\frac{\partial A^k}{\partial x^k} \Psi_0 = 0, \quad \left(\frac{\partial E^k}{\partial x^k} + j^0 \right) \Psi_0 = 0$$

by virtue of the field equations.

Now, within the framework of our theory, consider the following unitary transformation

$$\begin{aligned} \Phi &\rightarrow \Phi' = S \Phi S^{-1}, \quad \Psi_0 \rightarrow \Psi'_0 = S \Psi_0, \\ S &= e^{iR}, \quad R = -\frac{1}{2} \int_{-\infty}^{x^0} (E)^2 (dx)^4, \end{aligned} \quad (6.1)$$

where Φ stands for any field quantity of our theory.

From our eq. (2.17)'c, we obtain the following $[C.R]$ in the Heisenberg representation;

$$[E(x), E(x')] = 0, \quad (6.2)$$

where the two world points need not be simultaneous. By virtue of this $[C.R]$ we have evidently

$$E' = E \quad \text{and} \quad \left[R, \frac{dR}{dx^0} \right] = 0. \quad (6.3)$$

The new field equation of the transformed quantity Φ' turns out to be

$$\frac{d\Phi'}{dx^0} = i \left[\left(S \bar{H} S^{-1} - i \frac{dS}{dx^0} \cdot S^{-1} \right), \Phi' \right]. \quad (6.4)$$

On the other hand, in consequence of (6.3), it holds;

$$\frac{dS}{dx^0} \cdot S^{-1} = i \frac{dR}{dx^0} = -\frac{i}{2} \int (E')^2 d\vec{x},$$

so that the eq. (6.4) becomes

$$\frac{d\Phi'}{dx^0} = i \left[\bar{H}_F, \Phi' \right], \quad (6.5)$$

with Fermi's Hamiltonian H_F . In particular, for $\Phi' = A'_0$, we obtain the following equation,

$$\frac{dA'_0}{dx^0} = \frac{\partial A^{k'}}{\partial x^k} - E', \quad \text{i.e. } E' = E = (\text{Fermi's } E^0).$$

This is nothing but Fermi's equation. According to the transformation rule (6.1), our supplementary condition (2.20) becomes

$$\left(\frac{\partial A^{kl}}{\partial x^k}\right) \cdot \Psi'_0 = 0, \quad \left\{\frac{\partial E'^k}{\partial x^k} + j'^0\right\} \cdot \Psi'_0 = 0.$$

Thus the required proof of equivalence has been completed.

In concluding this paragraph let us return to the second case of §2 (page 90). In this case we must express S as a linear combination of the canonical quantities, where S is defined by

$$S = \frac{\partial A^k}{\partial x^k}. \quad (2.21)$$

The only possible form of S is

$$S = cE \quad (c \text{ is an arbitrary constant}).$$

If c is put equal to 1, then we arrive at Fermi's theory, whereas if c equal to zero, then our theory, treated as the first case, is derived. Among different choices of the value of the constant c , the case $c=0$ is distinguished from other cases in the fact that the Lagrangian in its original form is restored on the inverse Legendre transformation from the Hamiltonian obtained, only in that case of $c=0$.

§ 7. Application to the vector meson field⁸⁾

We encounter with the same kind of difficulty in case of the vector meson field as that of the radiation field. The Lagrange function of the system of the vector-meson and nucleon fields is taken, as usual, to be

$$L = L_m + L_n - V$$

where L_m belongs to the nucleon-field in free case, whereas L_m and V are given by,

$$L_m = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{x^2}{2} \gamma^{\mu\nu} \left(A_\mu - \frac{1}{x} \varphi_{,\mu} \right) \left(A_\nu - \frac{1}{x} \varphi_{,\nu} \right),$$

$$V = -f \cdot j^\mu \left(A_\mu - \frac{1}{x} \varphi_{,\mu} \right) + \frac{g}{2x} F_{\mu\nu} j^{\mu\nu} + \frac{g^2}{2x^2} j_{\mu\nu} j^{\mu\nu},$$

and

$$j^\mu = \psi^\dagger \gamma^\mu \psi, \quad j^{\mu\nu} = \frac{1}{2} \psi^\dagger (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \psi,$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \quad (7.1)^9$$

Here we describe the meson field with a vector A_μ and a redundant scalar field φ . The canonically conjugate quantities to A_μ and φ are defined by

8) In this paragraph, we shall refer all the quantities to an orthogonal rectilinear ξ -system.

9) Y. Miyamoto, Prog. Theor. Phys. **3** (1948), 124.

$$E^\mu = F^{\mu\dot{0}} + \frac{g}{x} j^{\bar{\mu}\dot{0}} \quad E^{\dot{0}} = 0,$$

$$\pi = (\varphi_{,\dot{0}} - x A_{\dot{0}}) - \frac{f}{g} j^{\dot{0}}. \quad (7.2)$$

Let us introduce a new quantity E^\bullet which is canonically conjugate to A_\bullet as was done in § 3. Solving (7.2) with respect to $A_{\mu,0}$, $\varphi_{,0}$, we obtain the following solution;

$$A_{\bar{\mu},\dot{0}} = E_{\bar{\mu}} + A_{\dot{0},\bar{\mu}} - \frac{g}{x} j^{\bar{\mu},\dot{0}}$$

$$A_{0,0} = A = \text{indeterminate},$$

$$\varphi_{,\dot{0}} = x A_{\dot{0}} + \pi + \frac{f}{x} j^{\dot{0}}.$$

If we put A equal to, with two arbitrary constants a and b ,

$$A \equiv A_{\bar{\mu},\bar{\mu}} - a\varphi - \frac{b}{2} E^{\dot{0}} \quad (7.3)$$

as in case of the radiation field, then we obtain the following Hamiltonian density

$$H = H_0 + V',$$

$$H_0 = \frac{1}{2} E^{\bar{\mu}} E^{\bar{\mu}} + E^{\bar{\mu}} \cdot A_{\bar{\mu}} + E^{\bullet} \cdot A_{\bar{\mu},\bar{\mu}} - a\varphi \cdot E^{\dot{0}} - \frac{b}{2} (E^{\dot{0}})^2$$

$$+ x\pi \cdot A_{\dot{0}} + \frac{1}{2} \pi^2 + \frac{1}{4} F_{\bar{\mu}\bar{\nu}} F^{\bar{\mu}\bar{\nu}} + \frac{1}{2} (xA_{\bar{\mu}} - \varphi_{,\bar{\mu}}) (xA_{\bar{\mu}} - \varphi_{,\bar{\mu}})$$

$$- i\psi^\dagger \gamma^{\bar{\mu}} \frac{\partial \psi}{\partial \xi^{\bar{\mu}}} - ix\psi^\dagger \psi,$$

$$V' = -fj^{\bar{\mu}} \left(A_{\bar{\mu}} - \frac{1}{x} \varphi_{,\bar{\mu}} \right) + \frac{f}{x} j^{\dot{0}} \pi + \frac{g}{2x} F_{\bar{\mu}\bar{\nu}} j^{\bar{\mu}\bar{\nu}}$$

$$- \frac{g}{x} E^{\bar{\mu}} j^{\bar{\mu}\dot{0}} + \frac{f^2}{2x^2} j^{\dot{0}} j^{\dot{0}} + \frac{g^2}{2x} j^{\bar{\mu}\dot{0}} j^{\bar{\mu}\dot{0}} + \frac{g^2}{2x^2} j_{\mu\nu} j^{\mu\nu}. \quad (7.4)$$

The nonvanishing $[C.R.]$'s related to the meson field are

$$[E^\mu A_\nu] = -i\delta^\mu_\nu \delta(\vec{\xi} - \vec{\xi}'), \quad [\pi, \varphi] = -i\delta(\vec{\xi} - \vec{\xi}'). \quad (7.5)$$

The equations of motion of the meson field derived from the Hamiltonian thus established, run as follows;

$$A_{\bar{\mu},\dot{0}} = E_{\bar{\mu}} + A_{\dot{0},\bar{\mu}} - \frac{g}{x} j^{\bar{\mu},\dot{0}}, \quad (7.6)a$$

$$A_{0,\dot{0}} = A_{\bar{\mu},\bar{\mu}} - a\varphi - bE^{\dot{0}}, \quad (7.6)b$$

$$\varphi_{,\dot{0}} = xA_{\dot{0}} + \pi + \frac{f}{x} j^{\dot{0}}, \quad (7.6)c$$

$$E_{,0}^{\bar{\mu}} = \frac{\partial E^0}{\partial \xi^{\bar{\mu}}} - \frac{\partial F_{\bar{\mu}\bar{\nu}}}{\partial \xi^{\bar{\nu}}} - x^2 A_{\bar{\mu}} + x \varphi_{,\bar{\mu}} + f \cdot j^{\bar{\mu}} - \frac{g}{x} \frac{\partial j^{\bar{\mu}\bar{\nu}}}{\partial \xi^{\bar{\nu}}}, \quad (7.6)d$$

$$E_{,0}^0 = \frac{\partial E^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} - x\pi, \quad (7.6)e$$

$$\pi_{,i} = a E^0 - \frac{\partial}{\partial \xi^{\bar{\mu}}} (x A_{\bar{\mu}} - \varphi_{,\bar{\mu}}) + \frac{f}{x} \frac{\partial j^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}}. \quad (7.6)f$$

Combinations of these equations give

$$(\square - x^2) A_{\bar{\mu}} + (1-b) \frac{\partial E^0}{\partial \xi^{\bar{\mu}}} + (x-a) \frac{\partial \varphi}{\partial \xi^{\bar{\mu}}} + f j_{\bar{\mu}} - \frac{g}{x} \frac{\partial j_{\bar{\mu}\bar{\nu}}}{\partial \xi^{\bar{\nu}}} = 0, \quad (7.7)a$$

$$(\square - ax) \varphi + (a-bx) E^0 = 0, \quad (7.7)b$$

$$(\square - ax) E^0 = 0 \quad (7.7)c$$

provided that the equation of continuity

$$\frac{\partial j^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} = 0$$

is taken into account. As one easily sees, the eq. (7.7)b turns out to be a consequence of the remaining eqs. (7.7)a (7.7)c and (7.6)b. Thus the eqs. (7.7) are not mutually independent.

Now our Lagrangian is invariant under the gauge transformation

$$\begin{aligned} \partial_G \psi &= \partial_G \psi^\dagger = 0, \\ \partial_G A_{\bar{\mu}} &= \frac{\partial \lambda}{\partial \xi^{\bar{\mu}}}, \quad \partial_G \varphi = x \lambda. \end{aligned} \quad (7.8)$$

The generating operator of this infinitesimal transformation is

$$G = \int_{\sigma} \left\{ E^0 \frac{\partial \lambda}{\partial \xi^0} + \left(x\pi - \frac{\partial E^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} \right) \lambda \right\} d\vec{x}, \quad (7.9)$$

and satisfies

$$\frac{dG}{d\xi^0} = 0 \quad (7.10)$$

as in case of the electromagnetic field.

The arbitrary function λ which is a scalar, must satisfy, in the present case, the following equation;

$$(\square - ax) \lambda = 0 \quad (7.11)$$

by virtue of the gauge invariance of (7.6)b, E^0 being assumed as gauge invariant. Actually the eq. (7.10) is derived by using (7.11) and the field equations.

Now we have to propose the following supplementary conditions

$$E^0 \Psi_0 = 0, \quad F \Psi_0 = 0 \quad \text{to be valid on } \sigma, \quad (7.12)$$

where

$$F = \frac{\partial E^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} - \kappa \pi.$$

From the eq. (7.10) we can easily show that the conditions (7.12) are mutually compatible and, moreover, consistent with the field equations. This amusing circumstance depends on the clever introduction of the redundant field φ .¹⁰⁾

Let us introduce a new quantity U_μ defined by

$$U_\mu \equiv A_\mu - \frac{1}{\kappa} \frac{\partial \varphi}{\partial \xi^\mu}, \quad (7.13)$$

then from (7.7)a and (7.7)b, we conclude

$$(\square - \kappa) U_\mu + \left(1 - \frac{a}{\kappa}\right) \frac{\partial E^0}{\partial \xi^\mu} + f j_\mu - \frac{g}{\kappa} \frac{\partial j_{\mu\nu}}{\partial \xi_\nu} = 0; \quad (7.14)$$

and the (7.6)b becomes

$$\frac{\partial U^\mu}{\partial \xi^\mu} = \frac{a}{\kappa} E^0. \quad (7.15)$$

Hence, provided that the indeterminate const. a is not equal to zero, we can replace (7.12) by

$$\frac{\partial U^\mu}{\partial \xi^\mu} \Psi_0 = 0, \quad \frac{\partial}{\partial \xi^0} \left(\frac{\partial U^\mu}{\partial \xi^\mu} \right) \Psi_0 = 0. \quad (7.12)'$$

In particular, if we put a and b equal to κ and zero respectively, then from (7.14) (7.15) we obtain,

$$(\square - \kappa^2) U_\mu + f j_\mu - \frac{g}{\kappa} \frac{\partial j_{\mu\nu}}{\partial \xi_\nu} = 0, \quad (7.14)''$$

$$\frac{\partial U^\mu}{\partial \xi^\mu} = E^0. \quad (7.15)'$$

The eqs. (7.7)b and (7.7)c, are, now, replaced by

$$(\square - \kappa^2) \varphi = 0, \quad (\square - \kappa^2) E^0 = 0,$$

and (7.11) runs as follows;

$$(\square - \kappa^2) \lambda = 0.$$

The eq. (7.14)' is the usual eq. of the meson field.

10) E. Stückelberg, *Helv. Phys. Acta* **11** (1938), 299.

K. Husimi, cf. footnote (2).

11) Of course other choices of a and b are also possible, for example, $a=b=0$, or $a=\kappa$, $b=1$. It can be shown, however, by means of the suitable unitary transformations, that these cases are equivalent to our case.

Next let us adopt the interaction representation. In this case, the equations of fields are those of the free fields, and by virtue of these we can eliminate the canonical momenta from the interaction energy density V' . Thus V' gets the following form;

$$V' = V + \frac{f^2}{2x^2} j^0 j^0 + \frac{g^2}{2x^2} \bar{j}^{\mu 0} j^{\mu 0}, \quad (7.16)$$

or, describing this with regard to the original Lorentz frame, we obtain;

$$V' = V + \frac{f^2}{2x^2} (n_k j^k) (n_l j^l) + \frac{g^2}{2x^2} (j_{kl} j^{kl}) (j^{kh} n_h). \quad (7.16)'$$

The normal-dependent terms, which are contrived previously through integrability conditions, emerge now quite naturally in our theory.¹²⁾

The Tomonaga-Schwinger eq. runs as follows;

$$i \frac{\partial \Psi(\tau)}{\partial \tau} = \bar{V}' \Psi(\tau), \quad \tau = -n_k x^k,$$

or

$$i \frac{\partial \Psi[\sigma]}{\partial \sigma_p} = V'_p \Psi[\sigma],$$

where V' can be expressed as follows by using the definition of U_μ ;

$$\begin{aligned} V' = & -f U_\mu j^\mu + \frac{g}{2x} U_{\mu\nu} j^{\mu\nu} + \frac{g^2}{2x^2} j_{\mu\nu} j^{\mu\nu} \\ & + \frac{f^2}{2x^2} (n_k j^k) (n_l j^l) + \frac{g^2}{2x^2} (n_l j^{kl}) (j_{kl} n^h). \end{aligned}$$

In the case where a and b are put equal to x and 0 respectively, the $[C.R.]$'s of the meson field are given by

$$[A_\mu, A'_\nu] = -i \gamma_{\mu\nu} \Delta(\xi - \xi') + i \int_{\sigma'}^{\sigma} \frac{\partial \Delta(\xi - \eta)}{\partial \eta^\mu} \frac{\partial \Delta(\eta - \xi')}{\partial \eta^\nu} (d\eta)^4,$$

$$[E^0, A'_\mu] = -i \frac{\partial \Delta(\xi - \xi')}{\partial \xi^\mu},$$

$$[\varphi, A'_\mu] = -ix \int_{\sigma'}^{\sigma} \Delta(\xi - \eta) \frac{\partial \Delta(\eta - \xi')}{\partial \eta^\mu} (d\eta)^4,$$

$$[\varphi, E^{0'}] = ix \Delta(\xi - \xi'),$$

$$[E^0, E^{0'}] = 0,$$

$$[\varphi, \varphi'] = -i \Delta(\xi - \xi') + ix^2 \int_{\sigma'}^{\sigma} \Delta(\xi - \eta) \Delta(\eta - \xi') (d\eta)^4,$$

12) cf. foot-note (9) and also S. Kanesawa and Z. Koba, Prog. Theor. Phys. 4 (1949), 297.

$$[U_\mu, U_\nu] = -i \left(A(\xi - \xi') \gamma_{\mu\nu} - \frac{1}{x^2} \frac{\partial^2 A(\xi - \xi')}{\partial \xi^\mu \partial \xi^\nu} \right). \quad (7.18)$$

In case of other choice of a and b , there appears the invariant delta function D .

Now we want to investigate an alternative type of the interaction part of the Lagrangian.

Let us adopt the Lagrangian of the following form, instead of (7.1);

$$L = L_m + L_n - W \quad (7.19)$$

where L_m and L_n are given by (7.1), but W is defined by

$$W = -f j^\mu A_\mu + \frac{g}{2x} F_{\mu\nu} j^{\mu\nu} + \frac{g^2}{2x^2} j^{\mu\nu} j_{\mu\nu}. \quad (7.19)'$$

The Hamiltonian density corresponding to (7.19) is

$$H' = H_0 + W',$$

$$W = -f A_\mu j^\mu + \frac{g}{2x} F_{\bar{\mu}\bar{\nu}} j^{\bar{\mu}\bar{\nu}} - \frac{g}{x} E^{\bar{\mu}} j^{\bar{\mu}0} + \frac{g^2}{2x^2} j^{\bar{\mu}0} j^{\bar{\mu}0} + \frac{g^2}{2x^2} j^{\mu\nu} j^{\mu\nu}, \quad (7.20)$$

and H_0 is the same one as in the former case.

The Lagrangian (7.19), now, admits the following gauge transformation, instead of (7.8):

$$\begin{aligned} \delta_G \phi &= i f \lambda \phi, & \delta_G \phi^\dagger &= -i f \lambda \phi, \\ \delta_G A_\mu &= \frac{\partial \lambda}{\partial \xi^\mu}, & \delta_G \varphi &= x \lambda, \end{aligned} \quad (7.21)$$

and the generating operator for (7.21) is given by

$$G = \int_\sigma \left\{ E^0 \frac{\partial \lambda}{\partial \xi^0} + \left(x\pi - \frac{\partial E^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} - f j^0 \right) \lambda \right\} d\xi. \quad (7.22)$$

In this case, the supplementary condition is

$$\begin{aligned} E^0 \Psi_0 &= 0, & F^0 \Psi_0 &= 0 & \text{on } \sigma, \\ F^0 &= \frac{\partial E^{\bar{\mu}}}{\partial \xi^{\bar{\mu}}} + f j^0 - x\pi. \end{aligned} \quad (7.23)$$

In the interaction representation, the field equations are the same as in the former case, (we have put a and b equal to x and zero respectively,) whereas the interaction energy density W' now turns into,

$$W' = W + \frac{g^2}{2x^2} (n_i j^{ki}) (n^k j_{kh}). \quad (7.24)$$

Now let us consider the elimination of the term depending on the nucleon field from the supplementary condition as was done in § 5.

Consider the following unitary transformation ;

$$\begin{aligned} \Psi(\tau) &\rightarrow \Psi'(\tau) = S\Psi(\tau), \\ S &= e^{iR}, \quad R = -\frac{f}{x} \int_0^\tau j^0 \cdot \varphi d\vec{\xi}. \end{aligned} \quad (7.25)$$

Then we obtain

$$SF'S^{-1} = \frac{\partial E^{\bar{\mu}}}{\partial \hat{\xi}^{\bar{\mu}}} - x\pi \equiv F, \quad SE^0 S^{-1} = E^0,$$

the supplementary conditions are transformed into

$$E^0 \Psi'(\tau) = 0, \quad F \Psi'(\tau) = 0; \quad \text{on } \sigma \quad (7.26)$$

and the Tomonaga-Schwinger eq. runs as follows ;

$$i \frac{d\Psi'(\tau)}{d\tau} = \bar{W}'' \Psi'(\tau),$$

where

$$\bar{W}'' = S\bar{W}'S^{-1} + i \frac{dS}{d\tau} S^{-1} \equiv \bar{\bar{W}}'. \quad (7.27)$$

Since the new expression for the interaction energy turns out to be just the same as the previous one \bar{V}' , the equivalence of the both theory has been proved.

In the latter theory it is an easy matter to make the meson field agree with the electromagnetic field by making x and g tend to zero, whereas in the former case it seems somewhat difficult because of the term $\frac{f}{x} \varphi_{,\mu} j^\mu$ in the interaction.

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Nuclear Disintegration caused by Cosmic-Rays

Yoichi FUJIMOTO and Satio HAYAKAWA

Department of Physics, Tokyo University and University of City Osaka

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The diffusion of the nucleon component through the atmosphere is investigated. They are classified into three groups according to their energy, and the various properties of their nuclear interactions are discussed.

§ 1. Introduction

During last several years, various authors investigated the nuclear interactions caused by cosmic-rays, using Wilson chambers, ionization chambers, or G-M counters. Although they brought to light some properties of such nuclear events, the complete understanding has not yet been obtained. The photographic experiments with highly sensitive emulsion have become very powerful on this subject, by which Bristol group has recently carried out detailed investigations.^{1) 2) 3)} Their results make us possible to clarify the various properties of nuclear interactions more directly in wider range of energies than ever, and thus we have now somewhat ample knowledges on them.

These nuclear events by cosmic rays are the only available data on high energy mesonic phenomena, which we have now at hand. In order to facilitate our understanding on these problems, we must, first of all, build a picture or a model for cosmic-ray nuclear interactions from various experimental data. Here, we shall be concerned with this preliminary analysis, and ask the kinds, the energies and the genetic relations of agents of these events. Its result will be discussed in more detail in a separate paper, comparing with the calculation on meson theories.

Firstly, we describe a survey of the experimental results of Bristol group, including the definition of some standards of energy for later use (§ 2). According to these standards of energy, we can classify the nuclear events and the nucleon component as tentatively named A , B and C (§ 3). The genetic relations between these components are discussed by solving a simplified diffusion problem, although there remains some ambiguity if we wish to get forth into quantitative points (§ 4). The radiationless process of B -component is, however, treated more unambiguously and we can get some informations complementary to the accelerator experiments (§ 5).

§ 2. Preliminaries

In the photographic experiment of Bristol group,^{1) 2)} they classified the tracks

of nuclear events into the following three groups according to their grain density; thin tracks of grain density $\lesssim 1.5 g_m$, grey tracks of $1.5 \sim 5 g_m$, and black tracks of $\gtrsim 5 g_m$, where g_m is the minimum observed value of the grain density. As they pointed out, these three groups can be interpreted as follows. i) The thin group consists of fast mesons and fast protons. Incident protons of the events are also included in it. ii) The main part of the grey group are protons with energies $25 \sim 330$ MeV, and the rest will be deuterons, slow meson and others. iii) Most of the evaporated protons, α -particles and other heavier nuclear fragments have rather low energies, and they make the black group.

Using the above classification of tracks, they specified the observed nuclear events by the number n_s of their thin secondary tracks (shower tracks) and the number N_h of the grey and black tracks (heavy tracks). Furthermore, the suffix n or p is attached, according as the agent is neutral or charged.

These n_s and N_h are closely related to the mechanism of nuclear interactions and their available energies. To see this, we suppose that one cosmic-ray nucleon hits on a Ag or Br nucleus in emulsion. It will make successive collisions with several nucleons in the nucleus. If it has a sufficiently large energy, a number of mesons will be produced, and we shall observe them as thin tracks. At the same time, collided nucleons may receive a considerable amount of recoil energy. They will also make several collisions in the nucleus, and finally be stopped, or run away from the nucleus and be detected as thin or grey tracks. The residual nucleus will be in a highly excited state due to these disturbances, and then it will evaporate protons, neutrons, α -particles and other nuclear fragments, which are recorded as black tracks.

If we could make an analogy between the electromagnetic and the mesonic interaction, the meson production might correspond to the bremsstrahlung and the knocked-out protons to the knocked-on electrons. Then a certain energy E_c can be introduced in the same way as the critical energy in the cascade theory, though it may have less meaning than in the latter case.

First, we consider the case of an incident energy larger than the critical energy E_c of nuclear collision. As most of thin tracks can be regarded as mesons,³⁾ their number n_s is closely related to the number of collisions with the available energy $\gtrsim E_c$. For example, their relation is linear, if mesons are always produced singly. In the case of genuine multiple production, it becomes more complicated.

When an incident energy is below E_c , the elastic nucleon-nucleon scattering will be more predominant than the meson production. In this case, we are mainly concerned with the grey and the black group, since most of knocked-out protons are classified as grey tracks. We shall be able to obtain various informations on nuclear transparency, by examining their relations.

Thus the determination of the critical energy E_c of nuclear collision becomes very important for us to interpret these nuclear events. For this purpose, we shall introduce further two standards for the energy of the nucleon component,

and determine their values comparing with each other.

One standard is the geomagnetic cut-off energy E_{cut} (or the knee) of the primary proton spectrum. At the top of the atmosphere, the energy spectrum of nucleons will have a remarkable discontinuity at E_{cut} , although contributions from primary α -particles may smear it out in some respects. Therefore, we can expect that the same circumstance will occur in the size-distribution curve of nuclear events observed in the stratosphere.

The other standard is the critical energy of the diffusion through the atmosphere. Protons of the nucleon component suffer ionization loss, whereas neutrons do not. From this ionization and the mean free path for the nuclear interaction, we can define the critical energy E_j of diffusion. Protons with energies $\gtrsim E_j$ make nuclear interactions, on an average, before they are stopped by ionization loss, while those with energies $\lesssim E_j$ are stopped before nuclear interactions. Therefore, we shall find below E_j considerable difference between the energy spectra of protons and neutrons. This will result in the different amount of contributions of neutrons and protons to the nuclear events of smaller size.

Furthermore, the comparison of the absolute intensity of nucleon component and the absolute frequency of nuclear events will give us another information on relations of the size of events and the energy of agents. For this purpose, we need some knowledges on the cross-section, which can be obtained from other experimental data.

§ 3. Classification of nuclear events and of nucleon component

From the considerations in § 2, we classify the nucleon component into the following three groups;

A-component with energies $E \gtrsim E_{cut}$. The ratio of numbers of neutrons and protons (we call it simply as the n - p ratio) of this component will be considerably smaller than unity at the top of the atmosphere, and will tend to unity at lower altitudes.

B-component with energies $E_{cut} \gtrsim E \gtrsim E_j$. They are produced in collisions of A -nucleons with air nuclei. Their n - p ratio will always be nearly 1, since protons and neutrons will be produced equally and the ionization loss does not play an important role in this case.

C-component with energies $E \lesssim E_j$. They are produced by A and B -component in nuclear collisions. Their n - p ratio will be considerably larger than 1 due to the ionization loss.

The geomagnetic cut-off energy E_{cut} will be ~ 2 BeV at the experimental station of Bristol group.* The critical energy E_j of diffusion depends on an assumed mean free path for collisions, but we may put $E_j \sim 1$ BeV without large error.

* Bristol group did not write in their paper the station of their experiment in the stratosphere. Here we suppose it will be near Jungfraujoch, where they performed their first experiment.

Next, we examine the nuclear events and determine which correspond to the above three groups of nucleons. The following table shows the n - p ratio of their agents calculated from the experimental data of Bristol group.^{1) 2)}

Table I Observed n - p ratio of agents of nuclear events^{1) 2)}

atmospheric depth \ n_s	0	1	≥ 2
45 g/cm ²	5.45 ± 0.87	0.91 ± 0.23	0.57 ± 0.19
690 g/cm ²	7.10 ± 0.43	1.14 ± 0.15	0.76 ± 0.14

We see from this table, that nuclear events of $n_s \geq 2$ (we denote it simply as $\geq 2_s$) are mainly caused by A -component, 1_s by B , and 0_s by C -component.

Now, we can find the following interpretation for these nuclear events,

$$\begin{aligned}
 0_s & \quad (n \text{ or } p) \rightarrow 0, \quad \text{or} \quad \rightarrow n, \\
 1_s & \quad (n \text{ or } p) \rightarrow p, \quad \text{or} \quad \rightarrow (p+n), (\pi+n), \\
 2_s & \quad (n \text{ or } p) \rightarrow (p+p), (p+\pi) \quad \text{or} \quad \rightarrow (\pi+\pi+n), (p+\pi+n), \\
 & \quad \text{etc.}
 \end{aligned}$$

Here p or n means a proton or a neutron with energy in thin track region, and 0 indicates a final state without such nucleons. π is a fast charged π -meson.

It can be shown that the events of the first mode are most frequent, as was pointed out by Bristol group.²⁾ (See also § 5.) The second mode may take place at most 20~30%. Strictly speaking, therefore, we must improve the above correspondence in the following ways,

$$\begin{aligned}
 A\text{-component} & \rightarrow \geq 2_s + (\text{the second mode of } 1_s), \\
 B\text{-component} & \rightarrow (\text{the first mode of } 1_s) + (\text{the second mode of } 0_s), \\
 C\text{-component} & \rightarrow (\text{the first mode of } 0_s).
 \end{aligned}$$

This remark is important for the comparison of their absolute frequency (cf. § 4).

Furthermore, we see that the critical energy E_c of nuclear collision will lie between the energies of agents of 1_s and 2_s . From the above consideration, we can suppose that E_c is between $E_{cut} \sim 2$ BeV and $E_f \sim 1$ BeV. Later, we shall discuss on them in more detail.

§ 4. Diffusion of the nucleon component

To make our discussion more quantitatively, we treat the diffusion of nucleon components through the atmosphere and ascertain our classification and correspondence on them.

In the course of calculation, we assume that the absorption mean free path

l_{abs} , the collision mean free path l_{col} for nuclear interactions, and the probability k for charge exchange in one collision are all constant for each nucleon component. We tentatively take their numerical values as follows,

$$\begin{aligned} l_{abs} &\approx 125 \text{ g/cm}^2 \text{ air,} \\ l_{col} &\approx \text{geometrical one, i.e. } 100 \text{ g/cm}^2 \text{ in emulsion or } 65 \text{ g/cm}^2 \text{ in air, (1)} \\ k &\approx 1/2. \end{aligned}$$

These are approximately verified by various experiments.

Diffusion of A-component Neglecting the ionization loss, the vertical intensity $A(t)$ of A -component can be expressed simply as,

$$A(t) = a \exp(-t/l_{abs}), \quad (2)$$

where a is the primary intensity at the top of the atmosphere. Its total intensity $\bar{A}(t)$ is, integrating over all directions,

$$\begin{aligned} \bar{A}(t) &= \int A(t/\cos \theta) d\Omega \\ &= 2\pi a [\exp(-t/l_{abs}) + (t/l_{abs}) \text{Ei}(-t/l_{abs})]. \end{aligned} \quad (3)$$

If we distinguish the proton intensity $p(t)$ and the neutron intensity $n(t)$, they are given by,

$$\begin{aligned} p(t) &= (a/2) [\exp(-t/l_{abs}) + \exp[-(t/l_{abs}) - (2kt/l_{col})]], \\ n(t) &= (a/2) [\exp(-t/l_{abs}) - \exp[-(t/l_{abs}) - (2kt/l_{col})]], \end{aligned} \quad (4)$$

under the initial conditions,

$$p(0) = a, \quad n(0) = 0.$$

Their total intensities $\bar{p}(t)$ and $\bar{n}(t)$ are also easily obtained.

Next we shall compare them with the experimental data. Table II shows the absolute frequency of nuclear events measured at two altitudes by Bristol group. Following the consideration in § 3, we can estimate from them the absolute frequency of nuclear events which are produced by A , B and C -component (we call them simply A , B and C -stars). We show in Table III their frequency obtained under the assumption that protons and neutrons are equally produced in a nuclear collision, i.e.

$$\begin{aligned} A\text{-stars} &\approx (\geq 2_s) + 2_s: (n \text{ or } p) \rightarrow (p+p \text{ or } p+n), (\pi+p \text{ or } \pi+n), \\ B\text{-stars} &\approx 2 \times (1_s - 2_s): (n \text{ or } p) \rightarrow (n \text{ or } p), \\ C\text{-stars} &\approx 0_s - (1_s - 2_s): (n \text{ or } p) \rightarrow 0. \end{aligned} \quad (5)$$

1) Stratosphere (45 g/cm^2). Assuming the numerical values (1) for the parameter, the n - p ratio for A -component is expected to be,

Table II.
Absolute frequency (per 1 day, 1 g emulsion) of nuclear events
measured by Bristol group.^{1) 2)}

type atmospheric depth	0_n	0_p	1_s	$\geq 2_s$
45 g/cm ²	167 \pm 7.9	30.7 \pm 3.4	46.5 \pm 4.2	27.0 \pm 3.2
690 g/cm ²	2.05 \pm 0.03	0.288 \pm 0.013	0.245 \pm 0.012	0.123 \pm 0.008

Table III.
Estimated absolute frequency of nuclear events produced
by *A*, *B* and *C*-component.

classification atmospheric depth	<i>A</i> -star	<i>B</i> -star	<i>C_n</i> -star*
45 g/cm ²	39.	70.	148.
690 g/cm ²	0.19	0.35	1.95

* *C_n*-star means the nuclear event produced by *C*-neutrons.
C_p-stars are discussed later (cf. § 5).

$$\bar{n}/\bar{p}=0.52 \quad \text{at } t=45 \text{ g/cm}^2,$$

which is nearly in agreement with the observed one, 0.57 ± 0.19 for $\geq 2_s$ (cf. Table I). As for the absolute intensity, we take the data obtained by the rocket experiments,⁴⁾

$$a=0.12/\text{sec. cm}^2 \text{ sterad.} \quad (6)$$

Then, we have the total intensity $\bar{A}(t)$ at this altitude, as,

$$\bar{A}(45 \text{ g/cm}^2)=2.67 \times 10^4/\text{cm}^2 \text{ day}, \quad (7)$$

which cause the nuclear events with the frequency,

$$A\text{-star at } 45 \text{ g/cm}^2=2.67 \times 10^2/\text{g. day.} \quad (8)$$

This should be compared with the observed frequency of *A*-stars, 39/g. day, which is much smaller than the expected one (8) (cf. Table III). We can find some possible explanations to account for this large discrepancy.

a) The assumed values (1) of l_{abs} and l_{col} may be wrong. But, as will be discussed later concerning the altitude dependence, we can not take a smaller value for l_{abs} . While, as is seen, the geometrical mean free path (1) is the minimum estimation for l_{col} , and its maximum estimation will be $\sim l_{abs}$, i.e. twice of the geometrical one. If we take $l_{col} \sim l_{abs}$, this modification decreases the expected frequency of *A*-stars by a factor ~ 2 and somewhat reduces the above discrepancy.

But, at the same time, it makes smaller the n - p ratio to 0.28, which again contradicts with the observed value. Therefore, we can say that the allowable maximum value for I_{tot} will be 1.5-times the geometrical one.

b) Recent experiments show that 20~30% of primary particles are α -particles or heavier nuclei, which have much larger absorption coefficient than protons.⁵⁾ Therefore, we should equate a in (2) to the primary proton intensity, and not to the total intensity of primary rays.

c) As is seen from Table I, the n - p ratio of events 1, may be somewhat smaller than 1. If this is really the case, we must attribute some of 1, of the first mode to A -stars. From its n - p ratio, at most 1/3 of 1, of the first mode can be regarded as A -stars, and thus we have the estimated maximum frequency of A -stars as follows (cf. Table III),

$$A\text{-stars}=39+70/3=62/\text{g. day.} \quad (9)$$

In this case, the frequency of B -stars becomes 2/3 of the previous value, i.e. 47/g. day.

d) Furthermore, we should not overlook the possibility that the considerable amount of back rays may exist at the top of the atmosphere, and may increase the apparent primary intensity.⁶⁾ The maximum estimation of the intensity of back-rays can be nearly the same order of magnitude as the primary rays.

e) Bristol group observed the nuclear events with $N_h \geq 3$. Therefore, we must add the frequency of missed events to their experimental value. Extrapolating their N_h -spectrum, we can estimate the contribution from these small stars may be 10%.

Taking into account the remarks a) and b), we have the minimum expected frequency of A -stars as,

$$(8) \times \frac{1}{1.5} \times 0.7 \approx 125/\text{g. day}, \quad (10)$$

while, the maximum estimation of its observed frequency is, from the remarks c) and e),

$$62 \times 1.1 = 68/\text{g. day.} \quad (11)$$

Furthermore, if we regard this somewhat still larger expected value as due to the back-rays (cf. remark d)), the discrepancy will not be so large as stated before.

2) Mountain altitude (690 g/cm²). We may tentatively take (9) as a standard for the intensity of A -component, since we do not know the precise value for the primary intensity. Then we obtain the expected frequency of A -stars at this altitude as follows, assuming the absorption mean free path l_{abs} to be 125 g/cm²,

$$A\text{-star at } 690 \text{ g/cm}^2 = 0.062/\text{g. day.} \quad (12)$$

It is nearly one third of the observed frequency of A -stars, 0.19/g. day (cf. Table III). If we adopt $l_{\text{abs}} \sim 150$ g/cm² instead of (1), i.e. 125 g/cm², we can get the

correct expected value equal to the observed one. This rather large absorption mean free path may be due to the difference in the energy spectra of A -component at two altitudes, as is seen from the comparison of n_s -spectra.*

The n - p ratio of this component is expected to be 1 at this altitude. The experimental value, 0.76 ± 0.14 , is somewhat smaller than 1. If this slight difference really exists, we must take at most $k \sim 0.3$, which again destroys the agreement obtained for the stratosphere data. Large statistical error prevents us to get a definite conclusion.

We have regarded that in the stratosphere a part of I_s of the first mode were due to A -component, whereas at the mountain altitude A -component responded only to $\geq 2_s$. It may look rather strange, but we can suppose that the different magnitude of contributions from B -component will result in such correspondences (cf. later discussion).

Diffusion of B-component If we assume that a direction of motion is conserved in A - B transmutations, we can easily calculate the intensity of B -component. Neglecting the ionization loss, we get its vertical intensity $B(t)$ at atmospheric depth t as follows,

$$B(t) = a\nu(t/l_{col}) \exp(-t/l_{abs}), \quad (13)$$

where ν is the number of B -nucleons produced in one collision. Its total intensity $\bar{B}(t)$ is, integrating over all directions,

$$\bar{B}(t) = 2\pi a\nu(t/l_{col})[-\text{Ei}(-t/l_{abs})]. \quad (14)$$

Comparing it with (3), we get the ratio of both components,

$$\bar{B}(t)/\bar{A}(t) = \nu \langle t \rangle / l_{col},$$

$$\langle t \rangle = \frac{-t \text{Ei}(-t/l_{abs})}{\exp(-t/l_{abs}) + (t/l_{abs}) \text{Ei}(-t/l_{abs})} = \begin{cases} 85 \text{ g/cm}^2 & \text{for } t = 45 \text{ g/cm}^2, \\ 780 \text{ g/cm}^2 & \text{for } t = 690 \text{ g/cm}^2. \end{cases}$$

While, we can determine this ratio from the experimental data. Equating both figures, we determine the unknown parameter ν .

1) Stratosphere. From the remark c), we estimate the frequency of B -stars as,

$$B\text{-sta}_1 = 47/\text{g. day}. \quad (16)$$

Thus their ratio is,

* The absorption mean free path $l_{abs} \sim 125 \text{ g/cm}^2$ was commonly ascertained by the experiments on hard showers and bursts, agents of which are supposed to be high energy A -component. The integral energy spectrum of nucleon-component will be, as is discussed later, nearly $1/E^2$ at lower altitudes, whereas its primary spectrum is less steep than this power law in lower energy region ($\gtrsim 5 \text{ BeV}$).⁷⁾ Therefore, we can suppose that the intensity of A -component can not be expressed by a simple exponential function as (2), but its lower energy part ($2 \sim 5 \text{ BeV}$) may show somewhat less steep altitude dependence (cf. discussion on B -component).

$$\bar{B}/\bar{A}=(16)/(9)=47/62=0.76. \quad (17)$$

Therefore, comparing it with (15), we have

$$\nu(l_{abs}/l_{col})=0.76/0.68=1.12. \quad (18)$$

2) Mountain altitude. In this case, we have from Table III,

$$B\text{-star}=0.35/\text{g. day.} \quad (19)$$

In the same way as above, we can estimate as,

$$\bar{B}/\bar{A}=(19)/\text{the value in Table III}=0.35/0.19=1.84, \quad (20)$$

and

$$\nu(l_{abs}/l_{col})=1.84/6.2=0.30. \quad (21)$$

We should not directly compare these two values of ν , (18) and (21), for the following reasons. First, as we discussed above, the energy spectrum of A -component may be somewhat different at the two altitudes. Therefore, the comparison of mean values of ν above obtained will have little meaning, and it seems necessary to treat more precisely taking into account the energy dependence of the multiplicity ν . Qualitatively, ν in (21) will be a mean value with larger weight for low energy A -component, since we adopted there the value in Table III as the frequency of A -stars. If we use (12) for its frequency at the lower altitude, we get the three times (21) for ν , which will be, on the contrary to the above, a mean value with larger weight for high energy A -component. These two determination of ν at the lower altitude will be its maximum and minimum estimation. We find somewhat still larger ν at the higher altitude.

Furthermore, contributions from heavier primaries should not be overlooked and can be estimated as follows. Here we denote the quantities concerning with heavier primaries by asterisk *. From the experiment of Bradt and Peters,⁵⁾ we may take into account only primary α -particles, the intensity of which can be written as,

$$A^*(t)=a^* \exp(-t/l_{abs}^*),$$

where

$$a^* \approx 0.1a, ^{\dagger)}$$

$$l_{abs}^* \approx 40 \text{ g/cm}^2.$$

The intensity $B^*(t)$ of B -nucleons produced by heavier primaries is obtained as,

$$B^*(t)=a^*\nu^* [l_{abs}/(l_{abs}-l_{abs}^*)] [\exp(-t/l_{abs})-\exp(-t/l_{abs}^*)]$$

^{\dagger}) The total intensity of heavier primaries is 20%~30% of that of protons, whereas the geomagnetic cut-off energy per nucleon for the former is a half of that for the latter. A maximum estimation of a^* from heavier primaries is 20%~30% of a , and its minimum one is 5%~7.5% which is obtained from the energy spectrum $\sim 1/E^2$ at higher energy region. We have adopted there a mean value, $a^* \approx 10\%$ of a .

$$=1.47a^*\nu^* [\exp(-t/l_{abs}) - \exp(-3.1t/l_{abs})],$$

where ν^* is the multiplicity of B -nucleons in one collision, and we have put $l_{col}^* \approx l_{abs}^*$.

Therefore, the ratio of B^* to A is,

$$B^*/A = 0.15\nu^* [1 - \exp(-2.1t/l_{abs})],$$

and

$$\bar{B}^*/\bar{A} = \begin{cases} 0.12\nu^* & \text{at } t = 45 \text{ g/cm}^2, \\ 0.15\nu^* & \text{at } t = 690 \text{ g/cm}^2. \end{cases}$$

Taking this into account, we should add it to the above obtained \bar{B}/\bar{A} . This modifies (18) and (21) as follows. Assuming $\nu^* \sim 4\nu$, we have,

$$\nu = \begin{cases} 1.12/[(l_{abs}/l_{col}) + 0.71] & \text{at } t = 45 \text{ g/cm}^2, \\ 0.30/[(l_{abs}/l_{col}) + 0.1] & \text{at } t = 690 \text{ g/cm}^2. \end{cases}$$

If we assume $(l_{abs}/l_{col}) \approx 2$, this results in $\nu = 0.41$ or 0.14 , and for $(l_{abs}/l_{col}) \approx 1.5$, ν is 0.51 or 0.19 . We see that the contributions from heavier primaries should not be neglected especially in the stratosphere, and they diminish the discrepancy in ν stated before.

Diffusion of C-component The treatment of C -component is not so simple as the above cases, since they are secondary or tertiary products. First, we shall be concerned only with C -neutrons, and later discuss C -protons taking into account the ionization loss. Furthermore, for simplicity, we assume that a direction is conserved in A - C or B - C transmutations, though this approximation will be worse than in the former case, i.e. A - B transmutations. Then we can get the intensity $C_A(t)$ of C -component produced by A -component in the same way as above,

$$C_A(t) = (\mu_A/\nu) B(t). \quad (22)$$

μ_A denotes the multiplicity in A - C transmutation. The contribution from B -component is also obtained. We have, denoting as μ_B the multiplicity in B - C transmutation,

$$C_B(t) = a\nu\mu_B \cdot (t^2/2l_{col}^2) \exp(-t/l_{abs}).$$

Thus its total intensity is,

$$\begin{aligned} \bar{C}_R(t) &= 2\pi a(\nu l_{abs}/l_{col}) \cdot (\mu_R t/2l_{col}) \exp(-t/l_{abs}) \\ &= (\mu_B l_{abs}/l_{col}) (\langle t \rangle / l_{abs}) \bar{B}(t), \end{aligned} \quad (23)$$

where

$$\langle t \rangle / l_{abs} = \frac{\exp(-t/l_{abs})}{-2\text{Ei}(-t/l_{abs})} = \begin{cases} 0.90 & \text{at } t = 45 \text{ g/cm}^2, \\ 6.3 & \text{at } t = 690 \text{ g/cm}^2. \end{cases}$$

Summing up both contributions, we have

$$\bar{C}(t)/\bar{B}(t) = (\mu_A/\nu) + \mu_B \langle t \rangle / l_{col}. \quad (24)$$

If we assume, as another limiting approximation, that angular divergence in A - C or B - C transmutation is isotropic, $\bar{C}_A(t)$ and $\bar{C}_B(t)$ can be obtained as follows,

$$\begin{aligned} \bar{C}_A(t) = (\mu_A/l_{col}) & \left\{ \int_0^t dt' \bar{A}(t-t') [\exp(-t'/l_{abs}) + (t'/l_{abs}) \text{Ei}(-t'/l_{abs})] \right. \\ & \left. + \int_t^\infty dt' \bar{A}(t+t') [\exp(-t'/l_{abs}) + (t'/l_{abs}) \text{Ei}(-t'/l_{abs})] \right\}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \bar{C}_B(t) = (\mu_B/l_{col}) & \left\{ \int_0^t dt' \bar{B}(t-t') [\exp(-t'/l_{abs}) + (t'/l_{abs}) \text{Ei}(-t'/l_{abs})] \right. \\ & \left. + \int_t^\infty dt' \bar{B}(t+t') [\exp(-t'/l_{abs}) + (t'/l_{abs}) \text{Ei}(-t'/l_{abs})] \right\}. \end{aligned} \quad (26)$$

Integrating them numerically, we have,

$$\bar{C}(t)/\bar{B}(t) = \begin{cases} 0.88(\mu_A/\nu) + 0.60\mu_B(l_{abs}/l_{col}) & \text{at } t = 45 \text{ g/cm}^2, \\ 0.49(\mu_A/\nu) + 3.1\mu_B(l_{abs}/l_{col}) & \text{at } t = 690 \text{ g/cm}^2. \end{cases} \quad (27)$$

Comparing (24) with (27), we find no serious difference between these two approximations.

Now, the experimental value of \bar{C}/\bar{B} can be obtained from Table III, and remark c):

$$\bar{C}/\bar{B} = \begin{cases} 148/47 = 3.2 & \text{in the stratosphere,} \\ 1.95/0.35 = 5.6 & \text{at the mountain altitude.} \end{cases} \quad (28)$$

Equating them to (24), we have

$$\left. \begin{aligned} \mu_A/\nu &= 2.8 \\ \mu_B l_{abs}/l_{col} &= 0.45 \end{aligned} \right\} \quad (\text{direction is conserved}). \quad (29)$$

If we use (27) instead of (24), they become

$$\left. \begin{aligned} \mu_A/\nu &= 2.7 \\ \mu_B l_{abs}/l_{col} &= 1.4 \end{aligned} \right\} \quad (\text{isotropic}). \quad (30)$$

Assuming $l_{abs} \approx 1.5 l_{col}$ and $\nu \approx 0.3$, the multiplicity μ_A and μ_B are,

	μ_A	μ_B
Direction is conserved	0.87	0.30
Isotropic	0.84	0.93

We can suppose thus $\mu_A \sim 0.8$ and $\mu_B \sim 0.6$, though we can not get a definite conclusion due to the crudeness of our treatment. Nearly the same number of C -protons are supposed to be emitted at the same time.

In order to treat the diffusion of C -protons and get the n - p ratio of C -component, we must take into account the ionization loss. It will be discussed later, comparing of its energy spectrum.

§ 5. Nuclear scattering

Fig. 1 shows integral N_h -spectrum of events 1_n , 1_p , 0_p and 0_n at different two altitudes obtained by Bristol group. As is seen at once, in the stratosphere three curves for 1_n , 1_p , 0_p resemble with each other. This shows, that these events are essentially the same phenomena, i.e. the penetration of fast nucleons through the nucleus, and the charge exchange probability k is $\sim 1/2$. At the lower altitude, the frequency of 0_p -stars becomes more than that of 1_n and 1_p . This is of course due to the increase of slow protons, which can not penetrate the nucleus.

Now if the meson is not produced, the incident nucleon loses its energy through secondary nucleons (black and grey nucleons). Powell and his cowor-

No. of events

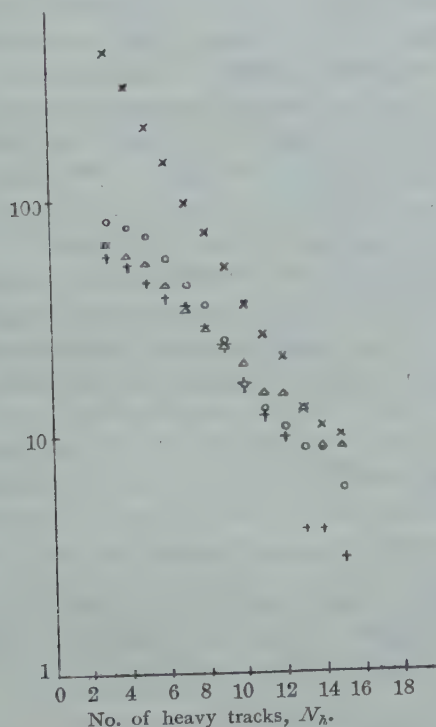


Fig. 1a Integral N_h -Spectrum in the stratosphere.²⁾

No. of events

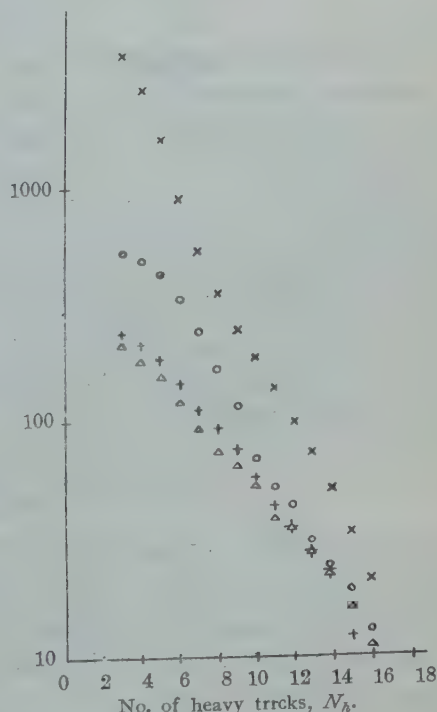


Fig. 1b Integral N_h -spectrum at the mountain altitude¹⁾

\times 0_n , \circ 0_p , $+$ 1_n , Δ 1_p

\times 0_n , \circ 0_p , $+$ 1_n , Δ 1_p

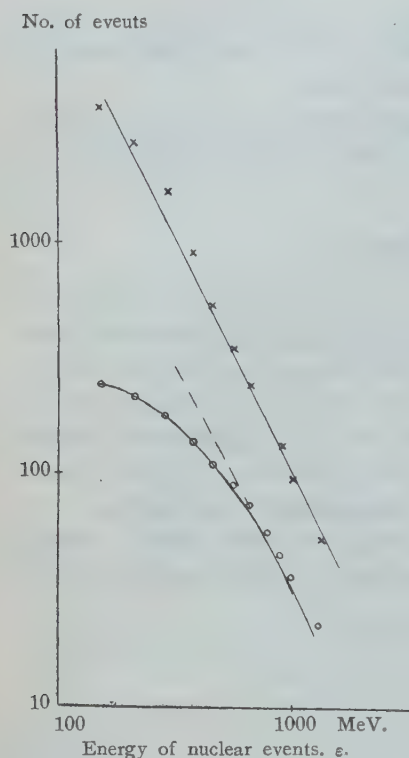


Fig. 2 Integral energy spectrum of nuclear events¹⁾.

× 0_s ○ 1_s

ted by dashed line, in Fig. 2. If this is the case, and assuming $\epsilon/E \sim 1/3$, the incident energy spectrum may be $\sim 1/E^2$ and ranges 0.9 BeV \sim 3.9 BeV.

These considerations on 1_p , 1_n -stars are supported by the following scattering data. Bristol group investigated the distribution of angle δ between the incident direction and the secondary thin tracks. In the case of single nucleon-nucleon scattering, the scattering angle δ is connected with the incident and final energies E_0 and E , in following way

$$\cos \delta = \left\{ \left(1 + \frac{2M}{E_0} \right) / \left(1 + \frac{2M}{E} \right) \right\}^{1/2}. \quad (31)$$

This relation holds approximately even in plural scattering inside a nucleus.

The experimental values of angle δ are in most cases confined between $10^\circ \sim 50^\circ$, and its distribution curve has a maximum at $\sim 30^\circ$. Large angle scatterings are seen in few cases. Lying aside these few large angle cases, which will be discussed later, the mean value for δ is,

$$\bar{\delta} = 33.6^\circ = 0.59, \quad \overline{\cos \delta} = 0.78. \quad (32)$$

Now we shall apply the formula (31). If we assume $E \approx E_0/2$, $\cos \delta$ becomes

kers estimated the energies of black and grey nucleons and find the relation between N_h and energy loss ϵ . As is seen from Fig. 1, the incident energies for 0_n stars are in most cases so low that the agent can not penetrate through the nucleus, or even if it penetrates its energy is degraded into grey region. Therefore we can obtain the spectrum of incident energy E for 0_n stars by identifying E with the loss ϵ itself. The curve shown in Fig. 2 is obtained in this way from the data at mountain altitude. It is well approximated by E^{-2} and ranges from 200 MeV \sim 1 BeV. This supports our views on C -component.

If we apply the same method for 1_s -curve, we can not get the power spectrum for loss ϵ as seen from Fig. 2. It is flatter in lower energy region and tends to E^{-2} in higher energy regions, 700 MeV \sim 1.3 BeV. This will be due to wrong $N_h - \epsilon$ relation, which was made to fit for 0_n stars. We can suppose that the above relations are considerably lower estimation for ϵ of small 1_s stars, and perhaps corrected spectrum will be $\sim 1/E^2$ as illustrated

$$\cos \delta = 0.78 \quad \text{or} \quad 0.83, \quad \text{for } E_0 = 1 \text{ or } 2 \text{ BeV.} \quad (33)$$

They are well in agreement with the experimental one (32). However, one may doubt that the assumed fractional loss $1/2$ is too large. But we can interpret the case of the larger values of δ than in (33) as the one in which the incident nucleon changes into a neutron and it accompanies a fast proton of smaller energy. Therefore the case of the larger δ will correspond to the smaller fractional loss than $1/2$.

From the definition of thin tracks, the secondary proton must have larger energy than $\gtrsim 330$ MeV. This determines the maximum value δ_{\max} for the scattering angle. On the other hand, we can estimate the minimum energy loss in penetration through the nucleus as ~ 200 MeV from the condition $N_h \geq 3$. This determines the minimum value δ_{\min} . Applying (31), we calculate δ_{\min} and δ_{\max} as shown in Table IV.

Table IV. Minimum and maximum scattering angle

incident energy (BeV)	δ_{\min}	δ_{\max}
1	19°	42°
2	9°	57°
4	3°	62°
experimental value	$\sim 10^\circ$	$\sim 50^\circ$

The result of Table IV supports our views on B -component.

The large angle scattering $\delta = 50^\circ \sim 90^\circ$ and the back scattering were found in several cases in the experiment. They amount nearly 40% of the small angle scatterings $\delta = 0^\circ \sim 50^\circ$ which we analyzed above. We are hard to explain these cases as nuclear scattering and suppose that they are meson production. Although there are some ambiguities in distinguishing between large and small scattering, we may conclude nearly 30% of 1_s star are associated with meson production.

Furthermore, we have some information on meson production in 1_s stars, from its N_h -distribution. Comparing precisely the N_h -spectrum of 0_p and 1_p stars in the stratosphere, we find that the frequency of 0_p -event is larger than that of 1_p stars for $N_h \geq 9$, while it becomes smaller for $N_h \leq 9$. The large N_h corresponds to the nuclear event with large energy. This slight difference $\sim 20\%$ in large N_h seems to be due to the meson production.

These estimation on meson production in 1_s -stars should be only qualitative. The large statistical error and the ambiguities of interpretation prevent us from further quantitative discussions. More direct experiments are hoped on this point.

If we know the energy spectrum of nucleon component, we can get the n - p -ratio of C -component. Assuming the integral spectrum as E^{-2} , which was ascertained above, we calculate them for an examples. The experimental values are shown in Table V.

Table V. n/p ratio of O_n stars at mountain altitude.

N_h	3	4	5	6	7~8	9~12	≥ 13
n/p ratio	27 ± 14.3	13 ± 6.2	5.3 ± 2.0	3.6 ± 1.5	2.5 ± 0.93	2.4 ± 0.98	1.45 ± 0.90

Example $N_h=5$. Incident energy is estimated as ~ 280 MeV. The proton is assumed to cause nuclear event, after traversing l_{col} . If we take l_{col} as $1.5 \times l_{g10}$, the energy of proton at its birth is ~ 560 MeV. Then we have

$$n/p = (560 \text{ MeV} / 289 \text{ MeV})^3 = 8.$$

As is seen, this n/p ratio is very sensitive to the value of l_{col} . If we wish n/p to be exactly 5.3, we have only to put $l_{col} = 1.15 \times l_{g10}$.

§ 6. Concluding remarks⁸⁾

We have analyzed the photographic data on nuclear events caused by high energy cosmic-rays, classifying the nucleon component into three groups according to their energies. Following characteristic features are found.

1) The cross-section for the nuclear interaction is nearly equal to the geometrical cross-section of the collided nucleus.

2) The integral energy spectrum of nucleon component is $\sim 1/E^2$ at lower altitudes.

3) For incident nucleons with energies below 2 MeV, nucleon-nucleon collisions are almost elastic, and collisions associated with meson production will be $\sim 20\%$.

4) The fractional energy loss in one nuclear collision is $\sim 1/3$ and it is nearly independent of incident energies.

5) The probability for charge exchange in one nuclear collision is $\sim 1/2$.

These are ascertained in the region of energies from several hundred MeV to several BeV, and allow us to infer the nuclear interaction at high energies. A main difficulty of our analysis is the inconsistency between the primary intensity of cosmic-rays and the absolute frequency of nuclear events observed in stratosphere, but later experiment⁹⁾ shows smaller primary intensity than before, and the discrepancy will not be so large as supposed at first. Contributions from heavier primaries also introduce various ambiguities. It seems necessary to analyze more precisely taking into account the energy spectrum of nucleons, when we will compare these results with the theoretical predictions. Also more detailed experiments are required for this purpose.

Previously, we calculated the problem on the penetration of fast nucleons through the nuclear matter.¹⁰⁾ Assuming the current nuclear potential which is well fitted to the neutron-proton scattering and deuteron binding energy, we find the cross-section which is nearly proportional to c^2/v^2 and tends to $\sim 5 \times 10^{-27}$

cm² per one nucleon at high energies. This result does not seem to be able to account for the observed nuclear events, and some new type of nuclear forces may be necessitated in high energy regions. We shall discuss on these problems in a separate paper.

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Addendum

- 1) If the absolute intensity of primary rays is revised as

$$\alpha = 0.070/\text{sec. cm}^2 \text{sterad}, \quad (6')$$

following the recent estimation of Van Allen et al⁹⁾, or

$$\alpha = 0.093/\text{sec. cm}^2 \text{sterad}, \quad (6'')$$

considering that in higher latitude where the experiment may be carried out, then some figures in § 4 are modified as follows (figures based on the latter estimation (6'') are represented in the brace).

$$\bar{A}(45 \text{ g/cm}^2) = 1.54(2.08) \times 10^4/\text{cm}^2 \text{ day}, \quad (7')$$

$$A\text{-stars at } 45 \text{ g/cm}^2 = 1.54(2.08) \times 10^2/\text{g. day}, \quad (8')$$

$$\text{the minimum expected frequency of } A\text{-stars} \approx 72(97)/\text{g. day}. \quad (10')$$

2) Taking into account the existence of α -particles in primaries, the diffusion of A -component is somewhat modified. The number of A -protons or -neutrons given rise from α -primaries is evaluated by

$$p(t) = \frac{1}{l_{\alpha b s}^*} \int_0^t 2\alpha^* \exp(-t/l_{\alpha b s}^*) p(t-t') dt',$$

where $p(t)$ is the solution of (4) with initial condition $p(0)=0$. Assuming $\alpha^* \approx 0.1\alpha$, we obtain

$$\bar{p}^*/\bar{p}(45 \text{ g/cm}^2) = 0.094, \quad \bar{n}^*/\bar{n}(45 \text{ g/cm}^2) = 0.30$$

and

$$\bar{A}^*/\bar{A}(45 \text{ g/cm}^2) = 0.14.$$

Accounting for the above, (10') must be modified as

$$82(111)/\text{g. day}.$$

The n/p ratio at the stratosphere increases to 0.37 from 0.31, the latter being derived from the proton primary hypothesis and $l_{001} = 1.5 l_{900}$.

On the Problem of Covariance in Quantum Electrodynamics, I

Jiro YUKAWA and Hiroomi UMEZAWA

Institute of Theoretical Physics, Nagoya University

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§ 1. Introduction and discussion

Various difficulties appearing in quantum electrodynamics have been considered to be due to the point model of elementary particles. Therefore, various attempts have been proposed to construct the theory of structure of elementary particles. As is well known, the theory of point electron contains the difficulty of infinite self-energy. In Dirac's theory of positron, though by virtue of the positron-theoretical subtraction the order of divergence was reduced, the energy of electron is still divergent. In order to overcome this difficulty, theories of extended model of electron and of point electron accompanied with some cohesive-force fields have been proposed. In the theory of extended model of electron, the stress-tensor does not necessarily vanish. In the relativistic field theory from the requirement of covariance of theory, an energy-momentum tensor density $T_{\mu\nu}$ of the total system of particle and field must satisfy the following transformation law;

$$E = \int T_{44} dv = \frac{\{T_{44}(0) dv(0) - \beta^2 \{T_{11}(0) dv(0)\}}{\sqrt{1-\beta^2}}, \quad (1a)$$

$$P = -\frac{i}{c} \int T_{14} dv = \frac{\{T_{44}(0) dv(0) - \{T_{11}(0) dv(0)\}}{c^2 \sqrt{1-\beta^2}} v, \quad (1b)$$

where quantities marked (0) refer to the rest system of the particle $dv = dv(0) \sqrt{1-\beta^2}$, and v is velocity of a particle moving in the x -direction.

The index 1 refers to the x -direction. The integrations are meant over all three-dimensional space.

Eq. (1) holds in classical theory as well as in quantum theory, provided that in the latter case we consider the energy and stress as expectation values referring to a specified state of the system.

From (1) we can see, that if the energy has a transformation property of inertia mass, the self-stress T_{11} , T_{22} , T_{33} always must vanish in the theory of point electron, moreover, the stress is to be zero and the energy must have a transformation property of inertia mass.

By the calculations in usual quantum electrodynamics, however, the self-stress of electron has a finite value which is not zero and moreover the self-energy of

electron has not the correct transformation property of inertia mass.

However, as the formalism of quantum electrodynamics is relativistically covariant, this circumstance seems quite strange to us.

Hence, it may be a very significant proposition of quantum electrodynamics that we would search for the cause of discrepancy.

The relativistically covariant formalism developed by Tomonaga and Schwinger¹⁾ and the w -method²⁾ which carry out the integration appearing in the perturbation theory in a relativistically invariant manner, gave the self-energy of electron with a transformation property of inertia mass. Particularly the latter showed that the defect of the usual calculating method for moving electron was due to indicating the domain of integration in relativistically non-invariant manner. However even these covariant theories could not make the self-stress of electron zero. This fact is clear from that even in the case of electron at rest a non-vanishing self-stress remains, because this situation is independent on the variation of the integration domain due to the motion of electron.

Recently Pais and Epstein³⁾ threw light upon the cause making the non-vanishing self-stress, as follows: let the electromagnetic self-energy of electron (of mass μ) $\delta\mu$, then the self-stress of electron at rest $\int T_{11}(0)dv(0)$ is given in the following equation:

$$\int T_{11}(0)dv(0) = -\frac{1}{3}\left(\delta\mu - \mu \frac{\partial \delta\mu}{\partial \mu}\right), \quad (2)$$

where to the e^2 -order-approximation in perturbation theory

$$\delta\mu = \frac{3e^2}{2\pi}\mu \left[\frac{1}{2} \log\left(\frac{2}{a^2}\right) - \frac{1}{6} \right], \quad (3)$$

and a is the radius of the electron.

$\delta\mu$ is not proportional to μ on account of the term $\log\left(\frac{2}{a^2}\right)$ and so we have the following value for (2) instead of zero:

$$\int T_{11}(0)dv(0) = -\frac{e^2}{2\pi}\mu. \quad (4)$$

However in the limit of $a \rightarrow 0$, the mass μ in above log-term also vanishes, $\delta\mu$ is proportional to μ , and in virtue of (2) the self-stress tends to zero. In this case, however, on account of the point model of electron the self-energy unfortunately diverges. This is the very reason why the non-vanishing self-stress destroys the covariance of theory (see (1)).

However in Pais' derivation of the formula (2) he used without any criticism the following relation,

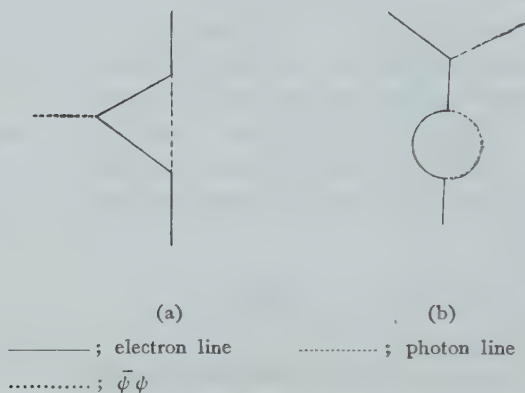
$$\langle \int \phi^* \beta \phi dv \rangle = \frac{\partial \langle H \rangle}{\partial \mu}. \quad (5)$$

But, Pais' formula (5) can not be used unconditionally, and then we shall justify it for the mass-type process of an electron with the following straightforward calculation. Now, we define h_1 and h_2 by the next two equations

$$h_1 \equiv \frac{i(\gamma \cdot p - R) - \mu}{\{(p - R)^2 + \mu^2\}}, \quad h_2 \equiv \frac{i(\gamma \cdot p) - \mu}{p^2 + \mu^2}. \quad (6)$$

Representing the interaction between electrons ψ and neutral field ϕ as $\bar{\psi} O \psi \phi$, the e^2 -correction of $\bar{\psi} \psi$, by the mass type processes (a) and (b), is

$$(\bar{\psi} \psi)_{2; \text{mass}} = -\frac{i}{4\pi^3} \int \bar{\psi}(p) [O h_1^2 O + O h_1 O h_2 O h_2 + h_2 O h_1 O] \psi(p) \Delta(k) d^4 k \quad (7)$$



where $\Delta(k)$ is a quantity independent of electrons.

From (6) and the equation of motion of $\psi(p)$, we have

$$\begin{aligned} \frac{\partial h_1}{\partial \mu} &= h_1^2, \\ \frac{\partial \psi(p)}{\partial \mu} &= h_2 \psi(p) + \psi(p) \lambda, \end{aligned} \quad (8)$$

where λ is an arbitrary regular function with the property of spinor. With the help of (8), we obtain

$$\begin{aligned} (\bar{\psi} \psi)_{2; \text{mass}} &= \frac{\partial}{\partial \mu} \left(-\frac{i}{4\pi^3} \int \bar{\psi}(p) O h_1 O \psi(p) \Delta(k) d^4 k \right) \\ &\quad + \frac{i}{4\pi^3} \int \{ \bar{\psi}(p) O h_1 O \psi(p) \lambda + \lambda^* \bar{\psi}(p) O h_1 O \psi(p) \} \Delta(k) d^4 k \\ &= \frac{\partial}{\partial \mu} (\delta \mu \bar{\psi} \psi) - \delta \mu (\bar{\psi} \psi \lambda + \lambda^* \bar{\psi} \psi) \\ &= \frac{\partial \delta \mu}{\partial \mu} \bar{\psi} \psi + \delta \mu (\bar{\psi} h_2 \psi + \bar{\psi} h_2 \psi) \end{aligned} \quad (9)$$

where in the second term, the denominator of h_2 is zero according to the equation of motion for $\phi(p)$ and this term is the singular one which should be expected from the fact that there may occur the emission and absorption of photon during the infinitely long time before the scattering in Dyson's formalism. Hence this term will be eliminated with mass renormalization. (According to the usual perturbation calculation, whether the renormalization is performed or not, does not affect the self-stress of an electron.) Therefore

$$\langle \bar{\psi}\psi \rangle_{2; \text{mass}} = \frac{\partial \delta\mu}{\partial \mu}. \quad (10)$$

Thus, it has been justified that Pais' formula (5) holds true for the mass type processes.

But, as we will discuss in detail in Addendum, we have two types of process which contribute to the self-stress of an electron such as mass term type and vacuum polarization type⁶⁾ respectively, and relation (5) holds true only for mass type. We will firstly discuss on the mass type which satisfies the relation (5), in § 2 and 3, and secondly discuss on the complete self-stress.

§ 2. The contribution to self-stress from M. T. type

Now the current quantum electrodynamics is assured to be relativistically invariant only on the basis of point model of electron. On the other hand, however, in the calculation of the self-energy, on account of its divergence, we must at the first step cut off the integration at some large momentum value l and at the second step, pass to the limit $l \rightarrow \infty$. However this cut-off procedure would contradict the point model of electron.

Such a very situation that in the calculations we must have a procedure contradicting the requirement of the quantum electrodynamics, gives us the non-vanishing self-stress and destroys the covariance of theory. *This confiction will be called "S₁ confiction" in the following.* Now, if we can give a finite value to the self-energy of electron on the basis of the current formalism, the procedure of cut-off is not needed, and therefore the contribution to self-stress from M.T. type may vanish. From the above discussion, if the theory satisfying the following two conditions in the frame of current formalism of field theory could be constructed, it would be the correct theory of point electron, having no S₁-confiction. That is;

- (i) The self-energy of electron has a transformation property of inertia mass.
- (ii) The self-energy of electron is finite.

Now we will examine this anticipation by means of the C-meson theory which satisfies the above conditions (i) and (ii) in the frame of the current field theory.¹⁾²⁾

Therefore in the C -meson theory it will be anticipated that the contribution to the self-stress of electron from M.T. type will vanish. On the other hand, in the renormalization method of Tomonaga and Schwinger, the self-energy to be amalgamated into is really infinite, and therefore an infinite mass term³⁾ is included in the Hamiltonian of electron. The term $\delta\mu$ is cut off in the calculation. As is illustrated above, this situation contradicts the requirement of the point model in current quantum electrodynamics. Therefore even in this theory, the self-stress does not vanish and has the value (4).

§ 3. The mass type self-stress of electron in the C -meson theory

In calculating the mass type process, we would adopt a method extended from Pais' method²⁾ and use the relation (5).

The energy momentum tensor $T_{\mu\nu}$ for the system of electrons interacting with the electromagnetic and C -meson fields are:

$$T_{\mu\nu} = T_{\mu\nu}^e + T_{\mu\nu}^{E.M.} + T_{\mu\nu}^c, \quad (11)$$

$$\text{with } T_{\mu\nu}^e = -\frac{1}{4i} (\bar{\psi} \partial_{(\mu} \gamma_{\nu)} \psi - \partial_{(\mu} \gamma_{\nu)} \bar{\psi} \psi), \quad (11')$$

$$T_{\mu\nu}^{E.M.} = \frac{1}{4\pi} \left[-F_{\mu\rho} F_{\nu\rho} + \frac{1}{4} \delta_{\mu\nu} F_{\rho\sigma}^2 \right], \quad (11'')$$

$$T_{\mu\nu}^c = \frac{1}{4\pi} \left[-\frac{\partial\phi}{\partial x_\mu} \frac{\partial\phi}{\partial x_\nu} + \frac{1}{2} \delta_{\mu\nu} \left(\frac{\partial\phi}{\partial x_\lambda} \frac{\partial\phi}{\partial x_\lambda} + x^2 \phi\phi \right) \right]. \quad (11''')$$

Here $\partial_{(\mu} \gamma_{\nu)} = \partial_\mu \gamma_\nu + \partial_\nu \gamma_\mu$, $\bar{\psi} = i\psi^* \beta$, $\partial_\mu = \partial/\partial x_\mu - ieA_\mu$. $T_{\mu\nu}^e$, $T_{\mu\nu}^{E.M.}$ and $T_{\mu\nu}^c$ are the electronic, electromagnetic and C -mesonic parts of the total energy momentum tensor $T_{\mu\nu}$ respectively. ψ is the electron wave function, A_μ the electromagnetic four-vector potential, ϕ the C -meson wave function. Further, $F_{\mu\rho} = \frac{\partial A_\rho}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\rho}$ where $F_{\mu\rho}$ is the electromagnetic field, x is the rest mass of C -meson.

Now, taking the trace of (11), we have

$$T_{\mu\mu}^e = \mu\psi^* \beta\psi - f\psi^* \beta\psi\phi, \\ T_{\mu\mu}^{E.M.} = 0, \quad (12)$$

$$T_{\mu\mu}^c = \frac{1}{4\pi} \left\{ \frac{\partial\phi}{\partial x_\lambda} \frac{\partial\phi}{\partial x_\lambda} + 2x^2 \phi\phi \right\}$$

where $-f\psi^* \beta\psi\phi$ is the term representing the interaction between electrons and C -meson field.

Let ϕ be expanded in Fourier series:

$$\phi = \sum_i \sqrt{\frac{2\pi}{\varepsilon_i}} (\xi_i^* e^{-i\varepsilon_i r} + \xi_i e^{i\varepsilon_i r}), \quad (13)$$

then

$$N \equiv \frac{\partial \phi}{\partial x_\lambda} \frac{\partial \phi}{\partial x_\lambda} + x^2 \phi \phi = \sum_i 4\pi \epsilon_i (\xi_i^* \xi_{-i}^* + \xi_i \xi_{-i}), \quad (14)$$

and the matrix element of N between the vacuum state and the state possessing two C -mesons \mathcal{U} and $-\mathcal{U}$ is:

$$(\mathcal{U}, -\mathcal{U} | N | \text{vac.}) = (2\epsilon_i) 2\pi. \quad (15)$$

Now, taking the expectation value of N for the state, one electron at rest present, we shall have

$$\langle N \rangle = \sum_i \frac{(\Psi_0 \Psi(\mathcal{U}, -\mathcal{U})) (\mathcal{U}, -\mathcal{U} | N | \text{vac.})}{(-2\epsilon_i)} + \text{hermite conj.} \quad (16)$$

In virtue of the interaction of electrons and C -meson field, $\Psi(\mathcal{U}, -\mathcal{U})$ deviates from the vacuum and hence it can be expanded in a power series of f .

In this case we restrict ourselves to the second order approximation of perturbation theory, and so we have only to do with the coefficient of f^2 term.

As the two C -mesons annihilate through the sea of electrons, we may easily verify that this coefficient becomes equivalent to the C -mesonic self-energy of electron.

Then the eq. (16) will be

$$\langle N \rangle = -4\pi \delta\mu_e \quad (17)$$

where $\delta\mu_e$ is the C -mesonic self-energy of electron.

Now, taking the expectation value of the second term of $T_{\mu\mu}^e$ (eq. 11') for the state in which one electron at rest is present, we shall have,

$$\langle -f \phi^* \beta \phi \rangle = (\Psi_0 \Psi(\mathcal{U})) (\mathcal{U} | -f \beta \phi | \text{vac}) + \text{hermite conj.} \quad (18)$$

If we expand $\Psi(\mathcal{U})$ in the power series in f and notice the term of first order in f , (18) is certainly equivalent to the C -mesonic self-energy of electron: that is

$$\langle -f \phi^* \beta \phi \rangle = \delta\mu_e. \quad (19)$$

Accordingly, from (12), (13) and (14),

$$\langle T_{\mu\mu} \rangle = \langle \mu \phi^* \beta \phi \rangle + \langle -f \phi^* \beta \phi \rangle + \frac{1}{4\pi} \langle N \rangle + \frac{1}{4\pi} \langle x^2 \phi \phi \rangle, \quad (20)$$

and by (17) and (19)

$$\langle T_{\mu\mu} \rangle = \langle \mu \phi^* \beta \phi \rangle + \frac{1}{4\pi} \langle x^2 \phi \phi \rangle. \quad (21)$$

Now, we have

$$\int T_{11} dv = \int T_{22} dv = \int T_{33} dv = S(O)$$

and

$$\int T_{\mu\mu} dv = 3S(O) + E(O) = 3S(O) + \mu + \delta\mu. \quad (22)$$

By (21) and (22), we have,

$$\langle \mu \int \phi^* \beta \phi dv \rangle + \langle \frac{x^2}{4\pi} \int \phi \phi dv \rangle = 3S(O) + \mu + \delta\mu.$$

By means of (5)

$$\langle \mu \int \phi^* \beta \phi dv \rangle = \mu \left(1 + \frac{\partial(\delta\mu)}{\partial\mu} \right),$$

so that

$$S(O) = -\frac{1}{3} \left(\delta\mu - \mu \frac{\partial(\delta\mu)}{\partial\mu} - \langle \frac{x^2}{4\pi} \int \phi \phi dv \rangle \right), \quad (23)$$

where $\delta\mu = \delta\mu_{E.M.} + \delta\mu_c$ and $\delta\mu_{E.M.}$ and $\delta\mu_c$ are the electromagnetic and C -mesonic self-energy of electron respectively. Using the relation $2e^2 = f^2$, $\delta\mu$ is the sum of only the finite parts of $\delta\mu_{E.M.}$ and $\delta\mu_c$.

That is, if we call $\delta\mu_{E.M.}^f$ and $\delta\mu_c^f$ as the finite parts,

$$\delta\mu = \delta\mu_{E.M.}^f + \delta\mu_c^f, \quad (24)$$

$$\delta\mu_{E.M.}^f = -\frac{f^2}{2\pi} \mu \frac{1}{4}, \quad (25)$$

$$\delta\mu_c^f = \begin{cases} -\frac{f^2}{2\pi} \mu \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{\delta^2 - 4} \log \left(\frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right) \right\}, & (\delta > 2) \\ \frac{f^2}{2\pi} \mu \frac{1}{4} (3 + 8 \log 2), & (\delta = 2) \\ -\frac{f^2}{2\pi} \mu \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{\delta^2 - 4} \left(\sin^{-1} \left(-\frac{\delta}{2} \right) + \frac{\pi}{2} \right) \right\}, & (\delta < 2) \end{cases}$$

where δ on the right hand of (26) means a mass ratio (x/μ) . Then, in the case of $\delta > 2$, computing $\delta\mu - \mu \frac{\partial(\delta\mu)}{\partial\mu}$ in eq. (23), (in the case $\delta \leq 2$ quite similar) we get,

$$\delta\mu - \mu \frac{\partial(\delta\mu)}{\partial\mu} = \frac{f^2}{2\pi} \mu \left\{ \delta^3 + \delta^2 (3 - \delta^2) \log \delta - \delta(1 - \delta^2) \sqrt{\delta^2 - 4} \log \left(\frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right) \right\}. \quad (27)$$

In the next place, we calculate the term $\langle \frac{x^2}{4\pi} \int \phi \phi dv \rangle$ by the second order perturbation calculation with coupling-constant e and f , when there is one electron present in vacuum.

Then, we have ;

$$\begin{aligned} & \langle \frac{x^2}{4\pi} \int \phi \phi dv \rangle_{p=0} \\ &= \frac{f^2}{2\pi} \mu \left\{ \delta^2 + \delta^2 (3 - \delta^2) \log \delta - \delta (1 - \delta^2) \sqrt{\delta - 4} \log \left[\frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right] \right\}. \end{aligned} \quad (28)$$

This is exactly equivalent to (27). Therefore, inserting (27) and (28) into (23), we have

$$S(O)_{M.T.} = -\frac{1}{3} \left(\delta \mu - \mu \frac{\partial(\delta \mu)}{\partial \mu} - \langle \frac{x^2}{4\pi} \int \phi \phi dv \rangle \right) = 0. \quad (29)$$

From the above calculation, our anticipation was now verified that the contribution to the self-stress of electron at rest from M.T. type process would vanish in *C*-meson theory, satisfying two above conditions (I) and (II).

This non-vanishing but finite *C*-mesonic self-stress of electron of Mass Type, as is shown in (29), just cancels the finite electromagnetic self-stress of electron of Mass Type.

The reason why under these circumstances the Mass Type self-stress of electron vanishes in the *C*-meson theory is the unnecessary of introducing an electron radius in order to have a finite self-energy of electron within the frame of current field theory. (*to be continued*)

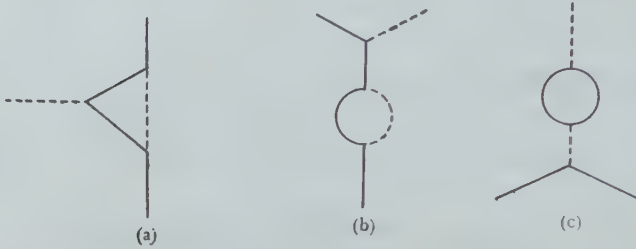
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- 5) In this connection, we should like to thank Mr. K. Sawada for his kind notice, by his private communication, that in addition to mass type contribution, vacuum polarization type process also contributes to self-stress.

Addendum

In our previous letter entitled "Relativistic Covariance in the Quantum Electrodynamics," we have pointed out on the basis of Pais¹⁾ argument, that the origin of the non-vanishing self-stress which destroys the covariance of field theory is the inconsistent procedure of introducing the electron radius in conflict with point model of the theory. This argument still now remains true. However it has been found that the contribution from the vacuum polarization type, which does not appear for the electron interacting with the electromagnetic fields, may destroy the covariance for the electron interacting with other general field, the origin of the failure of which is different from that of mass type. For the self-stress of an electron interacting with the electromagnetic field, the e^2 -

approximation of its expectation value is classified into the following three types:



—: an electron line, ----: a field line, interacting with an electron,: self-stress.

Calculating the expectation value for the Hamiltonian of the total system $\langle H \rangle$ by these diagrams, we get the self-energy of an electron by the electromagnetic field, to which (a) and (b) contribute, but (c) not, of course. Therefore we shall call (a) and (b), M.T. type and (c), V.P. type.

$$\langle H \rangle_{MT} = \delta\mu, \quad \langle H \rangle_{VP} = 0 \quad (1)$$

In our previous calculation of self-stress of an electron the following formula was used:

$$\langle \int \psi^* \beta \psi dv \rangle = \frac{\partial \langle H \rangle}{\partial \mu}, \quad (2)$$

In Pais' paper, this formula was used without critique in detail, but only the contribution from M.T. type satisfies this relation and that from V.P. type does not. For the self-stress, however, V.P. type may contribute.

In fact, the contribution from V.P. type, S_{VP} , is⁽²⁾

$$S_{VP} = \frac{1}{3} \langle \mu \int \psi^* \beta \psi dv \rangle_{VP} = \begin{cases} 0, & \text{vector, pseudoscalar,} \\ & \text{pseudovector coupling,} \\ -\frac{4f^2\mu}{3\pi^2 x^2} \int \frac{k^4 dk}{(k^2 + \mu^2)^{3/2}}, & \text{scalar coupling.} \end{cases} \quad (3)$$

For the contribution from M.T. type, the validity of the discussion in our previous letter still remains. The contribution to self-stress from V.P. type does not always vanish but, as in (3), diverges quadratically for scalar coupling. This term connects closely with the treatment of vacuum and its value depends on the procedure of integration (the problem of ambiguity). The result (3) is obtained by the straightforward integration.

Now, we can conclude that the difficulties of the self-stress derives from the following two conflicts:

(S₁) the introduction of the radius in conflict with the point model of the electron in the field theory.

(S_2) the destruction of the covariance by the vacuum polarization.

It can be expected that the theory without two conflicts (S_1) and (S_2), is the very theory, which has no difficulties of divergency, satisfies the relativistic covariance and moreover resolves difficulty of vacuum polarization.

In Tomonaga's and Schwinger's renormalization theory, (S_2) does not appear, but (S_1) appears, and so the finite self-stress is obtained. In F. Rohrlich's case,³⁾ in which the regulator method is applied, (S_1) and (S_2) did not appear, so that the self-stress vanished.

It might be expected that above discussion holds true not only for the electron but also any elementary particle. In the case of the meson interacting with Fermi particles, the conflicts (S_1) and (S_2) seem to mingle, as its self-energy is the vacuum polarization type.

The discussion for this case is being researched from the view point of this paper. We wish express our gratitude to Prof. S. Sakata and Mr. R. Kawabe for their kind interests and valuable discussions.

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Two Fluid Theory of Liquid Helium II. below 1°K

Sadao NAKAJIMA and Masao SHIMIZU

Physical Institute, Nagoya University

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§ 1. Introduction

In our previous report¹⁾ we gave the full discussion of the role played by Tisza's relation²⁾ $\rho_n/\rho = s/s_\lambda$, in the two fluid theory of liquid helium II. The main results there obtained are as follows. In the first place Tisza's relation is incompatible with the recent experimental results on the second sound velocity below 1°K.³⁾ The relation is, however, so deeply connected with the basic ideas of the two fluid theory that one can not discard it without refining the theory itself. Thus, for instance, H. London's equation for the fountain pressure is the direct consequence of the experimental fact proved by Kapitza⁴⁾ on the entropy flow through narrow channels. By means of the thermodynamical considerations¹⁾ on the mechano-caloric effect one can easily derive Tisza's relation, provided that the mass of the normal fluid should be conserved during adiabatic reversible processes. In these respects the hydrodynamical equations in Landau's theory⁵⁾ seems to be inconsistent with his phonon-roton model, in which Tisza's relation does not hold. Hence, in spite of their success in predicting the behavior of the second sound velocity at lower temperatures, it is necessary to reformulate the hydrodynamical equations in Landau's theory.

On the other hand, in the case of Tisza's model of Bose-Einstein liquid,²⁾ the phonon entropy should be also taken into account below 1°K, where the specific heat is found experimentally to obey T^3 -law. Here again we necessitate the general formalism which is not restricted by Tisza's relation.

Now, Gorter⁶⁾ and Usui⁷⁾ have recently developed the required formalism, avoiding ingeniously the use of Tisza's relation. We are applying this formalism to the construction of the two fluid theory below 1°K. Unfortunately the lack of the experimental data below 1°K prevents us from obtaining the conclusive result at present.

In the present paper we concern ourselves mainly with the second sound velocity and make several predictions on the related phenomena.

§ 2. Entropy flow

We consider the entropy flow along the line proposed by Usui.⁷⁾ For simplicity we assume that the two fluid components of liquid helium II are

always locally in equilibrium, the normal fluid concentration $x = \rho_n/\rho$ being a function of pressure and temperature determined by the condition of minimizing Gibbs free energy with respect to x . Thus we may use the total density $\rho = \rho_n + \rho_s$ and the concentration x as the independent thermodynamical variables. The entropy per unit mass s , for instance, is regarded as a function of ρ and x . Then the reversible entropy flow may be given by

$$\rho s \mathbf{V}_1 + (\partial s / \partial x)_\rho \rho_n \mathbf{V}_2, \quad (1)$$

where \mathbf{V}_1 is the velocity of the center of gravity and \mathbf{V}_2 the velocity of the normal fluid relative to \mathbf{V}_1 :

$$\rho \mathbf{V}_1 = \rho_n \mathbf{V}_n + \rho_s \mathbf{V}_s; \quad \mathbf{V}_2 = \mathbf{V}_n - \mathbf{V}_1.$$

The first term of eq. (1) represents the ordinary flow carried by the net mass current. The second term is characteristic of liquid helium II, i.e., the flow which arises from the internal convection even in the case of no net mass current.

On the basis of eq. (1), one can easily derive the generalized expressions for the fountain pressure and the second sound velocity respectively as follows:

$$\rho x (\partial s / \partial x)_\rho \text{ grad } T \quad (2)$$

and

$$c_2^2 = x(1-x) (\partial s / \partial x)_\rho (\partial T / \partial x)_\rho. \quad (3)$$

The latter expression is not identical with that used by Landau.⁵⁾ We have recalculated c_2 by means of eq. (3) in the case of Landau's model. The result will be described in § 4.

Now the entropy flow (1) is transcribed as

$$(s - x (\partial s / \partial x)_\rho) \rho \mathbf{V}_1 + (\partial s / \partial x)_\rho \rho_n \mathbf{V}_n. \quad (1')$$

The first term of this expression indicates that the entropy will be transferred by the superfluid flow even in the limiting case of very narrow capillaries ($\mathbf{V}_n \rightarrow 0$). It is therefore necessary for the validity of the experimental evidence proved by Kapitza⁴⁾ that

$$s = x (\partial s / \partial x)_\rho,$$

or, in integrated form,

$$s = x s_\lambda(\rho). \quad (4)$$

Then eq. (2) agrees with H. London's equation.

Thus we may conclude that both the experimental evidence of Kapitza and H. London's expression for the fountain pressure should be invalid below 1°K, where Tisza's relation (4) is incompatible with the experimental facts on the second sound velocity.

§ 3. The generalized model of Tisza

Tisza²⁾ has suggested that his model will be extended by including the phonon entropy:

$$s = s_\lambda x + s_{ph}. \quad (5)$$

Then the entropy flow (1') takes the form

$$s_\lambda \rho_n V_n + \rho s_{ph} V_1 + (\partial s_{ph} / \partial x)_p \rho_n V_2. \quad (6)$$

The first term of this expression is well known. The second term represents the fact that the phonon entropy is carried by the total mass current. The last term is the variation of the phonon entropy arisen from the internal convection. That is, even in the case of no net mass current, the internal convection causes the variation of the normal fluid concentration, which is followed immediately by the variation of temperature and therefore of the phonon entropy on the assumption of local equilibrium.

As we have pointed out in the last section, the phonon entropy invalidates H. London's equation.

In order to calculate c_2 by means of eqs. (3) and (5), we need the expressions for x , s_λ , and s_{ph} . x would be determined by the analysis of viscosity measurements and heat transfer experiments, as Gorter⁶⁾ has indicated. The available data, however, are lacking at present. According to eq. (5), the second sound velocity is given by

$$c_2^2 = [x(1-x)s_\lambda / (\partial x / \partial T)_p] + [x(1-x) / (\partial s_{ph} / \partial T)_p (\partial x / \partial T)_p^2].$$

If $x \rightarrow 0$ as $T \rightarrow 0$, only the second term can be appreciable at lower temperatures. For instance, if we assume that

$$x = (T/T_\lambda)^r, \quad (7)$$

c_2 has a non-zero or infinite value at $T=0$, when $r \geq 4$, since the phonon entropy is proportional to T^3 .

§ 4. The second sound velocity

We have calculated c_2 by means of eq. (3). In the first place we have used the expressions for s and ρ_n given by Landau's model. The result of the calculation is indicated by the curve II in Fig. 1.

We have also used the generalized model of Tisza, (5). Above 1°K, we may consider that $x = s/s_\lambda$, which Tisza has approximated by eq. (7). We have extrapolated this equation to lower temperatures. The result of this case is shown by the curve III in Fig. 1, where we have used the following values: $r=6$, $s_\lambda=0.397$ cal./g. deg., $s_{ph}=0.75 \times 10^{-3} T^3$ cal./g. deg. The last value was taken from the experimental data on the specific heat below 1°K.

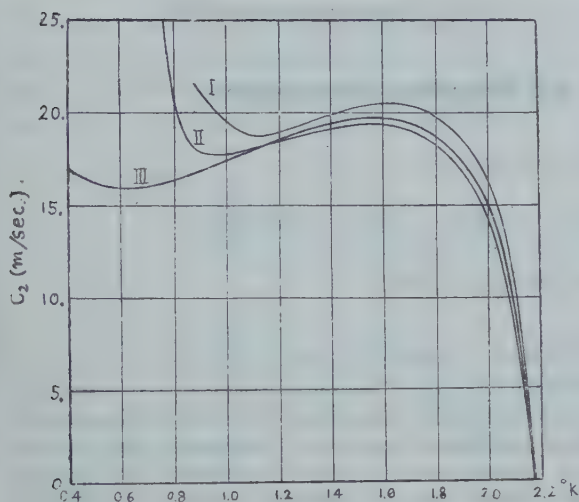


Fig. 1. The temperature dependence of the second sound velocity.

I: Experimental (M. H.); II: Landau's model;
III: Tisza's model.

phonon entropy is inversely proportional to the cube of the first sound velocity which increases as pressure increases. This pressure effect has been found recently by Maurer.⁸⁾ We are endeavouring to calculate the effect theoretically as well as to fit our curves to the experimental data more precisely. The results will be soon published in the following paper.

We are indebted to Associate Prof. T. Usui of Tokyo University for his valuable discussions. We express our hearty thanks also to Prof. R. Maurer who informed us kindly on his excellent experimental researches.

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On Tensor Forces and Saturation Requirements

Hideo KANAZAWA

Department of General Culture, Tokyo University

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The effects which the tensor forces exert upon the saturation properties of heavy nuclei have not yet been fully investigated. According to Volkoff,¹⁾ the occurrence of large tensor forces in the expression of the interaction of a pair of nucleons tends to favour collapsed configurations of a heavy nucleus and so such theories as involve an extreme predominance of the tensor forces are to be excluded, but about intermediate cases like the Rarita-Schwinger theory one can not say whether they satisfy the saturation conditions. The experiments concerning two-nucleon systems have been developed recently and the nuclear potentials which harmonize with these experimental results have been introduced. It may be of interest to consider whether these potentials are not contradictory with the saturation properties of heavy nuclei. We shall formulate the saturation properties of heavy nuclei. We shall formulate the saturation requirements to be fulfilled by the nuclear potentials involving tensor forces with the use of the individual nuclear model.

§ 1

As the nuclear potential we shall adopt the following one

$$\begin{aligned} V(1, 2) = & \{ a_0 + a_\sigma(\sigma^{(1)}\sigma^{(2)}) + a_\tau(\tau^{(1)}\tau^{(2)}) + a_{\sigma\tau}(\sigma^{(1)}\sigma^{(2)})(\tau^{(1)}\tau^{(2)}) \} J_1(r_{12}) \\ & + \{ b_0 + b_\sigma(\sigma^{(1)}\sigma^{(2)}) + b_\tau(\tau^{(1)}\tau^{(2)}) + b_{\sigma\tau}(\sigma^{(1)}\sigma^{(2)})(\tau^{(1)}\tau^{(2)}) \} S_{12} J_2(r_{12}) \\ = & V_{\text{central}}(1, 2) + V_{\text{tensor}}(1, 2), \end{aligned} \quad (1)$$

where S_{12} denotes $3(\sigma^{(1)}\mathbf{r})(\sigma^{(2)}\mathbf{r})/r^2 - (\sigma^{(1)}\sigma^{(2)})$ and the Coulomb potential is disregarded. We consider such stationary states as their eigenfunctions are single Slater-determinants of the following type²⁾

$$\psi = \frac{1}{\sqrt{A!}} \begin{vmatrix} \psi_{n_1}(Q^{(1)}) & \dots & \psi_{n_1}(Q^{(4)}) \\ \dots & \dots & \dots \\ \psi_{n_A}(Q^{(1)}) & \dots & \psi_{n_A}(Q^{(4)}) \end{vmatrix}, \quad (2)$$

$$\psi_n(Q) = \varphi_n(\mathbf{x}) v_\pm(\sigma_z') u_\pm(\tau_3'). \quad (3)$$

The expectation value of the nuclear interaction energy for the state represented by (2) may be written

$$\begin{aligned} \bar{V} &= \frac{1}{2} \sum_{i,k} \overline{V(i, k)} \\ &= \frac{1}{2} \sum_{n_1, n_2} \int \psi_{n_1}^*(Q^{(1)}) \psi_{n_2}^*(Q^{(2)}) V(1, 2) \{ \psi_{n_1}(Q^{(1)}) \psi_{n_2}(Q^{(2)}) - \psi_{n_2}(Q^{(1)}) \psi_{n_1}(Q^{(2)}) \}. \end{aligned} \quad (4)$$

Here the integral sign indicates summation over the spin and isotopic variables in addition to integration over space, while the summation over n_1 and n_2 independently extend over all the occupied individual nucleon states ψ_n . Now we shall calculate the expectation value of V_{tensor} . One finds

$$\bar{V}_{\text{tensor}} = {}_t\bar{V}_{+1} + {}_t\bar{V}_{-1} + {}_t\bar{V}_0, \quad (5)$$

$${}_t\bar{V}_{\pm 1} = \frac{1}{2} \sum_{n_1}^{(\pm)} \sum_{n_2}^{(\pm)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) {}_tV_{\pm 1} \{ \chi_{n_1}(1) \chi_{n_2}(2) - \chi_{n_2}(1) \chi_{n_1}(2) \},$$

$${}_t\bar{V}_0 = \sum_{n_1}^{(+)} \sum_{n_2}^{(-)} \int \chi_{n_1}^*(1) \chi_{n_2}^*(2) \{ {}_tV_0^{\text{ord}} \chi_{n_1}(1) \chi_{n_2}(2) - {}_tV_0^{\text{exch}} \chi_{n_2}(1) \chi_{n_1}(2) \}: \quad (6)$$

in these formulae the symbols $\sum^{(+)}$, $\sum^{(-)}$ represent summations extending over the occupied neutron states and occupied proton states respectively. Further we have put

$$\chi_n = \varphi_n(\mathbf{x}) v_{\pm}(\sigma_z'), \quad (7)$$

$${}_tV_{\pm 1} = \{ b_0 + b_{\tau} + (b_{\sigma} + b_{\sigma\tau})(\sigma^{(1)}\sigma^{(2)}) \} S_{12} J_2(r_{12}),$$

$${}_tV_0^{\text{ord}} = \{ b_0 - b_{\tau} + (b_{\sigma} - b_{\sigma\tau})(\sigma^{(1)}\sigma^{(2)}) \} S_{12} J_2(r_{12}), \quad (8)$$

$${}_tV_0^{\text{exch}} = 2 \{ b_{\tau} + b_{\sigma\tau}(\sigma^{(1)}\sigma^{(2)}) \} S_{12} J_2(r_{12}).$$

Using the following formulae

$$\begin{aligned} \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{\pm}(1) v_{\pm}(2) S_{12} v_{\pm}(1) v_{\pm}(2) &= 3 \cos^2 \theta - 1, \\ \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{-}(1) v_{+}(2) S_{12} v_{-}(1) v_{+}(2) &= \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{+}(1) v_{-}(2) S_{12} v_{-}(1) v_{+}(2) \\ &= -(3 \cos^2 \theta - 1), \\ \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{\pm}(1) v_{\pm}(2) (\sigma^{(1)}\sigma^{(2)}) S_{12} v_{\pm}(1) v_{\pm}(2) &= 3 \cos^2 \theta - 1, \\ \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{-}(1) v_{+}(2) (\sigma^{(1)}\sigma^{(2)}) S_{12} v_{-}(1) v_{+}(2) &= \sum_{\sigma_z^{(1)}, \sigma_z'^{(2)}} v_{+}(1) v_{-}(2) (\sigma^{(1)}\sigma^{(2)}) \\ S_{12} v_{-}(1) v_{+}(2) &= -(3 \cos^2 \theta - 1), \end{aligned} \quad (9)$$

(6) takes the form

$$\begin{aligned} {}_t\bar{V}_{\pm 1} &= \left[\frac{1}{2} \{ (\uparrow_{\pm} | \uparrow_{\pm})_t^{\text{ord}} + (\downarrow_{\pm} | \downarrow_{\pm})_t^{\text{ord}} - (\uparrow_{\pm} | \downarrow_{\pm})_t^{\text{ord}} \} (b_0 + b_{\tau} + b_{\sigma} + b_{\sigma\tau}) \right. \\ &\quad \left. - \left[\frac{1}{2} \{ (\uparrow_{\pm} | \uparrow_{\pm})_t^{\text{exch}} + (\downarrow_{\pm} | \downarrow_{\pm})_t^{\text{exch}} \} - (\uparrow_{\pm} | \downarrow_{\pm})_t^{\text{exch}} \right] (b_0 + b_{\tau} + b_{\sigma} + b_{\sigma\tau}) \right], \quad (10) \\ {}_t\bar{V}_0 &= \left[(\uparrow_{+} | \uparrow_{-})_t^{\text{ord}} + (\downarrow_{+} | \downarrow_{-})_t^{\text{ord}} - (\uparrow_{+} | \downarrow_{-})_t^{\text{ord}} - (\downarrow_{+} | \uparrow_{-})_t^{\text{ord}} \right] (b_0 - b_{\tau} + b_{\sigma} - b_{\sigma\tau}) \end{aligned}$$

$$-\left[(\uparrow_+|\uparrow_-)_{\text{exch}} + (\downarrow_+|\downarrow_-)_{\text{exch}} - (\uparrow_+|\downarrow_-)_{\text{exch}}(\downarrow_+|\uparrow_-)_{\text{exch}}\right] \cdot 2(b_\tau + b_{\sigma\tau}),$$

where one has put

$$(\uparrow_+|\uparrow_+)_{\text{ord}} = \int \uparrow\rho_+(\mathbf{x}^{(1)})(3\cos^2\theta - 1)J_2(r_{12})\uparrow\rho_+(\mathbf{x}^{(2)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)}, \quad (11)$$

$$(\uparrow_+|\uparrow_+)_{\text{exch}} = \int \uparrow\rho_+(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})(3\cos^2\theta - 1)J_2(r_{12})\uparrow\rho_+(\mathbf{x}^{(2)}, \mathbf{x}^{(1)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)}, \text{ etc.}$$

with

$$\uparrow\rho_+(\mathbf{x}) = \uparrow\sum^{(+)}\varphi_n^*(\mathbf{x})\varphi_n(\mathbf{x}), \quad \uparrow\rho_+(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \uparrow\sum^{(+)}\varphi_n^*(\mathbf{x}^{(1)})\varphi_n(\mathbf{x}^{(2)})$$

and similarly $\uparrow\rho_+$, $\uparrow\rho_-$, the symbols \uparrow, \downarrow denoting the spin orientations.

§ 2

When all the nucleon spins are saturated,

$$\uparrow\rho_{\pm} = \downarrow\rho_{\pm} = \frac{2}{1}\rho_{\pm}. \quad (12)$$

So in this case we get

$${}_i\bar{V}_{\pm i} = {}_i\bar{V}_0 = 0, \quad \bar{V} = \bar{V}_{\text{central}}. \quad (13)$$

The tensor forces having no effect in this case, we get the saturation condition enunciated by Breit and Feenberg²⁾

$$a_0 \geq 0. \quad (I)$$

Next we consider the case in which the nucleon spins are all parallel. In this case

$${}_i\bar{V}_{\pm i} = \frac{1}{2}\{(\uparrow_{\pm}|\uparrow_{\pm})_{\text{ord}} - (\uparrow_{\pm}|\uparrow_{\pm})_{\text{exch}}\}(b_0 + b_\tau + b_\sigma + b_{\sigma\tau}),$$

$${}_i\bar{V}_0 = (\uparrow_+|\uparrow_-)_{\text{ord}}(b_0 - b_\tau + b_\sigma - b_{\sigma\tau}) - (\uparrow_+|\uparrow_-)_{\text{exch}} \cdot 2(b_\tau + b_{\sigma\tau}). \quad (14)$$

In particular, consider a nucleus consisting of equal numbers of neutrons and protons, then $\uparrow\rho_+ = \uparrow\rho_- = \frac{1}{2}\rho$ and so

$$\bar{V}_{\text{central}} = \frac{1}{2}(a_0 + a_\pi)I_c - \frac{1}{4}(a_0 + 3a_\tau + a_\sigma + 3a_{\sigma\tau})X_c, \quad (15)$$

$$\bar{V}_{\text{tensor}} = \frac{1}{2}(b_0 + b_\sigma)I_t - \frac{1}{4}(b_0 + 3b_\tau + b_\sigma + 3b_{\sigma\tau})X_t, \quad (16)$$

where

$$I_c = \int \rho(\mathbf{x}^{(1)})J_1(r_{12})\rho(\mathbf{x}^{(2)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)},$$

$$X_c = \int \rho(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})J_1(r_{12})\rho(\mathbf{x}^{(2)}, \mathbf{x}^{(1)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)}, \quad (17)$$

$$I_t = \int \rho(\mathbf{x}^{(1)})(3\cos^2\theta - 1)J_2(r_{12})\rho(\mathbf{x}^{(2)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)},$$

$$X_t = \int \rho(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})(3\cos^2\theta - 1)J_2(r_{12})\rho(\mathbf{x}^{(2)}, \mathbf{x}^{(1)})d\mathbf{v}^{(1)}d\mathbf{v}^{(2)}. \quad (18)$$

But we must prevent these configurations from exhibiting non-saturation binding. We get the corresponding condition

$$(a_0 + a_\sigma)I_c + (b_0 + b_\sigma)I_t \geq 0. \quad (\text{II})$$

Further we consider nuclei containing only neutrons.

$$\begin{aligned} \bar{V}_{\text{tensor}} = & \left[\frac{1}{2} \{ (\uparrow_+ | \uparrow_+)_{\text{ord}} + (\downarrow_+ | \downarrow_+)_{\text{ord}} \} - (\uparrow_+ | \downarrow_+)_{\text{ord}} \right] (b_0 + b_\tau + b_\sigma + b_{\sigma\tau}) \\ & - \left[\frac{1}{2} \{ (\uparrow_+ | \uparrow_+)_{\text{exch}} + (\downarrow_+ | \downarrow_+)_{\text{exch}} \} - (\uparrow_+ | \downarrow_+)_{\text{exch}} \right] (b_0 + b_\tau + b_\sigma + b_{\sigma\tau}). \end{aligned} \quad (19)$$

As $\bar{V}_{\text{tensor}} = 0$ when all the spins are saturated, the instability condition is²⁾

$$a_0 + a_\tau \geq 0. \quad (\text{III})$$

When all the spins are parallel,

$$\bar{V}_{\text{central}} = \frac{1}{2} (a_0 + a_\tau + a_\sigma + a_{\sigma\tau}) (I_c + X_c), \quad (20)$$

$$\bar{V}_{\text{tensor}} = \frac{1}{2} (b_0 + b_\tau + b_\sigma + b_{\sigma\tau}) (I_t + X_t). \quad (21)$$

So we get the following condition

$$(a_0 + a_\tau + a_\sigma + a_{\sigma\tau})I_c + (b_0 + b_\tau + b_\sigma + b_{\sigma\tau})I_t \geq 0. \quad (\text{IV})$$

As compared with the above-mentioned first saturation conditions the second saturation requirements are not so clear cut on account of their dependency on the accuracy of the method. Volz formulated the second saturation requirement considering light α -nuclei. These nuclei have the spins all saturated, so we need not consider tensor forces owing to (13) and Volz's result hold in our case. His result is²⁾

$$a_0 \approx 0. \quad (\text{I})'$$

Further he has obtained the following inequality²⁾ considering even nuclei in states with saturated nucleon spins:

$$a_\tau \leq a_\tau, \quad a_\tau \approx 0.1. \quad (\text{V})$$

We shall adopt as the effective potentials for the configurations 3S and 1S the following forms³⁾ respectively:

$$\begin{aligned} ^3V &= -\{ J_1(r_{12}) + \gamma S_{12} J_2(r_{12}) \}, \\ ^1V &= -J_1(r_{12}). \end{aligned} \quad (22)$$

From (1)

$$\begin{aligned} ^3V &= (a_0 + a_\sigma - 3a_\tau - 3a_{\sigma\tau})J_1(r_{12}) + (b_0 + b_\sigma - 3b_\tau - 3b_{\sigma\tau})S_{12}J_2(r_{12}), \\ ^1V &= (a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau})J_1(r_{12}). \end{aligned} \quad (23)$$

Comparing (23) with (22) we get

$$a_0 + a_\sigma - 3a_\tau - 3a_{\sigma\tau} = a_0 - 3a_\sigma + a_\tau - 3a_{\sigma\tau} = -1,$$

$$\text{or} \quad a_\sigma = a_\tau, \quad (\text{IV})$$

$$\text{and} \quad b_0 + b_\sigma - 3b_\tau - 3b_{\sigma\tau} = -\gamma.$$

For example we take up some nuclear potentials and investigate whether they satisfy the saturation requirements. First we consider the following one:

$$V = \frac{1}{3}(\tau^{(1)}\tau^{(2)}) \{ (\sigma^{(1)}\sigma^{(2)})J_1(r_{12}) + \gamma S_{12}J_2(r_{12}) \}.$$

It is easily seen that this potential satisfies the conditions except (IV). The left hand side of (IV) is

$$-\frac{1}{3}(I_\sigma + \gamma I_\tau) = \frac{1}{3} \int \rho(x^{(1)}) \{ J_1(r_{12}) + \gamma(3\cos^2\theta - 1)J_2(r_{12}) \} \rho(x^{(2)}) dv^{(1)} dv^{(2)}.$$

If we take $J_1 = J_2$, the above quantity is non-negative so far as $\gamma \leq 1$. So in this case the potential satisfies the saturation requirements. But the potentials

$$V = -\frac{1}{2}(1 - P_\sigma \cdot P_\tau) \{ J_1(r_{12}) + \gamma S_{12}J_2(r_{12}) \}$$

$$\text{and}^4) \quad V = \left\{ \frac{3W}{4} + \left(\frac{B}{2} - \frac{W}{4} \right) (\sigma^{(1)}\sigma^{(2)}) - \left(\frac{B}{2} + \frac{W}{4} \right) (\tau^{(1)}\tau^{(2)}) - \frac{W}{4} (\sigma^{(1)}\sigma^{(2)}) (\tau^{(1)}\tau^{(2)}) \right\}$$

are contradictory with the saturation requirements.

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On the Absorption of Negative π -Meson by Proton, II

E. Yamada, Y. Nagahara
and
S. Ogawa

*Institute of Theoretical Physics,
Nagoya University*

January 12, 1951

Previously we have studied about the absorption of the π^- -meson by the proton, accompanied with emission of photon or π^0 -meson¹⁾. But we considered only the case of scalar type π -meson with scalar type coupling there. Neutral π -meson is now assumed to be of scalar type according to the fact that it decays into two photons, but about the charged one, we cannot say almost anything affirmative. So, we have treated the same problem with the various types of π -meson.

The effect of binding of π^- -meson around the proton by Coulomb interaction is so small that we have treated the π -meson as free, using the Feynman-Dyson's method. The anomalous magnetic moment of the nucleons are treated phenomenologically, and this part is added to the ordinary interaction in the Lagrangian. Then, it has been shown that the Dyson's theorem for the scalar type meson theory is valid for the photon emitting process but not for the π^0 -meson emitting one.

If the π^- -meson is initially at rest, the process in which the meson emits photon, is forbidden in the case of scalar type theory, but in the vector type theory, the interaction of the electromagnetic fields with the spin of the meson contributes to this process. In this process, however, the meson density at the position of the proton decreases and, when the meson is bound around the proton, the contribution of this process is considered to be smaller than the other, so we have

TABLE I.

π^-			S		PS		V	PV
π^0			S	V	PS	PV	V	PV
S	S	f_n^2	1.2 10^4	1.4	5.7 10	26.	0.40	5.0 10^3
		f_s^2	9.3 10^4	8.1 10^{-7}	3.2 10^{-2}	13.	3.7 10^{-2}	3.5 10^{-4}
	V	f_n^2	—**	—	—	—	—	—
		f_s^2	3.4 10^{-3}	3.0 10^{-7}	0.34	60.3	3.2 10^{-2}	1.3 10^{-3}
P S	P S	f_n^2	5.0 10^{-}	2.8 10^{-2}	8.6 10^{-4}	1.5	1.5	1.6
		f_s^2	2.8	1.0 10^{-2}	1.5	2.4 10^{-2}	3.5	3.5
	P V	f_n^2	1.5 10^{-2}	1.3 10^{-6}	8.2	53.	1.8	1.6 10^{-2}
		f_s^2	0.58	5.5 10^{-5}	1.3	1.3	7.6 10^{-3}	8.5 10^{-3}

* About this point, we will discuss in other place: S. Ogawa, E. Yamada and Y. Nagahara
Prog. Theor. Phys. (in press).

** — means that, in this case π^0 -emitting process is forbidden in this order of approximation.

The calculation is so elementary that we only have the results in Table I. In this table, the numerical values are the square of the coupling constants of π^0 -meson with the nucleon which make the theoretical results fit with the experimental results²⁾, in the lowest approximation of the perturbation. f_s or f_n is the coupling constant, respectively corresponding to the case where the interaction includes τ_3 or not. Here, we have taken the value $10.6m_e$ for the mass difference $\mu_{\pi^-} - \mu_{\pi^0}$.

Little is known about the correctness of the perturbation method for the nucleon-meson system. Even if such approximation is allowed, we have no knowledge about that magnitude of the meson-nucleon coupling constant, so we cannot say which type of the meson is adequate. To determine it, it is necessary to compare with the other process including the same coupling. However, it may be able to say that the theory, in which f deviates too much from order 1, is not permissible.

Finally we should like to express our sincere thanks to Prof. S. Sakata for his interest in this work.

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Some Remarks on the Non-local Field Theory

Z. Tokuoka and Y. Katayama

Department of Physics, Kyoto University

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The non-local field theory, proposed by H. Yukawa¹⁾, has two remarkable possibilities which the ordinary theory hasn't. The first of them is a possibility of free from

divergence difficulties, and the second is that of a new interpretation of spins of elementary particles. The latter is a formal result of the fundamental equations of non-local fields. As Fierz²⁾ pointed out, these equations describe not only the behaviours of particle of a given spin, but also that of a group of particles of various spins. But the former has not been studied decisively. Starting from Yukawa's fundamental equations, however, the divergence difficulties would remain obviously, unless particles of various spins satisfy some cancellation relations fortunately. Moreover, the infinite freedoms of spin may raise the difficulties³⁾.

The methods of avoiding these duplicate difficulties have not been known, but the most natural method⁴⁾ seems the reduction of infinite freedoms of spin into irreducible parts in the fundamental equations at the outset. It seems that this reduction can be made for real fields in the present form, but not for virtual fields, and this fact is the very origin of the difficulties. Starting from this assumption, Yennie⁵⁾ and Rayski⁶⁾ showed the possibilities of avoiding the divergence difficulties. However, they have some mistakes and considered partly, and not all of the self-energies of three particles (considering difference of charge) interacting with each other converge. Moreover, their methods are too artificial. In this letter, we discuss whether the difficulties of ordinary field theories can be removed, assuming that we can restrict spins (l -freedoms).

We define a associated local field quantity $a(X)$ from a non-local $A(x, r) = (x + \frac{r}{2} | A | x - \frac{r}{2})$ by

$$a(x) = \int dr \left(x + \frac{r}{2} \cdot A \mid x - \frac{r}{2} \right) \quad (1)$$

and consider a simplest possible interaction Lagrangian between a complex spinless field A and a real spinless B , as $L = gA^*BA$. g is a interaction constant. Then dropped this term.

$$L(x) = \int dr \left(x + \frac{r}{2} \right) L \left(x - \frac{r}{2} \right) \\ = O a^*(x) b(x) a(x) = O L_{div} a(x). \quad (2)$$

O is a operator which operates on a^* , a and b . This shows that non-local field introduces a (semi-convergence) factor in local field interaction

From equation (2), we can easily take calculation following to the covariant formalism. In our case, the order of divergence does not differ from ordinary theories except a factor, because vacuum fields contribute in ordinary way, which differs from Yukawa-Yennie's considerations. A analytic form of a operator O depends sensibly on a type of non-local interaction Lagrangians, and then it influences on the divergency of self-energies. The results are listed in Table I. In this Table, *div* means that a divergence of a same order to ordinary theories exist, because in this case the factor O plays no role. And $A^*BA = (x' | A^* | x'') (x''' | B | x''') (x'''' | A | x'')$, and so on.

Table I. Self-Energy

Type of Lagrangian	A+	A-	B
A^*BA	<i>div</i>	<i>div</i>	<i>conv</i>
A^*AB	<i>conv</i>	<i>div</i>	<i>div</i>
BA^*A	<i>conv</i>	<i>conv</i>	<i>div</i>

From these results we know that a factor O plays a convergence factor only for one particle, and can not expect that one gets converging results for each particles simultaneously. To remove these defects we are obliged to give up a matrix character of non-local quantity, and then we can construct a Lagrangian which gives converging results for each particle formally as follows;

$$L(x) = g \int A^*(x + \frac{r_2}{2} + \frac{r_3}{2}, r_1) B(x + \frac{r_1}{2} - \frac{r_3}{2}, r_2) A(x - \frac{r_3}{2} - \frac{r_2}{2}, r_3) dr_1 dr_2 dr_3. \quad (3)$$

This is very trivial. Anyhow, we can not expect a powerful result for divergence problems, even if we restrict spins.

- 1) H. Yukawa, Phys. Rev. **77** (1950), 219; Part II., to appear shortly. We express our thanks to Prof. H. Yukawa for having sent us kindly a copy of his work before publication.
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On the Difference between Local and Non-local Fields

H. Yukawa *

Columbia University, New York City

February 7, 1951

Recently Hara and Shimazu¹⁾ claimed that non-local fields could be reduced to local fields by a simple unitary transformation. This is certainly not true, whenever we take into account the interaction between two non-local fields. For instance, a product $W \equiv UV$ of two non-local field operators U and V can be represented by a function $W(X, r)$ of X_μ and r_μ . This function takes, in general, non-zero values even for those values of r_μ , which do not satisfy the condition $r_\mu r^\mu - \lambda^2 = 0$. The only thing which should be kept in mind is that $W(X, r)$ is not equal to the simple product $U(X, r)V(X, r)$.²⁾ Thus it is impossible to reduce W to a local operator by a transformation of the type

$$W' = T^{-1} W T$$

where

$$T = \exp(-\lambda \partial/\partial L)$$

with $L \equiv r_\mu r^\mu$.

Just contrary to the statement by these authors, the strange fact that some of the characteristic features of non-local field theory were completely overlooked by the experts of local field theories seems to indicate the novelty of the notion of non-local field, even though we don't know yet whether it will prove to be fruitful.

* On leave of absence from Kyoto University.

- 1) O. Hara and H. Shimazu, Prog. Theor. Phys. **5** (1950), 1055.
- 2) The case of the interaction between a non-local field and a local spinor field, which was taken up by these authors, is obviously exceptional in that the interaction operator contains only one non-local field operator *linearly*.

Errata [5 (1950), 1045 (L)]

T. Kinoshita, Note on the Infrared Catastrophe

The last part of the letter (P. 1046) should be read as follows :

Such a translation, however, reduces the possible "displaced poles"⁴⁾ in the integrand of (1) to the ordinary poles and thus the result of this integration may be different from the original integral (1) by a finite contribution arising from the residue of these poles. One can see with ease that the first term in the square bracket of (2) appears just in this way and would have been missed if one would try to evaluate (1) using the Feynman's method without care.

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Electronic States of C_2 -Molecule, I

—Interaction between p -Electrons—

Gentarō ARAKI, Simpei TUTIHASHI and Watarō WATARI

Department of Industrial Chemistry, Kyoto University

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Electronic states of a C_2 -molecule are discussed by the method of atomic orbitals without taking into account an effect of inner shells of the component C-atoms. Adiabatic potentials are numerically computed for all states which arise from configurations consisting of two atoms with two p -electrons each. Binding energies and equilibrium distances for attractive states are evaluated from these potentials. It is found that $(2p\sigma^2, 2p\sigma^2)^1\Sigma_g^+$ and $(2p\sigma^2, 2p\sigma\ 2p\pi)^3\Pi_u$ are the deepest and first excited state respectively, and that $(2p\sigma\ 2p\pi, 2p\sigma\ 2p\pi)^1\Delta_g$ is a highly excited state in contradiction to the Pauling theory. However the present study is the first step and the result is not conclusive for the C_2 -molecule. We shall find in the next paper that the order of the states will be changed when we shall take into account the effect of $2s$ -shells.

Introduction

The nature of the single bond was quantum-mechanically disclosed by many authors by working out quantitative theories of the ss - and sp - bonds. However the quantum-mechanical nature of the double and triple bonds has not yet been fully known. The double bond was first discussed by Hückel¹⁾ on the basis of quantum mechanics but his consideration was merely qualitative. The theory was later improved to some extent by Pauling²⁾, Penney³⁾, and Mulliken⁴⁾. It was shown that the double or triple bond was due to two or three p -electrons partly associated with s -electrons, though their discussion still remained within a qualitative limit.

To know the true nature of double bonds is of great importance especially from a chemical point of view. The present calculation is intended for this purpose. The simpler atoms which link with other atoms forming the double bonds are carbon, oxygen, and nitrogen atoms. Among them the double bonds formed between two carbon atoms are of special interest in the organic chemistry. However the real cases in organic substances are very complicated because of the effect of adjacent atoms. The simplest molecule with double pp -bonds is the C_2 -molecule. This may be only a molecule which enables us to study quantitatively the nature of its double bonds. This is the reason why we begin with the study of this molecule notwithstanding that it has not yet been chemically isolated.

To simplify the problem we first neglect all inner shells and consider an ideal system consisting of two atoms of atomic number 2 with two p -electrons each. We shall consider the effect of $2s$ -shells in the next paper. The nature of a single pp -bond was quantitatively studied by Bartlett⁵⁾. He found that the

deepest state is ${}^3\Sigma_u^+$ in contradiction to Pauling's assumption that a bond is formed by an electron pair in a singlet state. This result suggests us the necessity of close inspection of Pauling's idea of double and triple bonds though Bartlett's molecule with a single pp -bond does not exist really. However we shall find in the next paper that Penney's conclusion according to Pauling's idea will ultimately have a qualitative validity though the quantitative structure of the bonds will be very complicated. This is due to an effect of $2s$ -shells.

According to Pauling's idea no attractive state arises from $(2p\sigma^2, 2p\sigma^2)$ but the calculation shows the deepest attractive state is $(2p\sigma^2, 2p\sigma^2) {}^1\Sigma_g^+$ when inner shells are neglected altogether. This, as well as Bartlett's result, means that, strictly speaking, Pauling's idea has no general validity. If an effect of $2s$ -shells is taken into account the order of various electronic states changes, and the deepest state is replaced by $(2p\sigma 2p\pi, 2p\sigma 2p\pi) {}^1\Delta_g$ in agreement with Penney's idea. This may show that Pauling-Penney's idea represents an average nature of the complex structure of electronic states, as has been generally believed.

§ 1. Classification of States

The electron configuration of the carbon atom in its normal states is given by $1s^2 2s^2 2p^2$. According to Pauling's idea²⁾ it may be concluded that the carbon atoms behave as divalent ones in the C_2 -molecule and the double bonds are formed by four p -electrons, one being a $\sigma\sigma$ -bond and the other being a $\pi\pi$ -bond.

In order to verify the approximate validity of this qualitative conclusion we have to examine electronic eigenstates of a system which consists of twelve electrons and two nuclei whose atomic number is six. This is a very complicated problem. We shall therefore first neglect the $1s$ and $2s$ shells by assuming the complete screening of these shells on the nuclear charge, and next we shall consider the effect of the inner shells. This is not only a way of simplifying the problem but also a way which enables us to know the contribution of each electron separately.

(i) **Axial Quantum Number and Multiplicity.** We shall consider an ideal molecule which consists of two atoms with the atomic number two, containing two p -electron each. The orbital functions of these electrons are completely specified by their orbital magnetic quantum number (m_l) which is equal to the axial quantum number of the orbitals.

The axial quantum numbers of two p -electrons which belong to one of two atoms are denoted by λ_1 and λ_1' and those of another atom by λ_2 and λ_2' . We specify a set of orbitals which are contained in a determinantal wave function of the molecule by $(\lambda_1 \lambda_1', \lambda_2 \lambda_2')$. An assemblage of states specified by a definite set $(\lambda_1 \lambda_1', \lambda_2 \lambda_2')$ will be referred to as an electron configuration or briefly a configuration. The singlet states only belong to the configuration $(\lambda_1 \lambda_1, \lambda_2 \lambda_2)$. The singlet and triplet states belong to $(\lambda_1 \lambda_1, \lambda_2 \lambda_2')$ if $\lambda_2' \neq \lambda_2$, and the singlet, triplet and quintet states to $(\lambda_1 \lambda_1', \lambda_2 \lambda_2')$ if $\lambda_1' \neq \lambda_1$ and $\lambda_2' \neq \lambda_2$. Orbitals are denot-

ed by σ , π and ϖ according as their axial quantum number are equal to 0, +1 and -1. Special configurations are denoted by these symbols too. For example, $(\sigma^2, \pi \varpi)$ means (0 0, 1 -1) or that $\lambda_1 = \lambda_1' = 0$, $\lambda_2 = 1$, and $\lambda_2' = -1$.

The axial quantum number and multiplicity can easily be found for a given configuration. They are shown in Table I, where A denotes the axial quantum number of the molecule, and the negative- A part of the table is omitted.

Table I

A	configuration	state	configuration	state	configuration	state
4	(π^2, π^2)	1Γ				
3			$(\pi^2, \sigma\pi)$	$3,1\theta$		
2	(σ^2, π^2)	1Δ	$(\pi^2, \pi\varpi)$	$3,1\Delta$	$(\sigma\pi, \sigma\pi)$	$5,3,1\Delta$
1			$(\pi^2, \sigma\varpi)$	$3,1\Pi$	$(\sigma\pi, \pi\varpi)$	$5,3,1\Pi$
1			$(\sigma^2, \sigma\pi)$	$3,1\Pi$		
0	(π^2, ϖ^2)	1Σ	$(\sigma^2, \pi\varpi)$	$3,1\Sigma$	$(\sigma\pi, \sigma\varpi)$	$5,3,1\Sigma$
0	(σ^2, σ^2)	1Σ			$(\pi\varpi, \pi\varpi)$	$5,3,1\Sigma$

(ii) **Parity and Reflexion Symmetry.** The parity and the reflexion symmetry of a state can be found by writing its wave function explicitly where the latter property is considered for Σ states only. For this purpose we shall adopt some abbreviations for wave functions.

If an orbital of the k -th electron belongs to the a -atom we shall add a suffix a to the symbol of the orbital and write k as its argument where we shall distinguish two atoms by a and b . The spin magnetic quantum number of the one-electron state will be indicated by an upper suffix + or - according as it is equal to +1/2 or -1/2. For example, $\pi_a^+(k)$ stands for the one-electron wave function of the k -th electron whose state belongs to +1/2 of the spin magnetic quantum number and $2p\pi$ of the a -atom. A determinantal wave function of the molecule will be denoted by its first row. For example, we shall write as follows:

$$(\pi_a^+ \pi_a^- \sigma_b^+ \sigma_b^-) \equiv \sum_P \epsilon_P P \pi_a^+(1) \pi_a^-(2) \sigma_b^+(3) \sigma_b^-(4)$$

$$\equiv \begin{vmatrix} \pi_a^+(1) & \pi_a^-(1) & \sigma_b^+(1) & \sigma_b^-(1) \\ \pi_a^+(2) & \pi_a^-(2) & \sigma_b^+(2) & \sigma_b^-(2) \\ \pi_a^+(3) & \pi_a^-(3) & \sigma_b^+(3) & \sigma_b^-(3) \\ \pi_a^+(4) & \pi_a^-(4) & \sigma_b^+(4) & \sigma_b^-(4) \end{vmatrix} \quad (1.1)$$

where P denotes the permutation of electronic numbers, ϵ_P is equal to +1 or -1 according as P is even or odd, and \sum means the sum over all permutations. In our abbreviation a special caution is necessary regarding the order of orbitals contained in the parenthesis.

We shall now consider the transformation of wave functions induced by the inversion at the centre of the molecule and the reflexion (for Σ states) with

respect to a plane containing molecular axis.

If the origin of Cartesian coordinates is the centre of inversion, the molecular axis is the z -axis, and the distance of two nuclei is R , the inversion induces the following transformation :

$$\left. \begin{aligned} x &\longleftrightarrow -x, & y &\longleftrightarrow -y \\ z - R/2 &\longleftrightarrow -(z + R/2) \\ \sqrt{x^2 + y^2 + (z - R/2)^2} &\longleftrightarrow \sqrt{x^2 + y^2 + (z + R/2)^2}. \end{aligned} \right\} \quad (1.2) (a)$$

Therefore, by the inversion, $2p$ orbitals transform as follows :

$$\left. \begin{aligned} \pi_a(k) &\longleftrightarrow -\pi_b(k) \\ \varpi_a(k) &\longleftrightarrow -\varpi_b(k) \\ \sigma_a(k) &\longleftrightarrow -\sigma_b(k) \end{aligned} \right\} \quad k=1,2,3,4. \quad (1.2) (b)$$

The even and odd states for the inversion will be distinguished by g and u respectively.

By the reflexion with respect to the xz -plane the orbitals transform as follows :

$$\left. \begin{aligned} \sigma_a(k) &\longleftrightarrow \sigma_a(k), & \sigma_b(k) &\longleftrightarrow \sigma_b(k) \\ \pi_a(k) &\longleftrightarrow \varpi_a(k), & \pi_b(k) &\longleftrightarrow \varpi_b(k) \end{aligned} \right\} \quad k=1,2,3,4. \quad (1.3)$$

The symmetric and anti-symmetric states for the reflexion will be distinguished by $+$ and $-$ respectively.

On the basis of these transformation properties of orbitals we can immediately distinguish the parity and reflexion symmetry of wave functions of the states in the second and third column of Table I. The correspondence between states and wave functions in these cases is given by

$$(\pi^2, \pi^2) \quad {}^1\Gamma_g \quad (\pi_a^+ \pi_a^- \pi_a^+ \pi_b^-) \quad (1.4)$$

$$(\sigma^2, \pi^2) \quad {}^1\Delta_{g,u} \quad (\pi_a^+ \pi_a^- \sigma_b^+ \sigma_b^-) \pm (\pi_b^+ \pi_b^- \sigma_a^+ \sigma_a^-) \quad (1.5)$$

$$(\pi^2, \varpi^2) \quad {}^1\Sigma_g^+ \quad (\pi_a^+ \pi_a^- \varpi_b^+ \varpi_b^-) + (\pi_b^+ \pi_b^- \varpi_a^+ \varpi_a^-) \quad (1.6) (a)$$

$$(\pi^2, \varpi^2) \quad {}^1\Sigma_g^- \quad (\pi_a^+ \pi_a^- \varpi_b^+ \varpi_b^-) - (\pi_b^+ \pi_b^- \varpi_a^+ \varpi_a^-) \quad (1.6) (b)$$

$$(\sigma^2, \sigma^2) \quad {}^1\Sigma_g^+ \quad (\sigma_a^+ \sigma_a^- \sigma_b^+ \sigma_b^-) \quad (1.7)$$

$$(\pi^2, \sigma\pi) \quad {}^3\Phi_{g,u} \quad (\pi_a^+ \pi_a^- \sigma_b^+ \pi_b^+) \pm (\pi_b^+ \pi_b^- \sigma_a^+ \pi_a^+) \quad (1.8)$$

$$\begin{aligned} (\pi^2, \sigma\pi) \quad {}^1\Phi_{g,u} \quad & (\pi_a^+ \pi_a^- \sigma_b^+ \pi_b^-) - (\pi_a^+ \pi_a^- \sigma_b^- \pi_b^+) \\ & \pm \{ (\pi_b^+ \pi_b^- \sigma_a^+ \pi_a^-) - (\pi_b^+ \pi_b^- \sigma_a^- \pi_a^+) \} \end{aligned} \quad (1.9)$$

$$(\sigma^2, \pi\varpi) \quad {}^3\Sigma_{g,u}^- \quad (\sigma_a^+ \sigma_a^- \pi_b^+ \varpi_b^+) \pm (\sigma_b^+ \sigma_b^- \pi_a^+ \varpi_a^+) \quad (1.10)$$

$$\begin{aligned} (\sigma^2, \pi\varpi) \quad {}^1\Sigma_{g,u}^+ \quad & (\sigma_a^+ \sigma_a^- \pi_b^+ \varpi_b^-) - (\sigma_a^+ \sigma_a^- \pi_b^- \varpi_b^+) \\ & \pm \{ (\sigma_b^+ \sigma_b^- \pi_a^+ \varpi_a^-) - (\sigma_b^+ \sigma_b^- \pi_a^- \varpi_a^+) \} \end{aligned} \quad (1.11)$$

where $+$ and $-$ of the double signs in the wave functions correspond to g and u in the state symbols respectively.

The Δ and Π states in the third column of Table I are not written here because their parities and wave functions are similar to those of the Φ state. The wave functions of $(\pi^2, \pi\varpi)^3\Delta_{g,u}$ and $^1\Delta_{g,u}$ states can be obtained from those of $(\pi^2, \sigma\pi)^3\Phi_{g,u}$ and $^1\Phi_{g,u}$ respectively by replacing σ with ϖ . The wave functions of $(\pi^2, \sigma\varpi)^3\Pi_{g,u}$ and $^1\Pi_{g,u}$ can be obtained from those of $(\pi^2, \pi\varpi)^3\Delta_{g,u}$ and $^1\Delta_{g,u}$ respectively by replacing the non-paired π with σ . Those of $(\sigma^2, \sigma\pi)^3\Pi_{g,u}$ and $^1\Pi_{g,u}$ can be obtained from those of $(\pi^2, \sigma\pi)^3\Phi_{g,u}$ and $^1\Phi_{g,u}$ respectively by replacing the paired π with σ . The triplet wave functions (1.8) and (1.10) are written for the highest spin magnetic quantum number.

We shall next consider wave functions of the states in the fourth column of Table I. The parity and reflexion symmetry of $(\sigma\pi, \sigma\varpi)^5\Sigma$ can be easily found. They are given by

$$(\sigma\pi, \sigma\varpi)^5\Sigma_g^+ (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^+) + (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^+) \quad (1.12)(a)$$

$$(\sigma\pi, \sigma\varpi)^5\Sigma_u^- (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^+) - (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^+) \quad (1.12)(b)$$

where the wave functions are written for the highest spin magnetic quantum number.

If we denote the spin magnetic quantum number of the molecule by M_s , the wave functions which belong to $(\sigma\pi, \sigma\varpi)$ and $M_s=1$ are divided into two invariant linear manifolds for the inversion and reflexion. Their bases transform according to the following scheme:

$$\left. \begin{array}{ccc} (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^-) & \longleftrightarrow & (\sigma_a^+ \varpi_a^+ \sigma_b^+ \pi_b^-) \\ \downarrow & \uparrow & \downarrow \uparrow \\ (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^-) & \longleftrightarrow & (\sigma_b^+ \varpi_b^+ \sigma_a^+ \pi_a^-) \end{array} \right\} \quad (1.13)(a)$$

$$\left. \begin{array}{ccc} (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^+) & \longleftrightarrow & (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^+) \\ \downarrow & \uparrow & \downarrow \uparrow \\ (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) & \longleftrightarrow & (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+) \end{array} \right\} \quad (1.13)(b)$$

where arrows of full and dotted lines represent respectively the transformations induced by the inversion and reflexion. In each manifold we have two linearly independent $^3\Sigma$ functions as follows:

$$(\sigma\pi, \sigma\varpi)^3\Sigma_g^- (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^-) - (\sigma_a^+ \varpi_a^+ \sigma_b^+ \pi_b^-) \\ + (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^-) - (\sigma_b^+ \varpi_b^+ \sigma_a^+ \pi_a^-) \quad (1.14)(a)$$

$$(\sigma\pi, \sigma\varpi)^3\Sigma_g^- (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^+) - (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^+) \\ + (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) - (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+) \quad (1.14)(b)$$

$$(\sigma\pi, \sigma\varpi)^3\Sigma_u^+ (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^-) + (\sigma_a^+ \varpi_a^+ \sigma_b^+ \pi_b^-) \\ - (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^-) - (\sigma_b^+ \varpi_b^+ \sigma_a^+ \pi_a^-) \quad (1.14)(c)$$

$$(\sigma\pi, \sigma\varpi)^3\Sigma_u^+ (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^+) + (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^+) \\ - (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) - (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+)$$

$$-(\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) - (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+). \quad (1.14) (d)$$

The wave functions which belong to $(\sigma\pi, \sigma\varpi)$ and $M_s = +1$ contains eight linearly independent functions. Among them four linearly independent $^3\Sigma$ functions, which are given by (1.14) (a) - (d), and two linearly independent $^5\Sigma$ functions are included. We have therefore two $^3\Sigma$ functions besides the above mentioned four $^3\Sigma$'s. They can be obtained by linearly combining the above mentioned two sets as follows:

$$\begin{aligned} (\sigma\pi, \sigma\varpi) \ ^3\Sigma_g^+ & \quad (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^-) - (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^+) \\ & \quad + (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^-) - (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) \\ & \quad + (\sigma_a^+ \varpi_a^+ \sigma_b^+ \pi_b^-) - (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^+) \\ & \quad + (\sigma_b^+ \varpi_b^+ \sigma_a^+ \pi_a^-) - (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+) \end{aligned} \quad (1.14) (e)$$

$$\begin{aligned} (\sigma\pi, \sigma\varpi) \ ^3\Sigma_u^- & \quad (\sigma_a^+ \pi_a^+ \sigma_b^+ \varpi_b^-) - (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^+) \\ & \quad - (\sigma_b^+ \pi_b^+ \sigma_a^+ \varpi_a^-) + (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^+) \\ & \quad - (\sigma_a^+ \varpi_a^+ \sigma_b^+ \pi_b^-) + (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^+) \\ & \quad + (\sigma_b^+ \varpi_b^+ \sigma_a^+ \pi_a^-) - (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^+). \end{aligned} \quad (1.14) (f)$$

The wave functions which belong to $(\sigma\pi, \sigma\varpi)$ and $M_s = 0$ are divided into four invariant linear manifolds for the inversion and reflexion. Their bases transform according to the following scheme:

$$\left. \begin{array}{ccc} (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^-) & \longleftrightarrow & (\sigma_a^+ \varpi_a^+ \sigma_b^- \pi_b^-) \\ \downarrow \quad \uparrow & & \downarrow \quad \downarrow \\ (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^-) & \longleftrightarrow & (\sigma_b^+ \varpi_b^+ \sigma_a^- \pi_a^-) \end{array} \right\} \quad (1.15) (a)$$

$$\left. \begin{array}{ccc} (\sigma_a^+ \pi_a^- \sigma_b^- \varpi_b^+) & \longleftrightarrow & (\sigma_a^+ \varpi_a^- \sigma_b^- \pi_b^+) \\ \downarrow \quad \uparrow & & \downarrow \quad \uparrow \\ (\sigma_b^+ \pi_b^- \sigma_a^- \varpi_a^+) & \longleftrightarrow & (\sigma_b^+ \varpi_b^- \sigma_a^- \pi_a^+) \end{array} \right\} \quad (1.15) (b)$$

$$(\sigma_a^+ \pi_a^- \sigma_b^+ \varpi_b^-) \longleftrightarrow (\sigma_a^+ \varpi_a^- \sigma_b^+ \pi_b^-) \quad (1.15) (c)$$

$$(\sigma_a^- \pi_a^+ \sigma_b^- \varpi_b^+) \longleftrightarrow (\sigma_a^- \varpi_a^+ \sigma_b^- \pi_b^+). \quad (1.15) (d)$$

The linear combinations of these twelve linearly independent functions contain two linearly independent $^5\Sigma$'s and six linearly independent $^3\Sigma$'s. We have therefore four linearly independent $^1\Sigma$ functions. They can be obtained by linearly combining (1.15) (a) and (b), or (a), (c), and (d) as follows:

$$\begin{aligned} (\sigma\pi, \sigma\varpi) \ ^1\Sigma_g^+ & \quad (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^-) - (\sigma_a^+ \pi_a^- \sigma_b^- \varpi_b^+) \\ & \quad - (\sigma_a^- \pi_a^+ \sigma_b^+ \varpi_b^-) + (\sigma_a^- \pi_a^- \sigma_b^+ \varpi_b^+) \\ & \quad + (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^-) - (\sigma_b^+ \pi_b^- \sigma_a^- \varpi_a^+) \\ & \quad - (\sigma_b^- \pi_b^+ \sigma_a^+ \varpi_a^-) + (\sigma_b^- \pi_b^- \sigma_a^+ \varpi_a^+) \end{aligned} \quad (1.16) (a)$$

$$\begin{aligned}
 (\sigma\pi, \sigma\varpi) \quad {}^1\Sigma_g^+ & \quad (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^-) - (\sigma_a^+ \pi_a^- \sigma_b^+ \varpi_b^-) \\
 & \quad - (\sigma_a^- \pi_a^+ \sigma_b^- \varpi_b^+) + (\sigma_a^- \pi_a^- \sigma_b^+ \varpi_b^+) \\
 & \quad + (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^-) - (\sigma_b^+ \pi_b^- \sigma_a^+ \varpi_a^-) \\
 & \quad - (\sigma_b^- \pi_b^+ \sigma_a^- \varpi_a^+) + (\sigma_b^- \pi_b^- \sigma_a^+ \varpi_a^+) \quad (1.16) (b)
 \end{aligned}$$

$$\begin{aligned}
 (\sigma\pi, \sigma\varpi) \quad {}^1\Sigma_u^- & \quad (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^-) - (\sigma_a^+ \pi_a^- \sigma_b^- \varpi_b^+) \\
 & \quad - (\sigma_a^- \pi_a^+ \sigma_b^+ \varpi_b^-) + (\sigma_a^- \pi_a^- \sigma_b^+ \varpi_b^+) \\
 & \quad - (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^-) + (\sigma_b^+ \pi_b^- \sigma_a^- \varpi_a^+) \\
 & \quad + (\sigma_b^- \pi_b^+ \sigma_a^+ \varpi_a^-) - (\sigma_b^- \pi_b^- \sigma_a^+ \varpi_a^+) \quad (1.16) (c)
 \end{aligned}$$

$$\begin{aligned}
 (\sigma\pi, \sigma\varpi) \quad {}^1\Sigma_u^- & \quad (\sigma_a^+ \pi_a^+ \sigma_b^- \varpi_b^-) - (\sigma_a^+ \pi_a^- \sigma_b^+ \varpi_b^-) \\
 & \quad - (\sigma_a^- \pi_a^+ \sigma_b^- \varpi_b^+) + (\sigma_a^- \pi_a^- \sigma_b^+ \varpi_b^+) \\
 & \quad - (\sigma_b^+ \pi_b^+ \sigma_a^- \varpi_a^-) + (\sigma_b^+ \pi_b^- \sigma_a^+ \varpi_a^-) \\
 & \quad + (\sigma_b^- \pi_b^+ \sigma_a^- \varpi_a^+) - (\sigma_b^- \pi_b^- \sigma_a^+ \varpi_a^+) \quad (1.16) (d)
 \end{aligned}$$

The wave functions of $(\sigma\pi, \pi\varpi)II$ can be obtained from those of $(\sigma\pi, \sigma\varpi)\Sigma$ by interchanging σ with π and discarding the reflexion symmetry.

We shall now consider $(\pi\varpi, \pi\varpi)\Sigma$. There is only a function which belongs to the quintet, namely

$$(\pi\varpi, \pi\varpi) \quad {}^5\Sigma_g^+ \quad (\pi_a^+ \varpi_a^+ \pi_b^+ \varpi_b^+). \quad (1.17)$$

The wave functions which belong to $(\pi\varpi, \pi\varpi)$ and $M_s=1$ contain four linearly independent functions. One ${}^5\Sigma$ is included in their linear combinations. We have therefore three linearly independent ${}^3\Sigma$'s. The wave functions which belong to $(\pi\varpi, \pi\varpi)$ and $M_s=0$ contain six linearly independent functions. One ${}^5\Sigma$ and three ${}^3\Sigma$'s are included. We have therefore two ${}^1\Sigma$'s.

The wave functions which belong to $(\pi\varpi, \pi\varpi)$ and $M_s=0$ are divided into three invariant linear manifolds for the inversion and reflexion. Their bases transform according to the following scheme:

$$(\pi_a^+ \varpi_a^- \pi_b^+ \varpi_b^-) \longleftrightarrow (\pi_a^- \varpi_a^+ \pi_b^- \varpi_b^+) \quad (1.18) (a)$$

$$(\pi_a^+ \varpi_a^+ \pi_b^- \varpi_b^-) \longleftrightarrow (\pi_a^- \varpi_a^- \pi_b^+ \varpi_b^+) \quad (1.18) (b)$$

$$(\pi_a^+ \varpi_a^- \pi_b^- \varpi_b^+) \longleftrightarrow (\pi_a^- \varpi_a^+ \pi_b^+ \varpi_b^-) \quad (1.18) (c)$$

where (a) is invariant for the inversion and (b) is invariant for the reflexion. These three linear manifolds contain one ${}^3\Sigma$ each. Their parity, reflexion symmetry, and wave functions are given by

$$(\pi\varpi, \pi\varpi) \quad {}^3\Sigma_g^- \quad (\pi_a^+ \varpi_a^- \pi_b^+ \varpi_b^-) - (\pi_a^- \varpi_a^+ \pi_b^- \varpi_b^+) \quad (1.19) (a)$$

$$(\pi\varpi, \pi\varpi) \quad {}^3\Sigma_u^+ \quad (\pi_a^+ \varpi_a^+ \pi_b^- \varpi_b^-) - (\pi_a^- \varpi_a^- \pi_b^+ \varpi_b^+) \quad (1.19) (b)$$

$$(\pi\varpi, \pi\varpi) {}^3\Sigma_u^- (\pi_a^+ \varpi_a^- \pi_b^- \varpi_b^+) - (\pi_a^- \varpi_a^+ \pi_b^+ \varpi_b^-). \quad (1.19) (c)$$

Two linearly independent ${}^1\Sigma$'s can be obtained by linearly combining two sets of (1.18) as follows:

$$\begin{aligned} (\pi\varpi, \pi\varpi) {}^1\Sigma_g^+ & (\pi_a^+ \varpi_a^- \pi_b^+ \varpi_b^-) - (\pi_a^+ \varpi_a^+ \pi_b^- \varpi_b^-) \\ & + (\pi_a^- \varpi_a^+ \pi_b^- \varpi_b^+) - (\pi_a^- \varpi_a^- \pi_b^+ \varpi_b^+) \end{aligned} \quad (1.20) (a)$$

$$\begin{aligned} (\pi\varpi, \pi\varpi) {}^1\Sigma_g^+ & (\pi_a^+ \varpi_a^- \pi_b^+ \varpi_b^-) - (\pi_a^+ \varpi_a^- \pi_b^- \varpi_b^+) \\ & + (\pi_a^- \varpi_a^+ \pi_b^- \varpi_b^+) - (\pi_a^- \varpi_a^+ \pi_b^+ \varpi_b^-). \end{aligned} \quad (1.20) (b)$$

The wave functions of $(\sigma\pi, \sigma\pi) \Delta$ can be obtained from those of $(\pi\varpi, \pi\varpi) \Sigma$ by replacing ϖ with σ and discarding the reflexion symmetry.

We have thus been able to completely classify all states of our molecule. A summary of the result is shown in Table II where "double" and "triple" indicate doubly and trebly degenerate states.

Table II. States arising from $(2p^2, 2p^2)$

$(\pi^2, \pi^2) {}^1\Gamma_g$		
	$(\pi^2, \sigma\tau) {}^3\Phi_g$ ${}^3\Phi_u$ ${}^1\Phi_g$ ${}^1\Phi_u$	
$(\sigma^2, \pi^2) {}^1\Delta_g$ ${}^1\Delta_u$	$(\pi^2, \pi\varpi) {}^3\Delta_g$ ${}^3\Delta_u$ ${}^1\Delta_g$ ${}^1\Delta_u$	$(\sigma\pi, \sigma\pi) {}^5\Delta_g$ ${}^3\Delta_g$ ${}^3\Delta_u$ (double) ${}^1\Delta_g$ (double)
	$(\pi^2, \sigma\varpi) {}^3\Pi_g$ $(\sigma^2, \sigma\pi) {}^3\Pi_g$ ${}^3\Pi_u$ ${}^3\Pi_u$ ${}^1\Pi_g$ ${}^1\Pi_g$ ${}^1\Pi_u$ ${}^1\Pi_u$	$(\sigma\pi, \pi\varpi) {}^5\Pi_g$ ${}^5\Pi_u$ ${}^3\Pi_g$ (triple) ${}^3\Pi_u$ (triple) ${}^1\Pi_g$ (double) ${}^1\Pi_u$ (double)
$(\pi^2, \varpi^2) {}^1\Sigma_g^+ (\sigma^2, \sigma^2) {}^1\Sigma_g^+$ ${}^1\Sigma_u^-$	$(\sigma^2, \pi\varpi) {}^3\Sigma_g^-$ ${}^5\Sigma_u^-$ ${}^1\Sigma_g^+$ ${}^1\Sigma_u^+$	$(\sigma\pi, \sigma\varpi) {}^5\Sigma_g^+$ $(\pi\varpi, \pi\varpi) {}^5\Sigma_g^+$ ${}^5\Sigma_u^-$ ${}^3\Sigma_g^+$ ${}^3\Sigma_g^-$ (double) ${}^3\Sigma_g^-$ ${}^3\Sigma_u^+$ (double) ${}^3\Sigma_u^+$ ${}^3\Sigma_u^-$ ${}^3\Sigma_u^-$ ${}^1\Sigma_g^+$ (double) ${}^1\Sigma_g^+$ ${}^1\Sigma_u^-$ (double) (double)

§ 2. General Formulas for Electronic Energy

As has been assumed in the previous section we consider an ideal diatomic molecule which consists of four electrons and two nuclei of double charges. An electronic part of its Hamiltonian is given by

$$H = -\sum_{k=1}^4 \left(\frac{1}{2} 4_k + \frac{2}{r_{ak}} + \frac{2}{r_{bk}} \right) + \sum_{k=j+1}^4 \sum_{j=1}^4 \frac{1}{r_{kj}} + \frac{4}{R} \quad (2.1)$$

where all quantities are measured in atomic units. In order to obtain general formulas for the adiabatic potential which characterizes the nature of binding of the molecule we have to solve secular equations corresponding to sub-matrices, of this Hamiltonian, with the definite multiplicity, axial quantum number, parity, and reflexion symmetry (Σ -states).

These properties of the states in a horizontal row of Table II are the same, but those of the states in different rows are different. Therefore, the states in different rows belong to different eigenvalues of the electronic energy, and for every row of the table we have a secular equation whose degree is given by a number of states written in that row. These degrees are shown in Table III.

Table III. Degrees of Secular Equations

states	$^1\Gamma_g$	$^3\Phi_g$	$^3\Phi_u$	$^1\Phi_g$	$^1\Phi_u$	$^5\Delta_g$	$^3\Delta_g$	$^3\Delta_u$	$^1\Delta_g$	$^1\Delta_u$	$^5\Pi_g$	$^5\Pi_u$	$^3\Pi_g$	$^3\Pi_u$	$^1\Pi_g$	$^1\Pi_u$
degrees	1	1	1	1	1	1	2	3	4	2	1	1	5	5	4	4
states	$^5\Sigma_g^+$	$^5\Sigma_u^-$	$^3\Sigma_g^+$	$^3\Sigma_g^-$	$^3\Sigma_u^+$	$^3\Sigma_u^-$	$^1\Sigma_g^+$	$^1\Sigma_u^+$	$^1\Sigma_u^-$							
degrees	2	1	1	4	3	3	7	1	3							

The degrees are too high to solve the secular equations for $^3,1\Delta$'s, $^3,1\Pi$'s and $^3,1\Sigma$'s. However, fortunately we see from the result of our numerical calculation that the matrix elements connecting different configurations have only a very small contribution to the energy eigenvalue. Therefore we can confine our eigenvalue problem within a definite configuration. Then the secular equations are at most cubic, and their solution are facilitated in considerable degree by this reduction.

Especially the equations for the states in the first two columns of Table II are all linear, and their roots are at once given by

$$W = \frac{(\phi, H\phi)}{(\phi, \phi)} \quad (2.2)$$

where the parenthesis means a Hermitian inner product. For the states in the third column the equations are linear, quadratic for doubly degenerate states, and cubic for trebly degenerate states. When the equation is quadratic its roots are given by

$$W = \frac{1}{2A} (B \pm \sqrt{B^2 - 4AC}) \quad (2.3)$$

where

$$A = \mathcal{A}_{11}\mathcal{A}_{22} - \mathcal{A}_{12}^2 \quad (2.4a)$$

$$B = \mathcal{A}_{22}H_{11} + \mathcal{A}_{11}H_{22} - 2\mathcal{A}_{12}H_{12} \quad (2.4b)$$

$$C = H_{11}H_{22} - H_{12}^2 \quad (2.4c)$$

$$H_{kj} = (\phi_k, H\phi_j) \quad k=1,2 \quad (2.4d)$$

$$\mathcal{A}_{kj} = (\phi_k, \phi_j) \quad k=1,2. \quad (2.4e)$$

If we substitute the wave functions given in the previous section for ψ of (2.2), we have the energies of the states in the first column of Table II as follows:

$$(\pi^2, \pi^2) \quad {}^1\Gamma_g \quad W = \frac{Q(\pi^2, \pi^2) - 2J_n(\pi_a\pi_b; \pi_a\pi_b) + K_n(\pi_a\pi_b; \pi_a\pi_b)}{1 - 2S_1^2 + S_1^4} \quad (2.5)$$

$$(\sigma^2, \pi^2) \quad {}^1\mathcal{A}_g \quad W = \frac{Q(\sigma^2, \pi^2) - 2J_0(\sigma_a\pi_b; \sigma_a\pi_b) + K_n(\sigma_a\pi_b; \sigma_a\pi_b) - 2K_{\lambda n}(\sigma_a\pi_b; \sigma_a\pi_b)}{1 + S_0^2 - S_1^2} \quad (2.6a)$$

$$(\sigma^2, \pi^2) \quad {}^1\mathcal{A}_u \quad W = \frac{Q(\sigma^2, \pi^2) - 2J_0(\sigma_a\pi_b; \sigma_a\pi_b) - K_n(\sigma_a\pi_b; \sigma_a\pi_b) + 2K_{\lambda n}(\sigma_a\pi_b; \sigma_a\pi_b)}{1 - S_0^2 - S_1^2} \quad (2.6b)$$

$$(\pi^2, \varpi^2) \quad {}^1\Sigma_g^+ \quad W = \frac{Q(\pi^2, \varpi^2) - 2J_0(\pi_a\varpi_b; \pi_a\varpi_b) + K_n(\pi_a\varpi_b; \pi_a\varpi_b) - 2K_{\lambda n}(\pi_a\varpi_b; \pi_a\varpi_b)}{1 + S_1^4} \quad (2.7a)$$

$$(\pi^2, \varpi^2) \quad {}^1\Sigma_u^- \quad W = \frac{Q(\pi^2, \varpi^2) - 2J_0(\pi_a\varpi_b; \pi_a\varpi_b) - K_n(\pi_a\varpi_b; \pi_a\varpi_b) + 2K_{\lambda n}(\pi_a\varpi_b; \pi_a\varpi_b)}{1 - S_1^4} \quad (2.7b)$$

$$(\sigma^2, \sigma^2) \quad {}^1\Sigma_g^+ \quad W = \frac{Q(\sigma^2, \sigma^2) - 2J_n(\sigma_a\sigma_b; \sigma_a\sigma_b) + K_n(\sigma_a\sigma_b; \sigma_a\sigma_b)}{1 - 2S_0^2 + S_0^4} \quad (2.8)$$

where

$$Q(\varphi\psi, \xi\eta) = (\varphi_a \phi_a \xi_b \eta_b, H\varphi_a \phi_a \xi_b \eta_b) \quad (2.9)$$

$$J_n(\varphi_a \phi_b; \xi_c \eta_d) = (\varphi_a \phi_b \xi_c \eta_d, H\varphi_a \phi_b \xi_c \eta_d) \quad (2.10a)$$

$$J_0(\varphi_a \phi_b; \xi_c \eta_d) = (\varphi_a \phi_b \xi_c \eta_d, H\varphi_a \phi_b \eta_d \xi_c) \quad (2.10b)$$

$$K_n(\varphi_a \phi_b; \xi_c \eta_d) = (\varphi_a \phi_b \xi_c \eta_d, H\varphi_b \phi_a \xi_d \eta_c) \quad (2.11a)$$

$$K_{\lambda n}(\varphi_a \phi_b; \xi_c \eta_d) = (\varphi_a \phi_b \xi_c \eta_d, H\varphi_a \phi_b \xi_d \eta_c) \quad (2.11b)$$

$$S_0 = (\sigma_a, \sigma_b) \quad (2.12a)$$

$$S_1 = (\pi_a, \pi_b) = (\varpi_a, \varpi_b) \quad (2.12b)$$

φ, ϕ, ξ , or η denotes one of σ, π and ϖ , and a, b, c , or d denotes one of a and b . Further J and K mean single and double exchange integrals respectively, n means an exchange of nuclei, λ means an exchange of axial quantum numbers, o means an exchange of orbital functions, and the following abbreviations are here used; $Q(\varphi^2, \xi^2)$ is written instead of $Q(\varphi\varphi, \xi\xi)$, and $\varphi\psi\xi\eta$ is written in place of $\varphi(x_1)\psi(x_2)\xi(x_3)\eta(x_4)$. The right sides of (2.9), (2.10), (2.11) and (2.12) mean Hermitian inner products of two functions written in front and behind of commas.

If Ψ and Φ are two wave functions their Hermitian inner product is defined by

$$(\Psi, \Phi) = \int \Psi^* \Phi \quad (2.13)$$

where the integral sign means a multiple space integral with respect to all space coordinates contained in the functions. Energies of other non-degenerate states are given by similar forms.

Next we consider doubly degenerate states, for example $(\pi\varpi, \pi\varpi)^1 \Sigma_g^+$. Their wave functions are given by (1.20). The secular matrix for these states are given by

$$\begin{aligned} H_{11} = & Q(\pi\varpi, \pi\varpi) - J_n(\pi_a\pi_b; \varpi_a\varpi_b) + K_n(\pi_a\pi_b; \varpi_a\varpi_b) \\ & + J_0(\pi_a\varpi_b; \pi_a\varpi_b) - J_0(\pi_b\varpi_b; \pi_a\varpi_a) + K_{\lambda n}(\varpi_a\pi_b; \pi_a\varpi_b) \\ & - (\pi_a\varpi_a \pi_b\varpi_b, H\pi_b\varpi_b\varpi_a\pi_a) - (\pi_a\varpi_b\varpi_a\pi_b, H\pi_b\varpi_b\pi_a\varpi_a) \\ & - (\pi_a\varpi_b\varpi_a\pi_b, H\pi_a\varpi_a \pi_b\varpi_b) \end{aligned} \quad (2.14a)$$

$$\begin{aligned} H_{22} = & Q(\pi\varpi, \pi\varpi) - J_n(\pi_a\pi_b; \varpi_a\varpi_b) + K_n(\pi_a\pi_b; \varpi_a\varpi_b) \\ & - J_0(\pi_a\varpi_b; \varpi_a\pi_b) + J_0(\pi_a\varpi_a; \pi_b\varpi_b) - K_{\lambda n}(\varpi_a\pi_b; \pi_a\varpi_b) \\ & + (\pi_a\varpi_a \pi_b\varpi_b, H\pi_b\varpi_b\varpi_a\pi_a) - 2(\pi_a\varpi_b\varpi_a\pi_b, H\pi_b\varpi_b\pi_a\varpi_a) \end{aligned} \quad (2.14b)$$

$$\begin{aligned} H_{12} = & \frac{1}{2}Q(\pi\varpi, \pi\varpi) - 2J_n(\pi_a\pi_b; \varpi_a\varpi_b) + \frac{1}{2}K_n(\pi_a\pi_b; \varpi_a\varpi_b) \\ & + J_0(\pi_a\varpi_a; \pi_b\varpi_b) + \frac{1}{2}K_0(\pi_a\varpi_a \pi_b\varpi_b) + \frac{1}{2}K_0(\pi_a\varpi_b; \varpi_a\pi_b) \\ & + K_{\lambda n}(\varpi_a\pi_b; \pi_a\varpi_b) \\ & - 3(\pi_a\varpi_b\varpi_a\pi_b, H\pi_b\varpi_b\pi_a\varpi_a) + 2(\pi_a\varpi_a\pi_b\varpi_b, H\pi_a\varpi_b\varpi_a\pi_b) \\ & + (\pi_a\varpi_a\pi_b\varpi_b, H\pi_b\varpi_b\varpi_a\pi_a) \end{aligned} \quad (2.14c)$$

$$A_{11} = A_{22} = 1 - S_1^2 + S_1^4 \quad (2.15a)$$

$$A_{12} = \frac{1}{2} - 2S_1^2 + \frac{1}{2}S_1^4 \quad (2.15b)$$

where a common factor is omitted, and

$$K_0(\varphi_a \psi_b; \xi_c \eta_d) = (\varphi_a \psi_b \xi_c \eta_d, H\psi_b \varphi_a \eta_d \xi_c). \quad (2.16)$$

If H is equal to unity in these expressions only the first three terms of each elements remain without vanishing and all other terms vanish because of the orthogonality of π and ϖ . In fact the latter terms are not equal to zero because H is not equal to unity, but they may be still small compared with the first three. The result of our numerical calculation shows that they are smaller than 5% of Q . Therefore we first neglect these small terms. Then, if we substitute (2.14) and (2.15) in (2.3), we have two roots of the secular equation as follows:

$$W = \frac{Q(\pi\varpi, \pi\varpi) \pm 2J(\pi\pi; \varpi\varpi) + K(\pi\pi; \varpi\varpi)}{1 \pm 2S_1^2 + S_1^4} \quad (2.17)$$

where the plus sign of the numerator corresponds to the plus sign of the deno-

minator, the minus to the minus sign, and the abbreviations

$$J(\varphi\psi; \xi\eta) = J_n(\varphi_a\psi_b; \xi_a\eta_b) \quad (2.18a)$$

$$K(\varphi\psi; \xi\eta) = K_n(\varphi_a\psi_b; \xi_a\eta_b) \quad (2.18b)$$

are used because there remain only J and K with an exchange of nuclei in this case.

If we content ourselves with an approximation of neglecting integrals which include in their integrands $\sigma(x_i)\pi(x_i)$, $\sigma(x_i)\varpi(x_i)$, or $\pi(x_i)\varpi(x_i)$ regarding a definite electron, i , expressions of W are all reduced to simple ones as has been seen in the above mentioned example of $(\pi\varpi, \pi\varpi)^1\Sigma_g^+$. Especially secular equations for trebly degenerate states split into equations of lower degrees which can be readily solved. The influence of the neglected terms will be discussed in the next section, and their quantitative contributions will be fully taken into account later on in Part II. Thus, for all states in Table II, we have, in this approximation, general formulas for electronic energy as follows:

$$(\pi^2, \pi^2)^1\Gamma_g \quad \frac{Q(\pi^2, \pi^2) - 2J(\pi\pi; \pi\pi) + K(\pi\pi; \pi\pi)}{1 - 2S_1^2 + S_1^4} \quad (2.19)$$

$$(\pi^2, \sigma\pi)^3\rho_g^1\Phi_g \quad \frac{Q(\pi^2, \sigma\pi) - \{J(\pi\pi; \sigma\pi) + J(\sigma\pi; \pi\pi)\} + K(\sigma\pi; \pi\pi)}{1 - (S_0S_1 + S_1^2) + S_0S_1^3} \quad (2.20a)$$

$$(\pi^2, \sigma\pi)^3\Phi_u^1\Phi_u \quad \frac{Q(\pi^2, \sigma\pi) + \{J(\pi\pi; \sigma\pi) - J(\sigma\pi; \pi\pi)\} - K(\sigma\pi; \pi\pi)}{1 + (S_0S_1 - S_1^2) - S_0S_1^3} \quad (2.20b)$$

$$(\sigma\pi, \sigma\pi)^5\Delta_g \quad \frac{Q(\sigma\pi, \sigma\pi) - \{J(\pi\pi; \sigma\sigma) + J(\sigma\sigma; \pi\pi)\} + K(\sigma\sigma; \pi\pi)}{1 - (S_0^2 + S_1^2) + S_0^2S_1^2} \quad (2.21a)$$

$$(\pi^2, \pi\varpi)^3\Delta_g \quad \frac{Q(\pi^2, \pi\varpi) - \{J(\pi\varpi; \pi\pi) + J(\pi\pi; \pi\varpi)\} + K(\pi\pi; \pi\varpi)}{1 - 2S_1^2 + S_1^4} \quad (2.21b)$$

$$(\sigma\pi, \sigma\pi)^3\Delta_g \quad \frac{Q(\sigma\pi, \sigma\pi) - \{J(\pi\pi; \sigma\sigma) + J(\sigma\sigma; \pi\pi)\} + K(\sigma\sigma; \pi\pi)}{1 - (S_0^2 + S_1^2) + S_0^2S_1^2} \quad (2.21c)$$

$$(\pi^2, \pi\varpi)^3\Delta_u \quad \frac{Q(\pi^2, \pi\varpi) - \{J(\pi\varpi; \pi\pi) - J(\pi\pi; \pi\varpi)\} - K(\pi\pi; \pi\varpi)}{1 - S_1^4} \quad (2.21d)$$

$$(\sigma\pi, \sigma\pi)^3\Delta_u \quad \frac{Q(\sigma\pi, \sigma\pi) \pm \{J(\pi\pi; \sigma\sigma) - J(\sigma\sigma; \pi\pi)\} - K(\sigma\sigma; \pi\pi)}{1 \pm (S_0^2 - S_1^2) - S_0^2S_1^2} \quad (2.21e)$$

$$(\sigma^2, \pi^2)^1\Delta_g \quad \frac{Q(\pi^2, \sigma^2) + K(\sigma\pi; \sigma\pi)}{1 + S_0^2S_1^2} \quad (2.21f)$$

$$(\pi^2, \pi\varpi)^1\Delta_g \quad \frac{Q(\pi^2, \pi\varpi) - \{J(\pi\varpi; \pi\pi) + J(\pi\pi; \pi\varpi)\} + K(\pi\pi; \pi\varpi)}{1 - 2S_1^2 + S_1^4} \quad (2.21g)$$

$$(\sigma\pi, \sigma\pi)^1\Delta_g \quad \frac{Q(\sigma\pi, \sigma\pi) \pm \{J(\pi\pi; \sigma\sigma) + J(\sigma\sigma; \pi\pi)\} + K(\sigma\sigma; \pi\pi)}{1 \pm (S_0^2 + S_1^2) + S_0^2S_1^2} \quad (2.21h)$$

$$(\sigma^2, \pi^2)^1 \Delta_u \quad \frac{Q(\pi^2, \sigma^2) - K(\sigma\pi; \sigma\pi)}{1 - S_0^2 S_1^2} \quad (2.21k)$$

$$(\pi^2, \pi\varpi)^1 \Delta_u \quad \frac{Q(\pi^2, \pi\varpi) - \{J(\pi\varpi; \pi\pi) - J(\pi\pi; \pi\varpi)\} - K(\pi\pi; \pi\varpi)}{1 - S_1^4} \quad (2.21l)$$

$$(\sigma\pi, \pi\varpi)^5 \Pi_g \quad \frac{Q(\sigma\pi, \pi\varpi) - \{J(\pi\pi; \sigma\varpi) + J(\sigma\varpi; \pi\pi)\} + K(\pi\pi; \sigma\varpi)}{1 - (S_0 S_1 + S_1^2) + S_0 S_1^3} \quad (2.22a)$$

$$(\sigma\pi, \pi\varpi)^5 \Pi_u \quad \frac{Q(\sigma\pi, \pi\varpi) + \{J(\pi\pi; \sigma\varpi) - J(\sigma\varpi; \pi\pi)\} - K(\pi\pi; \sigma\varpi)}{1 + (S_0 S_1 - S_1^2) - S_0 S_1^3} \quad (2.22b)$$

$$(\pi^2, \sigma\varpi)^3 \Pi_g \quad \frac{Q(\pi^2, \sigma\varpi) + K(\sigma\pi; \varpi\pi)}{1 + S_0 S_1^3} \quad (2.22c)$$

$$(\sigma^2, \sigma\pi)^3 \Pi_g \quad \frac{Q(\sigma^2, \sigma\pi) - \{J(\sigma\pi; \sigma\sigma) + J(\sigma\sigma; \sigma\pi)\} + K(\sigma\sigma; \sigma\pi)}{1 - (S_0^2 + S_0 S_1) + S_0^3 S_1} \quad (2.22d)$$

$$(\sigma\pi, \pi\varpi)^3 \Pi_g \quad \frac{Q(\sigma\pi, \pi\varpi) \pm \{J(\pi\pi; \sigma\varpi) + J(\sigma\varpi; \pi\pi)\} + K(\pi\pi; \sigma\varpi)}{1 \pm (S_0 S_1 + S_1^2) + S_0 S_1^3} \quad (2.22e)$$

$$(\pi^2, \sigma\varpi)^3 \Pi_u \quad \frac{Q(\pi^2, \sigma\varpi) - K(\sigma\pi; \varpi\pi)}{1 - S_0 S_1^3} \quad (2.22f)$$

$$(\sigma^2, \sigma\pi)^3 \Pi_u \quad \frac{Q(\sigma^2, \sigma\pi) - \{J(\sigma\pi; \sigma\sigma) - J(\sigma\sigma; \sigma\pi)\} - K(\sigma\sigma; \sigma\pi)}{1 - (S_0^2 - S_0 S_1) - S_0^3 S_1} \quad (2.22g)$$

$$(\sigma\pi, \pi\varpi)^3 \Pi_u \quad \frac{Q(\sigma\pi, \pi\varpi) \pm \{J(\pi\pi; \sigma\varpi) - J(\sigma\varpi; \pi\pi)\} - K(\pi\pi; \sigma\varpi)}{1 \pm (S_0 S_1 - S_1^2) - S_0 S_1^3} \quad (2.22h)$$

$$(\pi^2, \sigma\varpi)^1 \Pi_g \quad \frac{Q(\pi^2, \sigma\varpi) + K(\sigma\pi; \varpi\pi)}{1 + S_0 S_1^3} \quad (2.22k)$$

$$(\sigma^2, \sigma\pi)^1 \Pi_g \quad \frac{Q(\sigma^2, \sigma\pi) - \{J(\sigma\pi; \sigma\sigma) + J(\sigma\sigma; \sigma\pi)\} + K(\sigma\sigma; \sigma\pi)}{1 - (S_0^2 + S_0 S_1) + S_0^3 S_1} \quad (2.22l)$$

$$(\sigma\pi, \pi\varpi)^1 \Pi_g \quad \frac{Q(\pi\pi, \sigma\varpi) \pm \{J(\pi\pi; \sigma\varpi) + J(\sigma\varpi; \pi\pi)\} + K(\pi\pi; \sigma\varpi)}{1 \pm (S_0 S_1 + S_1^2) + S_0 S_1^3} \quad (2.22m)$$

$$(\pi^2, \sigma\varpi)^1 \Pi_u \quad \frac{Q(\pi^2, \sigma\varpi) - K(\sigma\pi; \varpi\pi)}{1 - S_0 S_1^3} \quad (2.22n)$$

$$(\sigma^2, \sigma\pi)^1 \Pi_u \quad \frac{Q(\sigma^2, \sigma\pi) - \{J(\sigma\pi; \sigma\sigma) - J(\sigma\sigma; \sigma\pi)\} - K(\sigma\sigma; \sigma\pi)}{1 - (S_0^2 - S_0 S_1) - S_0^3 S_1} \quad (2.22p)$$

$$(\sigma\pi, \pi\varpi)^1 \Pi_u \quad \frac{Q(\sigma\pi, \pi\varpi) \pm \{J(\pi\pi; \sigma\varpi) - J(\sigma\varpi; \pi\pi)\} - K(\pi\pi; \sigma\varpi)}{1 \pm (S_0 S_1 - S_1^2) - S_0 S_1^3} \quad (2.22q)$$

$$(\sigma\pi, \sigma\varpi)^5 \Sigma_g^+ \quad \frac{Q(\sigma\pi, \sigma\varpi) - \{J(\pi\varpi; \sigma\sigma) + J(\sigma\sigma; \pi\varpi)\} + K(\sigma\sigma; \pi\varpi)}{1 - (S_0^2 + S_1^2) + S_0^2 S_1^2} \quad (2.23a)$$

$$(\pi\varpi, \pi\varpi)^5 \sum_g^+ \frac{Q(\pi\varpi, \pi\varpi) - 2J(\pi\pi; \varpi\varpi) + K(\pi\pi; \varpi\varpi)}{1 - 2S_1^2 + S_1^4} \quad (2.23b)$$

$$(\sigma\pi, \sigma\varpi)^5 \sum_u^- \frac{Q(\sigma\pi, \sigma\varpi) - \{J(\pi\varpi; \sigma\sigma) - J(\sigma\sigma; \pi\varpi)\} - K(\sigma\sigma; \pi\varpi)}{1 - (S_0^2 - S_1^2) - S_0^2 S_1^2} \quad (2.23c)$$

$$(\sigma\pi, \sigma\varpi)^3 \sum_g^+ \frac{Q(\sigma\pi, \sigma\varpi) - \{J(\pi\varpi; \sigma\sigma) + J(\sigma\sigma; \pi\varpi)\} + K(\sigma\sigma; \pi\varpi)}{1 - (S_0^2 + S_1^2) + S_0^2 S_1^2} \quad (2.23d)$$

$$(\sigma^2, \pi\varpi)^3 \sum_g^- \frac{Q(\sigma^2, \pi\varpi) + K(\sigma\pi; \sigma\varpi)}{1 + S_0^2 S_1^2} \quad (2.23e)$$

$$(\sigma\pi, \sigma\varpi)^3 \sum_g^- \frac{Q(\sigma\pi, \sigma\varpi) \pm \{J(\pi\varpi; \sigma\sigma) + J(\sigma\sigma; \pi\varpi)\} + K(\sigma\sigma; \pi\varpi)}{1 \pm (S_0^2 + S_1^2) + S_0^2 S_1^2} \quad (2.23f)$$

$$(\pi\varpi, \pi\varpi)^3 \sum_g^- \frac{Q(\pi\varpi, \pi\varpi) - 2J(\pi\pi; \varpi\varpi) + K(\pi\pi; \varpi\varpi)}{1 - 2S_1^2 + S_1^4} \quad (2.23g)$$

$$(\sigma\pi, \sigma\varpi)^3 \sum_u^+ \frac{Q(\sigma\pi, \sigma\varpi) \pm \{J(\pi\varpi; \sigma\sigma) - J(\sigma\sigma; \pi\varpi)\} - K(\sigma\sigma; \pi\varpi)}{1 \pm (S_0^2 - S_1^2) - S_0^2 S_1^2} \quad (2.23h)$$

$$(\pi\varpi, \pi\varpi)^3 \sum_u^+ \frac{Q(\pi\varpi, \pi\varpi) - K(\pi\pi; \varpi\varpi)}{1 - S_1^4} \quad (2.23k)$$

$$(\sigma^2, \pi\varpi)^3 \sum_u^- \frac{Q(\sigma^2, \pi\varpi) - K(\sigma\pi; \sigma\varpi)}{1 - S_0^2 S_1^2} \quad (2.23l)$$

$$(\sigma\pi, \sigma\varpi)^3 \sum_u^- \frac{Q(\sigma\pi, \sigma\varpi) - \{J(\pi\varpi; \sigma\sigma) - J(\sigma\sigma; \pi\varpi)\} - K(\sigma\sigma; \pi\varpi)}{1 - (S_0^2 - S_1^2) - S_0^2 S_1^2} \quad (2.23m)$$

$$(\pi\varpi, \pi\varpi)^3 \sum_u^- \frac{Q(\pi\varpi, \pi\varpi) - K(\pi\pi; \varpi\varpi)}{1 - S_1^4} \quad (2.23n)$$

$$(\pi^2, \varpi^2)^1 \sum_g^+ \frac{Q(\pi^2, \varpi^2) + K(\pi\varpi; \pi\varpi)}{1 + S_1^4} \quad (2.23p)$$

$$(\sigma^2, \sigma^2)^1 \sum_g^+ \frac{Q(\sigma^2, \sigma^2) - 2J(\sigma\sigma; \sigma\sigma) + K(\sigma\sigma; \sigma\sigma)}{1 - 2S_0^2 + S_0^4} \quad (2.23q)$$

$$(\sigma^2, \pi\varpi)^1 \sum_g^+ \frac{Q(\sigma^2, \pi\varpi) + K(\sigma\pi; \sigma\varpi)}{1 + S_0^2 S_1^2} \quad (2.23r)$$

$$(\sigma\pi, \sigma\varpi)^1 \sum_g^+ \frac{Q(\sigma\pi, \sigma\varpi) \pm \{J(\pi\varpi; \sigma\sigma) + J(\sigma\sigma; \pi\varpi)\} + K(\sigma\sigma; \pi\varpi)}{1 \pm (S_0^2 + S_1^2) + S_0^2 S_1^2} \quad (2.23s)$$

$$(\pi\varpi, \pi\varpi)^1 \sum_g^+ \frac{Q(\pi\varpi, \pi\varpi) \pm 2J(\pi\pi; \varpi\varpi) + K(\pi\pi; \varpi\varpi)}{1 \pm 2S_1^2 + S_1^4} \quad (2.23t)$$

$$(\sigma^2, \pi\varpi)^1 \sum_u^+ \frac{Q(\sigma^2, \pi\varpi) - K(\sigma\pi; \sigma\varpi)}{1 - S_0^2 S_1^2} \quad (2.23u)$$

$$(\pi^2, \varpi^2)^1 \sum_u^- \frac{Q(\pi^2, \varpi^2) - K(\pi\varpi; \pi\varpi)}{1 - S_1^4} \quad (2.23v)$$

$$(\sigma\pi, \sigma\pi)^1 \sum_u^- \frac{Q(\sigma\pi, \sigma\pi) \pm \{J(\pi\pi; \sigma\sigma) - J(\sigma\sigma; \pi\pi)\} - K(\sigma\sigma; \pi\pi)}{1 \pm (S_0^2 - S_1^2) - S_0^2 S_1^2} \quad (2.23w)$$

Table IV Numerical Values of Electronic Energy
in Atomic Units

States	Nuclear distance in atomic units			
	1.5	2.0	2.5	3.0
(π^2, π^2) $1\Gamma_g$	—	-0.384	-0.756	-0.979
$(\pi^2, \sigma\pi)$ $3\Phi_g$	—	-0.903	-1.102	-1.177
$(\pi^2, \sigma\pi)$ $3\Phi_u$	-0.489	-0.950	-1.160	-1.261
$(\pi^2, \sigma\pi)$ $1\Phi_g$	—	-0.903	-1.102	-1.177
$(\pi^2, \sigma\pi)$ $1\Phi_u$	-0.489	-0.950	-1.160	-1.261
$(\sigma\pi, \sigma\pi)$ $5\Delta_g$	-0.868	-1.228	-1.378	-1.420
$(\pi^2, \pi\pi)$ $3\Delta_g$	—	-0.510	-0.821	-1.000
$(\sigma\pi, \sigma\pi)$ $3\Delta_g$	-0.868	-1.228	-1.378	-1.420
$(\pi^2, \pi\pi)$ $3\Delta_u$	-0.259	-0.752	-1.010	-1.141
$(\sigma\pi, \sigma\pi)$ $3\Delta_u$	-0.614	-1.111	-1.301	-1.346
$(\sigma\pi, \sigma\pi)$ $3\Delta_u$	-1.314	-1.548	-1.642	-1.541
(σ^2, π^2) $1\Delta_g$	-0.974	-1.417	-1.419	-1.410
$(\pi^2, \pi\pi)$ $1\Delta_g$	—	-0.510	-0.821	-1.000
$(\sigma\pi, \sigma\pi)$ $1\Delta_g$	-0.975	-1.419	-1.571	-1.510
$(\sigma\pi, \sigma\pi)$ $1\Delta_g$	-0.868	-1.228	-1.378	-1.420
(σ^2, π^2) $1\Delta_u$	-1.075	-1.355	-1.460	-1.410
$(\pi^2, \pi\pi)$ $1\Delta_u$	-0.259	-0.752	-1.010	-1.141
$(\sigma\pi, \pi\pi)$ $5\Pi_g$	-0.492	-0.899	-1.092	-1.202
$(\sigma\pi, \pi\pi)$ $5\Pi_u$	-0.489	-0.950	-1.160	-1.261
$(\pi^2, \sigma\pi)$ $3\Pi_g$	-0.864	-1.180	-1.303	-1.341
$(\sigma^2, \sigma\pi)$ $3\Pi_g$	-1.401	-1.661	-1.649	-1.542
$(\sigma\pi, \pi\pi)$ $3\Pi_g$	-0.492	-0.899	-1.092	-1.202
$(\sigma\pi, \pi\pi)$ $3\Pi_g$	-1.004	-1.275	-1.389	-1.399
$(\pi^2, \sigma\pi)$ $3\Pi_u$	-0.796	-1.179	-1.303	-1.341
$(\sigma^2, \sigma\pi)$ $3\Pi_u$	-1.545	-1.711	-1.712	-1.610
$(\sigma\pi, \pi\pi)$ $3\Pi_u$	-0.489	-0.950	-1.160	-1.261
$(\sigma\pi, \pi\pi)$ $3\Pi_u$	-0.812	-1.218	-1.340	-1.360
$(\pi^2, \sigma\pi)$ $1\Pi_g$	-0.864	-1.180	-1.303	-1.341
$(\sigma^2, \sigma\pi)$ $1\Pi_g$	-1.401	-1.661	-1.649	-1.542
$(\sigma\pi, \pi\pi)$ $1\Pi_g$	-0.492	-0.899	-1.092	-1.202
$(\sigma\pi, \pi\pi)$ $1\Pi_g$	-1.004	-1.275	-1.389	-1.399
$(\pi^2, \sigma\pi)$ $1\Pi_u$	-0.796	-1.179	-1.303	-1.341
$(\sigma^2, \sigma\pi)$ $1\Pi_u$	-1.545	-1.711	-1.712	-1.610
$(\sigma\pi, \pi\pi)$ $1\Pi_u$	-0.489	-0.950	-1.160	-1.261
$(\sigma\pi, \pi\pi)$ $1\Pi_u$	-0.812	-1.218	-1.340	-1.360
$(\sigma\pi, \sigma\pi)$ $5\Sigma_g^+$	-0.868	-1.228	-1.378	-1.420

$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$5\sum_g^+$	-0.245	-0.607	-0.845	-1.001
$(\sigma\pi, \sigma\bar{\sigma})$	$5\sum_u^-$	-1.351	-1.562	-1.651	-1.601
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_g^+$	-0.868	-1.228	-1.378	-1.420
$(\sigma^2, \pi\bar{\sigma})$	$3\sum_g^-$	-0.974	-1.417	-1.419	-1.410
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_g^-$	-0.868	-1.228	-1.378	-1.420
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_g^-$	-0.975	-1.419	-1.571	-1.510
$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$3\sum_g^-$	-0.245	-0.607	-0.845	-1.001
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_u^+$	-1.351	-1.562	-1.651	-1.601
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_u^+$	-0.614	-1.111	-1.301	-1.346
$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$3\sum_u^+$	-0.259	-0.752	-1.010	-1.141
$(\sigma^2, \pi\bar{\sigma})$	$3\sum_u^-$	-1.075	-1.355	-1.460	-1.410
$(\sigma\pi, \sigma\bar{\sigma})$	$3\sum_u^-$	-1.351	-1.562	-1.651	-1.601
$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$3\sum_u^-$	-0.259	-0.752	-1.010	-1.141
$(\pi^2, \bar{\sigma}^2)$	$1\sum_g^+$	-0.644	-1.001	-1.151	-1.215
(σ^2, σ^2)	$1\sum_g^+$	-1.820	-1.929	-1.872	-1.722
$(\sigma^2, \pi\bar{\sigma})$	$1\sum_g^+$	-0.974	-1.417	-1.419	-1.410
$(\sigma\pi, \sigma\bar{\sigma})$	$1\sum_g^+$	-0.868	-1.228	-1.378	-1.420
$(\sigma\pi, \sigma\bar{\sigma})$	$1\sum_g^+$	-0.975	-1.419	-1.571	-1.510
$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$1\sum_g^+$	-0.245	-0.607	-0.845	-1.001
$(\pi\bar{\sigma}, \pi\bar{\sigma})$	$1\sum_g^+$	-0.615	-1.054	-1.207	-1.271
$(\sigma^2, \pi\bar{\sigma})$	$1\sum_u^+$	-1.075	-1.355	-1.460	-1.410
$(\pi^2, \bar{\sigma}^2)$	$1\sum_u^-$	-0.450	-0.814	-1.051	-1.160
$(\sigma\pi, \sigma\bar{\sigma})$	$1\sum_u^-$	-0.614	-1.111	-1.301	-1.346
$(\sigma\pi, \sigma\bar{\sigma})$	$1\sum_u^-$	-1.351	-1.562	-1.651	-1.601

§ 3 Adiabatic Potentials

The adiabatic potential between two atoms in the molecule is the eigenvalue of electronic energy as a function of a distance between two nuclei. In order to know explicitly this dependence on the distance we have to evaluate the eigenvalue for various distances. For this purpose we assume that the orbitals are hydrogen-like:

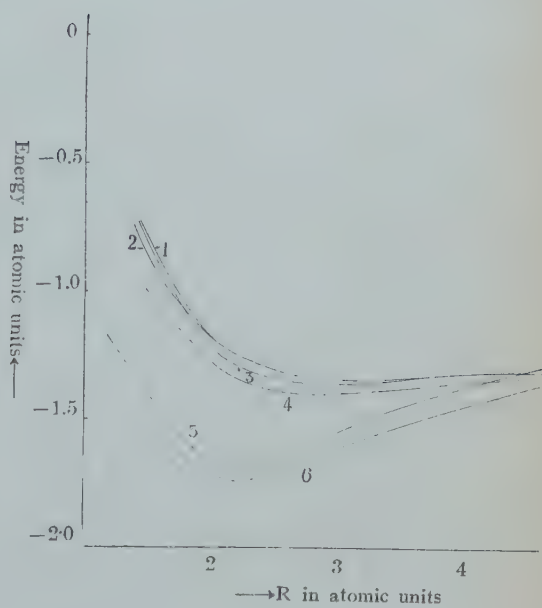
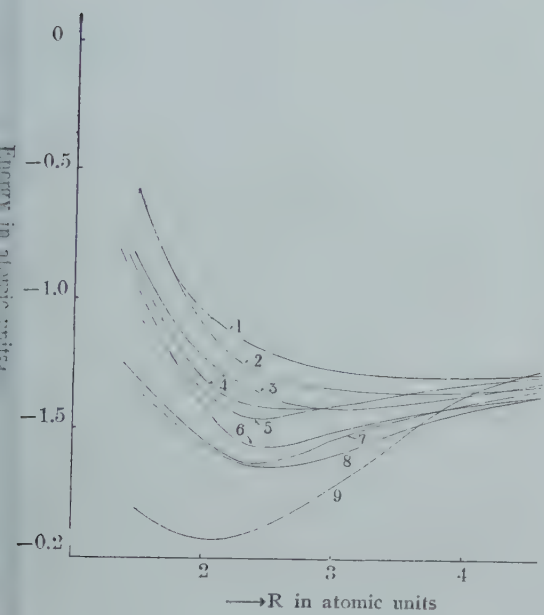
$$\sigma = N_\sigma z \exp(-\alpha r/2) \quad (3.1a)$$

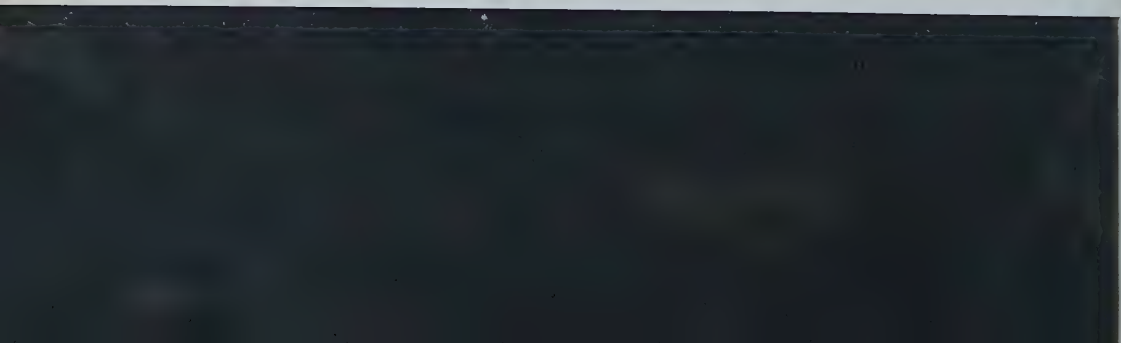
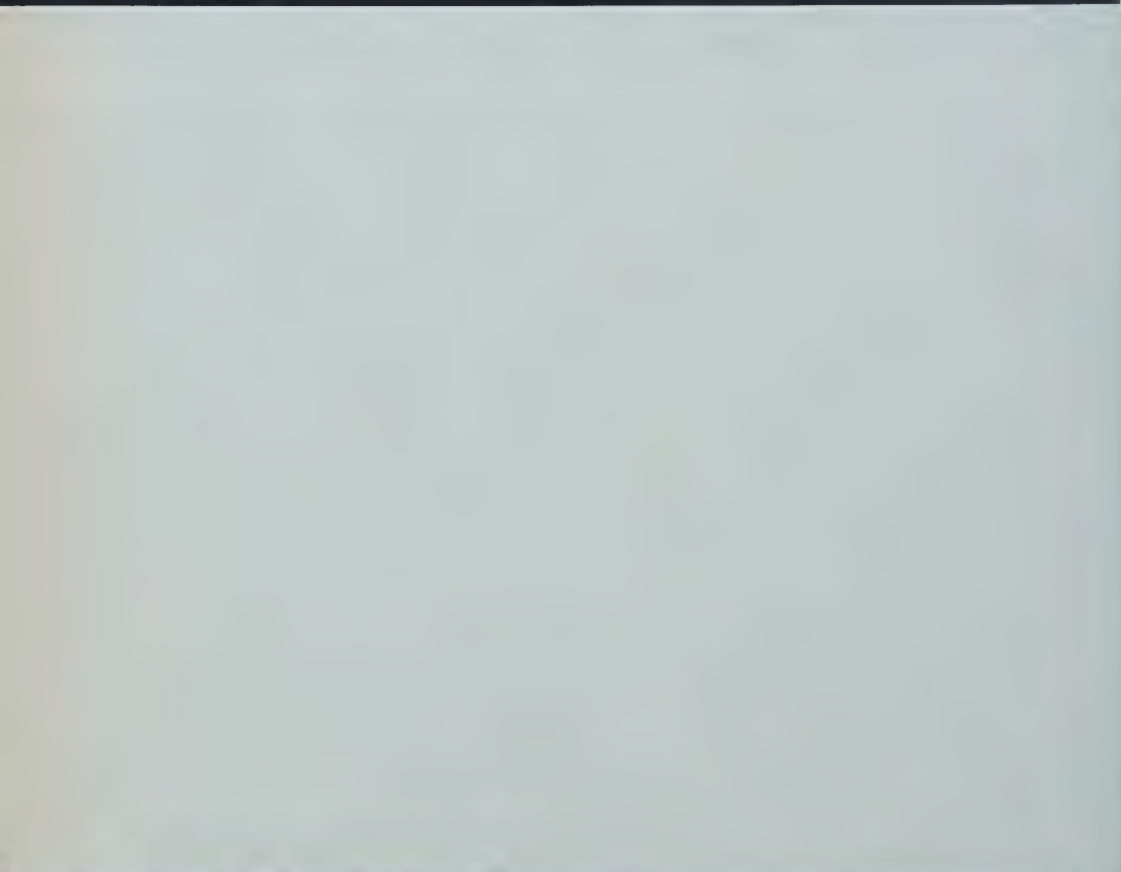
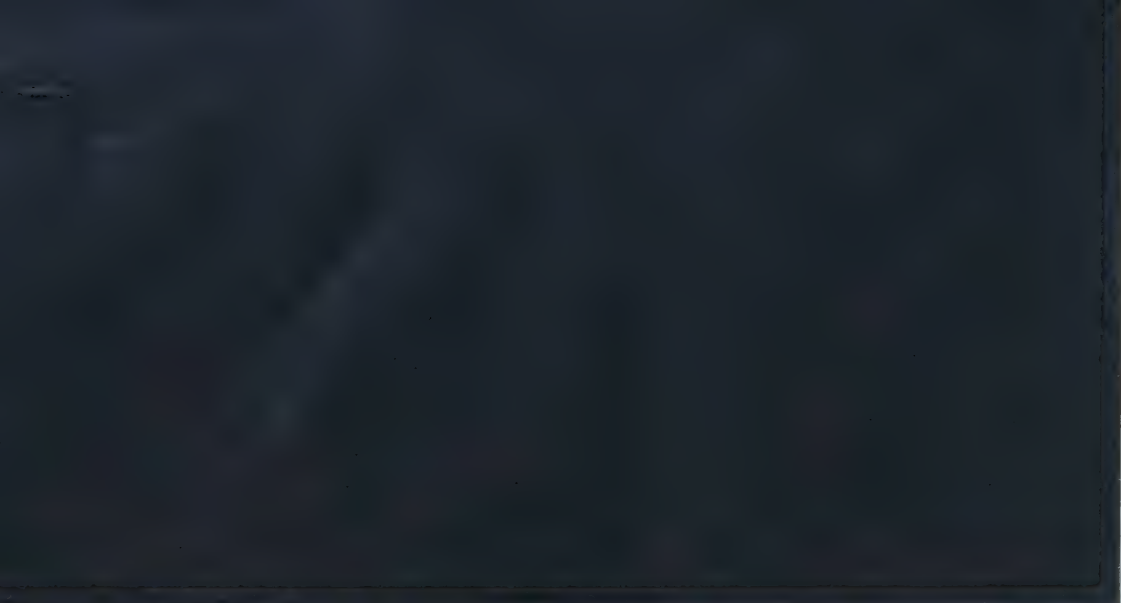
$$\pi = N_\pi (x + iy) \exp(-\alpha r/2) \quad (3.1b)$$

$$\bar{\sigma} = N_\pi (x - iy) \exp(-\alpha r/2) \quad (3.1c)$$

where N_σ and N_π are normalization constants, α is an effective charge of $2p$ -orbitals, x , y , and z are Cartesian coordinates, the origin of the coordinates is an atomic nucleus, and r is a distance of (x, y, z) from the origin.

The reduction of integrals Q , J and K was explained by Kotani, Amemiya and Simose⁶⁾, and numerical tables which were necessary for the evaluation of these integrals were given by them. We compute the numerical values of Q ,

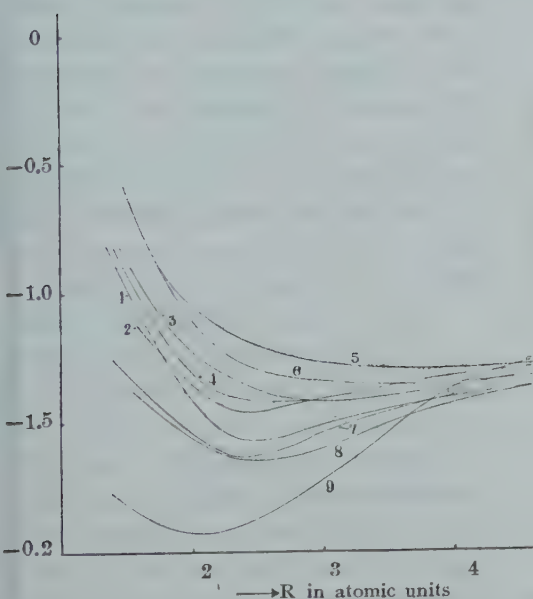




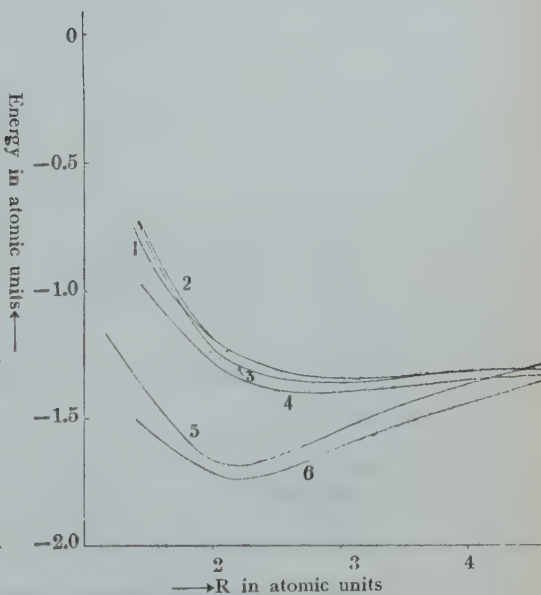
J and K by making use of their tables.* The result of evaluating the adiabatic potentials according to the general formulas given by (2.19)—(2.23) are shown

Table V Binding Energy

State	Binding energy in electron volts	Nuclear distance in atomic units
$(\sigma\pi, \sigma\pi) \ ^1\Delta_g$	6.48	2.53
$(\sigma\pi, \sigma\pi^*) \ ^1,^3\Sigma_g^+$	7.13	2.58
$(\sigma\pi, \sigma\pi) \ ^3\Delta_u$	8.89	2.50
$(\sigma\pi, \sigma\pi^*) \ ^1\Sigma_u^-$	9.49	2.48
$(\sigma^2, \sigma\pi) \ ^1,^3\Pi_g$	11.55	2.23
$(\sigma^2, \sigma\pi) \ ^1,^3\Pi_u$	13.01	2.15
$(\sigma^2, \sigma^2) \ ^1\Sigma_g^+$	19.42	1.98

Fig. I Adiabatic Potentials for Attractive Σ and Δ States.

- 1) $(\pi\pi^*, \pi\pi^*) \ ^1\Sigma_g^+$
- 2) $(\sigma\pi, \sigma\pi) \ ^3\Delta_u, (\sigma\pi, \sigma\pi^*) \ ^3\Sigma_u^+, \ ^1\Sigma_u^-$
- 3) $(\sigma\pi, \sigma\pi) \ ^5\Delta_g, \ ^3\Delta_g, \ ^1\Delta_g \ (\sigma\pi, \sigma\pi^*) \ ^5\Sigma_g^+, \ ^3\Sigma_g^+, \ ^3\Sigma_g^-, \ ^1\Sigma_g^+$
- 4) $(\sigma^2, \pi^2) \ ^1\Delta_g, \ (\sigma^2, \pi\pi^*) \ ^3\Sigma_g^-, \ ^1\Sigma_g^+$
- 5) $(\sigma^2, \pi^2) \ ^1\Delta_u, \ (\sigma^2, \pi\pi^*) \ ^3\Sigma_u^-, \ ^1\Sigma_u^+$
- 6) $(\sigma\pi, \sigma\pi) \ ^1\Delta_g, \ (\sigma\pi, \sigma\pi^*) \ ^3\Sigma_g^-, \ ^1\Sigma_g^+$
- 7) $(\sigma\pi, \sigma\pi) \ ^3\Delta_u$
- 8) $(\sigma\pi, \sigma\pi^*) \ ^5\Sigma_u^-, \ ^3\Sigma_u^+, \ ^3\Sigma_u^-, \ ^1\Sigma_u^-$
- 9) $(\sigma^2, \sigma^2) \ ^1\Sigma_g^+$

Fig. II Adiabatic Potentials for Attractive Π States.

- 1) $(\pi^2, \sigma\pi^*) \ ^3\Pi_u, \ ^1\Pi_u$
- 2) $(\pi^2, \sigma\pi^*) \ ^3\Pi_g, \ ^1\Pi_g$
- 3) $(\sigma\pi, \pi\pi^*) \ ^3\Pi_u, \ ^1\Pi_u$
- 4) $(\sigma\pi, \pi\pi^*) \ ^3\Pi_g, \ ^1\Pi_g$
- 5) $(\sigma^2, \sigma\pi) \ ^3\Pi_g, \ ^1\Pi_g$
- 6) $(\sigma^2, \sigma\pi) \ ^3\Pi_u, \ ^1\Pi_u$

* After we finished the evaluation we became aware of that our result for $S_{\psi\psi}, J_{\psi\psi}, K_{\psi\psi}, C_{\psi\psi\psi\psi}, C_{\psi\psi\psi\psi}, C_{\psi\psi\psi\psi}, D_{\psi\psi\psi\psi}, D_{\psi\psi\psi\psi}, L_{\psi\psi\psi\psi}, L_{\psi\psi\psi\psi},$ and $L_{\psi\psi\psi\psi}$ of Kotani, Amemiya and Simose is in agreement with the result of Bartlett and Furry⁷, where φ or ψ is one of σ, π and π^* .

in Table IV where $\alpha=2$ is assumed. The potential curves are plotted in Fig. I, Fig. II and Fig. III. The binding energies of the molecule and corresponding equilibrium distances between two nuclei are evaluated from these curves for attractive states. The result for several lower states is shown in Table V.

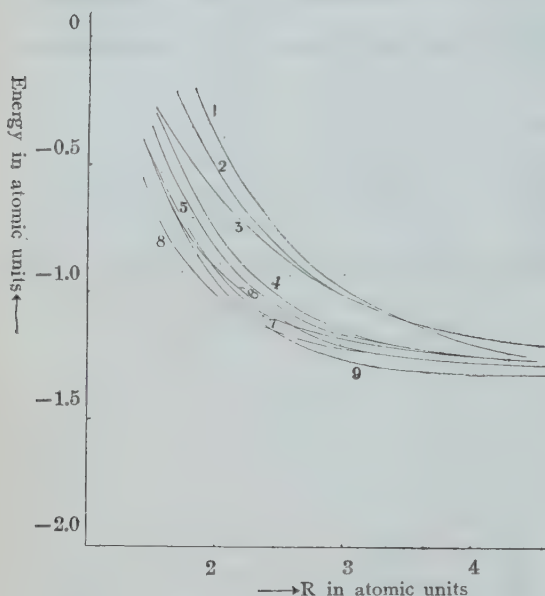


Fig. III. Adiabatic Potentials for Repulsive States.

- 1) $(\pi^2, \pi^2) {}^1\Gamma_g$
- 2) $(\pi^2, \pi\bar{\sigma}) {}^3\Delta_g, {}^1\Delta_g$
- 3) $(\pi\bar{\sigma}, \pi\bar{\sigma}) {}^5\Sigma_g^+, {}^3\Sigma_g^-, {}^1\Sigma_g^+$
- 4) $(\pi^2, \pi\bar{\sigma}) {}^3\Delta_u, {}^1\Delta_u$
- 5) $(\pi\bar{\sigma}, \pi\bar{\sigma}) {}^3\Sigma_u^+, {}^3\Sigma_u^-$
- 6) $(\pi^2, \bar{\sigma}^2) {}^1\Sigma_u^-$
- 7) $(\pi\sigma, \pi\bar{\sigma}) {}^5\Pi_g, {}^3\Pi_g, {}^1\Pi_g$
- 8) $(\pi^2, \sigma\pi) {}^3\Phi_g, {}^1\Phi_g$
- 9) $(\pi^2, \bar{\sigma}^2) {}^1\Sigma_g^+$
- 9) $(\pi^2, \sigma\pi) {}^3\Phi_u, {}^1\Phi_u$ $(\pi\sigma, \pi\bar{\sigma}) {}^5\Pi_u, {}^3\Pi_u, {}^1\Pi_u$

shells this phenomenon may become more remarkable because α becomes larger. A contribution to Q from interactions between σ 's is largest among those between various orbitals. Thus the deepest state is $(\sigma^2, \sigma^2) {}^1\Sigma_g^+$ in contradiction to the Pauling theory, as has been mentioned in the introduction. We can not compare this result with experiment⁸⁾ because the present calculation does not take into account 1s- and 2s-shells.

The present result that the deepest state is singlet is not in agreement with Hückel and Mulliken's theory¹⁴⁾ of molecular orbitals which concludes that the deepest state is triplet. According to Heitler and Poeschel¹⁵⁾ this disagreement can be removed by taking into account 2s-electrons. This case will be discussed in Part II.

In our approximation there is no interaction between different configurations except for an interaction between $(\sigma\pi, \sigma\bar{\sigma}) {}^1\Sigma_g^+$ and $(\sigma^2, \pi\bar{\sigma}) {}^1\Sigma_g^+$. The result of our numerical computation shows that the latter interaction is negligible. The neglect of integrals which vanish when H is assumed to be equal to unity may not give rise to any serious change in the result. This may be seen from the following example. If we take into account all terms, the binding energy for $(\sigma^2, \sigma\pi) {}^3\Pi_u$ becomes 13.15 ev, the corresponding nuclear distance is 2.25 atomic units, and those for $(\sigma^2, \sigma\pi) {}^1\Pi_u$ are approximately the same as the values given in Table V.

The order of potential heights are mainly determined by Q , while J and K give a small contribution. This is due to that α is large, and J and K decrease exponentially when αR increases whereas Q decreases as $1/\alpha R$, where R is a distance between two nuclei. If we take into account inner

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Effects of Nuclear Motion on the Fine and the Hyperfine Structure of Hydrogen, II

Takehiko ISHIDZU

*Department of Dynamics, 1st Faculty of Engineering,
Tokyo University, Tokyo*

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§ 5. Solution by expansion in power series in α^2 for general values of F

5.1. Expansion of Eqs. (7') in power series in α^2

Eqs. (7') may be solved for general values of F by expansion in power series in α^2 successively. Comparison of each term in each equation of (7') enables us the following expansions for the eigenvalue and the components of the eigenfunction^{*}):

$$\left. \begin{aligned} \varepsilon &= 1 + \alpha^2 \varepsilon^{[0]} + \alpha^4 \varepsilon^{[1]} + \dots, \\ \varphi &= \varphi^{[0]} + \alpha^2 \varphi^{[1]} + \dots, \\ \chi &= \alpha (\chi^{[0]} + \alpha^2 \chi^{[1]} + \dots), \\ \bar{\varphi} &= \beta \alpha (\bar{\varphi}^{[0]} + \alpha^2 \bar{\varphi}^{[1]} + \dots), \\ \bar{\chi} &= \beta \alpha^2 (\bar{\chi}^{[0]} + \alpha^2 \bar{\chi}^{[1]} + \dots). \end{aligned} \right\} \quad (18)$$

Substituting (18) in (7') and equating the coefficients of each power of α^2 to zero, one has the following set of equations with the notation: $\Psi^{[i]} = (\varphi^{[i]}, \chi^{[i]}, \bar{\varphi}^{[i]}, \bar{\chi}^{[i]})$ ($i=0, 1, \dots$)

$$H^{[0]} \Psi^{[0]} = 0, \quad (19a)$$

$$H^{[0]} \Psi^{[1]} + H^{[1]} \Psi^{[0]} = 0, \quad \text{and so on,} \quad (19b)$$

where

$$H^{[0]} = \begin{pmatrix} -\varepsilon^{[0]} + v & -(\sigma_e \mathcal{F}) & \beta(\sigma_M \mathcal{F}) & 0 \\ (\sigma_e \mathcal{F}) & -2 & 0 & 0 \\ -(\sigma_M \mathcal{F}) & 0 & -2 & 0 \\ -uS & -(\sigma_M \mathcal{F}) & \beta(\sigma_e \mathcal{F}) & -2(1+\beta) \end{pmatrix}, \quad (20a)$$

* Square brackets in the upper suffices are used in order to distinguish them from the round brackets used in the β -expansion before.

$$H^{[1]} = \begin{pmatrix} -\epsilon^{[1]} & 0 & 0 & -\beta u S \\ 0 & -\epsilon^{[0]} + v & \beta u S & \beta(\sigma_M \mathcal{V}) \\ 0 & u S & \beta(-\epsilon^{[0]} + v) & -\beta(\sigma_M \mathcal{V}) \\ 0 & 0 & 0 & \beta(-\epsilon^{[0]} + v) \end{pmatrix}, \quad (20b)$$

and for $i=2, 3, \dots$

$$H^{[i]} = \begin{pmatrix} -\epsilon^{[i]} & 0 & 0 & 0 \\ 0 & -\epsilon^{[i-1]} & 0 & 0 \\ 0 & 0 & -\beta \epsilon^{[i-1]} & 0 \\ 0 & 0 & 0 & -\beta \epsilon^{[i-1]} \end{pmatrix}.$$

Eq. (19a) yields

$$\left. \begin{aligned} & \{-(1+\beta)(\mathcal{V}^2/2) + (-\epsilon^{[0]} + v)\} \varphi^{[0]} = 0, \\ & 2\chi^{[0]} = (\sigma_M \mathcal{V}) \varphi^{[0]}, \\ & 2\bar{\varphi}^{[0]} = -(\sigma_M \mathcal{V}) \varphi^{[0]}, \\ & 2\bar{\chi}^{[0]} = -\{uS/(1+\beta) + (1/2)(\sigma_M \mathcal{V})(\sigma_M \mathcal{V})\} \varphi^{[0]}, \end{aligned} \right\} \quad (21)$$

where the second and third equations have been substituted in the remaining two, with the relation $(\sigma_M \mathcal{V})^2 = (\sigma_M \mathcal{V})^2 = \mathcal{V}^2$. The first of Eqs. (21) is equivalent to the Schrodinger wave equation with reduced mass $m/(1+\beta)$, or, alternatively, putting $z=1/(1+\beta)$, to the same equation with nuclear charge ze of which eigenvalue is $z\epsilon_n^{[0]}$; that is, it may be satisfied by the Schrodinger wave function with nuclear charge ze , $\varphi_{n,i,z}^{[0]}$, and then

$$\epsilon_n^{[0]} = -z/2n^2 = -(1/2n^2)(1+\beta)^{-1}, \quad (22)$$

showing just the Bohr's original reduced mass correction to the Rydberg energy $-a^2/2n^2$.

5.2. Energies of order a^4

Now we proceed to the next approximation. From Eq. (19b) one has

$$\left. \begin{aligned} & \epsilon^{[1]} \varphi^{[0]} = (-\epsilon^{[0]} + v) \varphi^{[1]} - (\sigma_M \mathcal{V}) \chi^{[1]} + \beta(\sigma_M \mathcal{V}) \varphi^{[1]} - \beta u S \bar{\chi}^{[0]}, \\ & 2\chi^{[1]} = (\sigma_M \mathcal{V}) \varphi^{[1]} + (-\epsilon^{[0]} + v) \chi^{[0]} + \beta u S \bar{\varphi}^{[0]} + \beta(\sigma_M \mathcal{V}) \bar{\chi}^{[0]}, \\ & 2\bar{\varphi}^{[1]} = -(\sigma_M \mathcal{V}) \varphi^{[1]} + u S \chi^{[0]} + \beta(-\epsilon^{[0]} + v) \bar{\varphi}^{[0]} - \beta(\sigma_M \mathcal{V}) \bar{\chi}^{[0]}. \end{aligned} \right\} \quad (23)$$

Substitution of the latter two with the latter three of Eqs. (21) in the first produces

$$\epsilon^{[1]} \varphi^{[0]} = \{-(\mathcal{V}^2/2z) + (-\epsilon^{[0]} + v)\} \varphi^{[1]} + H' \varphi^{[0]}, \quad (24)$$

where

$$\begin{aligned}
H' = & -(1/4) \{ (\sigma_e \nabla) (-\epsilon^{[0]} + v) (\sigma_e \nabla) + \beta^2 (\sigma_M \nabla) (-\epsilon^{[0]} + v) (\sigma_M \nabla) \} \\
& + (\beta/8\pi) \nabla^4 + (\beta/4) \{ (\sigma_e \nabla) u S (\sigma_M \nabla) + (\sigma_M \nabla) u S (\sigma_e \nabla) \\
& + (\sigma_e \nabla) (\sigma_M \nabla) u S + u S (\sigma_e \nabla) (\sigma_M \nabla) \} + (\beta \pi/2) u^2 S^2.
\end{aligned} \quad (24a)$$

Eq. (24) is a typical equation in usual perturbation theory and the energy $\epsilon^{[0]}$ of order u^4 is therefore given by the expectation value of the operator H' taken for the zeroth order eigenfunction $\varphi_{n,l,z}^{[0]}$, except the last term which, being proportional to $u^2 S^2$, shall be omitted hereafter as before.

For reduction of H' the following relations are useful:

$$\left. \begin{aligned}
(a\sigma)(\sigma b) &= (ab) + i[\sigma[a \times b]], \\
(\sigma \nabla) U(\rho) &= (dU/\rho d\rho)(\sigma \rho) + U(\rho)(\sigma \nabla), \\
(\sigma_e \mathbf{l})(\sigma_M \rho) &= (\sigma_M \rho)(\sigma_e \mathbf{l}) + i(\rho[\sigma_e \times \sigma_M]); \\
[[a \times b] \times c] &= b(ac) - a(bc), \\
[a \times [b \times c]] &= b(ac) - (ab)c,
\end{aligned} \right\} \quad (25)$$

where \mathbf{a} and \mathbf{b} in the first equation are any two commuting vectors with σ and those in the last two commute with each other; \mathbf{l} is defined by $\mathbf{l} = -i[\rho \times \nabla] = i[\nabla \times \rho]$. After somewhat lengthy reduction using these, H' may be transformed into

$$\begin{aligned}
H' = & -\frac{1+\beta^2}{4} \left\{ (-\epsilon^{[0]} + v) \nabla^2 + \frac{1}{\rho} \frac{dv}{d\rho} (\rho \nabla) \right\} + \frac{1}{4\rho} \frac{dv}{d\rho} \left\{ (\sigma_e \mathbf{l}) + \beta^2 (\sigma_M \mathbf{l}) \right\} \\
& + \frac{\beta}{4} \left\{ \frac{1}{2\pi} \nabla^4 + 2u \nabla^2 + \left(\frac{d^2 u}{d\rho^2} + \frac{3}{\rho} \frac{du}{d\rho} + \frac{u}{\rho^2} \right) + \left(\frac{4}{\rho} \frac{du}{d\rho} + \frac{2u}{\rho^2} \right) (\rho \nabla) \right. \\
& + \frac{2u}{\rho^2} (\rho \nabla)^2 - \left(\frac{1}{\rho} \frac{du}{d\rho} - \frac{u}{\rho^2} \right) (\sigma_e + \sigma_M, \mathbf{l}) - \frac{1}{2} \frac{d^2 u}{d\rho^2} (\sigma_e \sigma_M) \\
& \left. + \frac{1}{2} \left(\frac{d^2 u}{d\rho^2} - \frac{2}{\rho} \frac{du}{d\rho} + \frac{2u}{\rho^2} \right) \frac{1}{\rho^2} (\sigma_e \rho) (\sigma_M \rho) \right\}.
\end{aligned} \quad (24b)$$

Remembering that $-v = u = 1/\rho$ and $\nabla^4/2\pi = \nabla^2(-\epsilon^{[0]} + v) = (-\epsilon^{[0]} + v) \nabla^2 + 2\rho^{-3}(\rho \nabla)$ for our zeroth order functions $\varphi_{n,l,z}^{[0]}$, and collecting terms, we obtain finally

$$H' = H'_1 + H'_2 + H'_3 + H'_4$$

with

$$\left. \begin{aligned}
H'_1 &= -(1/4) (1+\beta)^2 [(-\epsilon^{[0]} + v) \nabla^2 + \rho^{-3} \{ (\rho \nabla) - (\sigma_e \mathbf{l}) \}], \\
H'_2 &= (\beta/4) [3(-\epsilon^{[0]} + v) \nabla^2 + 2\rho^{-1} \nabla^2 + 2\rho^{-3} \{ (\rho \nabla)^2 + (\rho \nabla) \}], \\
H'_3 &= (\beta/4) \rho^{-3} \{ 2(\sigma_M \mathbf{l}) - (\sigma_e \sigma_M) + 3\rho^{-2} (\sigma_e \rho) (\sigma_M \rho) \}, \\
H'_4 &= (\beta^2/4) \rho^{-3} (\sigma_M - \sigma_e, \mathbf{l}).
\end{aligned} \right\} \quad (24c)$$

The above result proves to be equivalent to that of Bechert and Meixner¹⁾; that is, H' in (24c) is identical with H except H_0 in their Eq. (2). For computing the expectation value of H they have neglected all terms with the second and higher powers of β , but, since the reduction in this section includes no approximation with respect to the powers in β , the dependence on β of terms in Eqs. (24c) should be considered as the correct one (see also § 6).

Computation of the expectation values of (24c) is straightforward by using the relations (9), (10), (15c), (15d), (22), and besides

$$\langle \rho^{-3} \rangle_{n,l,z} = z^3 / n^3 l(l + \frac{1}{2})(l + 1), \quad (26a)$$

$$\langle -\rho^{-3}(\rho \nabla) \rangle_{n,l,z} = \langle -\rho^{-2}(d/d\rho) \rangle_{n,l,z} = (2z^3/n^3) \delta_{l,0}, \quad (26b)$$

$$\text{and} \quad \rho^{-2}\{(\rho \nabla)^2 + (\rho \nabla)\} = \nabla^2 + l(l + 1)/\rho^2, \quad (26c)$$

where $\delta_{l,0}$ is the Kronecker's symbol. Since for the states $l=0$, however, they contain the diverging integral $\langle \rho^{-3} \rangle$ except $\langle H'_1 \rangle$ and hence special treatment is needed in this case, so we shall first evaluate these values in cases where l does not vanish.

H'_1 corresponds to H_a given by Eq. (5) minus H_0 of Bechert and Meixner, giving the fine structure energy with simple reduced mass correction:

$$\langle H'_1 \rangle = \frac{1}{2n^3} \left(\frac{3}{4n} - \frac{1}{k} \right) \bigg| (1 + \beta). \quad (27a)$$

H'_2 corresponds to H_b of Bechert and Meixner and reproduces the Darwin's correction term (1) exactly:

$$\langle H'_2 \rangle = -(\beta/8n^4)(1 + \beta)^{-3}. \quad (27b)$$

H'_3 and H'_4 which depend on σ_M give the hyperfine structure energies. H'_3 corresponds to H_c in Bechert and Meixner exactly, while H'_4 neglected by them corresponds to the one called the "Thomas-type" term by Breit and others^{2),3)} and is considered to express the effect of the proton's acceleration on the hyperfine structure.

Now, as is seen from the consideration given in § 3.4, the correct zeroth order eigenfunctions for these energies are given by $\Psi_1^{[0]} = \varphi_{+}^{[0]} + \varphi_{-}^{[0]}$ and $\Psi_2^{[0]} = \varphi_{-+}^{[0]} + \varphi_{+-}^{[0]}$. From a similar calculation to (15b) it is proved that $\int_0^\infty \rho^2 d\rho \cdot \rho^{-3} \varphi_{n,+ +,z} \varphi_{n,- -,z} = \int_0^\infty d\rho \cdot \rho^{-1} \varphi_{n,l,z}^{[0]} \varphi_{n,l \pm 2,z}^{[0]}$ vanishes, while $\int_0^\infty \rho^2 d\rho \cdot \rho^{-3} \varphi_{n,- +,z}^{[0]} \varphi_{n,+ -,z}^{[0]}$ is identical with the diagonal element $\int_0^\infty \rho^2 d\rho \cdot \rho^{-3} (\varphi_{n,l,z}^{[0]})^2$. Moreover from (10a) and (10b) $\langle H'_4 \rangle$ vanishes for $\Psi_1^{[0]}$ completely, while it gives for $\Psi_2^{[0]}$ exactly the same matrix as that of $\langle H'_3 \rangle$ except the factor β^2 . Consequently the matrices of the hyperfine splittings for a given F are given by, using (9c), (9d), (10b) and (26a),

$$\left. \begin{aligned}
 &\text{for } l=F+1 (l^{++} j^{+}), & -\frac{2}{n^3} \frac{1}{(2F+1)(2F+3)} \frac{\beta a^4}{(1+\beta)^3}, \\
 &\text{for } l=F-1 (l^{--} j^{-}), & \frac{2}{n^3} \frac{1}{(2F-1)(2F+1)} \frac{\beta a^4}{(1+\beta)^3}, \\
 &\text{for } l=F & \begin{aligned} &l^{++} j^{+} & l^{+-} j^{-} \\ &\begin{pmatrix} -2 & 1/\sqrt{F(F+1)} \\ 1/\sqrt{F(F+1)} & 2 \end{pmatrix} \cdot \frac{1}{n^3} \frac{1}{(2F+1)^2} \frac{\beta a^4}{(1+\beta)^2} \end{aligned}
 \end{aligned} \right\} \quad (28)$$

where the factor $(1+\beta)^3$ in denominators arising from the mean value $\langle \rho^{-3} \rangle$ is replaced by $(1+\beta)^2$ for $\Psi_s^{[50]}$ because of the presence of non-vanishing $\langle H'_4 \rangle$. Apart from these factors the formulas (28) agree exactly with the expressions (A17) for the hyperfine splittings with fixed nucleus shown in Appendix A.2. We may conclude, therefore, that the correction factor due to nuclear motion to the hyperfine structure is given by $(1+\beta)^{-3}$ for the two components $j=l+\frac{1}{2}$, $F=l+1$ and $j=l-\frac{1}{2}$, $F=l-1$ as Breit and Meyerott have already shown, but for the other two components $j=l\pm\frac{1}{2}$, $F=l$ by a different factor $(1+\beta)^{-2}$, provided that l does not vanish. Owing to this asymmetry in the correction which is called the "special mass effect" against the "regular mass effect" by Breit and others³⁾, the total hyperfine separation of the levels with definite values of l and j will be affected by a somewhat larger factor than $(1+\beta)^{-3}$ or $1-3\beta$; e.g. of the levels $n p_{1/2}$ by $(1-11\beta/4)$ instead of $1-3\beta$. For s levels ($l=0$), however, it will be proved in the next subsection that the term corresponding to H'_4 vanishes identically, so the correction factor to the hyperfine separation is given always by $(1+\beta)^{-3}$ as has been already established.

5.3. More exact calculation for s terms

As was mentioned above, the expectation value $\langle H' \rangle$ for s terms contains the diverging integral $\langle \rho^{-3} \rangle$ except $\langle H'_1 \rangle$, which has not appeared in the β -expansion procedure before. Coefficients of this integral do not vanish only for the last two terms of H'_3 , because other terms are proportional to \mathbf{L} . These two coefficients, however, cancel each other as is easily seen from the matrix elements of (9c) and (9d) for both cases $F=0$ and $F=1$, leaving an indefinite expression $0 \times \infty$. This fact suggests the necessity of more rigorous calculation for s terms, the approximation procedure by a^2 -expansion being insufficient in this case. Such rigorous calculation has been treated by Breit and Doermann⁴⁾ for a fixed nucleus, but not yet with moving nucleus. We shall give here such treatment briefly.

In order to obtain a rigorous expression for energy, elimination of the three smaller components χ , $\bar{\varphi}$ and $\bar{\chi}$ from Eqs. (7') must be carried out exactly; that is, we have not to neglect terms proportional to a^2 against terms of order unity

in course of elimination. But since our purpose is only to prevent the divergency at $\rho=0$ of $\langle \rho^{-3} \rangle$, it is sufficient to retain only terms proportional to $a^2 v$ which tends to infinity at $\rho=0$ with terms of order unity and that only in *one* factor of every term in the expression for energy. Elimination of the three functions may be performed quite algebraically, provided the non-commutability of their coefficients, giving rise to up to the order a^4

$$(H_0'' + H_1'' + H_2'' + H_3'' + H_4'')\varphi = 0$$

with

$$\left. \begin{aligned} H_0'' &= a^2(-\epsilon^{[0]} + v) - a^4 \epsilon^{[1]}, \\ H_1'' &= -\frac{a^2}{2} \left(\frac{1}{1-U} + \frac{\beta}{1-\beta U} \right) \mathcal{V}^2, \\ H_2'' &= -\frac{a^2}{2} \left\{ \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1}{1-U} + \frac{\beta}{1-\beta U} \right) (\rho \mathcal{V}) \right. \\ &\quad \left. - \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1}{1-U} \right) (\sigma_M \mathcal{L}) - \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{\beta}{1-\beta U} \right) (\sigma_M \mathcal{L}) \right\}, \\ H_3'' &= \frac{\beta a^4}{4} \left[\frac{1}{2} \left(\frac{1}{1-U} + \frac{\beta}{1-\beta U} \right) \mathcal{V}^4 + \frac{1}{1-\beta U} u S(\sigma_M \mathcal{V}) (\sigma_M \mathcal{V}) \right. \\ &\quad \left. + \frac{1}{1-U} (\sigma_M \mathcal{V}) \{ z(\sigma_M \mathcal{V}) u S + u S(\sigma_M \mathcal{V}) \} \right. \\ &\quad \left. + \frac{1}{1-\beta U} (\sigma_M \mathcal{V}) \{ \beta z(\sigma_M \mathcal{V}) u S + u S(\sigma_M \mathcal{V}) \} \right], \\ H_4'' &= \frac{\beta a^4}{4} \left[\frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1}{1-U} \right) (\sigma_M \rho) \left\{ \frac{1}{2} \mathcal{V}^2 (\sigma_M \mathcal{V}) + z(\sigma_M \mathcal{V}) u S + \right. \right. \\ &\quad \left. \left. u S(\sigma_M \mathcal{V}) \right\} + \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1}{1-\beta U} \right) (\sigma_M \rho) \left\{ \frac{\beta}{2} \mathcal{V}^2 (\sigma_M \mathcal{V}) + \right. \right. \\ &\quad \left. \left. \beta z(\sigma_M \mathcal{V}) u S + u S(\sigma_M \mathcal{V}) \right\} \right], \end{aligned} \right\} \quad (29)$$

where

$$U = a^2(-\epsilon^{[0]} + v)/2.$$

Eqs. (29) hold for general values of l , but for $l \approx 0$ no diverging integral appears, so that expansion of denominators such as $1-U$ or $1-\beta U$ may be carried out straightforwardly and yields, neglecting terms of higher order than a^4 , an expression identical with H' in (24c) except the energies of order a^2 .

For $l=0$ too, because of

$$H_1'' = -(a^2 \mathcal{V}^2 / 2z) - (a^4 / 4)(1 + \beta^2)(-\epsilon^{[0]} + v) \mathcal{V}^2$$

to the order a^4 , the part of order a^4 of $H_1'' + H_2'' + H_3''$ agrees with the expression of H' by (24c) in which the terms dependent on σ_M may be put to zero, because the convergency of integrals is assured in the former. It should be noted, however, that $\langle H_2' \rangle$ gives for $l=0$ from (26c)

$$\langle H_2' \rangle = -(\beta/8n^4)(1+\beta)^{-3} - (\beta/n^3)(1+\beta)^{-3}\delta_{l,0} \quad (27b')$$

instead of (27b). The second term will be shown soon below to be cancelled out by a term arising from $\langle H_4'' \rangle$. Reducing H_4'' to a similar form to Eqs. (24c) the expectation value of H_4'' for $l=0$ becomes to the order a^4

$$\frac{\beta \varepsilon a^4}{4} \left\langle \frac{1}{\rho^2} \frac{d}{d\rho} \left(\frac{1}{1-U} + \frac{\beta}{1-\beta U} \right) \{1 + (\sigma_e \sigma_M) - (\sigma_e \rho)(\sigma_M \rho) \rho^{-2}\} \right\rangle,$$

and, using

$$O(a^0) \cdot \left\langle \frac{1}{\rho^2} \frac{d}{d\rho} \left(\frac{1}{1-U} \right) \right\rangle = \{\varphi_{n,l,\varepsilon}^{[0]}(0)^2\} = \frac{4}{n^3(1+\beta)^3} \delta_{l,0}, \quad (30)$$

becomes

$$\langle H_4'' \rangle_{l=0} = (\beta a^4/n^3)(1+\beta)^{-3} \langle 1 + (\sigma_e \sigma_M) - (\sigma_e \rho)(\sigma_M \rho)/\rho^2 \rangle. \quad (31)$$

The first term of Eq. (31) cancels just the second term of (27b'); the second and the third terms give from (9c) and (9d)

$$\left. \begin{aligned} (2/3n^3) \cdot \beta a^4 / (1+\beta)^3 & \quad \text{for } F=1, \\ - (2/n^3) \cdot \beta a^4 / (1+\beta)^3 & \quad \text{for } F=0, \end{aligned} \right\} \quad (32)$$

which agree with the general formulas given by (A17) in Appendix, except the correction factor $(1+\beta)^{-3}$. In this case, as was mentioned above, terms corresponding to H_4' vanish, so the correction factor has proved to be given by $(1+\beta)^{-3}$ for $l^+=0$, $j^+=1/2$ ($F=0$) too, exceptionally unlike the general cases with non-vanishing l .

§ 6. Concluding remarks

The Breit's interaction Y given by (2a) is a quantum-mechanical equivalent to the first order approximation of the retarded potential between two charged particles, being of relative order of magnitude $v_1 v_2 / c^2$ compared with the Coulomb potential, where v_1 and v_2 are velocities of the two charged particles. Since the expectation value of the Coulomb potential between proton and electron is of order $mc^2 a^2$ and that of $v_1 v_2 / c^2 = p^2 / m M c^2 = (m/M)(p^2/m)/mc^2$ in our case is of order βa^2 , the expectation value of Y may be considered to be of order βa^4 in units of mc^2 . The exact retarded potential contains terms of the higher powers of βa^2 than Y relative to $mc^2 a^2$, so it would be meaningless to calculate the higher order energies than $\varepsilon^{[1]}$ in § 5. But since no approximation was made regarding the entrance of β , the dependence on β of our results in § 5 may be considered to be accurate.

Assuming that our results are accurate to every power of β , the energy proportional to β^2 , $\langle H_4' \rangle$, produces the asymmetric correction factor for the hyperfine splittings except of s terms. Experimental proof for our result, however, would be very difficult at present, for the total hyperfine separation of the $2p_{1/2}$ level having the largest separation except s levels amounts to only $(1/9)mc^2\beta\alpha^4 \approx 21\text{Mc/s}$, while difference between our result and the usual one with symmetric factor $(1+\beta)^{-3}$ or $1-3\beta$ becomes a relative amount of about $\beta/4 \approx 10^{-4}$, as was mentioned towards the end of § 5.2. Moreover, our treatment is restricted exclusively in the case of a proton obeying the Dirac equation. We can not see, therefore, whether the effect of nuclear motion on the energy due to the "Pauli part" of the proton's moment* gives the same asymmetric correction factor or not.

Finally we shall give a brief account for terms arising from a literal iteration of Y which have been omitted throughout in our calculations⁵⁾. The expectation value of Y^2 ought to be of order $(\beta\alpha^4)^2$ in units of mc^2 as is seen from what above mentioned, but because of the peculiar condition $\alpha_c^2 = \alpha_m^2 = 1$, it gives formally energy of order $\beta\alpha^4$ in the same units as shown in (24a). This fact, it seems to me, would mean that the expression of Y could not represent the first order approximation to retarded potentials correctly, and that the reason why the literal iteration of Y yields a result contradictory with experiment, would lie in this point. At any rate, more complete consideration concerning this point would be necessary and it is to be noted that Y. Nambu has recently discussed this problem from the field-theoretical ground⁶⁾.

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Appendix

Derivation of the relativistic hyperfine splittings formula for hydrogenlike atoms with fixed nucleus

(Read May 21, 1948)

A.1. Matrices of vector product operators

We shall say here that a vector operator \mathbf{T} belongs to \mathbf{J} , if its components satisfy the same commutation rule with respect to \mathbf{J} as the components of \mathbf{J} themselves satisfy, where \mathbf{J} is any angular momentum operator. Thus we have in dyadic form⁷⁾

$$[\mathbf{T}, \mathbf{J}] = -i \hbar \mathbf{T} \times \mathbf{J}, \quad (\text{A1})$$

* See Ref. (2), Section III.

where $[\quad]$ means the commutator, \times indicates the operation of forming the vector product and \mathbf{j} is the unit dyadic $\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j} + \mathbf{k}\mathbf{k}$. Now suppose that the vector resultant \mathbf{J} is composed from two commuting angular momenta \mathbf{J}_1 and \mathbf{J}_2 by $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$. We shall consider the problem of obtaining the matrices of any "volume product" of three vector operators $(\mathbf{P}\mathbf{Q}_1\mathbf{Q}_2) = (\mathbf{P} \cdot \mathbf{Q}_1 \times \mathbf{Q}_2)$ in a representation in which \mathbf{J}_1^2 , \mathbf{J}_2^2 , \mathbf{J}^2 and J_z are diagonal ($j_1 j_2 j m$ -scheme), when \mathbf{P} belongs to \mathbf{J}_2 , and \mathbf{Q}_1 and \mathbf{Q}_2 belong to \mathbf{J}_1 . Since the vector $\mathbf{Q} = \mathbf{Q}_1 \times \mathbf{Q}_2$ also belongs to \mathbf{J}_1 , it follows that

$$\begin{aligned} & (j_1 j_2 j m | (\mathbf{P}\mathbf{Q}_1\mathbf{Q}_2) | j_1' j_2' j m) \\ &= (j_1 j_2 j m | (\mathbf{P} \cdot \mathbf{Q}) | j_1' j_2' j m) \\ &= (j_2 | P | j_2') (j_1 | Q | j_1') W(j_1 j_2 j_1' j_2' j), \end{aligned}$$

where $(j_2 | P | j_2')$ and $(j_1 | Q | j_1')$ are the quantities defined in TAS (Ref. 7), § 9³, being independent of j and m . The quantities W are those given by TAS, § 12³, (2) and analogues of the W -functions introduced by Racah, differing from $W(j_1 j_2 j_1' j_2'; j_1)$ of Eq. (38) in his paper⁸) by factors including j_1 or j_2 alone. For the diagonal elements $j_1 = j_2'$, $j_2 = j_2'$, W becomes $W(j_1 j_2 j_1 j_2 j) = (j_1 j_2 j | (\mathbf{J}_1 \cdot \mathbf{J}_2) | j_1 j_2 j) = (1/2) \{ j(j+1) - j_1(j_1+1) - j_2(j_2+1) \}$. Herewith, it suffices to know the quantities $(j_1 | Q | j_1') = (j_1 | Q_1 \times Q_2 | j_1')$ in order to obtain the matrices of $(\mathbf{P}\mathbf{Q}_1\mathbf{Q}_2)$.

From the definition of $(j | Q | j')$ by TAS, § 9³, (11)

$$\left. \begin{aligned} (jm | Q_z | j+1 m) &= (j | Q | j+1) \sqrt{(j+1)^2 - m^2}, \\ (jm | Q_z | jm) &= (j | Q | j) m, \\ (jm | Q_z | j-1 m) &= (j | Q | j-1) \sqrt{j^2 - m^2}, \end{aligned} \right\} \quad (\text{A2})$$

where j_1 is replaced by j . Since $Q_z = Q_{1x}Q_{2y} - Q_{1y}Q_{2x} = (i/2)(Q_1^+Q_2^- - Q_1^-Q_2^+)$, denoting $Q^\pm = Q_x \pm iQ_y$, we have, using TAS, § 9³, (11)

$$\begin{aligned} & (jm | Q_z | j+1 m) \\ &= (i/2) \{ (jm | Q_1^+ | j m-1) (j m-1 | Q_2^- | j+1 m) - (jm | Q_1^- | j m+1) (j m+1 | Q_2^+ | j+1 m) \\ & \quad + (jm | Q_1^+ | j+1 m-1) (j+1 m-1 | Q_2^- | j+1 m) - (jm | Q_1^- | j+1 m+1) (j+1 m+1 | Q_2^+ | j+1 m) \} \\ &= (i/2) [(j | Q_1 | j) (j | Q_2 | j+1) \{ \sqrt{(j+m)(j-m+1)} (-)^{\sqrt{(j+m)(j+m+1)}} \\ & \quad - \sqrt{(j-m)(j+m+1)} \sqrt{(j-m)(j-m+1)} \} + (j | Q_1 | j+1) (j+1 | Q_2 | j+1) \\ & \quad \times \{ \sqrt{(j-m+1)(j-m+2)} \sqrt{(j-m+2)(j+m+1)} - (-)^{\sqrt{(j+m+1)(j+m+2)}} \\ & \quad \times \sqrt{(j+m+2)(j-m+1)} \}] \\ &= -i \{ j(j | Q_1 | j) (j | Q_2 | j+1) - (j+2) (j | Q_1 | j+1) (j+1 | Q_2 | j+1) \} \sqrt{(j+1)^2 - m^2}, \end{aligned}$$

and therefore, from the definition of (A2),

$$\begin{aligned} & (j | Q | j+1) = (j | Q_1 \times Q_2 | j+1) \\ &= i \{ (j+2) (j | Q_2 | j+1) (j+1 | Q_2 | j+1) - j(j | Q_1 | j) (j | Q_2 | j+1) \}. \end{aligned} \quad (\text{A3})$$

Similarly from the second and third equations of (A2), we have

$$\begin{aligned} & (j | Q_1 \times Q_2 | j) = i \{ -(2j+3) (j | Q_1 | j+1) (j+1 | Q_2 | j) \\ & \quad + (j | Q_1 | j) (j | Q_2 | j) + (2j-1) (j | Q_1 | j-1) (j-1 | Q_2 | j) \} \end{aligned} \quad (\text{A4})$$

and

$$\begin{aligned} & (j | Q_1 \times Q_2 | j-1) \\ &= i \{ (j+1) (j | Q_1 | j) (j | Q_2 | j-1) - (j-1) (j | Q_1 | j-1) (j-1 | Q_2 | j-1) \}. \end{aligned} \quad (\text{A5})$$

As an example of these formulas we can see at once for the orbital angular momentum $\mathbf{l} = -i\mathbf{r} \times \nabla$, where $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$,

$$(l | l | l) = 1, \quad (l | l | l \pm 1) = 0,$$

since

$$\left. \begin{aligned} (l:r:l+1) &= (l+1:r:l) = r/\sqrt{(2l+3)(2l+1)}, \\ (l:r:l) &= 0 \end{aligned} \right\} \quad (\text{A6})$$

and

$$\left. \begin{aligned} (l:\nabla:l+1) &= \{(\partial/\partial r) + (l+2)/r\}/\sqrt{(2l+3)(2l+1)}, \\ (l:\nabla:l) &= 0, \\ (l:\nabla:l-1) &= \{(\partial/\partial r) - (l-1)/r\}/\sqrt{(2l+1)(2l-1)}, \end{aligned} \right\}$$

where the integration of radial parts is not written explicitly.

A.2. *The relativistic formula for the hyperfine structure splittings of hydrogenlike atoms with fixed nucleus*

The relativistic Hamiltonian for a single electron in the Coulomb field of a nucleus with charge Ze is given by Eq. (3) in the text with $V = -Ze^2/r$, the nucleus being assumed to be a fixed centre of force. The additional energy due to the magnetic moment of the nucleus $\vec{\mu} = \mu_K g(I) \mathbf{I}$ is expressed by^(4,5)

$$H' = -e(\vec{a}_e \mathbf{A}), \quad \mathbf{A} = \vec{\mu} \times \mathbf{r}/r^3, \quad (\text{A7})$$

where μ_K is the nuclear magneton $e\hbar/2Mc$, \mathbf{I} denotes generally the spin vector of the nucleus in units of \hbar and $g(I)$ is the gyromagnetic ratio. H' is considered as a perturbation to H_0 and the energy of the first order is given by taking the expectation value of H' for the unperturbed states. For given j and m_j the zeroth eigenfunctions are given by

$$u(l \pm j m_j) = \begin{pmatrix} g_{\pm}(r) Y(l \pm j m_j) \\ i f_{\pm}(r) Y(l \mp j m_j) \end{pmatrix}, \quad (\text{A8})$$

using the notations defined in § 3.2 in the text, where $g_{\pm}(r)$, $f_{\pm}(r)$ differ from those given in (13a) only in the normalization factor, i.e. here $\int_0^{\infty} r^2 dr \{g_{\pm}^2(r) + f_{\pm}^2(r)\} = 1$. H' is written as

$$H' = -e\mu_K g(I) (\mathbf{I} \cdot \mathbf{r} \times \vec{a}_e)/r^3. \quad (\text{A7}')$$

Since, according to the definition of the preceding section, $\mathbf{r} \times \vec{\sigma}_e$ and therefore, $\mathbf{r} \times \vec{a}_e$ also belong to j , the matrix elements of the angular part of $(\mathbf{I} \cdot \mathbf{r} \times \vec{a}_e)$ are easily calculated by the formulas just obtained there. One has first

$$(l j I F_F, (\mathbf{I} \cdot \mathbf{r} \times \vec{a}_e) | l' j' I F_F) = (l j : \mathbf{r} \times \vec{a}_e : l' j') W(j I j' I F). \quad (\text{A9})$$

Taking into account that \vec{a}_e operates on $u(l \pm j m_j)$ in the same way as $\vec{\sigma}_e$ does, after interchanging their large and small components we find from (A8)

$$\left. \begin{aligned} (l \pm j : \mathbf{r} \times \vec{a}_e : l \pm j) &= i g_{\pm} f_{\pm} \{ (l \pm j : \mathbf{r} \times \vec{\sigma}_e : l \mp j) - (l \mp j : \mathbf{r} \times \vec{\sigma}_e : l \pm j) \}, \\ (l \pm j : \mathbf{r} \times \vec{a}_e : l' \mp j + 1) &= i \{ g_{\pm} f'_{\mp} (l \pm j : \mathbf{r} \times \vec{\sigma}_e : l' \mp j + 1) - f_{\pm} g'_{\mp} (l \mp j : \mathbf{r} \times \vec{\sigma}_e : l' \mp j + 1) \}, \end{aligned} \right\} \quad (\text{A10})$$

where a_e and σ_e are replaced by a and σ for simplicity. All other matrix elements vanish except those obtained by taking the complex conjugates of the second two above, since $(l:r:l')=0$ unless $l'=l \pm 1$. Using the formulas of TAS, § 11³, (8) and (A6), we can easily obtain

$$\left. \begin{aligned} (l \pm j : \mathbf{r} : l \mp j) &= (l \pm : \mathbf{r} : l \mp) / \sqrt{j(j+1)} = r/2j(j+1), \\ (l+j : \mathbf{r} : l' \mp j + 1) &= (l+ : \mathbf{r} : l' \mp) / \sqrt{(j+2)(j+1)} = r/2(j+1), \\ (l-j : \mathbf{r} : l' \mp j + 1) &= (l- : \mathbf{r} : l' \mp) / \sqrt{j(j+1)} = r/2(j+1), \end{aligned} \right\} \quad (\text{A11})$$

and

$$\left. \begin{aligned} (l+j : \sigma : l+j) &= -1/(j+1), \\ (l-j : \sigma : l-j) &= 1/j, \\ (l+j : \sigma : l' \mp j + 1) &= 1/(j+1). \end{aligned} \right\} \quad (\text{A12})$$

Therefore Eqs. (A10) become, on application of the formulas (A3), (A4) and (A5) to (A10) and substitution of the results (A11) and (A12) in it,

$$\left. \begin{aligned} (l \pm j | r \times a | l \pm j) &= \mp (g_{\pm} f'_{\pm}) \cdot (2j+1)/j(j+1), \\ (l \pm j | r \times a | l' \mp j+1) &= \pm (g_{\pm} f'_{\mp} + f_{\pm} g'_{\mp}) \cdot 1/2(j+1). \end{aligned} \right\} \quad (\text{A13})$$

Thus we have the non-vanishing matrix elements of the hyperfine splitting only between the states $j'=j \pm 1$ and $l'=l$ or $l'=l \pm 2$ other than the diagonal elements, and find from (A7') and (A9) the general formula for these matrix elements as follows:

$$\left. \begin{aligned} (l \pm j | I F M_F | H' | l \pm j I F M_F) &= \pm a \cdot \frac{2j+1}{j(j+1)} \cdot W(j I j I F) \cdot \int_0^{\infty} g_{\pm} f_{\pm} dr, \\ (l \pm j | I F M_F | H' | l' \mp j+1 I F M_F) &= \mp a \cdot \frac{1}{2(j+1)} \cdot W(j I j+1 I F) \cdot \int_0^{\infty} (g_{\pm} f'_{\mp} + f_{\pm} g'_{\mp}) dr, \end{aligned} \right\} \quad (\text{A14})$$

denoting $a = e \mu_K g(I)$.

In case of an idealized proton treated in the text we can put $I = \frac{1}{2}$ and $g(I) = 2$. Since

$$\left. \begin{aligned} W(j + \frac{1}{2} j + \frac{1}{2} F) &= -(F+3/2)/2, \\ W(j - \frac{1}{2} j - \frac{1}{2} F) &= (F-1/2)/2, \\ W(j \pm \frac{1}{2} j \mp \frac{1}{2} F) &= -\sqrt{F(F+1)}, \end{aligned} \right\}$$

the matrix elements of H' become for given F, M_F , for Ψ_1 defined by Eq. (11) in the text,

$$\left. \begin{aligned} (l^{++} j^{+} | H' | l^{++} j^{+}) &= -\frac{2(F+1)}{2F+1} a \cdot \int_0^{\infty} g_{++} f_{++} dr, \\ (l^{++} j^{+} | H' | l^{--} j^{-}) &= -\frac{\sqrt{F(F+1)}}{2F+1} a \cdot \int_0^{\infty} (g_{++} f_{--} + f_{++} g_{--}) dr, \\ (l^{--} j^{-} | H' | l^{--} j^{-}) &= -\frac{2F}{2F+1} a \cdot \int_0^{\infty} g_{--} f_{--} dr, \end{aligned} \right\} \quad (\text{A15a})$$

and for Ψ_2 ,

$$\left. \begin{aligned} (l^{-+} j^{+} | H' | l^{-+} j^{+}) &= \frac{2(F+1)}{2F+1} a \cdot \int_0^{\infty} g_{-+} f_{-+} dr, \\ (l^{-+} j^{+} | H' | l^{+-} j^{-}) &= \frac{\sqrt{F(F+1)}}{2F+1} a \cdot \int_0^{\infty} (g_{-+} f_{+-} + f_{-+} g_{+-}) dr, \\ (l^{+-} j^{-} | H' | l^{+-} j^{-}) &= \frac{2F}{2F+1} a \cdot \int_0^{\infty} g_{+-} f_{+-} dr. \end{aligned} \right\} \quad (\text{A15b})$$

It can be proved in the same way as in § 5.2 that the integral $\int_0^{\infty} (g_{++} f_{--} + f_{++} g_{--}) dr$ in (A15a) vanishes in the first approximation, while in the same approximation $g_{-+} = g_{+-}$, so that $\int_0^{\infty} (g_{-+} f_{+-} + f_{-+} g_{+-}) dr = \int_0^{\infty} g_{-+} f_{+-} dr + \int_0^{\infty} g_{+-} f_{-+} dr$. Again by a similar calculation one obtains at first the following exact expression for these integrals in general:

$$\int_0^{\infty} g_{\kappa} f_{\kappa} dr = -\frac{Z^3 a}{a_0^2} \cdot \frac{(N-x)^2 (2n'+2\gamma+1) - n' (2\gamma+n') (2n'+2\gamma-1)}{N^2 (N-x) (2\gamma-1) 2\gamma (2\gamma+1)}, \quad (\text{A16})$$

where $N = (n'^2 + 2n'\gamma + k^2)^{1/2}$, $n' = n - k$ and $\gamma = (k^2 - Z^2 a^2)^{1/2}$. Expanding (A16) in powers of a^2 one finds to relative order a^2

$$\int_0^{\infty} g_{\kappa} f_{\kappa} dr = \frac{a}{a_0^2} \cdot \left(\frac{Z}{n}\right)^3 \cdot \frac{1}{k(2\kappa+1)} (1 + a^2 A_1 + \dots) \quad (\text{A16}')$$

with

$$A_1 = \frac{n-k}{2n^2k(2k-1)} + \frac{2(n-k)}{n^2k} - \frac{x}{nk(2k-1)} + \frac{1}{2k^2} + \frac{4}{4k^2-1}.$$

Thus from (A15a) and (A15b) one obtains the expression in the first approximation for the hyperfine splittings of hydrogen atoms with an idealized proton, noting $a = 2e\mu_K = e^2\hbar/Mc = mc^2\beta a_0^2$, as follows:

$$\left. \begin{aligned} \text{for } l=F+1, j=F+\frac{1}{2} & -\frac{2}{n^3} \frac{1}{(2F+1)(2F+3)} \cdot mc^2\beta a^4, \\ \text{for } l=F-1, j=F-\frac{1}{2} & \frac{2}{n^3} \frac{1}{(2F-1)(2F+1)} \cdot mc^2\beta a^4, \\ \text{for } l=F, j^\pm=F\pm\frac{1}{2} & \left(\begin{array}{cc} j^+ & j^- \\ j^+ & -2 \\ j^- & 1/\sqrt{F(F+1)} \end{array} \right) \cdot \frac{1}{n^3} \frac{1}{(2F+1)^2} \cdot mc^2\beta a^4. \end{aligned} \right\} \quad (\text{A17})$$

Further we can see at once from (A16') that the relativistic correction to the hyperfine splittings are given by the term $a^2 A_1$ in general, which proves to become $3a^2/2$ especially for the ground state as has been given by Breit⁹⁾.

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Mesonic Processes in Two-Nucleon System

Yoichi FUJIMOTO and Yoshio YAMAGUCHI

*Department of Physics, University of Tokyo
and Osaka-City University*

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The production and annihilation of a meson in two nucleon system are discussed phenomenologically, leaving at first the matrix elements for production and annihilation undetermined. It is shown that when one adjusts the matrix element properly, the main features of the observed meson spectrum can be accounted for by the distortion of the nucleon wave functions by the nuclear and Coulomb forces. For the wave function of two nucleons, we use the method of distorted wave with shape independent approximation. Our calculation can be applied only to low energy phenomena, such as meson production near threshold. Some experiments on artificial mesons fulfil this condition, and can be compared with our results. There may be a possibility that the mesonic interaction is "non-local", or the wave function of a meson is largely distorted. However, we do not consider such cases in this paper.

§ 1. Introduction

We have previously discussed the process of meson production by protons or X -rays bombarded on target nuclei¹⁾ and meson capture by hydrogen or other nuclei²⁾. In these considerations, we assumed that the participating particles are all free and used the appropriate matrix elements derived from current meson theories with usual perturbation method. But these considerations are not quite free from objections in that the free particle model (Thomas-Fermi gas model) for nuclei is only a very poor approximation, which furthermore fails to take into account the interference effect of neighbouring nucleons inside a nucleus. It is also doubtful that the lowest order perturbation calculation based on any meson theory could explain the experimental data even qualitatively. Several efforts to include the higher order processes, such as phenomenological introduction of the anomalous magnetic moment for meson production by X -rays³⁾ or the fourth order process in nucleon scattering⁴⁾ seem unlikely to achieve any success.

To be free from these defects, we restrict ourselves with mesonic problems in two nucleon system. If the energies of the participating particles are not high, the distortion of the nucleon wave function is so large, that the free particle approximation can not be used at all. On the contrary, what is most essential in our problem is this distortion, so that the detailed nature of the meson is of secondary importance. This situation makes it possible to discuss the problem without referring any special meson theories, leaving the matrix elements of the meson production or annihilation undetermined, which can be afterward so adjusted

as to give the best agreement with the experimental result. Furthermore it is sufficient to use the method of distorted waves with the shape independent approximation⁵⁾. The matrix elements determined half empirically as mentioned above give directly the true mesonic interaction under elimination of the effect of nuclear forces. The type and coupling of a π -meson shall be sought as a form of selection rules.

Unfortunately, there are at present so few experimental results with sufficient accuracies, that we can not obtain a reliable selection rule for meson phenomena. But, we shall achieve in future more detailed information on mesons by the method described here.

§ 2 is devoted to the preliminary discussions on the two nucleon system, which are useful in later calculation. In § 3, we treat the process of meson production,

$$\begin{aligned} p + p &\rightarrow \pi^+ + n + p, \\ p + p &\rightarrow \pi^+ + d, \end{aligned} \quad (1.1)$$

and in § 4 the capture of π^- -meson,

$$\pi^- + d \rightarrow n + n, \quad (1.2)$$

$$\pi^- + d \rightarrow n + n + \gamma, \quad (1.3)$$

where p , n and π denote proton, neutron and π -meson. d is deuteron. Other applications, such as

$$\begin{aligned} \gamma + d &\rightarrow \pi^+ + n + n, \\ \gamma + d &\rightarrow \pi^- + p + p, \end{aligned} \quad (1.4)$$

are stated briefly in § 5. Comparison with some experimental data is given in § 6.

§ 2. Preliminary remark

We assume the point interaction on a meson and a nucleon. Even if it is not a case, i.e. "non-local interaction", our final results are approximately correct so far as the dimension of interaction region is smaller than the wave length of concerning particles in the process. For simplicity, we use the units $\hbar = c = 1$ hereafter.

A) The matrix element for processes (1.1) and (1.2)

Let $\Psi_I(\mathbf{r}_1, \mathbf{r}_2)$ be the spatial wave function of the system of two nucleons 1 and 2 in the state I . The state II is described by the wave function $\Psi_{II}(\mathbf{r}_1, \mathbf{r}_2)\phi(\mathbf{r})$ where Ψ_{II} concerns two nucleons and ϕ a meson. Then, the transition matrix element between I and II can be in general written as,

$$\Psi_{II}^*(\mathbf{r}_1, \mathbf{r}_2) \left\{ \sum_{j=1,2} \phi^*(\mathbf{r}_j) (II | S(j) | I) \right\} \Psi_I(\mathbf{r}_1, \mathbf{r}_2), \quad (2.1)$$

where $S(j)$ is a creation or annihilation operator on meson field, which may depend on the charge, spin, momentum of a nucleon and a meson.

To apply to the production (1.1), we take I as initial state A , and II as final state F . Furthermore, we assume the plane wave for a produced meson with momentum q ,

$$\phi(r) = \exp(-iqr).$$

This assumption is made only for simplicity. It is very probable that we must use a distorted wave also for the meson because of its strong interaction with the nucleons, and that this might have an appreciable effect on final results. But because we have at present no reliable knowledge about this distortion, we decided to ignore this fact. We expect that precise comparison with experiments will inform us on non-local interaction and deformation of meson wave.

For a capture process (1.2) by deuteron, we must take $I=F$ and $II=A$, inversely to the above. The meson wave function $\phi(r)$ should be inserted with that of bound state $\varphi_\pi(r)$. The first approximation of $\varphi_\pi(r)$ can be obtained by taking into account only Coulomb field of a nucleus.

Furthermore, we assume that the kinetic energy of relative motion of two nucleons be small in state II . This condition is approximately fulfilled in the case of artificial production of mesons; i.e. proton-proton collision of 340 MeV performed at Berkeley⁶⁾. Then we can apply shape independent approximation for the part of relative motion in $\Psi_{II}(r_1, r_2)$. In the special case of deuteron, this method has better accuracy. For example, in the case of meson capture by a deuteron, we have analogous calculation as photo-disintegration⁷⁾, inserting in $\Psi_{II}(r_1, r_2)$ the wave function of a deuteron.

In state I , where a meson is absent, the kinetic energy of two nucleons is usually very large ($\geq 2\mu$, μ is the rest energy of a π -meson). The phase shift in this state can be obtained from the analysis of high energy nucleon-nucleon scattering⁸⁾. It turns out to be much smaller than that in state II , and we can neglect it within an error of 10%. Thus we shall use a plane wave of $\Psi_I(r_1, r_2)$.

The Coulomb interaction between charged particles must be taken into account. This effect can be approximately estimated by the "Coulomb penetration factor"⁹⁾, $2\pi\eta/(\exp(2\pi\eta)-1)$; $\eta=c^2/v$. We can see it is usually negligible except between two protons in state II (i.e. with a emitted meson).

B) The matrix element for the processes (1.3) and (1.4)

We can apply the shape independent approximation for two nucleon system also in final states of the processes of the radiative meson capture by a deuteron (1.3) and the meson production by X-rays bombarded on a deuteron (1.4). It is because the kinetic energy of relative motion of nucleons are small both in initial and final states for these processes. The transition matrix element can in general be written as,

$$\Psi_{II}^*(r_1, r_2) \{ \sum_{j=1,2} \phi^*(r_j) (II | T(j) | I) \exp(ikr_j) \} \Psi_I(r_1, r_2). \quad (2.2)$$

I is a state of two nucleons and a photon (of momentum k), and II is a state of two nucleons and a meson.

§ 3. Meson production in proton-proton collision

Here, we shall discuss the process,

$$p + p \rightarrow \pi^+ + n + p.$$

For convenience, the center of mass system is used (total momentum is zero). As was shown in § 2, we can neglect the nuclear and Coulomb force in the initial state A , and approximate it with a plane wave,

$$\Psi_A(\mathbf{r}_1, \mathbf{r}_2) = \phi_A(\mathbf{r}) = \frac{1}{\sqrt{2}} (\exp(i\mathbf{p}_0\mathbf{r}) \pm \exp(-i\mathbf{p}_0\mathbf{r})). \quad (3.1)$$

The spatial wave function of a relative motion is always expressed by a small letter, $\phi(\mathbf{r})$ ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is a relative coordinate). From the conservation of momenta, the total momentum \mathbf{P} of two nucleons in final states F is related to that of mesons \mathbf{q} by,

$$\mathbf{P} = -\mathbf{q}.$$

After integration of the coordinates of a meson and the center of gravity of two nucleons, the matrix element (2.1) can be written simply as,

$$(F | S(1) + S(2) | A) \int d\mathbf{r} \phi_{\mathbf{r}}^*(\mathbf{r}) \cos(\mathbf{q}\mathbf{r}/2) \phi_A(\mathbf{r}), \quad (3.2a)$$

$$-i(F | S(1) - S(2) | A) \int d\mathbf{r} \phi_{\mathbf{r}}^*(\mathbf{r}) \sin(\mathbf{q}\mathbf{r}/2) \phi_A(\mathbf{r}), \quad (3.2b)$$

where, $\phi_{\mathbf{r}}(\mathbf{r})$ describes the part of relative motion, i.e.,

$$\Psi_{\mathbf{r}}(\mathbf{r}_1, \mathbf{r}_2) = \exp(i\mathbf{P}\mathbf{R}) \phi_{\mathbf{r}}(\mathbf{r}).$$

For a final state of $l=0$ (S -state), $\phi_{\mathbf{r}}(\mathbf{r})$ can be approximated by,

$$\phi_{\mathbf{r}}(\mathbf{r}) = \frac{\sin(pr + \delta) - \exp(-\beta r) \sin \delta}{pr}, \quad r = |\mathbf{r}|. \quad (3.3)$$

$\delta = \delta(p)$ means the phase shift, which can be expressed by the scattering length a and the effective range r_0 ,

$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 + \dots \quad (3.4)$$

The correction near the origin is introduced by the second term with parameter β , which is adjusted to fit the correct value of effective range. Assumption of constant value of β for energies affords an error of at most 10% in our case.

For the case of higher l value in the final state, $\phi_{\mathbf{r}}(\mathbf{r})$ is well approximated by the free wave function. Especially in the odd states of neutron-proton system,

$\delta=\delta(p)$ is experimentally ascertained to be very small⁸⁾. Therefore, their contributions to meson production are generally very small.

Meson production with deuteron formation can be treated in a similar way. $\psi_d(r)$ should be replaced by the deuteron wave function,

$$\begin{aligned}\psi_d(r) &= A(\exp(-\gamma r) - \exp(-\beta r))/r, \\ A &= \sqrt{2\gamma/4\pi(1-\gamma r_t)}. \end{aligned} \quad (3.5)$$

γ , the reciprocal of the radius of deuteron, is related to the binding energy E_d by

$$\gamma = \sqrt{ME_d},$$

and r_t is the triplet effective range. β in the second term of (3.5) is taken to be equal to that in (3.3) of triplet state. This means that the wave function of low energy triplet state near the origin is approximated by that of deuteron.

Inserting (3.1) and (3.3) or (3.5) into (3.2), we can get the matrix element. These results are discussed below. The matrix element consists of two parts, (3.2a) and (3.2b), which correspond with and without parity change. They are simply expressed by,

$$(F | S(1) \pm S(2) | A) = (F | \pm | A).$$

Triplet final state Without any error, we may neglect the final states of higher angular momentum other than 3S . So far as the matrix element $(F | + | A)$ is not much smaller than $(F | - | A)$, the most important contribution comes from the initial state of 1S . Then the transition probability for $^1S \rightarrow ^3S$ can be obtained simply,

$$\begin{aligned} & | (F | + | A) |^2 (\sin \delta/p)^2 \\ & \times \left[\frac{\beta^2 + p^2}{\{p_0 + q/2\}^2 - p^2} \{ (\mathbf{p}_0 + \mathbf{q}/2)^2 + \beta^2 \} + \frac{\beta^2 + p^2}{\{p_0 - q/2\}^2 - p^2} \{ (\mathbf{p}_0 - \mathbf{q}/2)^2 + \beta^2 \} \right]^2 \\ & \times 2\pi M p q q_0 d q_0 d\Omega_q / (2\pi)^3, \end{aligned} \quad (3.6)$$

where M and μ denote mass of nucleon and meson. q_0 is the total energy of a meson,

$$q_0 = \sqrt{q^2 + \mu^2},$$

and p is the relative momentum of two nucleons in final state. From the energy conservation, q , p and p_0 satisfy the following relation,

$$q^2/4M + p^2/M + q_0 = p_0^2/M.$$

Furthermore, $d\Omega_q$ means the differential of solid angle in the direction of \mathbf{q} .

The process with deuteron formation, $p + p \rightarrow \pi^+ + d$, can be analogously treated. Its transition probability is,

$$| (F | + | A) |^2 4A^2 \{ (\beta^2 - \gamma^2) / (p_0^2 + \gamma^2) (p_0^2 + \beta^2) \}^2 q q_0 d\Omega_q, \quad (3.7)$$

where

$$q^2/4M + q_0 = p_0^2/M + E_a.$$

We calculate the energy spectrum of mesons from (3.6) and (3.7), assuming matrix element $(F| + |A)$ to be constant. Fig. 1 shows their spectra at the direction of incident beam in laboratory system for 345 MeV proton-proton collision. The ordinate is the relative value of yields, normalized as the deuteron peak to be unity. If we could know the energy dependence of $|(F| + |A)|^2$, it should be multiplied on this curve.

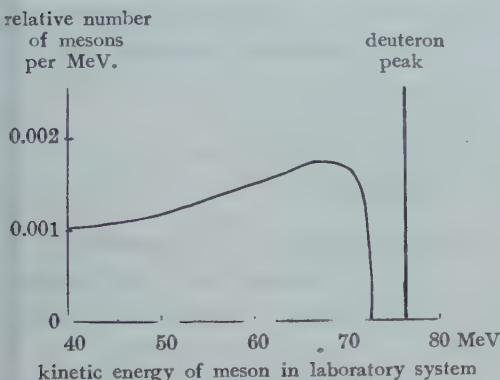


Fig. 1. The energy spectrum of meson at the direction of incident beam in the laboratory system for 345 MeV proton-proton collision; the case for 3S -final state is given. The ordinate is the relative value of yields, normalizing the deuteron peak as unity. The numerical values of various parameters adopted in this calculation are the same as used by Bethe and Longmire⁷⁾.

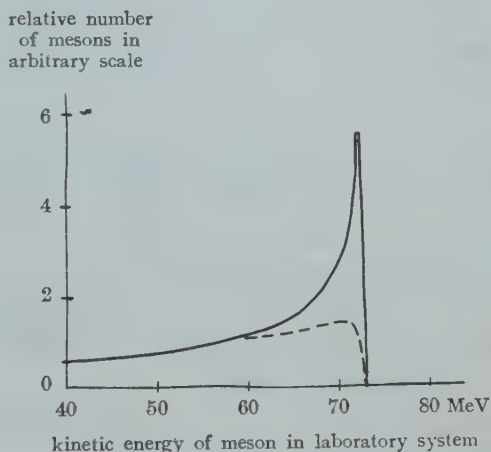


Fig. 2. The energy spectrum of mesons for 1S -final state. The condition is the same as in Fig. 1. The dotted curve shows the spectrum of neutral mesons, taking into account Coulomb force in final two protons.

Singlet final state Here we may be concerned mainly with final 1S -state. In the same way as the above, the initial state is decomposed into partial waves. As is seen at once, the transition of $^1S \rightarrow ^1S$ is forbidden in the first approximation. If the matrix elements $(F| + |A)$ and $(F| - |A)$ have the same order of magnitude, the transition of P to 1S becomes important. We have plotted in Fig. 2 the energy spectrum of π -mesons in this transition, assuming the constant matrix element.

For the production of the neutral meson π^0 ,

$$p + p \rightarrow p + p + \pi^0,$$

the final two nucleons can not be in 3S -state. Therefore, main contribution comes from a singlet final state 1S , and can be calculated in the similar way. The Coulomb interaction is not negligible for final two protons, and it is taken into account by the Coulomb penetration factor $2\pi\eta/(\exp(2\pi\eta) - 1)$, $\eta = e^2/v$. The

dotted line in Fig. 2 shows the energy spectrum of π^0 obtained in this way. It is remarkable that the resonance peak at zero energy 1S -state does not appear in this case. Thus, we shall observe much smaller yield of π^0 than π^+ in proton-proton collision.

§ 4. Capture of π^- -meson by deuteron¹⁰⁾

In discussing the first process of π^- -capture,

$$\pi^- + d \rightarrow n + n,$$

we can approximate the final nucleons as a plane wave (cf. § 2). For a transition without parity change, we can write down its probability per unit time as follows,

$$2\pi \sum_{\text{average on } A} \sum_{\text{sum up on } F} |(F|S(1)|A)|^2 |\varphi_\pi(0)|^2 (4\pi A)^2 \\ \times \{(\beta^2 - \gamma^2)/(\rho^2 + \gamma^2)(\rho^2 + \beta^2)\}^2 4\pi M\rho/(2\pi)^3,$$

where S is the operator depending spin and isotopic spin. The proton in a deuteron is denoted by 1, and the bound π^- -meson is assumed to be in S -state. In other cases, such as accompanying the parity change, we can easily write down the matrix element in the similar way.

The second mode,

$$\pi^- + d \rightarrow n + n + \gamma,$$

is now discussed. When an emitted γ -ray has rather small energy, we can approximate the final nucleons by a plane wave. For the case of large energy of γ -rays, the distortion of the final nucleon wave function must be taken into account. In this case, it may be sufficient to consider only 1S state as the final state.

The transition probability for emission of γ -rays with energy (k, dk) can be obtained as,

$$2\pi 2(8\pi^2 A)^2 \frac{1}{(2\pi)^6} \cdot M |\varphi_\pi(0)|^2 \\ \times \sum_{\text{average on } A} \sum_{\text{sum up on } F} |(F|T(1)|A)|^2 F(k) dk. \quad (4.2)$$

For a small energy of γ -rays, the plane wave approximation yields,

$$F(k) = \frac{k}{4} \left[\frac{2k\rho}{\{\gamma^2 + (\rho + k/2)^2\}\{\gamma^2 + (\rho - k/2)^2\}} + \frac{2k\rho}{\{\beta^2 + (\rho + k/2)^2\}\{\beta^2 + (\rho - k/2)^2\}} \right] \\ - \frac{2}{\beta^2 - \gamma^2} \ln \left\{ \left(\gamma^2 + \left(\rho + \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(\rho + \frac{k}{2} \right)^2 \right) / \left(\gamma^2 + \left(\rho - \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(\rho - \frac{k}{2} \right)^2 \right) \right\} \\ \pm \frac{1}{\gamma^2 + \rho^2 + k^2/4} \ln \left\{ \left(\gamma^2 + \left(\rho + \frac{k}{2} \right)^2 \right) / \left(\gamma^2 + \left(\rho - \frac{k}{2} \right)^2 \right) \right\}$$

$$\begin{aligned} & \pm \frac{1}{\gamma^2 + p^2 + k^2/4} \ln \left\{ \left(\beta^2 + \left(p + \frac{k}{2} \right)^2 \right) / \left(\beta^2 + \left(p - \frac{k}{2} \right)^2 \right) \right\} \\ & \mp \frac{1}{\gamma^2 + \beta^2 + 2p^2 + k^2/2} \ln \left\{ \left(\gamma^2 + \left(p + \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(p + \frac{k}{2} \right)^2 \right) \right. \\ & \quad \left. / \left(\gamma^2 + \left(p - \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(p - \frac{k}{2} \right)^2 \right) \right\} \Bigg], \quad (4.2a) \end{aligned}$$

where the double sign \pm indicates the final triplet and singlet state respectively. For a large energy of γ -rays, we get by the method of distorted waves,

$$\begin{aligned} F(k) = & \frac{\sin^2 \delta}{p} \left[\tan^{-1} \frac{\gamma k}{\gamma^2 + p^2 - k^2/4} - \tan^{-1} \frac{\beta k}{\beta^2 + p^2 - k^2/4} \right. \\ & - 2 \tan^{-1} \{ (\beta - \gamma) k / 2 ((\gamma + \eta)(\beta + \eta) + k^2/4) \} \\ & \left. + \frac{1}{2} \cot^2 \delta \ln \left\{ \left(\gamma^2 + \left(p + \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(p - \frac{k}{2} \right)^2 \right) / \left(\gamma^2 + \left(p - \frac{k}{2} \right)^2 \right) \left(\beta^2 + \left(p + \frac{k}{2} \right)^2 \right) \right\} \right]. \end{aligned} \quad (4.2b)$$

η means the β in (3.3) for 1S -state wave function.

Fig. 3 shows the curve of $F(k)$. It has a strong peak at high energy end, and its width is only a few Mev.

Finally, we remark that the process,

$$\pi^- + d \rightarrow n + n + \pi^0,$$

is forbidden from the conservation of parity and total angular momentum, when π^- is initially in a bound S -state and both π^- and π^0 have spin zero. Even in other cases, we can see that this process has much smaller probability than the above one, considering the available volume in phase space¹⁰⁾.

§ 5. Other applications

Production of π -mesons by X -rays bombarded on deuterons The kinetic energy of relative motion of two nucleons are small both in initial and final states, thus we can use our method throughout the whole process. Its results will give us best clue on mesonic selection rule, though the con-

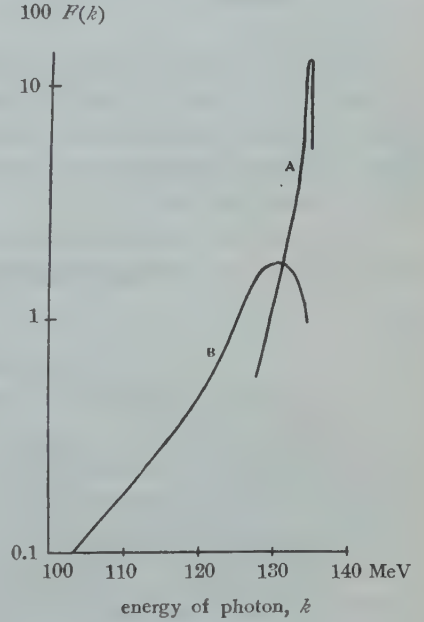


Fig. 3. The energy spectrum of photons from deuteron capturing π^- -meson. The ordinate is $100 \times F(k)$. The curve A is obtained by the plane wave approximation for singlet final state, and B by the method of distorted waves considering only 1S final state. (Assuming n - n force $= p$ - p force.)

tinuous spectrum of X -rays introduces some confusion. Unfortunately, no experimental results has published yet on this reaction, so we omit here our detailed calculation.

Neutron-deuteron scattering at high energies Applying our method for high energy n - d scattering, we can take into account the distortion of wave function of a slower pair of nucleons, besides the usual Born's approximation. Chew's detailed treatment on this reaction¹¹⁾ is essentially the same as ours.

§ 6. Conclusion

Previous treatments¹²⁾ for mesonic processes in two nucleon system were usually based on current meson theories and confined to Born's approximation. As is well known, a wave function of two nucleon system in low energy region can not be correctly obtained by Born's method. Therefore the method of distorted waves is used. Ambiguities in the choice of nuclear forces is avoided by the shape independent approximation. Some of our results can be compared with the experiment. They are irrespective of the type of meson.

1) The yield of π^0 in proton-proton collision is much smaller than that of π^+ .

2) The π^+ -meson spectra of triplet and singlet transitions in proton-proton collision are obtained. They have pronounced peaks of deuteron and zero energy 1S -state. Experiments seem to confirm these peaks, though they can not resolve them⁶⁾. Precise determination of their relative magnitude will infer us some mesonic selection rule.

3) The γ -ray spectrum from deuterons capturing π^- is obtained. It has a sharp peak at high energy end, which was confirmed by a recent experiment¹³⁾.

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Note on the Direct Interaction between Spinor Fields

Susumu KAMEFUCHI

Institute of Theoretical Physics, Nagoya University

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§ 1. Introduction

Renormalization procedure proposed by Tomonaga and Schwinger¹⁾ in quantum electrodynamics has been extensively applied by many authors²⁾ to other kinds of field theories, and it has become clear that in most cases this method is very successful in avoiding the various divergences encountered there. The investigations published up to the present are, however, restricted to the cases of interaction between spinor and Bose fields or that of two Bose fields, and so, for the interaction between two spinor fields, there still remain the questions; 1) what type of divergence appears in the higher order processes, and 2) to what extent the renormalization procedure is applicable to such divergences.

The investigations of this problem are the purpose of this note. As an example of such problems, we take Fermi-type interaction adopted by him in the theory of β -decay³⁾. Calculations are carried out by use of Tomonaga-Schwinger's covariant formalism.

§ 2. Second-order Hamiltonian

As the interaction energy density in the Heisenberg representation, we take the coupling of scalar type for brevity of calculation, i.e.,

$$H = g(\bar{\psi}_1\psi_2)(\bar{\varphi}_1\varphi_2) + c.c., \quad (1)$$

where ψ_1 , ψ_2 , φ_1 and φ_2 are the quantized wave functions of proton, neutron, electron and neutrino respectively. In the interaction representation the interaction Hamiltonian takes the same form as (1) (but with the ψ 's and φ 's satisfying the free field equations), since (1) contains no derivatives of wave fields.

It then follows by the same argument as given by Schwinger that the commutation relations take the following forms;

$$\left. \begin{aligned} \{\psi_{1\alpha}(X), \bar{\psi}_{1\beta}(X')\} &= \frac{1}{i} S_{1\alpha\beta}(X-X'), \\ \{\varphi_{1\alpha}(X), \bar{\varphi}_{1\beta}(X')\} &= \frac{1}{i} s_{1\alpha\beta}(X-X'), \\ \{\psi_{1\alpha}(X), \bar{\psi}_{2\beta}(X')\} &= \{\varphi_{1\alpha}(X), \bar{\varphi}_{2\beta}(X')\} = 0, \text{ etc.}, \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} \{\psi(X), \psi(X')\} &= \{\bar{\psi}(X), \bar{\psi}(X')\} = \{\varphi(X), \varphi(X')\} = \{\bar{\varphi}(X), \bar{\varphi}(X')\} = 0 \\ [\psi(X), \varphi(X')] &= 0, \text{ etc.} \end{aligned} \right\}$$

As to the commutation relations between ψ 's and φ 's, it is to be noticed that we may take anticommutators instead of commutators such as in (2). But this circumstance does not introduce any difference of the final results of the calculation, since ψ 's and φ 's always appear in pairs⁴⁾.

Schrödinger equation in this representation is

$$i\hbar c \frac{\partial \Psi[\sigma]}{\partial \sigma(X)} = H(X) \Psi[\sigma]. \quad (3)$$

From (1) and (2), we can easily verify the integrability condition for eq. (3),

$$\left[i\hbar c \frac{\partial}{\partial \sigma(X)} - H(X), i\hbar c \frac{\partial}{\partial \sigma(X')} - H(X') \right] = [H(X), H(X')] = 0 \quad (4)$$

for space-like $X-X'$.

In order to investigate the various divergent terms of the second order in g , we perform the usual Schwinger transformation,

$$\begin{aligned} \Psi[\sigma] &= U[\sigma] \Phi[\sigma], \quad U[\sigma] = e^{-iS[\sigma]}, \\ S[\sigma] &= \frac{1}{\hbar c} \int_{-\infty}^{\sigma} H(X') d\tau'. \end{aligned} \quad (5)$$

In the transformed Hamiltonian, we retain only the second order terms in g (neglecting the higher order ones), and so it becomes

$$\begin{aligned} H^{(2)} &= \frac{i}{2} [S, H(X)] \\ &= \frac{-i}{4\hbar c} \int_{-\infty}^{\infty} [H, H'] \varepsilon(X-X') d\tau' + \frac{-i}{4\hbar c} \int_{-\infty}^{\infty} [H, H'] d\tau'. \end{aligned} \quad (6)$$

The second term in (6) does not vanish in our present case. (The corresponding term in quantum electrodynamics vanishes so since $\int_{-\infty}^{\infty} H' d\tau' = 0$). This is due to the fact that already in the first order in g there occur the real processes satisfying the conservation law of energy and momentum. But this term, describing the mere iteration of two first order real processes, does not introduce any divergent term, since the degree of freedom in the intermediate states is limited by the conservation law of energy and momentum. As our purpose is merely to see the characters of divergences, it is allowed to omit this term from the following considerations. Then we may write

$$H^{(2)} = -\frac{ig^2}{4\hbar c} \int_{-\infty}^{\infty} [(\bar{\psi}_1 \psi_2)(\bar{\varphi}_1 \varphi_2), (\bar{\psi}_2' \psi_1')(\bar{\varphi}_2' \varphi_1')] - \widehat{X} \varepsilon(X-X') d\tau', \quad (7)$$

where \widehat{X} denotes the same expression as the preceding one but with the exchange of X and X' . (7) is further transformed as follows,

$$H^{\rho} = -\frac{ig^2}{8\hbar c} \int_{-\infty}^{\infty} ([\{(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')\}][(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')] + [(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')]\{(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')\}] - \overline{XX'})\epsilon(X-X')dw'. \quad (7')$$

Now we can split up the Hamiltonian (7') into various operators in the same manner as in Schwinger's paper¹⁾. Carrying out this procedure, (7') becomes

$$\begin{aligned} H^{\rho} = & (0,0; 0,0) \\ & + (1,0; 0,0) + (0,1; 0,0) + (0,0; 1,0) + (0,0; 0,1) \\ & + (1,1; 0,0) + (0,0; 1,1) \\ & + (1,0; 1,0) + (0,1; 0,1) + (1,0; 0,1) + (0,1; 1,0) \\ & + (1,1; 1,0) + (1,1; 0,1) + (1,0; 1,1) + (0,1; 1,1) \\ & + (1,1; 1,1), \end{aligned} \quad (8)$$

where (1,0; 1,0), for instance, implies the operator relating to one proton, no neutron, one electron and no neutrino.

The first line of (8) is merely an additional const. having no physical significance, four terms of the second line denote the self-energy terms of proton, neutron, electron and neutrino respectively, and two terms of the third line show the operators representing the scatterings of proton-neutron and electron-neutrino respectively, and so on

In the following section, we may carry out the practical calculation for each term and examine its divergence character.

§ 3. Calculations

i) *Self energy* (1,0; 0,0), etc.

As was mentioned above, the four terms of the second line of (8) denote the self-energy parts. As the theory is symmetrical with respect to $\psi_1, \psi_2; \varphi_1$ and φ_2 , it is sufficient to consider only the first term (1,0; 0,0). This part is expressed in the following form,

$$\begin{aligned} (1,0; 0,0) = & -\frac{ig^2}{8\hbar c} \int_{-\infty}^{\infty} ([\{(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')\}_{1,0} \langle [(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')] \rangle_0 \\ & + [(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')]_{1,0} \langle \{(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')\} \rangle_0] \\ & - \overline{XX'})\epsilon(X-X')dw', \end{aligned} \quad (9)$$

where the subscripts 1,0 refer to numbers of proton and neutron, and $\langle \rangle_0$ means the vacuum expectation value of the quantity in the bracket.

After lengthy but straightforward calculations, we can show that (9) takes the following form,

$$(1,0; 0,0) = \frac{1}{4} \{ [\bar{\psi}_1(x), \phi_1(x)] + [\phi_1(X), \psi_1(x)] \}, \quad (10)$$

$$\phi_1(X) = \frac{g^2}{\hbar c} \int_{-\infty}^{\infty} A_1(X-X') \psi_1(X') d\tau', \quad (11)$$

$$\begin{aligned} A_1(X) = & D_{1\lambda}^{(1)} D_{2\lambda}^{(1)} \bar{A}_{2\mu} \gamma_\mu - K_2 D_{1\lambda}^{(1)} D_{2\lambda}^{(1)} \bar{A}_2 - 4 \bar{D}_{1\lambda} \bar{D}_{2\lambda} \bar{A}_{2\mu} \gamma_\mu + 4 K_2 D_{1\lambda} \bar{D}_{2\lambda} \bar{A}_2 \\ & - x_1 x_2 D_1^{(1)} D_2^{(1)} \bar{A}_{2\mu} \gamma_\mu + 4 x_1 x_2 \bar{D}_1 \bar{D}_2 \bar{A}_{2\mu} \gamma_\mu + x_1 x_2 K_2 D_1^{(1)} D_2^{(1)} \bar{A}_2 - 4 x_1 x_2 K_2 \bar{D}_1 \bar{D}_2 \bar{A}_2 \\ & + \bar{D}_{1\lambda} D_{2\lambda}^{(1)} A_{2\mu}^{(1)} \gamma_\mu - K_2 \bar{D}_{1\lambda} D_{2\lambda}^{(1)} A_2^{(1)} + D_{1\lambda}^{(1)} \bar{D}_{2\lambda} A_{2\mu}^{(1)} \gamma_\mu - K_2 D_{1\lambda}^{(1)} D_{2\lambda} A_2^{(1)} \\ & - x_1 x_2 \bar{D}_1 D_2^{(1)} A_{2\mu}^{(1)} \gamma_\mu - x_1 x_2 D_1^{(1)} \bar{D}_2 A_{2\mu}^{(1)} \gamma_\mu + x_1 x_2 K_2 \bar{D}_1 D_2^{(1)} A_2^{(1)} + x_1 x_2 K_2 D_1^{(1)} \bar{D}_2 A_2^{(1)}, \end{aligned} \quad (12)$$

where $A_1, A_2, D_1, D_2; K_1, K_2, x_1, x_2$ are the invariant delta functions and reciprocal Compton wave lengths of proton, neutron, electron and neutrino respectively.

Introducing the Fourier transform $A_1(k) = \int A_1(X) e^{-ikX} d^4X$ of (12), $A_1(X)$, (11) becomes

$$\phi_1(X) = \frac{g^2}{\hbar c} A_1(p) \psi_1(X), \quad (11')$$

where it is to be understood that $p_\mu = \frac{1}{i} \frac{\partial}{\partial X_\mu}$.

Using the Schwinger's integral representations of the delta functions, $A_1(p)$ becomes

$$\begin{aligned} A_1(p) = & \frac{-i}{2} \frac{1}{(2\pi)^8} \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dc e^{iax_1^2 + ibx_2^2 + icK_2^2} \{ \varepsilon(a) + \varepsilon(b) + \varepsilon(c) + \varepsilon(a)\varepsilon(b)\varepsilon(c) \} \\ & \times \int d^4k \int d^4l e^{ial^2 + ibl^2 + ic(p-k-l)^2} \\ & \times [(-i\gamma_\mu) k_\lambda l_\lambda (p-k-l)_\mu + x_1 x_2 (-i\gamma_\mu) (p-k-l)_\mu + K_2 k_\lambda l_\lambda + x_1 x_2 K_2]. \end{aligned} \quad (13)$$

To evaluate the above expression (13), we first perform the integration over four vectors k and l by use of the formulas

$$\left. \begin{aligned} \int_{-\infty}^{\infty} \exp[iak_\lambda^2] d^4k &= \frac{i\pi^2}{a^2} \epsilon(a), \\ \int_{-\infty}^{\infty} k_\mu \exp[iak_\lambda^2] d^4k &= 0, \\ \int_{-\infty}^{\infty} k_\mu k_\nu \exp[iak_\lambda^2] d^4k &= -\delta_{\mu\nu} (\pi^2/2a^3) \epsilon(a), \\ \int_{-\infty}^{\infty} k_\mu^2 \exp[iak_\lambda^2] d^4k &= -(2\pi^2/a^3) \epsilon(a), \end{aligned} \right\} \quad (14)$$

and replace the operators $i\gamma_\mu p_\mu$ and p_μ^2 by $-K_1$ and $-K_1^2$ respectively as they operate on $\psi_1(X)$ which satisfies the free field equation. Then we get for $A_1(p)$ the following p -independent expression (therefore denoting it by $A_1(K_1)$)

$$A_1(K_j) = \frac{1}{2^7 \pi^4} \int_0^\infty dt_1 \int_0^\infty dt_2 \left[\left\{ 3K_1 \frac{t_1 t_2}{T^4} + 2K_2 \frac{t_2}{T^3} \right\} \int_{-\infty}^\infty \frac{\epsilon(x)}{x^3} e^{iBx} dx \right. \\ \left. + i \left\{ -K_1^3 \frac{t_1^2 t_2^2}{T^5} + x_1 x_2 K_1 \frac{t_1}{T^3} - K_1^2 K_2 \frac{t_1 t_2^2}{T^4} + x_1 x_2 K_2 \frac{1}{T^2} \right\} \int_{-\infty}^\infty \frac{\epsilon(x)}{x^2} e^{iBx} dx \right], \quad (15)$$

here we have introduced new variables x, t_1, t_2 defined by $a=x, b=t_1 x, c=t_2 x$ and we have put

$$T = t_1 + t_1 t_2 + t_2, \quad B = x_1^2 + t_1 x_2^2 + t_2 K_2^2 - \frac{t_1 t_2}{T} K_1^2.$$

Carrying out the integration of (15), and inserting it into (11'), we find that $\phi_1(X)$ is simply a multiple of $\psi_1(X)$,

$$\phi_1(X) = \delta M_1 c^2 \psi_1(X), \quad (11'')$$

$$\text{and} \quad \delta M_1 = \frac{-1}{2^7 \pi^4} \frac{g^2}{\hbar c^3} \left(\frac{K_1}{2} + K_2 \right) \frac{\log y}{x^2} + \left(\begin{array}{l} \text{lower order divergent terms} \\ \text{with respect to } x \text{ or } y \text{ alone} \end{array} \right) \\ + (\text{finite terms}), \quad (16)$$

where both x (dimension = (length)²) and y (dimensionless) tend to 0 independently.

From these results it is evident that self energy divergence can be renormalized into the original mass M_1 . It is to be noticed that the self-mass δM_1 contains two independent divergences, which is due to the fact that in the intermediate states there appear three particles and two of them can take arbitrary magnitude of energy and momentum.

ii) *Proton-neutron (Electron-neutrino) scattering* (1,1; 0,0), etc.

For the aforesaid reason, we consider only the first term (1,1; 0,0) of the third line of (8). This term can be written in the following form,

$$(1,1; 0,0) = -\frac{ig^2}{4\hbar c} \int_{-\infty}^\infty \{ (\bar{\psi}_1 \psi_2, \bar{\psi}_2' \psi_1')_{1,1} \langle [(\bar{\varphi}_1 \varphi_2), (\bar{\varphi}_2' \varphi_1')] \rangle_0 - (\bar{X} X') \} \epsilon(X - X') dw'. \quad (17)$$

Performing the tedious calculation in a similar manner as above, the divergent part can be shown to be

$$(1,1; 0,0) = \frac{-g^2}{8\pi^2 \hbar c} \left[\{ K_1^2 + K_2^2 - 2(x_1^2 + x_1 x_2 + x_2^2) \} (\bar{\psi}_1 \psi_2) (\bar{\psi}_2 \psi_1) \right. \\ \left. + (\bar{\psi}_1 \psi_2) \left(\frac{\partial \bar{\psi}_2}{\partial X_\mu} \frac{\partial \psi_1}{\partial X_\mu} \right) + \left(\frac{\partial \bar{\psi}_1}{\partial X_\mu} \frac{\partial \psi_2}{\partial X_\mu} \right) (\bar{\psi}_2 \psi_1) \right] \int_0^\infty \frac{\cos w}{w} dw. \quad (18)$$

(By use of the old method of perturbation calculation, instead of the terms with derivatives we obtain $-2x_1 x_2 (\bar{\psi}_1 \psi_2) (\bar{\psi}_2 \psi_1)^{(5)}$.) It is evident that such a divergence can be renormalized neither into the original mass M_1 nor into the coupling const. g of (1).

iii) *Nucleon-lepton scattering* (1,0; 1,0), etc.

Though the first two terms and two succeeding terms of the fourth line of

(8) have the slightly different forms, we may consider the first term only, since the calculations are performed in the same way in both cases.

Proceeding as above, we find

$$(1,0; 1,0) = -\frac{ig^2}{8\hbar c} \int_{-\infty}^{\infty} ([\{(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')\}_{1,0}[(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')]\}_{1,0} \\ + [(\bar{\psi}_1\psi_2), (\bar{\psi}_2'\psi_1')]\}_{1,0}\{(\bar{\varphi}_1\varphi_2), (\bar{\varphi}_2'\varphi_1')\}_{1,0}] - \overline{XX'})(\epsilon X - X') d\tau v', \quad (19)$$

and its diverging part becomes

$$(1,0; 1,0) = \frac{g^2}{64\pi^2\hbar c} \left\{ -\frac{1}{3} \left(\bar{\psi}_1 \gamma_\mu \frac{\partial}{\partial X_\nu} \psi_1 \right) \left(\bar{\varphi}_1 \gamma_\mu \frac{\partial}{\partial X_\nu} \varphi_1 \right) \right. \\ - \frac{1}{3} \left(\bar{\psi}_1 \gamma_\mu \frac{\partial}{\partial X_\nu} \psi_1 \right) \left(\bar{\varphi}_1 \gamma_\nu \frac{\partial}{\partial X_\mu} \varphi_1 \right) \\ + \left(K_2 + \frac{K_1}{3} \right) \left(\bar{\psi}_1 \frac{\partial}{\partial X_\nu} \psi_1 \right) (\bar{\varphi}_1 \gamma_\nu \varphi_1) + \left(x_2 + \frac{x_1}{3} \right) (\bar{\psi}_1 \gamma_\nu \psi_1) \left(\bar{\varphi}_1 \frac{\partial}{\partial X_\nu} \varphi_1 \right) \\ + \frac{1}{2} \left(K_2^2 + x_2^2 - \frac{K_1^2 + x_1^2}{3} \right) (\bar{\psi}_1 \gamma_\mu \psi_1) (\bar{\varphi}_1 \gamma_\mu \varphi_1) \\ \left. - (K_2 x_1 + K_1 x_2 + 2K_2 x_2 + \frac{1}{3} K_1 x_1) (\bar{\psi}_1 \psi_1) (\bar{\varphi}_1 \varphi_1) + c.c. \right\} \int_0^\infty \frac{\cos w}{w} dw. \quad (20)$$

This divergence is also unrenormalizable in any way.

The terms of fifth and sixth lines do not diverge and so they are not considered here.

§ 4. Conclusion

From the above results we may conclude as follows: for the direct interaction between spinor fields the renormalization procedure is applicable only to the self-energy divergence, and it can not eliminate the other divergences such as (1,1; 0,0), (1,0; 1,0), etc. (their orders of divergence are both logarithmic).

If we take, at the outset, such types of interaction terms as $g'(\bar{\psi}_1\psi_2)(\bar{\psi}_2\psi_1)$, $g''[(\bar{\psi}_1\psi_2)(\frac{\partial\bar{\psi}_2}{\partial X_\mu}\frac{\partial\psi_1}{\partial X_\mu}) + (\frac{\partial\bar{\psi}_1}{\partial X_\mu}\frac{\partial\psi_2}{\partial X_\mu})(\bar{\psi}_2\psi_1)]$, $g'''(\bar{\psi}_1\psi_1)(\bar{\varphi}_1\varphi_1)$ in addition to (1), the divergent terms (1,1; 0,0) (1,0; 1,0), shall be renormalized into the coupling const. g' , g'' , g''' , When we assume, in this way, the various possible coupling types in the original Hamiltonian and every divergent term appearing in the second order can be renormalized into any of the original coupling terms, the whole theory will, then, constitute a consistent closed system. But such an artificial method will probably be unpromising. (though we cannot conclude it definitely at present.) Because, we can expect that there appears any other type of divergence than that we have obtained above (i.e., terms with

higher derivatives of ψ 's and φ 's) and moreover the canonical field theory in the present form cannot treat correctly the interaction type involving time-derivative (such as Konopinski-Uhlenbeck type) as was pointed out by Fierz⁷⁾.

Similar difficulty will probably appear in the case of ordinary pair theories⁶⁾, since the character of divergence is mostly determined by the nature and number of several delta functions appearing in the evaluation of matrix elements. These circumstances give rise to the serious difficulties in the rigorous consideration of various models in the two-meson theory.

We have no consistent means, at present, but the regularization procedure⁸⁾ to avoid these defects entirely.

The author wishes to express his cordial thanks to Prof. S. Sakata and Mr. H. Umezawa for their continual encouragement and valuable guidance.

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Behavior of D -Function in Yukawa's Non-Local Field Theory

Yôrô ÔNO and Masao SUGAWARA

Physics Department, Hokkaido University, Sapporo

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§ 1. Introduction

Yukawa's non-local field theory has been subjected to further investigations by many authors and Yukawa himself. They mostly deal with interactions with other fields and how these interactions are constructed seems to play a decisive role in the future development of the theory. However, it is also worthwhile to examine the physical meaning of the theory as far as possible within the framework of free field. Considering that many difficulties of local field theory arise from the singularity of D -function on the light cone, we have investigated the behaviors of D -function and its associated functions in Yukawa's non-local field theory.

The D -function in local field theory has the following characteristics:

- (i) $(\partial^2/\partial X_\mu \partial X^\mu - x^2)D(X^\mu) = 0$,
- (ii) $(\partial D(X^\mu)/\partial X^4)|_{X^4=0} = \delta(\mathbf{X})$,
- (iii) $D(X^\mu) = 0$, for $X_\mu X^\mu > 0$.

The problem is to investigate how the above features are to be modified in non-local field theory.

§ 2. Commutation relations

For charged scalar field, field quantities are

$$\left. \begin{aligned} U(X^\mu, r^\mu) &= \int \cdots \int (dk)^4 (dl)^4 \bar{u}(k_\mu, l_\mu) \exp(i k_\mu X^\mu) \prod_\mu \delta(r_\mu + l_\mu), \\ U^*(X'^\mu, r'^\mu) &= \int \cdots \int (dk')^4 (dl')^4 \bar{u}^*(k'_\mu, l'_\mu) \exp(-i k'_\mu X'^\mu) \prod_\mu \delta(r'_\mu - l'_\mu) \end{aligned} \right\} \quad (1)$$

where notations are the same as in Yukawa's paper¹⁾. From the commutation relation¹⁾ of \bar{u} and \bar{u}^* :

$$\begin{aligned} [\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] &= \\ &= -\frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(k_\mu l^\mu) \delta(l_\mu l'^\mu - \lambda^2) \prod_\mu \delta(k_\mu - k'_\mu) \delta(l_\mu - l'_\mu). \end{aligned}$$

we obtain

$$\begin{aligned}
[U(X^\mu, r^\mu), U^*(X'^\mu, r'^\mu)] &= -\Pi \delta(r_\mu + r'_\mu) \delta(r_\mu r^\mu - \lambda^2) \cdot \\
&\cdot \int \cdot \int (dk)^4 \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(k_\mu r^\mu) \exp(ik_\mu (X^\mu - X'^\mu)) \\
&\equiv \Pi \delta(r_\mu + r'_\mu) \delta(r_\mu r^\mu - \lambda^2) D'(X^\mu - X'^\mu, r^\mu),
\end{aligned} \quad (2)$$

where

$$D'(X^\mu, r^\mu) = - \int \cdot \int (dk)^4 \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \delta(k_\mu r^\mu) \exp(ik_\mu X^\mu). \quad (3)$$

From equation (2) it can readily be seen that

1) U and U^* are commutable unless relative coordinates r^μ and r'^μ are mutually antiparallel;

2) its commutator depends only upon the difference of mean coordinates X^μ and X'^μ (homogeneous character of space).

$D'(X^\mu, r^\mu)$ differs from the conventional D -function only through the factor $\delta(k_\mu r^\mu)$, which must be introduced in order that D' may coincide with D in the limit $\lambda \rightarrow 0$. Due to this factor the integration domain of k_μ space is limited to the region perpendicular to the relative coordinate r^μ on the hyperbolic surface $k_\mu k^\mu + x^2 = 0$. Thus it can be expected that the direction of r^μ plays an important role in the behavior of D' -function.

§ 3. Properties of D' -function

We extend (3) to the following form:

$$D'(X^\mu, r^\mu, C) = - \frac{i}{8\pi^3} \int \cdot \int (dk)^4 \frac{k_4}{|k_4|} \delta(k_\mu k^\mu + x^2) \frac{1}{2} \delta(k_\mu r^\mu \pm C) \exp(ik_\mu X^\mu), \quad (4)$$

where

$$\frac{1}{2} \delta(k_\mu r^\mu \pm C) \equiv \frac{1}{2} \{ \delta(k_\mu r^\mu + C) + \delta(k_\mu r^\mu - C) \},$$

introducing an arbitrary real constant C . (4) reduces to (3) in the case $C=0$. This factor $\delta(k_\mu r^\mu \pm C)$ means that k_{II} (the component of \mathbf{k} in the direction of \mathbf{r}) takes only the value $\pm \frac{C}{\lambda}$ in the coordinate system $r^4 = 0$, where $k_\mu r^\mu \pm C = \mathbf{k} \cdot \mathbf{r} \pm C = k_{II} r \pm C$, $r_\mu r^\mu - \lambda^2 = \mathbf{r}^2 - \lambda^2 = r^2 - \lambda^2$ and hence $r = \lambda$. Owing to the factor $\delta(k_\mu k^\mu + x^2)$, the general D -function $D'(X^\mu, r^\mu, C)$ satisfies the equation (i) of § 1. Performing the integrations in the usual way, we get

$$\begin{aligned}
D'(X^\mu, r^\mu, C) &= \frac{i}{16\pi^3} \int d\mathbf{k} \frac{1}{K} \frac{1}{2} \delta(\mathbf{k} \cdot \mathbf{r} - K r^4 \pm C) \cdot \\
&\cdot \{ \exp(i(\mathbf{k} \cdot \mathbf{X} - K X^4)) - \exp(-i(\mathbf{k} \cdot \mathbf{X} - K X^4)) \} \quad (K = \sqrt{k_\perp^2 + x^2}) \\
&= \frac{1}{4\pi^2 \lambda} \cos \frac{C X_{II}}{\lambda} \int_0^\infty \frac{k_\perp dk_\perp}{\sqrt{k_\perp^2 + (C^2/\lambda^2) + x^2}} J_0(k_\perp |X_\perp|) \cdot \sin \sqrt{k_\perp^2 + (C^2/\lambda^2) + x^2} \cdot X^4
\end{aligned}$$

$$= \frac{1}{4\pi^2\lambda} \cos \frac{CX_{\parallel}}{\lambda} \frac{1}{|X_{\perp}|} \frac{\partial}{\partial |X_{\perp}|} |X_{\perp}| \int_0^{\infty} \frac{dk_{\perp}}{\sqrt{k_{\perp}^2 + (C^2/\lambda^2) + x^2}} J_1(k_{\perp}|X_{\perp}|) \cdot \sin \sqrt{k_{\perp}^2 + (C^2/\lambda^2) + x^2} \cdot X^4, \quad (5)$$

where the suffices \parallel and \perp mean respectively parallel and perpendicular components with reference to the direction of \mathbf{r} . The expression (5) corresponds to

$$D(X^\mu) = -\frac{1}{2\pi^2} \frac{1}{R} \frac{\partial}{\partial R} \int_0^{\infty} \frac{dk}{K} \cos kR \sin KX^4 \quad (R=|\mathbf{X}|) \quad (5')$$

in the conventional case. Further integration of (5) cannot indeed be performed, but in the special case $C=0$, $x=0$, we get

$$\begin{aligned} D'(X^\mu, r^\mu) &= \frac{1}{4\pi^2\lambda} \frac{X^4}{|X^4|} \frac{1}{\sqrt{X^{42} - |X_{\perp}|^2}} \quad \text{for } |X_{\perp}|^2 < |X^4|^2, \\ &= \pm \infty \quad \text{for } |X_{\perp}|^2 = |X^4|^2, \\ &= 0 \quad \text{for } |X_{\perp}|^2 > |X^4|^2, \end{aligned} \quad (6)$$

which corresponds to

$$D(X^\mu) = \frac{1}{4\pi R} \{ \delta(R - X^4) - \delta(R + X^4) \} \quad (6')$$

in the usual case. From (6) we see that i) D' -function is independent of X_{\parallel} and ii) its singularity does not lie on the light cone but on a surface $|X_{\perp}|^2 - |X^4|^2 = 0$, whose axis is in the direction of \mathbf{r} , as shown in Fig. 1. As this surface has the shape of wedge, we shall call this surface "light wedge". In the general case $C \neq 0$ and/or $x \neq 0$ the singular surface is also the light wedge, and $D'(X^\mu, r^\mu, C) = 0$ outside the light wedge, as will be shown in the following.

Inserting the relation

$$\begin{aligned} \frac{1}{2}(k_\mu r^\mu \pm C) &= \frac{1}{2\pi} \frac{1}{2} \int_{-\infty}^{+\infty} da \cdot \exp(i(k_\mu r^\mu \pm C)a) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} da \cdot \cos Ca \exp(ik_\mu r^\mu a) \end{aligned}$$

into (4), we obtain

$$D'(X^\mu, r^\mu, C) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} da \cdot \cos Ca D(X^\mu + ar^\mu) \quad (7)$$

whose integration domain is the straight line $X^\mu + ar^\mu$ in the four-dimensional X -space. Since the D -function in local theory is different from zero only inside the light cone, only that part of the straight line which is intercepted by the light cone contributes to the integration (see Fig. 2). The intersecting points are determined by the equation

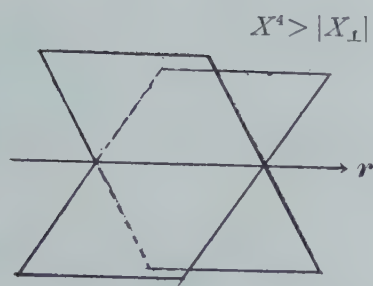


Fig. 1

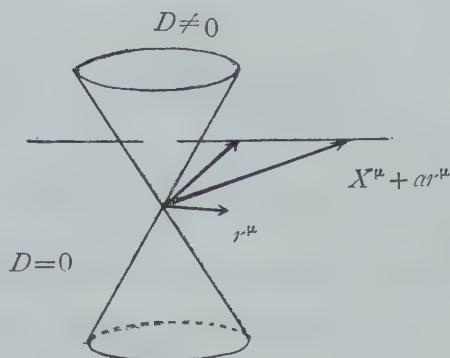


Fig. 2

$$(X_\mu + ar_\mu)(X^\mu + ar^\mu) = 0$$

$$(X_{II}a\lambda)^2 = |X^4|^2 - |X_\perp|^2 \quad (r^4 = 0).$$

If $|X^4|^2 < |X_\perp|^2$, all real a values do not satisfy the above equation, that is, $D'(X^\mu, r^\mu, C)$ vanishes outside the light wedge. In case $C=0$, $D'(X^\mu, r^\mu)$ takes the same value for all X^μ 's that terminate on the straight line $X^\mu + ar^\mu$. $D'(X^\mu, r^\mu, C=0)$ is thus independent of X_{II} .

§ 4. Infinite character of commutation relations

That $D'(X^\mu, r^\mu, C)$ depends on X_{II} only through the factor $\cos(CX_{II}/\lambda)$ means that two field quantities do not commute with each other, even if two world points are infinitely far apart from each other in the direction of relative coordinate. The relation (ii) of § 1 can be transformed into

$$\begin{aligned} \left. \frac{\partial D'(X^\mu, r^\mu, C)}{\partial X^4} \right|_{X^4=0} &= \frac{1}{4\pi^2 r} \cos \frac{rC}{\lambda^2} X_{II} \int_0^\infty dk J_0(k|X_\perp|) \cos \frac{\sqrt{C^2 + (k^2 + x^2)}\lambda^2}{\lambda^2} r^4 X_{II} \\ &= \frac{1}{2\pi\lambda} \cos \frac{C}{\lambda} X_{II} \cdot \delta(X_\perp) \quad (r^4=0), \end{aligned} \quad (8)$$

which indicates that the commutation relations have infinitesimal character in the direction perpendicular to relative coordinate r , while they have infinite character in the direction of relative coordinate r .

§ 5. \bar{D} -function

In order to examine the behavior of D' -function inside the light wedge, we define, after the example of Schwinger²⁾, the following \bar{D}' -function:

$$\bar{D}'(X^\mu, r^\mu, C) = \frac{1}{(2\pi)^4} P \int (dk)^4 \frac{e^{ik_\mu X^\mu}}{k_\mu k^\mu + x^2} \frac{1}{2} \delta(k_\mu r^\mu \pm C), \quad (9)$$

which satisfies

$$\left(\frac{\partial^2}{\partial X_\mu \partial X^\mu} - x^2\right) \bar{D}'(X^\mu, r^\mu, C) = -\delta(X^\mu, r^\mu, C), \quad (10)$$

where

$$\delta(X^\mu, r^\mu, C) = \frac{1}{(2\pi)^4} \int (dk)^4 \exp(ik_\mu X^\mu) \frac{1}{2} \delta(k_\mu r^\mu \pm C). \quad (11)$$

Performing the integrations in (11), we get

$$\delta(X^\mu, r^\mu, C) = \frac{1}{2\pi r} \delta(X_\perp) \delta\left(X^4 - \frac{r^4}{r} X_{II}\right) \cos \frac{C}{r} X_{II},$$

from which follows, by integration with respect to X^4 ,

$$\int dX^4 \delta(X^\mu, r^\mu, C) = \frac{1}{2\pi r} \delta(X_\perp) \cos \frac{C}{r} X_{II}, \quad (12)$$

which coincides with (8). The relation between D' - and \bar{D}' -functions is given by, as in the local case,

$$D'(X^\mu, r^\mu, C) = 2\varepsilon(X) \bar{D}'(X^\mu, r^\mu, C), \quad (13)$$

with

$$\varepsilon(X) = -\frac{\varepsilon_\mu X^\mu}{|\varepsilon_\mu X^\mu|}, \quad (13)$$

where ε^μ is, in the local case, an arbitray future vector ($\varepsilon_\mu \varepsilon^\mu < 0$, $\varepsilon^4 > 0$), while, in the non-local case, it has to fulfil the additional condition $\varepsilon_\mu r^\mu = 0$.

Carrying out the integrations in (9), following Schwinger, yields

$$\bar{D}'(X^\mu, r^\mu, C) = \frac{\sqrt{\pi^3}}{8\pi^4 \lambda} \cos \frac{C}{\lambda} X_{II} \int_0^\infty \frac{d\alpha}{\sqrt{\alpha}} \sin \left\{ (X^{42} - X_\perp^2) \alpha + \frac{1}{4\alpha} \left(x^2 + \frac{C^2}{\lambda^2} \right) + \frac{\pi}{4} \right\}. \quad (14)$$

From this equation we can readily see that $\bar{D}' = \infty$ for $X^{42} = X_\perp^2$, i.e. \bar{D}' -function is singular on the light wedge.

Incidentally it is to be remarked that, as D' is relativistic invariant and is an even function of X^μ , \bar{D}' should be in general a function of $A(= -X_\mu X^\mu)$, $B(= r_\mu X^\mu)$, λ , C , and x and, with respect to B , it should be an even function. Actually, $X^{42} - X_\perp^2 = A + B^2/\lambda^2$ and $X_{II} = B/\lambda$ and so the expression (14) has the required form.

§ 6. Generalization of D' -function

In order to get rid of the infinite character of D' -function in the direction of relative coordinate, we consider, introducing a weight factor $\rho(C)$, the following function:

$$\mathfrak{D}(X^\mu, r^\mu) = \int dC \rho(C) D'(X^\mu, r^\mu, C), \quad (15)$$

which reduces to Yukawa's D' -function when $\rho(C) = \delta(C)$.

The introduction of $\rho(C)$ is necessitated because of the requirement that the \mathfrak{D} -function should tend to the usual D -function in the limit $\lambda \rightarrow 0$. If we replace the above requirement by the one that the \mathfrak{D} -function is to tend only approximately to the D -function, we may take, for example,

$$\rho(C) = \frac{1}{\sqrt{\pi}} e^{-C^2}, \quad (16)$$

when we obtain, instead of (8),

$$\left. \frac{\partial \mathfrak{D}(X^\mu, r^\mu)}{\partial X^4} \right|_{X^4=0} = \frac{1}{2\pi\lambda} \frac{1}{\sqrt{\pi}} e^{-\frac{X_{II}^2}{4\lambda^2}} \delta(\mathbf{X}_\perp) \quad (r^4=0), \quad (17)$$

which has an extension λ in the direction r .

Assumption of the form (16) is only formal and, in order to seek for its physical meaning, considerations from different and more fundamental standpoint will be necessary.

§ 7. Conclusions

To summarize the conclusions deduced from the foregoing considerations, concerning the D' -function in Yukawa's non-local field theory:

i) The singular surface is not the light cone, but the *light wedge* whose axis lies in the direction of relative coordinate.

ii) D' -function has *singularities* on the light wedge and vanishes in the outside region.

iii) Commutation relations have infinitesimal character in the direction perpendicular to the relative coordinate, but *infinite character* in the direction of relative coordinate.

iv) In order to get rid of the infinite character, it is necessary to introduce a certain weight function of C .

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Pseudoscalar Meson Theory and Ground State of Deuteron

Gentaro ARAKI and Yukio MORI

Department of Industrial Chemistry, Kyoto University

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It is examined whether the two-nucleon adiabatic potential predicted by the pseudoscalar meson theory correctly describes the ground state of the deuteron or not, assuming an effective form of the potential. The energy eigenvalue and the corresponding eigenfunction are calculated according to the variational method. By comparing the theoretical binding energy of the deuteron with the observed value, the value of the coupling constant of mesons with nucleons is estimated to be $g^2=0.2$. The theoretical values of the electric quadrupole moment and the dipole magnetic moment of the deuteron are compared with experiment in Table II. The theory is in agreement with experiment in a very rough approximation.

Introduction

The validity of nuclear forces predicted by the meson theory is usually examined by comparing the theoretical conclusion regarding two-nucleon systems with experiment, and the strength of coupling between mesons and nucleons is so adjusted that the theoretically calculated binding energy of a deuteron agrees with the observed value. Such a comparison involved, however, an ambiguity because of a too strong singularity of a customary two-nucleon potential. It was found that the adiabatic potential has only a weak singularity so far as the scalar and the pseudoscalar meson theory are concerned.¹⁾²⁾³⁾ Further, it was shown that the customary potential is a non-relativistic approximation which is valid only for a large distance between two nucleons.²⁾

It must be of much interest that the deuteron problem is reexamined from such a new stand point. The present calculation is intended as the first step of this reexamination. It will be found that the pseudoscalar theory is satisfactory in a very rough approximation so far as the ground state of the deuteron is concerned. A rough value of a coupling constant between pseudoscalar mesons and nucleons will be obtained. Since there has been no evaluation of the constant, our result may be considered as giving a contribution to a knowledge of the constant though it still involves an ambiguity due to the method of approximation used in the calculation. In order to obtain a more definite conclusion a more advanced method having no ambiguity must be adopted. We hope that this will be considered later on. The brief account of the present calculation was already given in this journal.⁴⁾ The present paper is its detailed report.

§ 1. An Assumption of Effective Nuclear Forces

According to the pseudoscalar meson theory the *adiabatic* two-nucleon potential as a function of a relative position, \mathbf{x} , of nucleons is given by²⁾

$$W(\mathbf{x}) = -2T\rho_2^{(1)}\rho_2^{(2)}(2M/\mu)^2 J(r) \quad (1.1) (a)$$

where M and μ are masses of a nucleon and a meson respectively, $\rho_2^{(k)}$ ($k=1,2$) is Dirac's matrix of the k -th nucleon, T is a charge operator, and $J(r)$ is the Yukawa potential depending only on a distance, r , between nucleons. If we adopt a natural system of units, in which \hbar and c are equal to unity, $J(r)$ is given by

$$J(r) = (g^2/2)r^{-1}\exp(-\mu r) \quad (1.1) (b)$$

where g stands for $f_2 + (2Mf_1/\mu)$, and f_1 and f_2 denote constants of pseudoscalar and pseudovector couplings of mesons with nucleons.³⁾ The following calculation will be carried out according to the symmetrical theory in which T is given by

$$T = (\tau_1^{(1)}\tau_1^{(2)} + \tau_2^{(1)}\tau_2^{(2)} + \tau_3^{(1)}\tau_3^{(2)})/2 \quad (1.1) (c)$$

where the eigenvalue of τ_3 is equal to $+1$ in a neutron state and to -1 in a proton state. The result will be essentially the same in case of the charged theory, except for a numerical factor.

For the sake of brevity we shall use a system of mesonic units in which μ is equal to unity. In a non-relativistic approximation $W(\mathbf{x})$ reduces to a customary potential²⁾.

$$W(\mathbf{x}) = (2/3)T\{\sigma^{(1)}\sigma^{(2)} + \eta(r)A\}J(r) \quad (1.2) (a)$$

where

$$\eta(r) = 1 + 3r^{-1} + 3r^{-2} \quad (1.2) (b)$$

$$A = 3(\sigma^{(1)}\mathbf{x})(\sigma^{(2)}\mathbf{x})r^{-2} - \sigma^{(1)}\sigma^{(2)} \quad (1.2) (c)$$

This potential is valid only for $r \gg 1$. An asymptotic value of η is equal to 1 for very large r , where the approximation of (1.2) is very good, and η amounts to 7 when r is equal to 1, where the approximation of (1.2) is probably bad. In a region of smaller r , (1.2) may not be valid. We know from (1.1) that the rigorous potential increases as r^{-1} when r becomes very small. Therefore, if we assume that (1.2) (a) with a constant η represents effectively an approximate potential, the effective value of η may be between 1 and 7. For the ground state, which is triplet and even, of the deuteron this effective potential reduces to

$$W(\mathbf{x}) = -(1 + \eta A)J(r) \quad 1 \leq \eta \leq 7 \quad (1.3)$$

In the following, the ground state of the deuteron will be tentatively examined on the basis of this effective potential.* In the present stage it is not obvious whether such an assumption is adequate or not. In order to settle this point the exact solution of the deuteron problem is required.

§ 2 Method of Calculation

We shall search for the energy eigenvalue and the corresponding eigenfunc-

* The potential of Rarita and Schwinger may be considered as a sort of such a potential.⁵⁾

tion of a deuteron in its ground state according to the variational method. As is well known, this state consists of S and D . If ϑ denotes a normalized eigenfunction of 3S_1 , $A\vartheta$ represents a 3D_1 state of the same magnetic quantum number, its norm, $\|A\vartheta\|^2$, being equal to 8. Therefore the wave function of the ground state can be written as follows⁵⁾:

$$\psi(\mathbf{x}) = r^{-1}\{u(r) + v(r)A\}\vartheta \quad (2.1)$$

This function must satisfy the equation which is given by

$$\{-M^{-1}\Delta + W(\mathbf{x})\}\psi(\mathbf{x}) = E\psi(\mathbf{x}) \quad (2.2)$$

where Δ is a Laplacian with respect to \mathbf{x} . Therefore the asymptotic form of $u(r)$ and $v(r)$ is $\exp(-r\sqrt{-ME})$ because $W(\infty)$ vanishes. A simple calculation shows that the exponents of $u(r)$ and $v(r)$ at the origin ($r=0$) are 1 and 2 respectively or 4 and 3 respectively. Thus we assume the variational forms of $u(r)$ and $v(r)$ as follows:

$$u(r) = u_1(r) + cu_2(r), \quad v(r) = v_1(r) + cv_2(r) \quad (2.3)$$

$$u_1(r) = r\exp(-ar), \quad v_1(r) = \beta_1 r^2 \exp(-ar) \quad (2.4)(a)$$

$$u_2(r) = r^4 \exp(-ar), \quad v_2(r) = \beta_2 r^3 \exp(-ar) \quad (2.4)(b)$$

The approximate eigenvalue and the corresponding eigenfunction can be determined respectively as a minimum of an expectation value of the Hamiltonian:

$$\langle H \rangle = (\psi, H\psi) (\psi, \psi)^{-1} \quad (2.5)$$

and a corresponding wave function, where H is the Hamiltonian given by an operator in the brace of the left side of Eq. (2.2), and ψ is given by (2.1), (2.3) and (2.4).

The variational minimization of $\langle H \rangle$ is carried out in the following way. We first determine β_1 and β_2 of Eq. (2.4) for each given value of a in such a way that $(\psi_1, H\psi_1)(\psi_1, \psi_1)^{-1}$ and $(\psi_2, H\psi_2)(\psi_2, \psi_2)^{-1}$ are equal to their minimum values, $E_1(a)$ and $E_2(a)$ respectively, where $\psi_k(\mathbf{x}) = r^{-1}\{u_k(r) + v_k(r)A\}\vartheta$ ($k=1, 2$). We next determine c of Eq. (2.3) such that $\langle H \rangle$ given by Eq. (2.5) is equal to its minimum value, $E(a)$, for each set of β_1 , β_2 and a . We thus get a set of parameters, $\beta_1(a)$, $\beta_2(a)$, $c(a)$ and $E(a)$, which depend on the given values of a . We can finally find a minimum value, E_0 , of $E(a)$ and a corresponding value of a . The calculations are repeated for a sequence of values of the coupling constant g . In this way we can determine the eigenvalue, $E_0(g^2)$, of H as the function of g^2 , and the corresponding values of $\beta_1(g^2)$, $\beta_2(g^2)$, $c(g^2)$, and $a(g^2)$.

The value of g^2 is obtained as a root of the following equation:

$$E_0(g^2) = -0.0149 \text{ mesonic units} \quad (2.6)$$

The right side of this equation is equal to an empirical value, 2.17 Mev., of the binding energy of the deuteron where μ is assumed to be equal to 286 electron mass.

§ 3 Result and Comparison with Experiment

If we normalize the function given by Eq. (2.1) it can be written in the form

$$\phi(x) = \sqrt{1-\gamma^2} \phi_s(x) + \gamma \phi_D(x) \quad (3.1)$$

where ϕ_s and ϕ_D are the normalized functions which represents the 3S_1 and 3D_1 states respectively:

$$\|\phi_s\| = \|\phi_D\| = 1 \quad (\phi_s, \phi_D) = 0 \quad (3.2)$$

and γ^2 represents a probability of the 3D state. As was considered in the first section the effective value of γ can be assumed to be between 1 and 7. The result for these extreme cases is shown in Table I.

Table I Determined Values of Parameters

γ	g^2	a	β_1	β_2	γ^2
1	0.54	1.28	0.0865	0.1100	0.038
7	0.140	1.75	0.1770	0.1522	0.087

Since our method of calculation is of a very rough approximation we must content ourselves with a rough result, and too elaborate evaluations have no significance. We obtain a root of Eq. (2.6) by a graphical method. We recalculate $\langle H \rangle$ according to Eq. (2.5) by using values of a , β_1 , β_2 and γ which correspond to the value of this root, and compare it with E_0 . Repeating such computations we search for a value of g^2 which is consistent with Eq. (2.6). The value of g^2 in Table I give $\langle H \rangle - 0.0152$ as its value in case of $\gamma=1$ and -0.0149 in case of $\gamma=7$. The latter value is in good agreement with Eq. (2.6), but the former is inconsistent with the equation. The further recalculation changes the value of g^2 only in the next of its last figure which has no meaning so far as the present approximation is concerned. Therefore we regard the value of Table I as a root of Eq. (2.6), giving up the further calculation.

We next calculate the electric quadrupole moment, Q , and the magnetic dipole moment, μ_D , of the deuteron using these values of the parameters. The numerical data which are make use of in the computation of μ_D are 2.79245 and -1.91283 nuclear magnetons for the magnetic moments of a proton and a neutron respectively.⁶⁾ The result is compared with experiment in Table II, where Q is measured in 10^{-27}cm^2 and μ_D in nuclear magnetons.

Table II

				Q	μ_D
calc.	{	$g^2=0.54$		3.18	0.858
		$g^2=0.140$		2.72	0.830
obs.	(6) (7) (8)			2.766	0.8573

In view of the fact that the variational method can only give an eigenvalue of energy an upper limit but does not give the wave function a good approximation and that the calculation is based on the assumption of an effective nuclear potential, the result may be considered to show that the theory and the experiment are in agreement in this degree of approximation so far as the ground state of a deuteron is concerned. In case of $g^2=0.54 (\eta=1)$ the value of μ_D is in good agreement with the observed value, and the value of Q is too large, whereas in case of $g^2=0.140 (\eta=7)$ the value of Q is in good agreement with the observed value and the value of μ_D is too small.

The value of μ_D depends on the value of γ^2 and that of Q depends on the behaviour of the wave function. The value of g^2 for $\eta=1$ is larger than that for $\eta=7$. This means that the weight of the central part of the nuclear potential in the former case is larger than that for the latter case. Therefore the wave function is more diffuse (that is, a is smaller) in the former case than in the latter case, and the probability, γ^2 , of the 3D state is smaller in the former case. Consequently the values of Q and μ_D are larger in the former case.

The empirical value of g^2 may be between those two extreme values in Table II. The further significant precision can not be obtained in our method because of its nature of the rough approximation. Thus we must content ourselves with the above mentioned rough result so far as the present method is concerned. We may assume the value of g^2 to be equal to 0.2 because a too fine value is meaningless. In order to obtain a more definite conclusion it is necessary to work out with a more exact form of the potential and a more exact method of the solution. We hope that this will be carried out later on.

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On the Production of Mesons by X-rays

Kô AIZU, Yoichi FUJIMOTO and Hiroshi FUKUDA

Department of Physics, Tokyo University

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Recent experiments at Berkeley¹⁾ show firstly that, for free proton, the two processes (a) $p + \gamma \rightarrow n + \pi^+$ and (b) $p + \gamma \rightarrow p + \pi^0$ resemble each other as to the total cross section, energy and angular distributions, and secondly that, for bound neutron in nucleus, the process (c) $n + \gamma \rightarrow n + \pi^0$ occurs with comparative frequency with (c). These facts suggest that the magnetic moment of nucleon might play a more essential rôle than the charge current in this phenomena, especially when we compare the process (b) with (c). Thus we must take into account the effect of anomalous magnetic moment of nucleon (a.m.m.), which are neglected in the second order perturbation calculation in meson theory.²⁾ Moreover, according to the same theory, the cross section of (b) is smaller than that of (a) by the factor $(\kappa/M)^2$. Brueckner,²⁾ and Koba, Kotani and Nakai³⁾ calculated the higher order corrections in meson theory for charged mesons, and showed that the effect of a.m.m. is not large. But the a.m.m. which is calculated by this method can not predict its experimental values.⁴⁾ So we shall here examine the effect of a.m.m. for the above three processes by treating it phenomenologically, i.e. by adding Pauli interaction term to Hamiltonian. Of course, it is probable that the magnitude of a.m.m. will be a function of the energy of incident photon, but in our case its dynamical change will not be large, because the wave length of photon is not smaller than the dimension of the meson cloud which originates a.m.m.. Thus we may hope that our phenomenological treatment will give us some qualitative features of higher order corrections.

Our calculations are carried out for scalar(s.) and pseudoscalar (ps.) mesons with scalar ((s)), vector ((v)), pseudoscalar ((ps)) or pseudovector ((pv)) coupling. We use the following notations: M , κ , mass of nucleon and meson; p_0 , p , 4-momenta of nucleon before and after the collision; q , k , those of meson and photon. Other notations are similar to those of Feynman. The differential cross sections $d\sigma^\pm$, $d\sigma^0$ of π^\pm , π^0 mesons, in the case of no a.m.m., are given by

$$d\sigma^+ = \frac{e^2}{4\pi} \left\{ \frac{f^2/4\pi}{(2M_g/\kappa)^2/4\pi} \right\} \frac{d\omega}{p_0 k} \frac{|q|^2}{WE} \frac{d|q|}{d(W+E)} \frac{1}{4} \left[\frac{p k}{p_0 k} - \left(aM^2 - \frac{\kappa^2}{2} \right) \left(\frac{p_0}{p_0 k} - \frac{q}{q k} \right)^2 \right] \quad (1)$$

$$d\sigma^- = (p_0 k / p k)^2 d\sigma^+ \quad (2)$$

$$d\sigma^0 = (\sqrt{d\sigma^+} - \sqrt{d\sigma^-})^2 = (qk/pk)^2 d\sigma^+ \quad (3)$$

where f and g are usual coupling constants in "charged" or "neutral" meson theory, $a=2$ or 0 according to s . or ps . meson, and $p=(W, \mathbf{p})$ and $q=(E, \mathbf{q})$. In the energy range in the above experiments, we have $pk \sim p_0 k$ and $d\sigma^+ \sim d\sigma^-$ by (2) and thus $d\sigma^0$ is smaller than $d\sigma^+$ by the order of $(\kappa/M)^2$. The reason for this is that two matrix elements differing in the order of emission of meson and absorption of photon in Feynman diagram are cancelled each other.

In order to take into account a.m.m., we assume the magnetic interaction Hamiltonian^{5) 6)}

$$H = (1/2) (\epsilon/2M) \psi^+ (g_p \tau_y + g_n \tau_n) \gamma_{\lambda\sigma} \psi \cdot F_{\lambda\sigma} \quad (4)$$

where

$$F_{\lambda\sigma} = \partial_\lambda A_\sigma - \partial_\sigma A_\lambda, \quad \gamma_{\lambda\sigma} = (1/2i) (\gamma_\lambda \gamma_\sigma - \gamma_\sigma \gamma_\lambda)$$

and $g_p=1.8$, $g_n=-1.9$. The magnitude of the contribution to (1)–(3), caused by this Hamiltonian, is different according to the types of mesons, as is shown in the following table, in which it is compared with (1)–(3).

	π^\pm	π^0
scalar	comparable	small
pseudoscalar	small	comparable

The reason for this is as follows; the nucleon with a.m.m. can absorb photon both before and after the emission of meson, and the contribution of each of them to matrix elements is the same order as ordinary magnetic moment does. But the two terms have opposite signs for s . meson and the same signs for ps . meson, if the algebraic signs of a.m.m. are neglected. Hence the total effect is large for $\pi^\pm s$. and $\pi^0 ps$. mesons. For $\pi^0 s$. meson, the above consideration shows that it is comparable, but numerical calculation shows that it is accidentally small. The above result for ps . meson agrees with that obtained by higher order corrections.

The expressions corresponding to the square bracket in (1) are as follows; for $\pi^+ s$. meson in the process (a) with (s) coupling

$$\frac{pk}{p_0 k} - 2M^2 \left(\frac{p_0}{p_0 k} - \frac{q}{q k} \right)^2 + (g_p - g_n)^2 + (g_p - g_n) \frac{(p + p_0, k)}{p_0 k} \quad (5,a)$$

and, with (v) coupling

$$(g_p - g_n)^2 \frac{pk \cdot p_0 k}{4M^2}. \quad (5,b)$$

For $\pi^0 ps$. meson in the process (b) with (ps) coupling

$$\left(1+2g_p+\frac{g_p^2}{2}\right)\frac{(qk)^2}{p_0k \cdot pk} + g_p^2\left(\frac{qk}{M}-\frac{x^2}{2M^2}\right) + \frac{x^2}{2}\left(\frac{p_0}{p_0k}-\frac{p}{pk}\right)^2 \quad (6,a)$$

and, with (pv) coupling

$$\left(1+g_p+\frac{g_p^2}{2}\right)\frac{(qk)^2}{p_0k \cdot pk} + g_p^2\left(-\frac{qk}{M^2}-\frac{x^2}{2M^2}+\frac{pk \cdot p_0k}{M^2}\right) + \frac{x^2}{2}\left(\frac{p_0}{p_0k}-\frac{p}{p_0k}\right)^2 - g_p\frac{x^2}{M^2} \quad (6,b)$$

For the process (c), it is obtained by picking up the terms in g_p^2 in (6) and changing g_p in g_n .

For π^+ s. meson, the angular distribution of mesons becomes more flat than $\sin^2 \theta$ distribution which is caused by the charge current. For π^0 ps . meson, the first term in (6) is roughly three times the remaining terms, and so the energy distribution of mesons is not similar to that of π^+ ps . meson, and its angular distribution is fairly smaller forward than backward direction. The total cross section of ps . meson, modified by the bremsstrahlung spectrum of incident X-rays is given by

$$\sigma = \int_x^{em} \sigma(\epsilon) \frac{d\epsilon}{\epsilon} = \frac{e^2}{4\pi} \left\{ \frac{f^2/4\pi}{(2Mg/x)^2/4\pi} \right\} \frac{\pi}{M^2} \int_s^{em} \phi(x) dx \text{ cm}^2/\text{Mc Millan}$$

where

$$x = \epsilon/M, \quad \delta = x/M = 0.15$$

and ϵ is the energy of photon, and further $\phi(x)$ is given by :

for the process (a) with (s) and (ps) coupling

$$\sqrt{x^2 - \delta^2} \frac{1+x}{x^2(1+2x)^2}$$

for the process (b) with (s) coupling

$$\sqrt{x^2 - \delta^2} \left[\left(1+2g_p+\frac{g_p^2}{2}\right) \left\{ \frac{1}{(1+x)(1+2x)^2} + \frac{x^2 - \delta^2}{3x^2(1+x)^3} \right\} \right. \\ \left. + g_p^2 \left\{ \frac{1}{(1+2x)^2} - \frac{\delta^2}{2x^2(1+2x)} \right\} \right]$$

and with (ps) coupling

$$\sqrt{x^2 - \delta^2} \left[\left(1+g_p+\frac{g_p^2}{2}\right) \left\{ \frac{1}{(1+x)(1+2x)^2} + \frac{x^2 - \delta^2}{3x(1+x)^3} \right\} \right. \\ \left. + \frac{g_p^2 x}{(1+2x)^2} - \left(g_p + \frac{g_p^2}{2}\right) \frac{\delta^2}{2x^2(1+2x)} \right]$$

for the process (c), it is obtained from that for the process (b) by picking up the terms in g_p^2 and changing g_p in g_n . It is to be noticed that the forms of $\phi(x)$

are different for π^+ and π^0 mesons. In the above expressions, small terms is neglected within the error of 15%. The numerical values of them presented in the following table are calculated for $\epsilon=235$ and 330 Mev.

Table. The values of $\sigma/\frac{e^2}{4\pi} \left\{ \frac{f^2/4\pi}{(2M_\pi/\chi)^2/4\pi} \right\} \frac{\pi}{M_1}$

process	-(a)		(b)		(c)	
max. energy of photon in Mev	235	330	235	330	235	330
(ρ^s) coupling	0.13	0.3	0.05	0.15	0.026	0.07
(ρ^v) coupling	0.13	0.3	0.017	0.06	0.013	0.034
without a.m.m.	0.13	0.3	0.006	0.016	0	0
exp. values in $\text{cm}^2/\text{McMillan}$	0.9 $\times 10^{-28}$		1.3 $\times 10^{-28}$		4 $\times 10^{-29}$	

The experimental value in the last column is estimated by making the simple assumption $\sigma^0(\text{carbon}) = 6\sigma^0(\text{proton}) + 6\sigma^0(\text{neutron})$ and using the empirical relation $\sigma^0(\text{carbon}) = 8\sigma^0(\text{proton})$. Similar values are obtained for Li and Be. From the above results, it may be concluded that the neutral ρ^s meson theory with a.m.m. can give the reasonable total cross section for processes (b) and (c), if the couplings of charged and neutral mesons to nucleons are the same order. But, it is questionable that it can explain the energy and angular distributions. In this point, it must be remembered that our calculation is tentative and higher order corrections may give better results.⁸⁾ In order to get any definite conclusion, it is very desirable to make more precise measurements about the energy and angular distributions and the ratio of π^+ and π^0 mesons. But, if the above discrepancy is true, we must search for other possibilities, for example, the strong coupling approximation,⁸⁾ or a new type interaction.

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On the Problem of Covariance in Quantum Electrodynamics, II

Jiro YUKAWA and Hiroomi UMEZAWA

Institute of Theoretical Physics, Nagoya University

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§ 1. Introduction and summary

It is well known that, in quantum electrodynamics, there exist the divergence difficulties of mass type and vacuum polarization type. In the problem of self-stress of electron, these two types of divergence constructed their respective difficulties. In the case of Bose particle, however, it is expected that these two causes get entangled and cannot be treated separately. As for the energy-momentum tensor of photon, there appears only the vacuum polarization type, and the source, which destroys the covariance of the whole theory, spoils, at the same time, the requirement of its gauge invariance. Thus it is an interesting problem to investigate the above situation. Now we shall clear up such a situation of the energy momentum tensor of photon at first.

In general, the energy momentum tensor $\theta_{\mu\nu}$ is required to satisfy the following two conditions:

[1] conservation equation (the condition of covariance),

$$\frac{\partial \theta_{\mu\nu}}{\partial x_\nu} = 0 \quad (\text{in coordinate space}), \quad (1)$$

$$\Delta p_\nu \theta_{\mu\nu}(p', p) = 0 \quad (\text{in momentum space}).$$

[2] gauge invariance.

The condition [1] includes the requirement, as its special case, that the self-stress of electron at rest must be zero¹⁾²⁾. In the following, we call [1] the requirement of covariance. We shall calculate the e^2 -correction for energy-momentum tensor of photon interacting with electrons, investigate whether above conditions [1] and [2] are satisfied or not, and inquire those origins which [1] is not satisfied by, as well as which [2] is not satisfied by, if they cannot hold good, and then throw light on relation between these origins of difficulties and that of difficulty of self-energy.

In the investigation of self-stress of electron, the so-called Pais' formula³⁾

$$\langle \bar{\psi}\psi \rangle_{e^2} = \frac{\partial \langle \delta x \rangle_{e^2}}{\partial x_0} \quad (x_0: \text{rest mass of electron}) \quad (2)$$

holds true only for mass-type processes, but not in general for vacuum polarization-type process¹⁾. Therefore we will inquire whether above formula (2) is valid in the case of photon, since there appear only vacuum-polarization-type-processes.

§ 2. e^2 -correction of energy-momentum tensor of photon

Symmetric gauge invariant energy-momentum tensor density $\theta_{\mu\nu}$ of the whole system in which photon interacts with electron-positron, is represented in the interaction representation as follows:

$$\begin{aligned}\theta_{\mu\nu} = & \frac{1}{2}(F_{\mu\lambda} F_{\nu\lambda} + F_{\nu\lambda} F_{\mu\lambda} - \frac{1}{2}\delta_{\mu\nu} F_{\lambda\sigma}^2) \\ & + \frac{1}{2}(\bar{\psi}\gamma_{(\mu}\frac{\partial\psi}{\partial x_{\nu)}} - \frac{\partial\psi}{\partial x_{(\nu}}\gamma_{\mu)}\psi) \\ & - \frac{1}{2}ic(\bar{\psi}\gamma_{(\mu}\psi A_{\nu)}) \\ & + \frac{1}{2}ie\delta_{(\nu\lambda}\bar{\psi}(\gamma_{\mu)}\gamma_{\lambda}\gamma_{\lambda} + \gamma_{\lambda}\gamma_{\lambda}\gamma_{\mu)})\psi A_{\lambda},\end{aligned}\quad (3)$$

where $F_{\mu\lambda}$, and ψ are field strength of electromagnetic field A_ν , and wave function of electron respectively and $P_{(\mu} Q_{\nu)} = P_\mu Q_\nu + P_\nu Q_\mu$. In the following we shall calculate e^2 -correction of $\theta_{\mu\nu}$, where the diagrams contributed are classified into the following three types. (See Fig. 1).

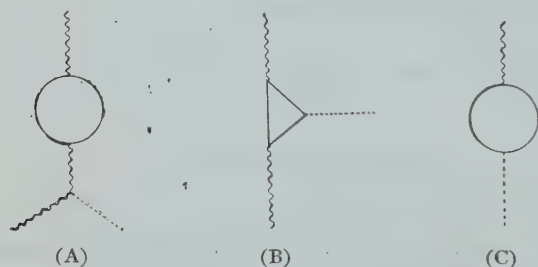


Fig. 1. ~~~~~: photon line, —: electron line,: assumed as external field.

The first line of (3) is connected with diagram (A), second line, diagram (B) and third line, diagram (C).

The results of fourth line of (3) cancel the spacelike surface-dependent terms of the results from other lines in the whole calculation.

The contributions due to (A)

(B) and (C), S^A , S^B , and S^C are obtained as follows:

$$\begin{aligned}S^A = & -\frac{1}{2}(S_1^A(\mu, \nu) + S_1^A(\nu, \mu) - \frac{1}{2}\delta_{\mu\nu}(S_2^A(\lambda, \sigma))^2), \\ S_1^A = & -\frac{e^2}{8}\int_{-\infty}^{\infty}\int dx_1 dx_2 \left(\frac{\partial A_\lambda(x_0)}{\partial x_{(\mu}^0} - \frac{\partial A_{(\mu}(x_0)}{\partial x_{\lambda}^0} \right) A_w(x) \left(\delta_{\lambda\rho} \frac{\partial D_F(x_0 - x_1)}{\partial x_{\nu}^0} \right. \\ & \left. - \delta_{\nu\rho} \frac{\partial D_F(x_0 - x_1)}{\partial x_{\lambda}^0} \right) \times T_F S_F(x_1 - x_2) \gamma_\rho S_F(x_0 - x_1) \gamma_\sigma,\end{aligned}$$

$$\begin{aligned}
 S_2^A &= -\frac{e^2}{8} \int_{-\infty}^{\infty} dx_1 dx_2 \left(\frac{\partial A_\sigma(x_0)}{\partial x_\lambda^0} - \frac{\partial A_\lambda(x_0)}{\partial x_\sigma^0} \right) A_\omega(x_2) \left(\delta_{\sigma\rho} \frac{\partial D_F(x_0-x_1)}{\partial x_\lambda^0} \right. \\
 &\quad \left. - \delta_{\lambda\rho} \frac{\partial D_F(x_0-x_1)}{\partial x_\sigma^0} \right) \times T_r S_F(x_1-x_2) \gamma_\rho S_F(x_2-x_1) \gamma_\omega, \\
 S^B &= \frac{1}{4} (S_1^B(\mu, \nu) + S_1^B(\nu, \mu) - S_2^B(\mu, \nu) - S_2^B(\nu, \mu)), \\
 S_1^B &= \frac{e^2}{16} \int_{-\infty}^{\infty} dx_1 dx_2 A_\rho(x_1) A_\omega(x_2) T_r S_F(x_1-x_2) \gamma_\omega S_F(x_2-x_0) \gamma_{(\mu} \frac{\partial S_F(x-x_0)}{\partial x_\nu^0} \gamma_{\rho)}, \\
 S_2^B &= \frac{e^2}{16} \int_{-\infty}^{\infty} dx_1 dx_2 A_\rho(x_1) A_\omega(x_2) T_r S_F(x_1-x_2) \gamma_\omega \frac{\partial S_F(x_0-x_1)}{\partial x_\nu^0} \gamma_{(\mu} S_F(x_0-x_1) \gamma_{\rho)}, \\
 S^C &= \frac{1}{2} (S^C(\mu, \nu) + S^C(\nu, \mu)), \\
 S^C &= -\frac{ie^2}{4} \int_{-\infty}^{\infty} dx_1 A_\nu(x_0) A_\lambda(x_1) T_r S_F(x_0-x_1) \gamma_\lambda S_F(x_1-x_0) \gamma_\mu, \\
 \theta e_{\mu\nu}^0 &= S^A + S^B + S^C.
 \end{aligned} \tag{4}$$

§ 3. Covariance and gauge invariance

In the following, we shall derive the condition to be required for energy-momentum tensor to satisfy above two conditions (1) and (2). Now, we define $K_{\mu\nu}(x-x')$ as follows:

$$K_{\mu\nu}(x-x') = T_r S_F(x-x') \gamma_\mu S_F(x'-x) \gamma_\nu. \tag{5}$$

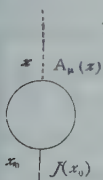
Moreover, e^2 -correction of vacuum induced current $\delta J_\rho(x_0)$, the diagram of which is shown in fig 2, is given by

$$\delta J_\rho(x_0) = -\frac{ie^2}{4} \int dx K_{\rho\mu}(x_0-x) A_\mu(x). \tag{6}$$

Then, in order to see whether the condition of covariance is satisfied, we differentiate eqs. (4) directly and we get

$$\frac{\partial \theta_{\mu\nu}(x_0)}{\partial x_\nu^0} = f_\mu \left(\frac{\partial \delta J_\lambda(x)}{\partial x_\lambda} \right) + \frac{1}{4} \left\{ \delta J_\lambda(x_0) F_{\mu\lambda}(x_0) + F_{\mu\lambda}(x_0) \delta J_\lambda(x_0) \right\}, \tag{7}$$

Fig 2.



$$(\gamma_\lambda \partial / \partial x_\lambda + x_0) S_F(x) = 2i\delta(x), \tag{8}$$

$$\square D_F(x) = 2i\delta(x).$$

In eq. (7), the first term, $f_\mu(x)$ vanishes, when $x=0$.

Since the source of field, $J_\lambda^{ext}(x_0)=0$ in the present problem, the term

$J_{\lambda}^{ext}(x_0) \delta F_{\mu\lambda}(x_0)$ does not appear on the right hand side of (7), where $\delta F_{\mu\lambda}(x_0)$ ϵ^2 -correction for field strength $F_{\mu\lambda}(x_0)$ due to the vacuum polarization. Thus, $\frac{\partial \theta_{\mu\nu}(x_0)}{\partial x_{\nu}^0}$ is not always zero.

Assuming the condition of gauge invariance for vacuum induced current,

$$\frac{\partial K_{\mu\nu}(x_0)}{\partial x_{\nu}^0} = 0, \quad (9)$$

we obtain $f_{\mu} \left(\frac{\partial \delta J_{\lambda}(x)}{\partial x_{\lambda}} \right) = 0$, since $\delta J_{\lambda}(x_0)$ satisfies the equation of continuity. Then, above equation (7) is rewritten,

$$\frac{\partial \theta_{\mu\nu}(x_0)}{\partial x_{\nu}^0} = \frac{1}{4} \left\{ \delta J_{\lambda}(x_0) F_{\mu\nu}(x_0) + F_{\mu\lambda}(x_0) \delta J_{\lambda}(x_0) \right\}. \quad (10)$$

This equation shows that a part of photonic energy is lost as the work due to induced current. But, if we make induced current gauge invariant, by the condition (9), the right hand side of (10) vanishes, since photon field does not induce the current. This situation shows that the cause destroying the condition of covariance (1) coincides mathematically as well as physically, with that, of non-vanishing photonic self-energy.

Secondly, we will investigate whether the gauge invariance of $\theta_{\mu\nu}$ is satisfied by the requirement (9) of gauge invariance for induced current. In the equation (4) we carry out gauge transformation

$$A_{\lambda}(x) \longrightarrow A_{\lambda}(x) + \frac{\partial \Lambda(x)}{\partial x_{\lambda}}, \quad (11)$$

$$\square \Lambda(x) = 0,$$

and transform the terms including $\Lambda(x)$, performing partial integration and using equation (7). The application of above condition (9) gives a gauge invariant form for S^A itself and S^B and S^C together where S^B and S^C don't become to be gauge invariant separately. Accordingly, the origin by which $\theta_{\mu\nu}$ destroys the conditions [1] and [2] has the same property as the one of non-vanishing photon self-energy has and so has the same ambiguities as the self-energy of photon.

The formalistic mixed field method (regulator) making induced current gauge invariant, may give $\theta_{\mu\nu}$ that satisfies conditions (1) and (2), if we suitably set the order of integration.

However above verification has been accomplished by differentiating singular functions and therefore there still exists ambiguities. The calculation of ϵ^2 -correction for $\theta_{\mu\nu}$ by transforming equations (4) into momentum space will be very complex and its result may still depend on the method of integration. Applying such an integration as neglecting the shift due to the transformation of variables, result of such an integration is

$$\theta_{\mu\nu}^2 = -\frac{e^2}{12\pi^2} [K_{\mu\nu} + L_{\mu\nu}] \lim_{l \rightarrow \infty} \log \frac{l + \sqrt{l^2 + x_0^2}}{x_0} \quad (12)$$

$$-\frac{e^2}{12\pi^2} [(K_{\mu\nu} + L_{\mu\nu}) \frac{1}{x_0} + \frac{1}{5x_0^2} k_\rho k_\omega A_\rho A_\omega R_\mu R_\nu],$$

where

$$K_{\mu\nu} = (k_\mu A_\lambda - k_\lambda A_\mu) (k_\nu A_\lambda - k_\lambda A_\nu) - \frac{\delta_{\mu\nu}}{4} (k_\lambda A_\sigma - k_\sigma A_\lambda)^2,$$

$$L_{\mu\nu} = k_\rho k_\omega A_\rho A_\omega \delta_{\mu\nu} + 2A^2 k_\mu k_\nu. \quad (12')$$

In (12'), $K_{\mu\nu}$, the last term of the second line and the first term in $L_{\mu\nu}$ are gauge invariant, but the second term in $L_{\mu\nu}$ is not gauge invariant.

Finally, we will see if Pais' formula

$$\langle \bar{\psi}\psi \rangle_{e^2} = \frac{\partial \langle \text{Self-Energy of photon} \rangle_{e^2}}{\partial x_0} \quad (13)$$

holds true or not, where $\bar{\psi}\psi$ is the trace of energy-momentum tensor of photon interacting with electrons. Evaluating the expectation value of $\bar{\psi}\psi$ in the e^2 -approximation, we obtain

$$\langle \bar{\psi}\psi \rangle_{e^2} = \frac{4e^2 x_0}{3\pi k}, \quad (14)$$

where k is momentum of a photon and only the (B) type process contributes. We differentiate the photonic self-energy W , obtained by similar calculations, with respect to rest mass of electron, x_0 , and then obtain

$$\frac{\partial \langle W \rangle_{e^2}}{\partial x_0} = \frac{4e^2 x_0}{3\pi k}. \quad (14')$$

Hence we have ascertained the validity of the formula (13) in the present case.

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“Integro-kaŭzeco” en konverĝa kvantumteorio de kampoj

F. J. BELINFANTE

Dept. of Physics, Purdue University, Lafayette Indiana, U. S. A.

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Field theory requires the exclusion of “unallowable” Feynman diagrams, that is, diagrams containing self-energy parts, vertex parts and similar. Omitting those, small changes are made in the factors of the matrix element integrands corresponding to each vertex or line of the diagram. A prediction of $\Psi(t_2)$ from $\Psi(t_1)$ requires integration of the allowable diagrams over the time between t_1 and t_2 only. The result is the paradoxical inequality (4).

It is then shown that the inequality (4) was to be expected, if one takes into account that Ψ has no objective meaning, but is determined by the experimental evidence going into the initial value for Ψ . As a new experiment abruptly changes Ψ , the meaning of $\Psi(t_1)$ as appearing in equations (2) and (3) is apparently different. The general equation (5) presents an “integro-causal” relation between prediction at t and knowledge at t_0 , as contrasted to the “immediate causal” relation between $\Psi(t)$ and $\Psi(t+dt)$ in the ordinary Schrödinger theory. The present theory leads in its differential form (6) to an “interaction operator with a memory”, $B_{t_0}(t)$.

§ 1. Eblo elimini diverĝaĵojn de la nuna kvantumteorio de kampoj

La nuna stato de kvantum-elektrodinamiko estas tia, ke oni povas kalkuli la efikon de la interagado de elektronoj kun la radiada kampo, sed ke por tiu celo oni uzas metodojn, kiuj ne formas parton de tute kontentiga kaj komplete kohereca teorio. Oni devas trakti sensignifajn diverĝajn integralojn kvazaŭ ili estas bone determinitaj kvantoj: Estas necese “renormigi” la elektronan mason kaj ŝargon per kvantoj infinitaj, kaj ankaŭ la metodo uzita por enkalkuli la efikon de la interagado kun laŭlongaj kaj skalaraj fotonoj estas duba. Ĉio ĉi estas kaŭzita de manko je konverĝeco de la integraloj respondantaj al virtualaj procezoj priskribataj per tiaj partoj de Feynman-diagramoj, kiajn Dyson nomas “mem-energiaj partoj” kaj “verticaj partoj”. En mezon-teorio ankaŭ aliaj partoj (ekzemple “disĵetataj partoj de alta ordo”) povas kaŭzi diverĝadojn.

Dum la lastaj jaroj evidentiĝis, ke oni povas atendi ke praktike oni povas kalkuli finitajn rezultojn por ĉiaj fizikaj fenomenoj, se oni unuflanke *elligas* tiajn “ne-permeseblajn” partojn el la Feynman-diagramoj, kiaj kondukas al infinitaĵoj, kaj aliflanke samtempe *iomete ŝanĝas* la faktorojn en la integrendoj en la Dyson-serio por la S -matrico, kiuj respondas al la verticoj kaj la lateroj de tiaj diagramoj (kaj kiuj tiel respondas al hipotezoj pri la interagado kaj, pri la komutaj rilatoj). Tiuj ŝanĝetoj de faktoroj nuntempe estas kalkulataj el la malnova konvencia teorio, (kiu per si mem kondukas al diverĝaj esprimoj por tiuj ŝanĝetoj), per certa metodo forigante la infinitaĵojn. Almenaŭ en la kazo de kvantum-elektrodinamiko tiu

metodo kondukas al rezultoj akordantaj kun eksperimentaj faktoj, kvankam la metodo mem logike ne estas kontentiga pro la diverĝeco de la kvantoj forigitaj.

Oni preferus *dekomenca* ellasi tiujn nepermeseblajn partojn, kaj *postulati* la ŝanĝetojn en la verticaj kaj lateraj faktoroj necesajn por klarigi la eksperimentajn rezultojn. Per tia postulatio oni certe perdas la eblecon “klarigi” tiujn ŝanĝetojn; sed aliflanke oni evitas la diverĝadojn entute. Krome oni tiel havus la okazon klarigi eksperimentajn rezultojn, ankaŭ se evidentigus ke la ŝanĝetoj antaŭdiritaj de la konvencia teorio (kun forigado de infinitaĵoj) ne sufiĉas por klarigi tiujn faktojn, sed ke malsimila ŝanĝeto sukcesus klarigi ilin. Tial tia postulatio plivastigus la eblecojn oferatajn en la kadro de la teorio.

Tamen ŝajnas ekzisti nepra obstaklo kontraŭ tia programo. Kompleta teorio ne nur devos ebligi la kalkuladon de la S -matrico por ŝanĝigo de la fizika stato inter la momentoj $t = -\infty$ kaj $t = +\infty$, sed ankaŭ devas ebligi nin kalkuli ŝanĝigon dum *finita* daŭro, tiel ke el la fizika situacio trovita je tempo $t = t_0$ oni povos antaŭdiri ion pri la probabla situacio je tempo $t = t_2$. Nun supozu, ke tia rilato estas donita per

$$\Psi(t_2) = U(t_2, t_0)\Psi(t_0), \quad (1)$$

kaj ke oni kalkulas la unitaran matricon $U(t_2, t_0)$ laŭ la metodo de Feynman kaj Dyson, integrante laŭ la verticoj de la diagramoj nur inter t_0 kaj t_2 , kaj samtempe ellasante ĉiujn “nepermeseblajn diagramojn”, kun konvena ŝanĝeto de verticaj kaj lateraj faktoroj. Inter la diagramoj ellasitaj troviĝas tiaj, en kiaj mem-energiaj partoj etendiĝas ambaŭflanke de iu momento t_1 elektita ie inter t_0 kaj t_2 . Ekzemple, oni ellasas diagramon, kiu prezentas elektronon, kiu virtuale elsendas fotonon antaŭ t_1 kaj reabsorbas ĝin post t_1 . Tamen, se oni kalkulas la ŝanĝiĝojn de t_0 ĝis t_1 aŭ de t_1 ĝis t_2 ,

$$\Psi(t_2) = U(t_2, t_1)\Psi(t_1), \quad (2)$$

$$\Psi(t_1) = U(t_1, t_0)\Psi(t_0), \quad (3)$$

tiam la operacianto $U(t_1, t_0)$ devas enhavi matricajn elementojn, kiuj kontribuas al la elsendado de tia fotono, kaj la operacianto $U(t_2, t_1)$ devas enhavi matricajn elementojn por la absorbado de fotono. (Se $\Psi(t_1)$ priskribas la fizikan situacion je la tempo t_1 , kaj se ĝi dependas nur de la okupigaj nombroj N je tiu tempo t_1 , tiam $\Psi(t_1)$ ne distingas virtualajn fotonojn de realaj fotonoj.) Tiel la postulatio de ellasado de nepermeseblaj Feynman-diagramoj konsekvencigas

$$U(t_2, t_0) \doteq U(t_2, t_1)U(t_1, t_0), \quad (4)$$

kio ŝajnas kontraŭa al la ekvacioj (1)-(2)-(3). En la sekvonta ĉapitro mi montios ke tamen la ne-cgalaĵo (4) estas tute korekta, kaj ke la kontraŭdiro kun (1)-(2)-(3) estas nur ŝajno kaŭzita de nekompleta formuligo de tiuj tri ekvacioj.

§ 2. La signifo de la kaŭzeca interrilato inter la valoroj de Ψ je nesamaj tempoj

Supozu ke de observantoj, A kaj B , kune faras eksperimenton pri skatolo plena je elementaj korpuskloj, kaj ke je la tempo $t=0$ ili kune decidas pri la Schrödinger-funkcionalo $\Psi_0(0)$ priskribanta la rezultojn trovitajn per la eksperimento. Se ili havas idealan konadon pri la Schrödinger-ekvacio kaj ĝia solvo, tiam el $\Psi_0(0)$ ili povos kalkuli antaŭdiraĵon $\Psi_0(t)$ pri la distribuo de probabloj je la tempo t .

Nun supozu ke B foriras por libertempa vojaĝo. Dum lia foresto, A faras novan eksperimenton je la tempo $t=1$ por kontroli la antaŭdiraĵon

$$\Psi_0(1) = U(1, 0)\Psi_0(0). \quad (3a)$$

Ĉu li trovos $\Psi_0(1)$? Certe ne! Ĉar en $\Psi_0(1)$ la ondopakajoj uzitaj en $\Psi_0(0)$ por priskribi la elementajn korpusklojn verŝajne jam multe etendiĝis kaj konfuziĝis. Aŭ eble pro interferado aperis jam pluraj maksimumoj en la probableca distribuo de iu korpusklo tra spaco. Tamen la eksperimento montros tian korpusklon ĉe unu loko aŭ ĉe alia, sed ne samtempe en du aŭ pluraj lokoj disaj. Tial la mezurita distribuo de probabloj estos pli koncentrigita ol la antaŭdiraĵo $\Psi_0(1)$. Oni atendas nur ke la rezulto de la eksperimento ne "*tute kontraŭdiros*" la antaŭdiraĵon; tio estas, ke oni ne trovos rezulton, por kiu oni kalkulis probablon nulan.

Post tiu eksperimento, A alĝustigos sian Schrödinger-funkcion kaj ŝanĝos ĝin abrupte de $\Psi_0(1)$ al nova valoro $\Psi_1(1)$ konforma al la eksperimentaj rezultoj. El la vidpunkto de statistika teorio tio signifas, ke li ŝanĝas la ensemblon de eblaj eksperimentaj rezultoj antaŭdireblaj, de la ensemblo priskribita per Ψ_0 al la ensemblo priskribita per Ψ_1 .

Poste, B hejmenvenas de sia libertempa vojaĝo, kaj A kaj B kune faras novan eksperimenton je la tempo $t=2$. Je tiu tempo, A antaŭdiras

$$\Psi_1(2) = U(2, 1)\Psi_1(1), \quad (2a)$$

kaj B , aŭdinte pri la eksperimento je $t=1$ sed ne sciante ĝian rezulton, antaŭdiras

$$\Psi_0(2) = U(2, 0)\Psi_0(0), \quad (1a)$$

kaj ambaŭ kune trovas la eksperimentan rezulton $\Psi_2(2)$, tute kontraŭdirante *nek* $\Psi_1(2)$ *nek* $\Psi_0(2)$. Ekvacioj (1a), (2a), (3a) evidente estas pli klara formuligo ol ekvacioj (1), (2), (3), de tio, kio estas principo antaŭdirebla per la matrico $U(t, t_0)$. Kaj en (1a), (2a), (3a) troviĝas nenio, kio povus kontraŭdiri la malegalajon (4).

La eraro farita en § 1 estis en la fakto, ke ni tie traktis la funkcionalon Ψ kvazaŭ ĝi estas iu fizika kvanto objektiva, kiu regule "propagiĝas" de t_0 tra t_1 al t_2 . Tio tute ne estas korekta. Objektivaj estas nur la rezultoj de eksperimentoj. Jam en la determinado de $\Psi_{t_0}(t_0)$ el la rezulto de eksperimentoj je la tempo kaj antaŭ la tempo t_0 envenas iom da subjektiveco, kiu respondas al elekto de iu ensemblo priskribonta nian ne-certan konadon pri la fizika situacio. Tamen ni

povas diri, ke $\Psi_{t_0}(t_0)$ havas almenaŭ iun limigitan objektivecon. Saman limigitan objektivecon do havas la antaŭdiraĵo

$$\Psi_{t_0}(t) = U(t, t_0) \Psi_{t_0}(t_0) \quad (5)$$

kalkulita el $\Psi_{t_0}(t_0)$ por tempo t . La faktoro Ψ aperanta tuj post la operacianto U tiam devas priskribi nian komencan konadon eksperimentan pri la fizika sistemo. Do $\Psi(t_1)$ en ekvacio (2) signifas la rezulton de mezurado $\Psi_{t_1}(t_1)$. Sed tiu mezurado abrupte ŝanĝis la funkcionalon Ψ de $\Psi_{t_0}(t_1)$ al $\Psi_{t_1}(t_1)$, tiel ke $\Psi_{t_1}(t_1)$ en ekvacio (2) tute ne egalas je $\Psi_{t_0}(t_1)$ en formulo (3), kvankam ni uzis la saman simbolon $\Psi(t_1)$ por ambaŭ kvantoj. Ekz., se $\Psi_{t_0}(t_0)$ priskribas unu solan elektronon, kaj se $\Psi_{t_0}(t_1)$ enhavas probablon por ekzisto de kroma (virtuala) fotono, tamen eksperimento ne montros tian fotonon, do $\Psi_{t_1}(t_1)$ ne plu enhavas tiun probablon.

Ĝenerale la rezulto de tiarabrupta ŝanĝo de Ψ dependas de la *speco* de eksperimento farita je t_1 , kaj tiu speco dependas de variabloj priskribantaj nian liberan volon, — variabloj, kiuj certe ne estas parto de nia fizika sistemo. Pro tio ankaŭ ne estas eble indiki iun unikan objektivan rilaton inter $\Psi_{t_0}(t_1)$ kaj $\Psi_{t_1}(t_1)$, nek estas eble kombini tian rilaton kun U kaj indiki unikan kaj objektivan *senperan* rilaton inter $\Psi_{t_0}(t_0)$ kaj $\Psi_{t_1}(t_1)$. Tial *por la funkcio* $\Psi_t(t)$ (kiu same kiel la Schrödinger-funkcio en la malnova teorio dependas de nur unu sola tempo t) *ne ekzistas iu objektiva regulo de propagado*, kaj ni *ne* povas esperi ekziston de iu Schrödinger-ekvacio kun Schrödinger-funkcio kaj Hamilton’a operacianto ambaŭ dependantaj de nur *unu* tempo t .

La kaŭzeca rilato $U(t, t_0)$ aperanta en (5) estas “longtempa” kaŭzeca rilato, kaj laŭ (4) ni *ne* devas esperi ke estus eble faktorigi tiun rilaton tiamaniere, ke nur la plej dekstra faktoro dependus de t_0 . Se ni diferencias ekvacion (5) laŭ t , ni trovas specon de “Schrödinger-ekvacio”,

$$i\hbar \partial \Psi_{t_0}(t) / \partial t = B_{t_0}(t) \Psi_{t_0}(t), \quad (6)$$

sed la operacianto

$$B_{t_0}(t) = i\hbar \{ \partial U(t, t_0) / \partial t \} \{ U(t, t_0) \}^{-1} \quad (7)$$

dependas ne nur de la “nuntempo” t , sed *ankaŭ de la pasinteco ĝis* t_0 . Ni do povas diri ke la interagado inter elementaj korpuskloj estas priskribita per operacianto $B_{t_0}(t)$, kiu *havas* “memoron”. Tial ni rajtas postulati, ke ĝi havu nulajn matric-elementojn por ĉiuj “ne-permeseblaj” procezoj. Se ni konsideras $B_{t_0}(t)$ kiel matricon, tiam ni povas konsideri la funkciojn $N(t')$ kun $t_0 \leq t' \leq t$ kiel ĝiajn matricajn indicojn, tiel ke B estas funkcionalo de la okupiĝaj nombroj rigardataj kiel funkcioj de tempo. Feynman-diagramo inter t_0 kaj t respondas al aro de funkcioj $N(t')$. Ni do postulu, ke B estu hermita mitrico kun matricaj elementoj $\langle N'(t') | B_{t_0}(t) | N''(t'') \rangle$, kiuj estas nulaj, se $N'(t')$ aŭ $N''(t'')$ priskribas nepermeseblan Feynman-diagramon. Do B estas hermita operacianto, tiel ke $U(t, t_0)$ estas unitara, kaj tiel ke B anstataŭas la Hamilton’an operacianton de la Schrödinger-ekvacio ankaŭ kiel operacianto por la interagada energio (en interagada reprezentaĵo) uzebla en la kalkulado de la valoro de la tuta energio. (Kalkulante la valoron de q -nombro

kiel la energio, aŭ kiel nomro 1 en la normigado de Ψ , oni ĉiam uzu la regulon, ke $((B\Psi)^*, \Omega\Psi)$ signifas nur $(\Psi^* B \Omega\Psi)$.

En la malnova Schrödinger'a teorio, kie $H(t)$ anstataŭis $B_{t_0}(t)$, ekzistis *senpera* kaŭzeca rilato inter nuntempo kaj *tuja estonteco*. En nia nova teorio ni havas *senperan* rilaton nur en la *integrita* formo (5), tiel ke en nia teorio "*integro-kaŭzeca*" rilato inter la konitaj faktoj je t_0 kaj la antaŭdiro pri probablecoj je t anstataŭas la konvencian "tujan kaŭzegan rilaton" de t al $t+dt$ de la Schrödinger'a teorio. En la nova teorio, la diferenciita rilato (6) inter $\Psi_{t_0}(t)$ kaj $\Psi_{t_0}(t+dt)$ ne plu estas "senpera", pro la "memoro" de B al tempoj *ekster* la intervalo de t al $t+dt$.

On the Cerenkov Radiation

Tosiya TANIUTI

Physics Department, Kobe University

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§ 1. Introduction

It has been observed by Cerenkov and others,¹⁾ that a fast electron travelling the transparent medium, whose velocity is greater than that of light in that medium, emits a directed visible radiation, and the direction of light is given by $\cos \theta = 1/\beta n$, where θ is the angle between the direction of light and the incident electron; $\beta = v/c$, v the velocity of the electron; c , the velocity of light in the vacuum; and n , the refractive index of that medium. The theoretical explanations of this radiation have been given by several authors not only classically but also quantum-electrodynamically, on each different standpoint of view.²⁾ Many of them, however, introduced the refractive index n of the medium phenomenologically and so the mechanism of the radiation seems to be somewhat obscure. Especially, in the quantum-electrodynamics, the energy momentum relations of photon in the medium³⁾ are given unsatisfactorily on the phenomenological considerations. For an example, when the electric field strength \mathbf{E} is developed as usual

$$\mathbf{E} = \int_{i=1}^3 \mathbf{e}_i(k) (E_i(k) \exp(i\mathbf{k}\mathbf{r}) + E_i^*(k) \exp(-i\mathbf{k}\mathbf{r})) d\mathbf{k}$$

with the polarization vectors \mathbf{e}_i , the refractive index n of the dispersive medium may be defined as follows

$$\mathbf{D} = \int_{i=1}^3 \mathbf{e}_i(k) \epsilon(k) (E_i(k) \exp(i\mathbf{k}\mathbf{r}) + E_i^*(k) \exp(-i\mathbf{k}\mathbf{r})) d\mathbf{k} \quad n(k) = \sqrt{\epsilon(k)}$$

where \mathbf{D} is the electric displacement vector and we consider the medium as a non-magnetic one. But the refractive index n , thus defined, which is the function of the wave number vector \mathbf{k} , can not be compared with $n(\omega)$ defined by means of the function of the frequency ω as usual as in the classical electromagnetic theory, unless any correspondence between \mathbf{k} and ω is given.

That is to say, the relation between the frequency ω and the corresponding wave number vector \mathbf{k} in the dispersive medium, is given by $\omega = ck/n(k)$ not by $\omega = ck$ as in the vacuum, and on the other hand the so-called refractive index $n(\omega)$

may be expressed by $n(\omega) = 1 + a/(\omega_0^2 - \omega^2)$ aside the damping term, in the classical electron theory, and also we should notice that it is the latter which is used in experimental results.

Of course, this obscurity about the nature of the refractive index will vanish as soon as the atomistic consideration about the medium is taken into account, what means to give any functional dependence of n on k .

However, in this phenomena, not the individual but the total atom's polarization seems to be essential and also this is the reason why the phenomenological treatments are successful. Under these statuses it is the easiest way to start from the semi-phenomenological standpoint. In section 2 and 3, the semi-phenomenological electrodynamics in the medium is discussed and in § 4 and § 5, this applied to the Cerenkov radiation.

§ 2. Semi-phenomenological electrodynamics in the medium

At the first, we consider the polarization of the group of atoms in the physically small space region, instead of individual atom; then \mathbf{P} , the polarization vector, may be obtained by taking the average of these atom's ones over the small space region under consideration. In this manner, the polarization vector \mathbf{P} can be seen as the field quantity defined at every point in the medium, in the macroscopic meaning, and as is well-known, the equation of motion for \mathbf{P} may be given by⁴⁾

$$\frac{d^2 \mathbf{P}}{dt^2} + \nu_0^2 \mathbf{P} = \frac{a}{4\pi} \mathbf{E} \quad (1)$$

where $\nu_0^2 = \omega_0^2 - a/2$, ω_0 : eigenfrequency of atoms in the harmonic oscillator model. $a = 4\pi N e^2 / m$, N is the number of atoms per unit volume, m , the electron mass, e its charge, The equation (1) will be considered as the limiting case of the equation

$$-\Delta \mathbf{P} + \frac{\partial^2 \mathbf{P}}{\partial t^2} + \nu_0^2 \mathbf{P} = (a/4\pi) \mathbf{E}, \quad \text{when } \Delta \mathbf{P} \rightarrow 0.$$

The following Maxwellian Equations have to be added to (1) as the fundamental equations of our total system.

$$\left. \begin{aligned} \text{rot } \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c} \left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} \right), \\ \text{rot } \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} &= 0, \\ \text{div } \mathbf{E} &= 4\pi (\rho - \text{div } \mathbf{P}), \\ \text{rot } \mathbf{H} &= 0 \end{aligned} \right\} \quad (2)$$

which may be derived from the following Lagrangian

$$L = \frac{1}{2} (\mathbf{E}^2 - \mathbf{H}^2) - 4\pi(\rho - \text{div} \mathbf{P})\varphi + \frac{4\pi}{c} \left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} \right) \mathbf{A} + 2\pi \left\{ \left(\frac{\partial \mathbf{P}}{\partial t} \right)^2 - \nu_0^2 \mathbf{P}^2 \right\}.$$

The Hamiltonian of this system may be given easily by the above Lagrangian in the usual manner. We introduce the Fourier expansions of the field variables

$$\begin{aligned} \mathbf{A} = & \frac{c}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \{ \mathbf{e}_\lambda(k) (q_\lambda(k, t) \exp(i\mathbf{k}\mathbf{r}) + \text{comp. conj.}) \\ & + \mathbf{e}_o(k) (q_o(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c.) \} d\mathbf{k} \end{aligned} \quad (3)$$

where λ and σ are the transversal and the longitudinal index respectively. (for simplicity, we considered the linearly polarized case), and similarly

$$\varphi = \frac{c}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \{ a(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c. \} d\mathbf{k}, \quad (4)$$

$$\begin{aligned} \mathbf{P} = & \frac{1}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \{ \mathbf{e}_\lambda(k) (\xi_\lambda(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c.) \\ & + \mathbf{e}_o(k) (\xi_o(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c.) \} d\mathbf{k}, \end{aligned} \quad (5)$$

$$\rho = \frac{1}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \{ \sigma(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c. \} d\mathbf{k}, \quad (6)$$

$$\begin{aligned} \mathbf{J} = & \frac{1}{\sqrt{8\pi^2}} \int_{-\infty}^{\infty} \{ \mathbf{e}_\lambda(k) (s_\lambda(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c.) \\ & + \mathbf{e}_o(k) (s_o(k, t) \exp(i\mathbf{k}\mathbf{r}) + c.c.) \} d\mathbf{k} \end{aligned} \quad (7)$$

with the following conditions^{b)}

$$\begin{aligned} \mathbf{e}_\lambda(k) &= \mathbf{e}_\lambda(-k), & \mathbf{e}_o(k) &= -\mathbf{e}_o(-k), & \sigma^*(-k) &= \sigma(k), \\ q_\lambda(k) &= q_\lambda^*(-k), & a^*(-k) &= a(k), & \sigma^*(k) &= \sigma(-k), \\ q_\lambda(k) &= q_\lambda^*(k), & a^*(k) &= a(-k), & S_\lambda^*(k) &= S_\lambda(-k), \\ & & & & S_\lambda^*(-k) &= S_\lambda(k) \end{aligned}$$

and the Lorentz conditions $i\dot{\mathbf{a}}(k) = ckq_\lambda(k)$ for each k .

We then find the total Hamiltonian has a form;⁶⁾

$$\begin{aligned} H = & \int_0^\infty \left\{ p_\lambda p_\lambda^* + \omega^2 q_\lambda q_\lambda^* - 4\pi(s_\lambda q_\lambda^* + s_\lambda^* q_\lambda) + 16\pi^2 \frac{\sigma\sigma^*}{k^2} \right. \\ & + 16\pi^2 \left(\xi_o \xi_o^* + \frac{i}{k} (\sigma \xi_o^* - \sigma^* \xi_o) \right) + \frac{a}{16\pi^2} (\eta_\lambda^* - 4\pi q_\lambda) (\eta_\lambda - 4\pi q_\lambda^*) \\ & \left. + \frac{16\pi^2}{a} \nu_0^2 \xi_\lambda \xi_\lambda^* + \frac{a}{16\pi^2} \eta_o \eta_o^* + \frac{16\pi^2}{a} \nu_0^2 \xi_o \xi_o^* \right\} d\mathbf{k} \end{aligned} \quad (8)$$

where η_λ , η_o , and p_λ etc. are canonical conjugate momentums for ξ_λ , ξ_o , and q_λ etc. respectively and they are given by

$$p_\lambda = \dot{q}_\lambda^*, \quad \eta_\lambda^* = \frac{16\pi^2}{a} \dot{\xi}_\lambda + 4\pi q_\lambda, \quad \eta_\sigma^* = \frac{16\pi^2}{a} \dot{\xi}_\sigma$$

and their complex conjugates.

§ 3. In the case of the electron absent

In this case, we can put $\mathbf{J} = \rho = 0$, and then (8) becomes

$$H_0 = \int \left\{ p_\lambda p_\lambda^* + \omega^2 q_\lambda q_\lambda^* + \frac{a}{16\pi^2} (\eta_\lambda^* - 4\pi q_\lambda) (\eta_\lambda - 4\pi q_\lambda^*) + \frac{16\pi^2}{a} \nu_0^2 \dot{\xi}_\lambda \dot{\xi}_\lambda^* \right. \\ \left. + \frac{a}{16\pi^2} \eta_\sigma \eta_\sigma^* + \frac{16\pi^2}{a} \nu_0^2 \dot{\xi}_\sigma \dot{\xi}_\sigma^* + 16\pi^2 \dot{\xi}_\sigma \dot{\xi}_\sigma^* \right\} dk. \quad (8)'$$

The last three terms represent the energy of the longitudinal part of the polarization field itself and these may be equated to zero.

Then, by the method of the contact transformation, (8)' can be represented as the aggregate of the independent harmonic oscillators. Although this contact transformation will be easily performed, we prefer the primitive way for the purpose of the application to the Cerenkov effect. From (8)', it follows immediately

$$\left. \begin{aligned} \ddot{q}_\lambda + \omega^2 q_\lambda &= 4\pi \dot{\xi}_\lambda, \\ \ddot{\xi}_\lambda + \nu_0^2 \xi_\lambda &= -\frac{a}{4\pi} \dot{q}_\lambda. \end{aligned} \right\} \quad (9)$$

For the normal modes ω' of these coupled oscillators, we get

$$\omega_i'^2 = \frac{1}{2} \{ \omega^2 + \nu_0^2 + a \pm \sqrt{(\omega^2 + \nu_0^2 + a)^2 - 4\pi\omega^2\nu_0^2} \} \quad (10)$$

$i=1, 2$, for $-$, $+$ respectively.

Introducing the refractive index n_i for each ω_i defined by

$$n_i^2(\omega_i') = 1 + \frac{a}{\nu_0^2 - \omega_i'^2} \quad (11)$$

and using the secular equation of (9) for ω_i' , we also obtain

$$\omega_i'^2 = \omega^2 / n_i^2(\omega_i'). \quad (12)$$

There exist some relations between ω' , ω and n_i which are shown in Fig. 1 and 2, indicating that n_i are the functions of ω_i' in the usual manner, but not of k .

We introduce new field variables $\Xi_{\lambda i}, \Xi_{\lambda i}^* (i=1, 2)$, which satisfy the equations

$$\ddot{\Xi}_{\lambda i} + \omega_i'^2 \Xi_{\lambda i} = 0 \quad (13)$$

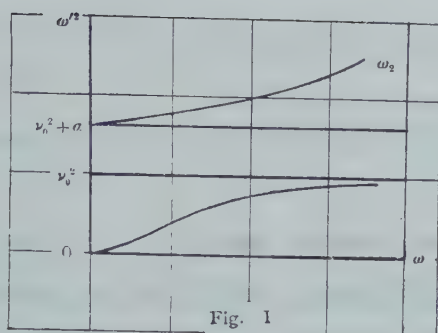


Fig. 1

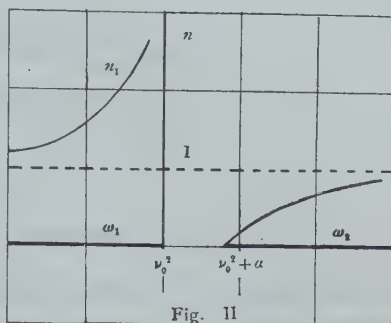


Fig. 11

and their complex conjugates and give the following relations for them

$$q_\lambda = \sum_{i=1}^2 x_i \Xi_{\lambda i}, \quad \bar{q}_\lambda = \sum_{i=1}^2 \mu_i \Xi_{\lambda i} \quad (14)$$

etc. where

$$x_i = 4\pi i \omega'_i x_{0i}, \quad \mu_i = \frac{1}{4\pi(\omega/n_i) \sqrt{n_i^2 + \omega_i^2 a / (\nu_0^2 - \omega_i'^2)^2}} \quad (15)$$

We find the following expression for the total energy of this system

$$\int \left(\sum_{i=1}^2 2\omega_i' \Xi_{\lambda i} \Xi_{\lambda i}^* \right) dk \quad (8'')$$

(hereafter, the primes of ω'_i will be omitted for simplicity.)

The equivalence of (8)' and (8)'' can be immediately verified by a direct calculation, except the longitudinal part in (8)'.

In order to quantize this system, we introduce new canonical conjugate variables

$$P_{\lambda i} = -i\omega_i (\Xi_{\lambda i} - \Xi_{\lambda i}^*), \quad Q_{\lambda i} = \Xi_{\lambda i} + \Xi_{\lambda i}^* \quad (i=1,2) \quad (16)$$

satisfying the commutation relations

$$\left. \begin{aligned} [P_{\lambda i}(k), Q_{\lambda j}(k')] &= \delta_{ij} \delta_{kk'}, \\ [P_{\lambda i}(k) P_{\lambda j}(k')] &= [Q_{\lambda i}(k), Q_{\lambda j}(k')] = 0. \end{aligned} \right\} \quad (17)$$

Then (8)'' becomes

$$H_0(k) = \sum_{i=1}^2 \frac{1}{2} (P_{\lambda i}^2 + \omega_i^2 Q_{\lambda i}^2).$$

The matrix elements of $Q_{\lambda i}$ are given by

$$Q_{\lambda i}(k)_{n,n+1} = Q_{\lambda i}^*(k)_{n+1,n} = \sqrt{\frac{\hbar(n+1)}{2\omega_i}} \exp(-i\omega/n_i t). \quad (18)$$

From (15), (16) and (18), we obtain also

$$\left. \begin{aligned} q_{\lambda n, n+1} &= \sum_{i=1}^2 x_{0i} \sqrt{\frac{\hbar(n+1)}{2\omega_i}} \exp(-i\omega/n_i t), \\ q_{\lambda n+1, n}^* &= \sum_{i=1}^2 x_{0i}^* \sqrt{\frac{\hbar(n+n)}{2\omega_i}} \exp(-i\omega/n_i t). \end{aligned} \right\} \quad (19)$$

§ 4. Cerenkov radiation, classical theory

In the presence of a free electron with constant velocity, (9) are modified as follows

$$\ddot{q}_\lambda + \omega^2 q_\lambda = 4\pi(\dot{\xi}_\lambda + s_\lambda), \quad (9a)'$$

$$\dot{\xi}_\lambda + \nu_0^2 \xi_\lambda = -\frac{a}{4\pi} \dot{q}_\lambda, \quad (9b)'$$

$$\ddot{\xi}_\sigma + \nu_0^2 \xi_\sigma = -\frac{ai}{k} \sigma \quad (9c)'$$

where

$$s_\lambda = \frac{ev}{\sqrt{32}\pi^2} (ne_\lambda) \exp(-ik_z vt), \quad \sigma = \frac{e}{\sqrt{32}\pi^2} \exp(-ik_z vt)$$

because of the relations $\mathbf{J} = nev\delta(x)\delta(y)\delta(z-vt)$, $\rho = e\delta(x)\delta(y)\delta(z-vt)$, \mathbf{n} , the direction of the movement of the electron, v , the velocity of it. Using the abbreviations $b = ev/(\sqrt{2}\pi)(ne_\lambda)$, $\nu = -k_z v$; (9a' b') are written down

$$\ddot{q}_\lambda + \omega^2 q_\lambda - 4\pi\dot{\xi}_\lambda = b e^{i\nu t}, \quad (9a)''$$

$$\ddot{\xi}_\lambda + \nu_0^2 \xi_\lambda + (a/4\pi)\dot{q}_\lambda = 0. \quad (9b)''$$

The solutions of these equations are

$$q_\lambda = \sum_{i=1}^2 c_i 4\pi i \omega_i \exp\{i(\omega_i t + \varphi)\} + \{b/(\omega^2 - \nu^2 n_i^2)\} \exp(i\nu t),$$

$$\xi_\lambda = \sum_{i=1}^2 c_i (\omega^2 - \omega_i^2) \exp\{i(\omega_i t + \varphi)\} - \{i\nu ab/(4\pi(\nu_0^2 - \nu^2)(\omega^2 - \nu^2 n_i^2))\} \exp(i\nu t)$$

with arbitrary constants c_i .

The above solutions indicate that for $|\omega| \simeq \nu = k_z v$, the resonance occurs, that is to say, the radiation can be observed in the direction $\cos \theta = \frac{1}{\beta n}$ where θ is the angle between \mathbf{k} and \mathbf{n} .

Now we consider this process stepwisely.

At the first step, we can impose the following equation

$$\ddot{q}_\lambda^{(1)} + \omega^2 q_\lambda^{(1)} = b \exp\{i\nu t\}.$$

For the solution of this equation, we get

$$q_{\lambda}^{(1)} = \{b/(\omega^2 - \nu^2)\} \exp(i\nu t)$$

Substituting this into the another equation

$$\ddot{\xi}_{\lambda}^{(1)} + \nu_0^2 \xi_{\lambda}^{(1)} = -(a/4\pi) \dot{q}_{\lambda}^{(1)}$$

we obtain

$$\xi_{\lambda}^{(1)} = -\{i\nu ab/4\pi(\nu_0^2 - \nu^2)(\omega^2 - \nu^2)\} \exp(i\nu t).$$

At the second step, substituting this into the following equation

$$\ddot{q}_{\lambda}^{(2)} + \omega^2 q_{\lambda}^{(2)} = 4\pi \dot{\xi}_{\lambda}^{(1)}$$

we get for the second order solution of q_{λ}

$$q_{\lambda}^{(2)} = \{a\nu^2/(\omega^2 - \nu^2)(\nu_0^2 - \nu^2)(\omega^2 - \nu^2)\} b \exp(i\nu t).$$

Repeating this process, we get finally

$$q_{\lambda} = \sum_{n=1}^{\infty} q_{\lambda}^{(n)} = \frac{b}{\omega^2 - \nu^2} \left\{ 1 + \frac{a\nu^2}{(\omega^2 - \nu^2)(\nu_0^2 - \nu^2)} + \left(\frac{a\nu^2}{(\omega^2 - \nu^2)(\nu_0^2 - \nu^2)} \right)^2 + \dots \right\} \exp(i\nu t) = \frac{\nu_0^2 - \nu^2}{(\omega^2 - \nu^2)(\nu_0^2 - \nu^2) - a\nu^2} b \exp(i\nu t).$$

The resonance condition requires that the denominator of the above q_{λ} should be zero, namely

$$(\nu_0^2 - \nu^2) \left\{ \omega^2 - \nu^2 \left(1 + \frac{a}{\nu_0^2 - \nu^2} \right) \right\} = 0$$

and therefore we get again, $\cos \theta = 1/\beta n_i(\nu)$.

Physically speaking above calculations, at the first, the eigenfield of the incident electron induces atomic polarizations; at the second process, these induced atomic polarizations excite the vacuum electromagnetic field, moreover the electromagnetic field, thus excited, reacts to the atomic polarizations and then these mutual actions and reactions between the electromagnetic field and the atomic polarizations, lead to the Cerenkov radiation after the infinite process and this fact also indicates that the perturbational consideration is valid only in the infinite order in this case. The radiated energy is calculated by using the formulation performed in § 3, in which the equations (9'a) (9'b) can be rewritten as follows

$$\ddot{\Xi}_{\lambda 1} + \omega_1^2 \Xi_{\lambda 1} = 4\pi x_{01} s_{\lambda}.$$

As for $\Xi_{\lambda 2}$, the resonance condition is not satisfied and it may be omitted in the calculation of the radiated energy. Then radiated energy is given by

$$\begin{aligned}
 U &= \int_0^T \int_0^{\infty} 4\pi (\mathbf{x}_{01} S_{\lambda} \dot{\mathbf{E}}_{\lambda 1} + \mathbf{x}_{01}^* S_{\lambda}^* \dot{\mathbf{E}}_{\lambda 1}^*) dt d\mathbf{k} \\
 &= e^2 v t \int_0^{\infty} \left(1 - \frac{1}{\beta^2 n_1^2(\omega_1)}\right) k |\mathbf{x}_{01}|^2 d\mathbf{k}.
 \end{aligned}$$

From (11), it is verified that $|\mathbf{x}_{01}|^2 \omega d\omega = \omega_1 d\omega_1$ and then it becomes finally

$$U = (e^2 v / c^2) t \int_{\frac{1}{\beta n_1} < 1}^{\nu_0} \left(1 - \frac{1}{\beta^2 n_1^2}\right) \omega_1 d\omega_1.$$

This result coincides with the quite phenomenological one as Frank and Tamm's,⁷⁾ aside the upperlimit of the integration domain. However, in the usual phenomenological treatment performed by some authors,⁽⁷⁾ the integral variable is the wave number vector \mathbf{k} , not ω , and the functional dependence of n to k is not given and may be different from that of $n(\omega_i)$ to ω_i , and so the connection to the result deduced from the classical electron theory, which uses only $n(\omega)$ as the refractive index, is not so obvious as in our case. Mathematically speaking, the connection between two formulations obtained in one hand by Fourier developments of field variables about k and on the other, about ω_i , may be clearly expressed in our treatment. The remaining equation (9'c) gives the longitudinal propagation of the polarization wave and it will be easily seen that this longitudinal flow comes from the propagation of the coulombic disturbance of the atomic polarization caused by the coulomb field of the incident electron and can not be observed as the radiation. In our discussion of the Cerenkov radiation, therefore it may be omitted.

§ 5. Cerenkov radiation; quantum mechanical treatment⁸⁾

In the quantum mechanical treatment of this radiation, we should start from the following Hamiltonian.

$$H_{total} = \int_0^{\infty} \sum_{i=1}^2 \frac{1}{2} (P_{ki}^2 + \omega_i^2 Q_{ki}^2) dk + \alpha (\mathbf{p} - e \mathbf{A}_{tr}) + \mu \beta$$

where we used energy momentum unit and $\mu = mc^2$. By virtue of the ordinary first order perturbation, and using the formulae (19), we obtain the following radiated energy in the extreme relativistic case, *i. e.* $E \sim p$

$$U = e^2 c^2 t \int_0^{\infty} |\mathbf{x}_{01}|^2 \left(1 - \frac{1}{n_1^2(\omega_1)}\right) \frac{k}{c} dk = \frac{e^2}{c} t \int_{\frac{1}{\beta n_1} < 1}^{\nu_0} \left(1 - \frac{1}{n_1^2(\omega_1)}\right) \omega_1 d\omega_1.$$

This result coincides with the classical one in which v is put equal to c . As

the energy-momentum conservation, we also get from the perturbation calculation,

$$\cos \theta = \frac{1}{\beta n_1} + \frac{\left(1 - \frac{1}{n_1^2}\right) \hbar c k}{2p} \approx \frac{1}{\beta n_1}.$$

In conclusion, the mechanisms of this radiation became somewhat clearer from the atomistic standpoint, and in the resultant formulae, the natural cut-off appeared because of the two valued correspondence between energy and momentum of the mixed field of electromagnetic and polarization field.

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Non-hermitian Operators and Eigenfunction Expansions

Giiti IWATA

Institute of Physics, Tokyo University

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For the present, hermitian operators only play chief role in theoretical physics. Being long neglected, non-hermitian operators however are supposed to partake a certain role in physical theories. Annihilation or creation operators are clearly non-hermitian. Hitherto we are interested in eigenfunction expansions over some real interval. In the following, we consider eigenfunction expansions over some closed curve in the complex plane.

§ 1. Expansions in finite dimensions

$A = \|a_{ik}\|$ be an (n, n) matrix, a_1, \dots, a_n different eigenvalues of A . Then we have two systems of eigenvectors $(f_1, \dots, f_n), (e_1, \dots, e_n)$ each satisfying

$$Af_i = a_i f_i, \quad e_i A = a_i e_i, \quad i = 1, \dots, n$$

where f_i are $(n, 1)$ matrices, e_i $(1, n)$ matrices. For any two indices i, j , we have

$$e_i A f_j = a_i e_i f_j = a_j e_i f_j, \quad (a_i - a_j) e_i f_j = 0.$$

Therefore we get

$$e_i f_j = \delta_{ij}$$

adjusting numerical factors of e_i .

Any vector f of the type f_i may be expanded as a linear combination of f_i where coefficients of f_i are obtainable by multiplying f by e_i from the left.

Expansion:
$$f = \sum_i f_i (e_i f).$$

Analogously we have an expansion of vector e of the type e_i in terms of e_i ,

$$e = \sum_i (e f_i) e_i.$$

§ 2. Operators in the complex domain

Let z, \bar{z} be two operators, hermite conjugate to each other, related by the commutation relation $\bar{z}z - z\bar{z} = 1$. Eigenvalues of z are complex numbers. Adopting the z -diagonal representation, the inner product of two functions $\phi(z), \psi(z)$ is defined as

$$(\phi\psi) = \frac{1}{2\pi i} \int \phi(z)\psi(z)dz,$$

C being some curve on the complex plane.

We assume $\phi(z)$, $\psi(z)$ be one-valued functions on C . Then we have

$$(\phi z \psi) = (\phi, z \psi) = \frac{1}{2\pi i} \int_C \phi(z) z \psi(z) dz = (z \phi, \psi),$$

$$(\phi \bar{z} \psi) = (\phi, \frac{d}{dz} \psi) = \frac{1}{2\pi i} \int_C \phi(z) \psi'(z) dz = \frac{1}{2\pi i} \int_C \phi'(z) \psi(z) dz = (-\frac{d}{dz} \phi, \psi).$$

In general we have

$$(\phi L(z, \bar{z}) \psi) = (\phi, L(z, \frac{d}{dz}) \psi) = (\bar{L}(z, \frac{d}{dz}) \phi, \psi)$$

where \bar{L} is adjoint to L .

If we have for any ϕ

$$(\phi L \phi_\lambda) = \lambda (\phi \phi_\lambda), \quad (1)$$

we call λ eigenvalue of L , ϕ_λ eigen ϕ corresponding to eigenvalue λ . Analogously, when we have for any ψ

$$(\phi_\lambda L \psi) = \lambda (\phi_\lambda \psi), \quad (2)$$

ϕ_λ is called eigen ϕ . This definition seems more legitimate than the usual definition, for expectation values e.g. $(\phi L \psi)$ only have physical meanings. If we denote any two eigenvalues of L by λ, μ , we have

$$(\phi_\lambda L \phi_\mu) = \lambda (\phi_\lambda \phi_\mu) = \mu (\phi_\lambda \phi_\mu).$$

Therefore we get orthogonal relations

$$(\phi_\lambda \phi_\mu) = 0 \quad \text{for } \lambda \neq \mu.$$

When eigenvalues of L are discrete, we have expansions of any ψ or ϕ in terms of ϕ_λ or ϕ_λ respectively as follows

$$\psi = \sum \phi_\lambda (\phi_\lambda \psi), \quad \phi = \sum (\phi \phi_\lambda) \phi_\lambda,$$

where $(\phi_\lambda \phi_\lambda)$ are normalized to be unity. Sometimes we use Dirac notation and denote ϕ_λ by $(z|\lambda)$, ϕ_λ by $(\lambda|z)$, $(\lambda|z)$ being not necessarily complex conjugate to $(z|\lambda)$.

§ 3. Harmonic oscillator

Put $2z = q - ip$, $2\bar{z} = q + ip$, then q and p are hermitian operators related by the commutation relation $qp - pq = 2i$.

We know in quantum mechanics that the operator $n = z \bar{z} = (q^2 + p^2 - 2)/4$ has eigenvalues 0, 1, 2, ..., and eigen $\phi_n = (q|n) = (n|q)$ corresponding to eigenvalue n

$$(q|n) = \exp\{-q^2/4\} H_n(q) / n!^{1/2} (2\pi)^{1/2},$$

which satisfy Weber's equation

$$\left\{ \frac{d^2}{dq^2} + \left(n + \frac{1}{2} - \frac{q^2}{4} \right) \right\} (q|n) = 0.$$

If we adopt the z -diagonal representation and choose a closed curve encircling once $z=0$ as C , eigen ψ_n, ϕ_n will be determined by the relation

$$\begin{aligned} (\phi | n | \psi_n) &= n (\phi | \psi_n) && \text{for any } \phi, \\ (\phi_n | n | \psi) &= n (\phi_n | \psi) && \text{for any } \psi. \end{aligned}$$

In this case we can determine ψ_n, ϕ_n by

$$n\psi_n = n\psi_n, \quad \phi_n n = n\phi_n.$$

which may be transcribed as

$$z \frac{d}{dz} \psi_n = n \psi_n, \quad -z \frac{d}{dz} \phi_n = n \phi_n.$$

Solutions of these equations are

$$\psi_n = C z^n, \quad \phi_n = C' z^{-n-1}, \quad C, C' \text{ constants.}$$

Because ψ, ϕ are one-valued, n must be integral. Further we assume ψ be regular at $z=0$. Then we have $n=0, 1, 2, \dots$. Normalization of ψ, ϕ requires $CC'=1$. For further applications we choose $C=n!^{-1/2}, C'=n!^{1/2}$ and we get

$$\psi_n = (z|n) = z^n n!^{-1/2}, \quad \phi_n = (n|z) = z^{-n-1} n!^{1/2}$$

We verify easily that

$$\begin{aligned} (\phi_m | \psi_n) &= \frac{1}{2\pi i} \int (m|z) (z|n) dz = \delta_{mn}, \\ \sum_{n=0}^{\infty} (z|n) (n|t) &= \sum_{n=0}^{\infty} z^n t^{-n-1} = 1/(t-z) \quad |z| < |t|, \\ \psi = \sum_{n=0}^{\infty} \psi_n (\phi_n | \psi) &= \sum_{n=0}^{\infty} z^n \cdot \frac{1}{2\pi i} \int_C z^{-n-1} \psi(z) dz \quad \psi \text{ regular at } z=0, \\ \phi = \sum_{n=0}^{\infty} (\phi | \phi_n) \phi_n &= \sum_{n=0}^{\infty} z^{-n-1} \frac{1}{2\pi i} \int_C z^n \phi(z) dz \quad \phi=0 \text{ at } z=\infty. \end{aligned}$$

We can build up transformation functions $(z|q), (q|z)$ from $(z|n), (z|n), (q|n) (n|q)$, as follows

$$\begin{aligned} (z|q) &= \sum_{n=0}^{\infty} (z|n) (n|q) = \exp \left\{ -\frac{q^2}{4} + qz - \frac{z^2}{2} \right\} / (2\pi)^{1/4}, \\ (q|z) &= \sum_{n=0}^{\infty} (q|n) (n|z) = \exp \left(-\frac{q^2}{4} \right) \sum_{n=0}^{\infty} H_n(q) z^{-n-1} / (2\pi)^{1/4} \end{aligned}$$

each satisfying respectively

$$\begin{aligned} \left((|z\rangle \mathbf{q}(z|q)) \right) &= q \left((|z\rangle (z|q)) \right), & \left((|q\rangle \mathbf{z}(q|z)) \right) &= z \left((|q\rangle (q|z)) \right), \\ \left((z|q) \mathbf{z}(q|) \right) &= z \left((z|q) (q|) \right), & \left((q|z) \mathbf{q}(z|) \right) &= q \left((q|z) (z|) \right), \end{aligned}$$

where

$$\left((z|q) (q|) \right) = \int_{-\infty}^{\infty} (z|q) (q|) dq \quad \text{etc.}$$

To complete the list of transformation functions we add the followings

$$\begin{aligned} (q|p) &= e^{i\eta p/2} / (4\pi)^{1/2}, & (p|q) &= e^{-i\eta p/2} / (4\pi)^{1/2}, \\ (n|p) &= \exp\left(-\frac{p^2}{4}\right) i^n H_n(p) n!^{1/2} (2\pi)^{1/4}, & (p|n) &= \exp\left(-\frac{p^2}{4}\right) i^{-n} H_n(p) n!^{1/2} (2\pi)^{1/4}. \end{aligned}$$

§ 4. Eigenfunctions of the hypergeometric differential equations

Hypergeometric differential equation

$$x(1-x)y'' + \{\gamma - (a+\beta+1)x\}y' - a\beta y = 0$$

has solutions represented by Riemann's P function

$$y = P \left\{ \begin{matrix} 0 & \infty & 1 \\ 1 & a & 0 \\ 1-\gamma & \beta & \gamma-a-\beta \end{matrix} \middle| x \right\}.$$

Assuming $a+\beta$ be a non-integral constant ν , we put

$$L = x(1-x)\bar{x}^2 + \{\gamma - (\nu+1)x\}\bar{x}, \quad \bar{x}x - x\bar{x} = 1,$$

$$Ly = x(1-x)y'' + \{\gamma - (\nu+1)x\}y' = \lambda y,$$

then
$$\bar{L}z = \{x(1-x)z\}'' - \{\gamma - (\nu+1)x\}z' = \lambda z.$$

We shall choose a curve encircling once $x=\infty$ as C , and determine eigenfunctions of the differential equation, one-valued over C .

An index a at $x=\infty$ must be integral. Therefore eigenvalues of L are

$$\lambda = n(\nu - n)$$

and eigenfunctions corresponding to eigenvalue $n(\nu - n)$ are

$$y_n = x^{-n} F(n, n-\gamma+1, 2n-\nu+1, x^{-1}),$$

$$z_n = x^{n-1} F(1-n, \gamma-n, \nu-2n+1, x^{-1}),$$

z_n are polynomials in x if we assume $n > 0$.

We have here a remarkable result

$$\left. \sum_{n=1}^{\infty} z_n(x) y_n(t) = \frac{1}{t-x} \quad |x| < |t|, \right\} \quad (3)$$

with $(z_m z_n) = 0, \quad (y_m y_n) = 0, \quad (z_m y_n) = \delta_{mn}.$

The result may be proved by rearranging series expansions of y_n, z_n in t, x . From (3) we get expansions of any $z(x), y(x)$ in terms of $z_n(x), y_n(x)$ as follows

$$z(x) = \frac{1}{2\pi i} \int \frac{z(t)}{t-x} dt = \sum_{n=1}^{\infty} z_n(x) \left(z(t) y_n(t) \right) \quad x \text{ inside } C,$$

$$y(x) = \frac{1}{2\pi i} \int \frac{y(t)}{x-t} dt = \sum_{n=1}^{\infty} y_n(x) \left(z_n(t) y(t) \right) \quad x \text{ outside } C.$$

§ 5. Eigenfunctions of Legendre's equations

Let

$$Ly = \frac{d}{dx} (x^2 - 1) \frac{dy}{dx} = \lambda y$$

then

$$\bar{L}z = \frac{d}{dx} (x^2 - 1) \frac{dz}{dx} = \lambda z.$$

In this case $L = \bar{L}$. We choose the same curve C as above. Eigenvalues are $\lambda = n(n+1)$, $n=0, 1, 2, \dots$

Eigenfunctions:

$$y_n = P_n(x), \quad z_n = (2n+1) Q_n(x),$$

where $P_n(x), Q_n(x)$ are Legendre's functions of the first or of the second kind. The relations

$$\left. \begin{aligned} (y_m y_n) &= 0, \quad (z_m z_n) = 0, \quad (z_m y_n) = \delta_{mn}, \\ \sum_{n=0}^{\infty} z_n(t) y_n(x) &= \sum_{n=0}^{\infty} (2n+1) P_n(x) Q_n(t) = 1/(t-x) \end{aligned} \right\} \quad (3) \text{ bis}$$

are well known¹⁾

§ 6. Eigenfunctions of Bessel's equations

Let

$$Ly = x \frac{d}{dx} x \frac{dy}{dx} + x^2 y = \lambda y,$$

then

$$\bar{L}z = \frac{d}{dx} x \frac{d}{dx} xz + x^2 z = \lambda z,$$

C : a curve encircling once $x=0$.

Eigenvalues $\lambda=n^2, \quad n=0,1,2,\dots,$
 $y_n=J_n(x).$

There are no solutions having index $-n-1$ to $\bar{L}z-n^2z=0$. We must start from the basic equations (1), (2). We assume the index of z_n be $-n-1$ to preserve $(z_n y_n)=1$ and z_n be expandable as

$$z_n = \frac{2^n n!}{x^{n+1}} \left\{ 1 + c_1 x + c_2 x^2 + \dots + c_n x^n \right\}.$$

Then we can determine coefficients c_i so that we may have

$$(\bar{L} - n^2)z_n = \begin{cases} x & n=0 \\ 2n & n=1,3,5,\dots \\ 2x & n=2,4,6,\dots \end{cases}$$

and we can have

$$(\bar{L} z_m, y_n) = n^2 (z_n y_m) \quad \text{for any } m \geq 0$$

z_n are called Neumann's polynomials²⁾ except for constant factors. There are classical relations

$$\left. \begin{aligned} (y_m y_n) &= 0, & (z_m z_n) &= 0, & (z_m y_n) &= \delta_{mn}, \\ \sum_{n=0}^{\infty} z_n(t) y_n(x) &= \frac{1}{t-x} & |x| &< |t|. \end{aligned} \right\} \quad (3) \text{ ter}$$

If we start from the differential equation

$$Ly = \frac{d}{dx} x^2 \frac{dy}{dx} + (2\nu-1)x \frac{dy}{dx} + x^2 y = \lambda y,$$

we have

$$\begin{aligned} \text{eigenvalues} \quad & \lambda = (n+\nu)^2 - \nu^2, \\ \text{eigenfunctions} \quad & y_n(x) = x^{-\nu} J_{\nu+n}(x), \\ & z_n(x) = A_{n,\nu}(x) \end{aligned}$$

which satisfy

$$\frac{d}{dx} x^2 \frac{d}{dx} z_n - (2\nu-1) \frac{d}{dx} x z_n + (x^2 - \lambda_n) z_n = g_{n,\nu}(x)$$

where $g_{n,\nu}(x)$ are polynomials³⁾ of the first degree in x . The same relations as (3) hold analogously. Let $L = x/(1-x^2) \cdot \bar{x} x \bar{x}$, then we get expansions in Kapteyn series.⁴⁾

§ 7. Eigenfunctions of the confluent hypergeometric equations

$$L = x^2 \bar{x}^2 - \frac{x^2}{4} + kx$$

Differential equation to y

$$x^2 y'' + \left\{ -\frac{x^2}{4} + kx - \lambda \right\} y = 0,$$

C : a curve encircling once $x=0$.

Eigenvalues $\lambda = n(n+1), \quad n=0, 1, 2, \dots$.

Eigenfunctions

$$y_n = x^{1+n} e^{-x/2} \left\{ 1 + \frac{n+1-k}{1!(2n+2)} x + \frac{(n+1-k)(n+2-k)}{2!(2n+2)(2n+4)} x^2 + \dots \right\},$$

$$z_n = x^{-2-n} e^{x/2} \left\{ 1 + \frac{k-n}{1! 2n} x + \frac{(k-n)(k-n+1)}{2! 2n(2n-1)} x^2 + \dots + \frac{(k-n) \dots (k-1)}{n! 2n \dots (n+1)} x^n \right\},$$

z_n satisfying

$$(x^2 z_n)'' + \left\{ -\frac{x^2}{4} + kx - n(n+1) \right\} z_n = \frac{e^{x/2}}{x} \frac{(k-n) \dots (k-1) k}{2n!}.$$

There are similar relations as (3)

$$(y_m y_n) = 0, \quad (z_m z_n) = \delta_{m,n}, \quad \sum_{n=0}^{\infty} z_n(t) y_n(x) = \frac{x e^{(t-x)/2}}{t(t-x)} \quad |x| < |t|.$$

If we take

$$L = x \bar{x}^2 - \frac{x}{4} - \frac{r(r+1)}{x}, \quad r=0, 1, 2, \dots$$

C : a curve encircling once $x=\infty$.

Differential equation to y

$$x y'' - \left(\frac{x}{4} + \frac{r(r+1)}{x} \right) y = \lambda y.$$

Eigenvalues $\lambda = r+n, \quad n=1, 2, \dots$

Eigenfunctions

$$y_n = e^{-x+n} x^{r+n} \left(1 + \frac{(1-n)(2r+n)}{1! x} + \frac{(1-n)(2-n)(2r+n)(2r+n-1)}{2! x^2} + \dots \right),$$

$$z_n = e^{x/2} x^{-r-n-1} \left(1 + \frac{n(2r+n+1)}{1! x} + \frac{n(n+1)(2r+n+1)(2r+n+2)}{2! x^2} + \dots \right),$$

$$\sum_{n=0}^{\infty} z_n(t) y_n(x) = e^{(t-x)/2} \left(\frac{x}{t} \right)^{r+1} / (t-x).$$

This result may be applied to the eigenvalue problem for hydrogen atom.⁵⁾
Let

$$L = \bar{x}^2 + \frac{c}{x} - \frac{r(r+1)}{x^2}.$$

To determine eigenvalues and eigenfunctions of the differential equations

$$y'' + \left(\frac{c}{x} - \frac{r(r+1)}{x^2} \right) y = \lambda y$$

under the condition that y be one-valued around $x = \infty$, we change the variable from x to $t = \sqrt{4\lambda} x$, and we have

$$\frac{d^2 y}{dt^2} + \left\{ -\frac{1}{4} + \frac{c}{\sqrt{4\lambda}} \frac{1}{t} - \frac{r(r+1)}{t^2} \right\} y = 0,$$

whence we get, using the above result

$$\frac{c}{\sqrt{4\lambda}} = r + n, \quad n = 1, 2, \dots$$

therefore we rediscover the famous formula

$$\lambda = \frac{c^2}{4(r+n)^2}, \quad n = 1, 2, \dots$$

Another curious example is the case of harmonic oscillator. If we determine solutions of Weber's equation

$$y'' + \left(n + \frac{1}{2} - \frac{x^2}{4} \right) y = 0$$

under the condition that y be one-valued around $x = \infty$, we get

$$n = 0, 1, 2, \dots,$$

$$y_n = \exp\left(-\frac{x^2}{4}\right) \cdot x^n \left\{ 1 - \frac{n(n-1)}{2x^2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot x^4} - \dots \right\},$$

$$z_n = \exp\left(\frac{x^2}{4}\right) \cdot x^{-n-1} \left\{ 1 + \frac{(n+1)(n+2)}{2x^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot x^4} + \dots \right\},$$

$$\sum_{n=0}^{\infty} y_n(x) z_n(t) = \exp\{(t^2 - x^2)/4\} \cdot (t - x) \quad |x| < |t|.$$

More curious is the fact that the expectation values of x^{2m} (e.g. $m=1$) calculated by two ways are equal.

$$\langle x^2 \rangle_1 = \int_{-\infty}^{\infty} \left\{ \exp\left(-\frac{x^2}{4}\right) H_n(x) / n!^{1/2} (2\pi)^{1/4} \right\}^2 x^2 dx = 2n + 1,$$

$$\langle x^2 \rangle_2 = \frac{1}{2\pi i} \int_c x^2 y_n(x) z_n(x) dx = 2n + 1.$$

From the preceding results we can guess the existence of a certain property concerning eigenfunction expansions in the complex plane.

§ 8. Eigenfunction expansions of $1/(t-x)$

We consider here an operator

$$L = p_0(x)\bar{x}^2 + p_1(x)\bar{x} + p_2(x) \quad (4)$$

where we assume in the neighbourhood of $x=0$

$$p_0(x) = x^2(1 + cx + \dots),$$

$$p_1(x) = x(1 - \nu + \dots), \quad \nu : \text{no integer.}$$

$$p_2(x) = \mu + \dots.$$

The differential equation to y

$$Ly = p_0(x)y'' + p_1(x)y' + p_2(x)y = \lambda y \quad (5)$$

is assumed to have no singular point other than $x=0$ inside the closed curve C encircling $x=0$.

The indicial equation at $x=0$ is

$$\rho^2 - \nu\rho + \mu = \lambda.$$

If we require y be regular inside the curve C , we have

$$\rho = 0, 1, 2, \dots, n, \dots$$

so that we get $\lambda_n = \mu - \nu n + n^2, \quad n = 0, 1, 2, \dots,$

$$y_n = x^n(a_{nn} + a_{n,n+1}x + a_{n,n+2}x^2 + \dots), \quad a_{nn} = 1.$$

The adjoint equation to (5) is

$$\bar{L}z = (p_0(x)z)'' - (p_1(x)z)' + p_2(x)z = \lambda_n z$$

whose indicial equation at $x=0$ has two roots $-n-1, n-\nu-1$. We denote the solution corresponding to the index $-n-1$ by v_n which is in general an infinite ascending power series in x .

$$v_n = x^{-n-1}(c_{nn} + c_{n-1,n}x + \dots + c_{0n}x^n + \dots), \quad c_{nn} = 1.$$

We cut off the terms higher than x^{-1} in this series and denote the remained part by z_n

$$z_n = x^{-n-1}(c_{nn} + c_{n-1,n}x + \dots + c_{0n}x^n)$$

which does not satisfy the differential equation

$$(\bar{L} - \lambda_n)z = 0,$$

but

$$(\bar{L} - \lambda_n)z_n = \delta_0 + b_1x + \dots.$$

The right side does not contain any term of negative power in x .

We have from the above the following relations

$$(y_m y_n) = 0, \quad (z_m z_n) = 0, \quad (y_m z_n) = \delta_{mn},$$

the first two are obvious. The last is deduced from the relation

$$((\bar{L} - \lambda_n)z_n, y_m) = 0, \quad \text{i.e.} \quad (\bar{L}z_n, y_m) = \lambda_n(z_n y_m).$$

Further we can deduce from $(y_m z_n) = \delta_{mn}$, $m=0, 1, \dots, n$

$$(y_0 z_n) = a_{00} c_{0n} + a_{01} c_{1n} + \dots + a_{0n} c_{nn} = 0,$$

$$(y_1 z_n) = a_{11} c_{1n} + \dots + a_{1n} c_{nn} = 0,$$

$$(y_n z_n) = a_{nn} c_{nn} = 1.$$

If we put

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots \\ 0 & a_{11} & a_{12} & \dots \\ 0 & 0 & a_{22} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad C = \begin{pmatrix} c_{00} & c_{01} & c_{02} & \dots \\ 0 & c_{11} & c_{12} & \dots \\ 0 & 0 & c_{22} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$X = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} y_0(x) \\ y_1(x) \\ y_2(x) \\ \vdots \end{pmatrix}, \quad T = (t^{-1}, t^{-2}, t^{-3}, \dots),$$

$$Z = (z_0(t), z_1(t), z_2(t), \dots),$$

we have

$$Y = AX, \quad Z = TC, \quad \text{and} \quad AC = 1.$$

Therefore we get $C = A^{-1}$ and

$$ZY = TCAX = TX,$$

that is

$$\sum_{n=0}^{\infty} z_n(t) y_n(x) = \sum_{n=0}^{\infty} x^n t^{-n-1} = 1/(t-x) \quad |x| < |t|. \quad (6)$$

The last problem is to determine $(\bar{L} - \lambda_n) z_n$. Multiplying the equation (6) by L_x or \bar{L}_t from the left, we have

$$L_x \frac{1}{t-x} = \sum_{n=0}^{\infty} z_n(t) \lambda_n y_n(x),$$

$$\bar{L}_t \frac{1}{t-x} = \sum_{n=0}^{\infty} \bar{L}_t z_n(t) y_n(x),$$

whence we get

$$\bar{L}_t \frac{1}{t-x} - L_x \frac{1}{t-x} = \sum_{n=0}^{\infty} (\bar{L}_t - \lambda_n) z_n(t) y_n(x). \quad (7)$$

From (4) we have

$$L_x \frac{1}{t-x} = p_0(x) \left(\frac{d}{dx} \right)^2 \frac{1}{t-x} + p_1(x) \frac{d}{dx} \frac{1}{t-x} + p_2(x) \frac{1}{t-x},$$

$$\bar{L}_t \frac{1}{t-x} = \left(\frac{d}{dt} \right)^2 \frac{p_0(t)}{t-x} - \frac{d}{dt} \frac{p_1(t)}{t-x} + p_2(t) \frac{1}{t-x}.$$

Expanding $p_r(t)$ in Taylor series in $t-x$,

$$p_r(t) = p_r(x) + (t-x) p_r'(x) + \frac{(t-x)^2}{2} p_r''(x) + \dots$$

On the Absorption of the Negative π -meson by Deuteron

Shuzo OGAWA, Eiji YAMADA and Yukio NAGAHARA

Institute of Theoretical Physics, Nagoya University

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We have analyzed the various processes which are induced by the absorption of the negative π -meson by the deuteron. The negative π -meson is assumed to be bound in K -orbit. The results indicate that, as for the model of meson, the vector and the pseudovector types are not favourable. The π -meson of pseudoscalar type is most favourable.

§ 1. Introduction

In the analysis of the processes in which π -mesons take part, it is generally difficult to support the validity of the obtained results because of the complicated correlations among the meson model, the approximation method and the theory in itself.¹⁾ However, we would rather accept such standpoint that, in the concrete analysis of the various processes guided by the present theory, we search the key perspecting the characters of the difficulties. The problem here treated is one of the works with such intentions.

On the π -meson production by the nucleon- and photon-nucleon collision various analysis have been done.²⁾ By these works, the pseudoscalar π -meson theory is shown to be advantageous. Previously, we have studied the absorption of the negative π -meson by the proton³⁾ and found that the pseudoscalar theory was also favourable in this case. But because of the arbitrariness of the coupling constants of the π -meson with the nucleon, we could not give any affirmative conclusion.

It must be noted that in the deuteron case, here treated, the obtained results do not depend upon the magnitude of the coupling constant, but do only upon the type of the interaction between the meson and the nucleon.

With absorption process of the π -meson in K -orbit, we can expect the following three processes tentatively,

$$\begin{array}{ll}
 \nearrow N+N & \text{(i)} \\
 \pi^- + D \rightarrow N+N+\gamma & \text{(ii)} \\
 \searrow N+N+\pi^0 & \text{(iii).}
 \end{array} \tag{1}$$

(i) is the process in which the deuteron disintegrates into two neutrons, getting the rest energy of the π^- -meson, and (ii) is the one which accompanies the photon emission. The process (iii) is allowed only if the rest energy of the neutral π -

meson is less than that of the negative π -meson minus the binding energy of the deuteron and the neutron-proton mass difference. Recent experiment shows that $m_\pi - m_{\pi^0}$ is almost of ten times of the electron mass, so the process (iii) is not always forbidden. But, on account of the smallness of $m_\pi - m_{\pi^0}$, the momentum of the emitted π^0 -meson becomes small. Accordingly, the contribution to the probability of the transition (iii) from the final state density comes out to be smaller compared with that in the case (ii). So, we shall not perform the exact calculation on the process (iii), but only give the order estimation about the relative magnitude of the probability of process (iii), comparing with that of (ii).

The experiments on these processes have been done by Panofsky et al.⁴⁾ The roughness of the obtained results does not enable us to affirm the existence of the process (iii). And it has been known that the processes (i) and (ii) occur with the comparable magnitude.*

Before entering into the calculation, we give some preliminary discussions. The perturbation method is used. Process (i) is of the first order and the processes (ii) and (iii) are of the second. Moreover, for the process (i), the effects of the third order correction are studied. Our aim is pointed to the analysis of the π -meson model, that is, of the meson-nucleon interaction, and so the relativistic treatment becomes necessary. However, the initial state—deuteron in our problem—is the bound one, so that such a treatment becomes seriously difficult. So, we treat the states of the nucleons in Pauli's approximation. Considering the energy of the associated nucleon to be at most one half of the rest energy of the π -meson, we may expect that such a treatment is valid. The state of the nucleon is expressed by

$$\Psi = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad \phi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \quad (2)$$

then, we can write in Pauli's approximation,

$$\bar{\psi} = (-i/2M)(\sigma \text{ grad})\phi; \quad M; \text{ nucleon mass.} \quad (3)$$

Some remarks are needed concerning to the wave function of the deuteron and of the two neutrons after disintegration. In fact, it has become clear that the assumption upon the type of the internucleonic potential bring rather large influence on the results, from the theoretical analysis on the γ -disintegration of the deuteron.⁵⁾ On the other hand, by the analysis of the nucleon-nucleon scattering, it has been clear out that the long-tailed potential give good results.⁶⁾ We take, accordingly, the solution for Serber-Hulthen potential as the ground state of the deuteron.

* Private conversation with Prof. H. Yukawa, in August 1950.

Because of the comparatively small associated energy, the state of the neutrons after disintegration cannot be treated as free. Any accurate knowledge, however, has not yet been obtained on the neutron-neutron potential. Accordingly, from the analogy with the results on the analysis of low energy ($\lesssim 10$ Mev) nucleon-nucleon scattering,—charge independence of the nuclear potential—, we assume the neutron-neutron potential to be also of Serber-Hulthen type. There is, of course, no support on that assumption up to the energy treated here (~ 100 Mev).⁷ But when we study the relative probability of the processes like (1), the small variations of the nuclear potential may result in small correction. The states of the neutrons after disintegration are separated into the states $^1S, ^3P, ^1D$ etc. which are the solutions in the above potential. And we perform the calculation only about the allowed transition to the lowest level, whose probability takes the most part of the total transition probability. The error caused by this simplification is of some 10% in magnitude.

§ 2. $\pi^- + D \rightarrow N + N$; First order process

As the nucleon- π^- -meson interaction, we take only scalar, vector, pseudo-scalar and pseudovector meson theory, respectively. For instance, for the pseudo-scalar theory, we take

$$H'_{p.s.} = f \bar{\Psi}^* \beta \gamma_5 \Psi \phi.$$

According to Pauli's approximation, it results in

$$H'_{p.s.} = f \{ \psi^* \bar{\psi} - \bar{\psi}^* \psi \} \phi = f \{ \psi^* \phi (-i/2M) (\sigma \text{ grad}) \phi - (i/2M) (\text{grad } \phi^*) \sigma \phi \}.$$

As equivalent form, we take the following

$$H'_{p.s.} = f(i/2M) \psi^* (\sigma \cdot \text{grad } \phi) \phi. \quad (4)$$

Similarly, for other meson types we obtain the followings,

$$\begin{aligned} H'_s &= f \psi^* \phi, & H'_v &= (if/M) \psi^* \sigma U \cdot \sigma \text{ grad } \phi, \\ H'_{p.v.} &= f \psi^* \sigma U \psi. \end{aligned} \quad (5)$$

ϕ and U are the wave functions of π -meson of scalar type and of vector type, respectively.

From the facts that the deuteron is in 3S state and that two neutrons exist in the final state, there occur some restrictions on the transition allowed by the interaction (4) and (5). In the scalar theory, for instance, the interaction having no σ -dependent term, only the transition to the triplet state is to be allowed. If the final state is a triplet one, its parity must be odd. Considering that the parity of the deuteron is even and π -meson has a spherical-symmetric wave-function, such a transition must be forbidden. Allowed transition to the lowest level are as follows,

$$\text{scalar: } ^3S \rightarrow n0,$$

$$\text{vector: } ^3S \rightarrow ^3P,$$

pseudoscalar: $^3S \rightarrow ^3P$, pseudovector: $^3S \rightarrow ^1S$,

To calculate the matrix element, we separate the wave function of the nucleon into two parts, that is, into σ -space and into the ordinary space, letting it to be $\psi = \chi(\sigma)u(r)$. (r is relative coordinate.) The matrix elements result in

$$\begin{aligned} H'_{P.S} &= f(2\pi/\kappa)^{1/2} (i/2M) (\chi^3 \sigma_k \chi^3) (\phi_0/a) (4\pi/3) \int_0^\infty v_0 u_1 r^2 dr^*, \\ H'_V &= f(2\pi/\kappa)^{1/2} (i/M) (\chi^3 \sigma_v \sigma_k \chi^3) \phi_0 (4\pi/3) \int_0^\infty u_1 (\partial/\partial r) v_0 r^2 dr, \\ H'_{P.V} &= f(2\pi/\kappa)^{1/2} (\chi^1 \sigma_v \chi^3) \phi_0 4\pi \int_0^\infty u_0 v_0 r^2 dr. \end{aligned} \quad (7)$$

Here, we have neglected the binding energy of the π -meson compared with its rest energy. And we have replaced ϕ by ϕ_0 (value of ϕ at the origin), for ϕ is almost constant in the region of the deuteron's extent. k is the relative momentum of the neutron after disintegration. σ_k and σ_v are the k - and U -component of the spin vector σ . For convenience of the calculation of the matrix in σ -space, the projection operators which select the triplet and the singlet are introduced,

$$A_s = (1 - \sigma^{(1)} \sigma^{(2)})/4, \quad A_T = (3 + \sigma^{(1)} \sigma^{(2)})/4. \quad (8)$$

Using H' in (7), the probability can be obtained as

$$\begin{aligned} W &= 2\pi |H'|^2 \rho_F, \quad \rho_F = k \sqrt{M^2 + k^2} / 2(2\pi)^3 \cdot d\Omega, \\ k &= [M(\kappa - \epsilon - \Delta M)]^{1/2}. \end{aligned} \quad (9)$$

ϵ and ΔM are the binding energy of the deuteron and the neutron-proton mass difference, respectively. M is the neutron mass.

If we assume the mass of the π^- -meson to be 278 electron mass, two neutrons whose kinetic energy is about 11 MeV, are to be emitted simultaneously in the opposite directions in the above process.

§ 3. $\pi^- + D \rightarrow N + N + \gamma$

In this case, the transitions are allowed for all meson types and the lowest final state is 1S . The process is of the second order and the explicit virtual states are expressed as,

$$\begin{array}{ccc} & \nearrow \left\{ \begin{array}{l} P' + \gamma + \pi^- + N \\ [P] + N' + N + P \end{array} \right. & (1) \\ D + \pi^- & \longrightarrow & \\ & \searrow \left\{ \begin{array}{l} N' + \gamma + P + \pi^- \\ N' + N \end{array} \right. & (2) \\ & \nearrow \left\{ \begin{array}{l} N' + N + \pi^- + \pi^+ \\ D + \gamma + \pi^- \end{array} \right. & (3) \end{array} \quad \longrightarrow N + N + \gamma. \quad (10)$$

* We take $\hbar = c = 1$ unit.

The dash means the particles which take part in the emission and the absorption of the protons in the virtual state, and [] means a hole of the negative state. In the virtual state, we treat the nucleon as free. Our processes are associated with the absorption of the π^- -meson, so we cannot separate the photon-nucleon interaction into the dipole, quadrupole ... interaction as in the case of the photo-disintegration. For instance, in the pseudoscalar theory, the contribution from the negative state nucleon becomes distinctly large. We should treat the process one-body-problematically.

On the scalar type meson theory, the interaction in the transition (12) is,

$$H' = e\bar{\Psi}^* \alpha \mathbf{A} \Psi + \frac{ie}{4\pi} \phi \mathbf{A} \text{grad } \phi + \mu_p \bar{\Psi}^* \boldsymbol{\sigma} \mathbf{H}(\mathbf{x}_1) \Psi + \mu_n \bar{\Psi}^* \boldsymbol{\sigma} \mathbf{H}(\mathbf{x}_2) \Psi + f \bar{\Psi}^* \mathbf{O} \Psi \phi,$$

$$\mathbf{O} = \beta: \text{ for scalar, } \quad \mathbf{O} = \beta \gamma_5: \text{ for pseudoscalar.} \quad (11)$$

\mathbf{x}_1 and \mathbf{x}_2 are the position of the proton and of the neutron, respectively. We have introduced the anomalous magnetic moment of the nucleon, phenomenologically, indicating it as μ . Process (3) in (12) is of little contributions to the probability, because of the small binding of the π^- -meson. On the vector type meson theory, however, though the above discussion is not valid, we might be able to neglect the process (3) according to the reason discussed below. On one hand, if the π^- -meson is on the orbit after radiating γ -ray in the virtual state, the contribution of the space integral to the matrix element becomes quite small. And on the other hand, if the π^- -meson is out of orbit in that case, the density of the π^- -meson's wave function comes out to be small, effecting to make the absorption of the π^- -meson by the proton difficult. Though we cannot examine such an effect quantitatively, it may be not far from the reality.

We can calculate the matrix element of the interaction (11) explicitly. For a example, we take the pseudoscalar case. In this case, the process (1) in (12) gives the largest contribution. Hereafter, we shall assume $2 \gg \mathbf{x}/M$, and we neglect $O(\mathbf{x}/M)$ compared with unity. In this approximation, only the process (1) is effective. Noting the initial, virtual and final states of the proton, as Ψ_i , Ψ_v and Ψ_f , respectively, we set similarly to the treatment in § 2,

$$\begin{aligned} \Psi_i &= \sum_p \chi_p^i v_p^i, & \Psi_v &= \sum_p \chi_p^v v_p^v = \chi_p^v \exp(i\mathbf{k}' \cdot \mathbf{x}_1), \\ \Psi_f &= \sum_p \chi_p^f v_p^f, & \rho &= 1, 2, 3, 4. \end{aligned} \quad (12)$$

Then, the matrix element is

$$H' = f e (2\pi/\mathbf{x})^{1/2} (2\pi/\mathbf{k})^{1/2} (\chi_f^\dagger \beta \gamma_5 \chi_i^\dagger \cdot \chi_v^v u_c \chi^c) \phi_0 \int v^f e^{i\mathbf{k}' \cdot \mathbf{r}} d\mathbf{r}$$

* In the calculation of $|\langle \chi_f \mathbf{O} \chi_i \rangle|^2$ with these operators, we must take the following normalization, corresponding to the singlet or triplet initial state,

$$\begin{aligned} |\langle \chi_f \mathbf{O} \chi_i \rangle|^2 &= S_p \mathbf{O} \mathbf{A} \mathbf{O} \mathbf{A}_s, \\ \text{or} \quad |\langle \chi_f \mathbf{O} \chi_i \rangle|^2 &= \frac{1}{3} S_p \mathbf{O} \mathbf{A} \mathbf{O} \mathbf{A}_T. \end{aligned}$$

$$\times \int e^{-i(\mathbf{k}' + \mathbf{k}/2) \cdot \mathbf{r}} v^i d\mathbf{r}' \times \left(\frac{d\mathbf{k}'}{(2\pi)^3} \right) \left(\frac{1}{\mathbf{z} - \sqrt{M^2 + (\mathbf{q} - \mathbf{k}/2)^2} + E_{|\mathbf{k} - \mathbf{q}|}} \right) \quad (13)$$

$|\mathbf{k}| = k$ is the energy of the photon in the final state and \mathbf{q} is the momentum of the proton in the relative coordinate. $E_{|\mathbf{k} - \mathbf{q}|}$ is the proton's energy in the virtual state, including its sign. We replace ϕ by ϕ_0 according to the same reason as discussed in § 2. This simplification may be valid on the assumption that the various transitions in the virtual state occur in the region of the K -shell of the π^- -meson.*

Further approximation is made. Space function v_j can be replaced by the plane wave assigned to the momentum \mathbf{q} except in the nearest neighbour of the origin where the nuclear force has a strong effect. Accordingly, the most contribution of the first integral of (13) comes from the region $|\mathbf{k}'| \sim |\mathbf{q}|$. And so we can take the denominator in (13) to be nearly constant in the integration on \mathbf{k}' . It gives simply $\delta(\mathbf{r} - \mathbf{r}')$. The calculation in the spin space is as follows;

$$\begin{aligned} (\chi^j \beta \gamma_5 \chi^i \cdot \chi^v a_c \chi^t) \frac{1}{E_0 + E_p} &= (\chi^j \cdot \beta \gamma_5 [E_0 - \boldsymbol{\sigma} \mathbf{p} - \beta M] a_c \chi^t) \times (1/E_0^2 - E_p^2) \\ &\cong (-1/2M) (\chi \boldsymbol{\sigma}_e \chi_0), \\ E_0 &= \mathbf{z} - \sqrt{M^2 + (\mathbf{q} - \mathbf{k}/2)^2}, E_p = E_{|\mathbf{k} - \mathbf{q}|}. \end{aligned} \quad (14)$$

Here, we have used the approximation $1 \gg (\mathbf{z}/M)$. $\boldsymbol{\sigma}_e$ is the component of the spin vector $\boldsymbol{\sigma}$ in the polarization direction of the emitted photon. In the last line of (14), χ is 2-dimensional spinor in Pauli's spin space. Thus, the interaction becomes,

$$H'_{FS} = f e (2\pi/\mathbf{z})^{1/2} (2\pi/k)^{1/2} (-1/2M) (\chi \boldsymbol{\sigma}_e \chi_0) \phi_0 \int u_0(q) e^{-i\mathbf{k}\mathbf{r}/2} v_0 d\mathbf{r} \quad (15)$$

v_0 and u_0 are the wave function of the ground state of the deuteron and the S -wave function of the wave number q .

The probability is

$$dW = 2\pi |H'|^2 \rho_F. \quad (16)$$

The density of the final states is,

$$\begin{aligned} \rho_F &= k^2 dK | (2\pi) | \cdot d\Omega_k \cdot \frac{q^2 d\Omega_q}{(2\pi)^3} (\partial q / \partial E_F), \\ E_F &= E_D + \mathbf{z} = \sqrt{M^2 + (\mathbf{q} - \mathbf{k}/2)^2} + \sqrt{M^2 + (\mathbf{q} + \mathbf{k}/2)^2} + k. \end{aligned} \quad (17)$$

$\mathbf{q} - \mathbf{k}/2$ and $-\mathbf{q} - \mathbf{k}/2$ are the final momenta of the two neutrons in the center of mass system (in this case, laboratory system).

* Strictly speaking, it is the three-body-problem. Because of the difficulty in such problem, we rather depend on the physical picture.

§ 4. Numerical estimations and the process accompanied with the π^0 -emission

I) We shall begin with numerical analysis of the results obtained in the foregoing paragraphs. The spectra of the γ -ray emitted in the process $D + \pi^- \rightarrow N + N + \gamma$ are presented in Fig. I, being compared with Panofsky's results. Taking

into account the uncertainty of the experiments, the consistency is satisfactory. Still more a few remarks on the spectra are given. The spectra of the emitted γ -ray accompanied with the absorption of scalar π^- -meson and of pseudovector π^- -meson are of same type. Concerning to vector and pseudoscalar π -meson, the situations are similar. The former spectrum differs from the latter only within the experimental errors. From Fig. I, it is clear that the majority of the emitted γ -rays have the energy greater than 110 Mev. Accordingly, the probability that the neutron is emitted with the energy more than 30 Mev is only of magnitude less than 1%. This situation is favourable to discriminate the above process from the process $\pi^- + D \rightarrow N + N$ in the experimental researches.

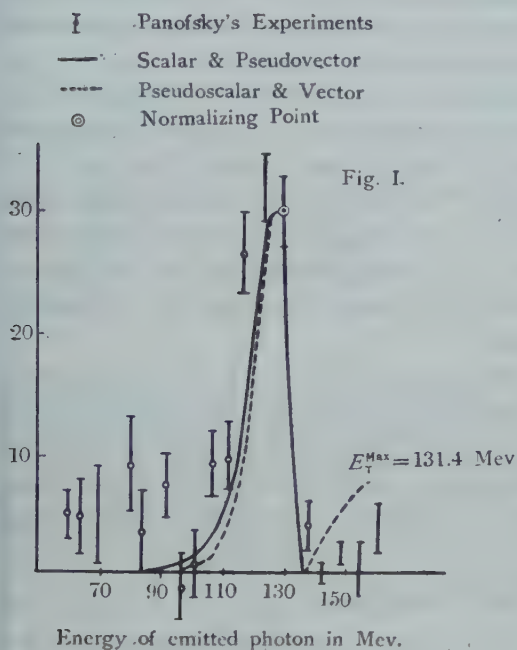


Fig. I

The ratios of W_{2N} (probability of the process $\pi^- + D \rightarrow N + N$) to $W_{D\gamma}$ (probab. of the process $\pi^- + D \rightarrow N + N + \gamma$) are presented in Table I. In these calculations, the coupling constants of the π -meson with the nucleon cancel out. Accordingly, the indefiniteness caused by the magnitude of the coupling constants vanishes. As is mentioned in the introduction, these discussions, of course, is valid only in the region in which the value of the coupling constant allows the perturbation method. From Table I, we can see that the results for any mesonic type are inconsistent with the experimental result which indicates $W'_{2N}/W_{D,\gamma} \sim 1$. The following considerations, however, may be needed. Considering the coupling constants only, we may expect $W_{2N}/W_{D,\gamma} \sim 1/e^2 = 137$. The situations that $W'_{2N}/W_{D,\gamma}$ deviates far from 10^2 are accidental, being caused by the peculiarity of the interaction of the π -meson with the nucleon. For

Meson Type	Ratio $W'_{D,\gamma}/W_{P,\gamma}$
S.	0
P.S.	0.02
V.	$5 \cdot 10^2$
P.V.	$7 \cdot 10^6$

Table I.

instance, in the case of pseudoscalar π^- -meson, the operator $\beta\gamma_5$ which is included in the 1st approximation makes the value of W_{2N} quite small. In this case, higher order correction to the process $\pi^- + D \rightarrow N + N$ may be needed and be expected to contribute considerably. For the case of scalar π^- -meson, similar affairs are expected. In the following paragraph, we shall analyse the 3rd order correction, which does not include such peculiarity as in the 1st approximation. From the view point of the perturbation method, these considerations are unnecessary for the π^- -meson of vector type. And so, for the vector type meson, the consistency with the experiment are not expected.

Further, the ratios of $W_{D,\gamma}$ to $W_{P,\gamma}$ (probab. of the process $P + \pi^- \rightarrow N + \gamma$) are given in Table II. It is to be noted that the values given in Table II must not be compared with the ratio of the process $D + \pi^- \rightarrow N + N + \gamma$ to the process $P + \pi^- \rightarrow N + \gamma$ given by Panofsky's result. Because Panofsky's result remains unchanged, even if the probabilities of all transitions caused by the absorption of π^- -meson by the proton increase by a arbitrary factor on the whole.

Meson Type	Ratio $W_{D,\gamma}/W_{P,\gamma}$
<i>S</i>	0.85
<i>P.S</i>	0.81

Table II

II) We have mentioned only that the process $D + \pi^- \rightarrow N + N + \pi^0$ is allowed energetically. Here, we estimate its probability relative to $W_{D,\gamma}$ based on the density of the final states. In the calculation of $W_{D,\gamma}$, the density of the final states is given by (20). And most contribution to the probability comes from the region $k \sim q \sim x$. On the other hand, the density of the final states in the transition $D + \pi^- \rightarrow N + N + \pi^0$ becomes

$$l^2 dl / (2\pi)^3 \cdot d\Omega_l \cdot q^2 d\Omega_q / (2\pi)^3 \cdot (\partial q / \partial E). \quad (18)$$

l is the momentum of the emitted π^- -meson. Now, the mass difference of π^- -meson and π^0 -meson is $\sim 10 m_e$ (m_e : electron mass). Taking into account of the proton-neutron mass difference and the binding energy of the deuteron, the available kinetic energy of the emitted π^0 -meson is only $\sim 4 m_e$. This indicates $l_{max} \sim \sqrt{8x_0 m_e}$ (x_0 : π^0 -meson mass). Accordingly,

$$[\rho_F(\pi^- + D \rightarrow N + N + \pi^0) / \rho_F(\pi^- + D \rightarrow N + N + \gamma)] < (l_{max}/x)^4 \sim 0.1\%. \quad (19)$$

The results in (19) support the discussions in § 1. on the process or $\pi^- + D \rightarrow N + N + \pi^0$. Hitherto we have not treated the matrix element in detail. The contribution of the process $\pi^- + D \rightarrow N + N + \pi^0$, however, is of magnitude less than 10% compared with that of $\pi^- + D \rightarrow N + N + \gamma$, even if we take the coupling constant g_0 of the π^0 -meson with the nucleon such as $g_0^2 \sim 1$.

§ 5. $\pi^- + D \rightarrow N + N$: 3rd order correction

As mentioned above, the fact that the probability of the process $\pi^- + D \rightarrow N + N$ calculated in the 1st order becomes unexpectedly small in the case of the scalar type meson, is accidentally caused by the character of the interaction

treated in the 1st order approximation, and so we may need to calculate the 3rd order correction to this process. We treat this problem in this paragraph.

The ordinary perturbation method is generally not so perspective in the treatments of the higher order correction. We follow Feynman-Dyson formalism. F - D formalism, however, is based on the treatments of the free field. Accordingly, its application to the problem which includes the transition from the bound states such as deuteron may lack some strictness. However, we may expect the validity of this procedure on the point of the qualitative discussions. We give only the order estimation of the result, even if quantitative estimation is carried out.

The lowest corrections to the process that one of two nucleons system absorbs the π^- -meson are expressed by Fig. II.

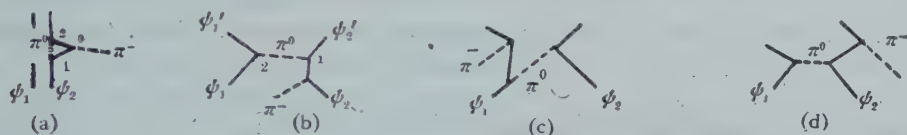


Fig. II.

We dropped the figures which give the mass term correction in Fig. II. ϕ_1 and ϕ_2 are the wave functions of neutron and proton in the initial states respectively. If the initial and final wave functions of the nucleons are free, the calculation can be performed strictly. In our problem, initial state is bound one. However, the effect of the binding to such correction is expected to be small. In fact, the binding energy of the deuteron is quite small compared with the rest energy of the nucleon. The main part of the 3rd order correction may be obtained from the diagrams in Fig. II.

Process (a): This process, as is well known, belongs to Lamb shift type. The corrected interaction caused by this process is,

$$H_a = i \frac{f f_N f_P}{8} \int \int dx_1 dx_2 \phi_N(x_1) \omega S_F(x_0 - x_2) \omega S_F(x_1 - x_2) \omega \phi_P(x_1) \Delta_F(x_2 - x_1) \phi(x_0) \quad (20)$$

$$\omega = \gamma_5 \text{ for pseudoscalar, } \omega = 1 \text{ for scalar,}$$

the derivation of which follows the neglect of the motion of the neutron in the deuteron. Then, the calculation is straightforward and is presented only its results.

$$H_a = \frac{if}{4} \frac{(\bar{f}^2 - 4f^2)}{4\pi^2} \bar{\psi}_N(x_0) \omega \phi_P(x_0) \phi(x_0) [F(M, x, x_0) + O(\square^2 - x^2/M^2)]^*. \quad (21)$$

F is the numerical constant which includes logarithmic divergence, and is neglected for the coupling renormalization. The operator $\square^2 - x^2/M^2$, being separated from

* $\bar{f} = (f_N^0 + f_P^0)/3$, $\Delta f = (f_N^0 - f_P^0)/2$. f^0 is the coupling constant of π^0 -meson with the nucleon.

time component, reduces to $4/M^2$ which operates on the wave function of the π^- -meson. In our problem the binding of the π^- -meson is so small that the contribution (24) is quite small.

Process (b): From the view point of the character of the corrected interaction, this diagram is equivalent with (c) and (d). And we pick up this diagram as a sample and calculate it. The S -matrix for this process is,

$$S_b = i \frac{ff_N^{02}}{4} \cdot \int dx^0 \int dx_1 \int dx_2 \bar{\psi}(x_2) \bar{\psi}(x_1) \omega^2 \omega^1 S_F(x_0 - x_1) \times \\ \times \omega' A_F(x_1 - x_2) \phi_2(x_0) \phi_1(x_2) \phi(x_0). \quad (22)$$

In order to express the initial and final state not so explicitly, we take the following approximation. The special variations of $\bar{\psi}(x_0) \phi(x_1)$ are replaced by the deuteron state,* whose explicit expression is unnecessary. Then, time integral is performed. In the neglect of the binding energy, it results in,

$$S_b \cong i \frac{ff_N^{02}}{4(2\pi)^2} \int d\mathbf{r}_0 d\mathbf{r} \int d\mathbf{k} \phi^*(\mathbf{r}_0 \mathbf{r}_2) \beta' \omega' \frac{\exp[i\mathbf{k}(\mathbf{r}_0 - \mathbf{r}_2)]}{k^2 + 3/4\chi^2} \times \\ \times \beta^2 \omega^2 \frac{M(1-\beta) + \chi}{2M\chi} \beta' \omega' \phi(\mathbf{r}_0 \mathbf{r}_2) \phi. \quad (23)$$

As is well known, $\int \beta' \omega' \frac{\exp[i\mathbf{k}(\mathbf{r}_0 - \mathbf{r}_2)]}{k^2 + 3/4\chi^2} \beta^2 \omega^2 d\mathbf{k}$ is the nuclear force potential caused by the exchange of π -meson in the 2nd approximation. We represent it by $V(\mathbf{r}_1 - \mathbf{r}_2)$. Taking out the interaction from (7), we obtain,

$${}^3H'_{PS} = \frac{\pi}{2} ff_N^{02} \left\{ \phi^* \left[\sigma \frac{i}{2M} \text{grad} \left(\frac{1}{\chi} V_{PS} \right) \right] \phi \right\} \phi, \\ {}^3H'_S = \frac{\pi}{2} ff_N^{02} \left\{ \phi^* \cdot \frac{\chi}{2M} \left(\frac{1}{\chi} V_S \right) \phi \right\} \phi. \quad (24)$$

Pauli's approximation (3) is used in the derivation of (24). For the 3rd order correction of the absorption process of scalar π^- -meson, we take into account only the correction due to the exchange of scalar π^- -meson. For the pseudoscalar case, it is the same. As is seen from (24), the correction in the case of scalar π^- -meson, does not include any σ -dependent term. Accordingly, for the scalar π^- -meson, the process $\pi^- + D \rightarrow N + N$ is forbidden even in the 3rd order correction.

For the case of the pseudoscalar π^- -meson, in (24) $\text{grad}(V_\chi)$ appears in the place of $\text{grad}\phi$ in (4). The fact that the contribution of the process $\pi^- + D \rightarrow N + N$ in the first approximation becomes quite ineffective is caused by the lax binding of the π^- -meson. The breadth of the wave function ϕ of the bound π^- -

* Such procedure may be not taken as the precision of the approximation. In fact, for the process which includes the bound state, the reconstruction of $F-D$ formalism is heeded. The above procedure is valid only as an approximation.

meson is about compton wave length of π^- -meson multiplied by $1/e^2 \sim 100$. Taking into account that V/α spreads in the region of compton wave length of the π^- -meson, ${}^3H'_{PS}/H'_{PS} \cong (f_N^0)^2/e^2$. Accordingly, if $(f_N^0)^2 \sim 0.1$, the value of W_{2N}/W_{DT} can be consistent with the experiment.

In the calculation of the 3rd order correction above, we have assumed that the absorbed π -meson is of identical type to the exchanged one. This assumption is not always necessary. In the absorption process of the scalar π^- -meson, the correction due to the exchange of another type meson, for instance, of τ -meson may occur. If the obtained V includes the operator* not commutable with $\sigma^{(1)}\sigma^{(2)}$, the process $\pi^- + D \rightarrow N + N$ is allowed for the case of the scalar π^- -meson in the 3rd order transition.

§ 6. Summary

From the analysis above, we can see that the cases of scalar and pseudo-scalar meson may be consistent with the experiment by Panofsky et al. Especially, the pseudoscalar meson is most favourable because of its consistency by itself.

Our conclusion, however, may be restricted by the validity of the perturbation method. The method indicates some doubtness in the application to the problems which correlate to the structure of the elementary particle, for instance, to the problem of the anomalous magnetic moment of the nucleon. On the other hand, in the process in which real mesons are emitted and absorbed, the analysis based on the perturbation method gives not always unreasonable results. The problem of the meson production by the γ -ray may seem to support this method. Accordingly, we may expect that our conclusion is not far from reality.

In conclusion, we should like to express our deep gratitude to Prof. S. Sakata for his valuable suggestions. And we are also indebted to Mr. Yamaguchi and Mr. Fujimoto for their kind discussions.

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* Such operators make the transition triplet \leftrightarrow singlet possible.

On the Spin of Neutrino

Ken-ichi Ono

*Department of Physics, Faculty of Science,
University of Tokyo*

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§ 1. Introduction

Within about fifteen years since Fermi's theory of the β -disintegration was presented, this theory has come out, we may say, through two "crisis". The first crisis was the period when the spectrum for allowed transitions was believed to disagree with the result of Fermi's theory, and when the $K-U$ theory was presented. The main reason why the spectrum for allowed transitions was misassigned to the $K-U$ type, was that the maximum of the spectrum was displaced to the lower energy side by absorption, back scattering and energy loss caused by the insufficiency of source thinness. Afterwards spectra of allowed transition type were observed in case of In^{114} , N^{12} , and C^{11} . Then the general view mentioned above was corrected, and many people have come to believe that the spectrum for allowed transitions belongs to Fermi type. Even though nowadays, when techniques in the experiments have developed so much that the very thin source is applicable, there exists remarkable disagreement between experimental data and theoretical values in the energy region lower than 50–100 kev. It should, however, be explained by the secondary effects, e.g. screening effects of core electrons on the nuclear potential. Therefore it is the most plausible to think that is no fundamental discrepancy between experimental data and theoretical values on this point.

The second crisis broke out several years ago when it was considered by many people, mainly among American physicists, that the spectrum for forbidden transitions become also Fermi's allowed transition type. Unfortunately, as all the spectra of forbidden β -ray on Au^{198} , K^{40} , P^{32} , Na^{24} , C^{14} and Be^{10} observed in these days showed the allowed transition type consistently, it became to be believed that the forbidden type spectrum did not exist principally. The only exceptional case was RaE . Considering in this case some special causes, many people regarded the β -ray spectrum as an allowed transition type, both for allowed transitions and for forbidden transitions. However, after the forbidden-type spectra were observed on Y^{90} , Y^{91} , Sr^{89} , Sr^{90} , and Cs^{137} and Cl^{38} , these thoughts were abandoned.

Of course, there exist detail problems concerning with individual radioactive elements, and many of them are not yet solved. For example, it seems interesting

how to explain the long life of C^{14} . But, generally speaking, Fermi's theory is successful in the explanation of experimental data. Therefore, it is plausible to think that the remaining problems should be solved within the frame of Fermi's theory, and there is no evidence that the Fermi's theory itself should be modified.

By the reasons mentioned above, it seems that we have arrived at a stable state. We should, however, be rather prudent, because in the theory of β -disintegration, we have, as well known, to introduce the undiscovered particle, that is, neutrino. In Fermi's theory we assume that the neutrino is a neutral light particle which has spin $1/2$ and satisfies the Dirac equation. Are there no alternatives in this assumption? If there are any alternatives, what are the concrete forms of them?

When we consider the problems related to mesons besides the proper β -disintegration, the problem concerning with the model of neutrino looks rather practical and significant for us. According to the meson theory of the β -disintegration, the disintegration process of π mesons to μ meson competes with the β -disintegration process of π meson. If we take a small coupling constant in order to prevent the β -disintegration of π meson, the life time of nucleonic β -disintegration becomes too long. Therefore nowadays we should regard at least the allowed transition as the process due to a direct interaction.

However, if we suppose the spin of π meson to be zero—this is very probable—and the spin of neutrino larger than $1/2$, it would be possible to take a larger coupling constant in conformity with the life time of the nucleonic β -disintegration, because the disintegration process of π meson is forbidden on account of the conservation law of angular momentum. In this paper we shall investigate the possibility to assume the spin of neutrino larger than $1/2$. It becomes clear that the case of spin $1/2$ is the only one case possible.

§ 2. Field with spin $3/2$

The field with spin $3/2$ is described by two spinors of the third rank $a_{\rho\sigma}^{\lambda}$ and $b_{\rho\sigma}^{\dot{\lambda}}$. When there is no other fields, $a_{\rho\sigma}^{\lambda}$ and $b_{\rho\sigma}^{\dot{\lambda}}$ must satisfy the next equations:

$$p^{\nu\rho}a_{\rho\sigma}^{\lambda}=mc b_{\sigma}^{\dot{\nu}\dot{\lambda}}, \quad p^{\nu\rho}b_{\rho\sigma}^{\dot{\lambda}}=mca_{\sigma}^{\lambda}, \quad (1)$$

where m and c are respectively the rest mass of this field and the light velocity. Moreover, in order to preserve the spin of the field as $3/2$, we must impose the symmetricity property upon the spinors as a supplementary condition, that is,

$$a_{\rho\sigma}^{\lambda}=a_{\sigma\rho}^{\lambda}, \quad b_{\rho\sigma}^{\dot{\lambda}}=b_{\sigma\rho}^{\dot{\lambda}}. \quad (2)$$

Except in case of spin $1/2$, any field with zero rest mass has the freedom of a kind of gauge transformation. In our case of spin $3/2$, the following gauge transformation is permissible when the rest mass vanishes: At first we introduce two spinor functions c_{ρ} and $d^{\dot{\nu}}$, which satisfy the following equations:

$$p^{\dot{\nu}\rho}c_\rho=0, \quad p_{\dot{\nu}\rho}d^{\dot{\nu}}=0. \quad (3)$$

Then we define $n_{\rho\sigma}^{\dot{\lambda}}$ and $m_\sigma^{\dot{\lambda}\dot{\nu}}$ by the equations,

$$n_{\rho\sigma}^{\dot{\lambda}}=p_\rho^{\dot{\lambda}}c_\sigma+p_\sigma^{\dot{\lambda}}c_\rho, \quad m_\sigma^{\dot{\lambda}\dot{\nu}}=p_\sigma^{\dot{\lambda}}d^{\dot{\nu}}+p_\sigma^{\dot{\nu}}d^{\dot{\lambda}}. \quad (4)$$

Evidently these two symmetrical spinors of the third rank satisfy the equation (1), when $m=0$. Nevertheless, since the energy-momentum tensor and the current vector derived from these spinors become identically zero, $n_{\rho\sigma}^{\dot{\lambda}}$ and $m_\sigma^{\dot{\lambda}\dot{\nu}}$ do not represent any physically significant state of the spin 3/2 field. Using these $n_{\rho\sigma}^{\dot{\lambda}}$ and $m_\sigma^{\dot{\lambda}\dot{\nu}}$, we can define a gauge transformation as follows:

$$a'_{\rho\sigma}^{\dot{\lambda}}=a_{\rho\sigma}^{\dot{\lambda}}+n_{\rho\sigma}^{\dot{\lambda}}, \quad b'_{\sigma}^{\dot{\lambda}\dot{\nu}}=b_{\sigma}^{\dot{\lambda}\dot{\nu}}+m_{\sigma}^{\dot{\lambda}\dot{\nu}}. \quad (5)$$

By this transformation the energy-momentum tensor and the current vector are obviously invariant. Moreover, we can define another gauge invariant quantity, which might be called "Field strength" of this field, as follows:

$$a_{\tau\rho\sigma}=p_{\tau\dot{\lambda}}a_{\rho\sigma}^{\dot{\lambda}}, \quad b^{\dot{\tau}\dot{\nu}\dot{\lambda}}=p^{\sigma\dot{\tau}}b_{\sigma}^{\dot{\lambda}\dot{\nu}}. \quad (6)$$

Of course, the preceding discussions are applied only on the case $m=0$. When $m \neq 0$, such problems concerning with a gauge transformation would not take place. Now, the mass of the neutrino seems very small. Actually we know that it is negligibly small comparing with the electron mass. However, it is not evident that the mass of the neutrino rigorously vanishes. Nevertheless, we would like to propose the next assumption:

The interaction term, which gives rise to the β -disintegration, involves only $a_{\tau\rho\sigma}$ and $b^{\dot{\tau}\dot{\nu}\dot{\lambda}}$ of Eq. (6) as far as concerning with the neutrino. Accordingly, the interaction is automatically gauge invariant when the mass of the neutrino is precisely zero. In other words, we would assume that only such interaction takes place, that does not change its form spontaneously when the mass of neutrino is reduced to zero. From this assumption, the interaction Hamilton function is determined uniquely as we show in the followings:

Using the spinorial wave function of an electron u_τ and $v^{\dot{\tau}}$, complex conjugates of them, and $a_{\tau\rho\sigma}$ and $b^{\dot{\tau}\dot{\nu}\dot{\lambda}}$ defined above, we can construct a symmetrical spinor of the second rank S :

$$S_{\rho\sigma}=v^{*\dot{\tau}}a_{\tau\rho\sigma}, \quad S^{\dot{\nu}\dot{\lambda}}=u_\tau^*b^{\dot{\tau}\dot{\nu}\dot{\lambda}}. \quad (7)$$

As well known, a symmetrical spinor of the second rank is equivalent to an antisymmetrical tensor of the second rank. Since the tensorial representations are rather familiar for us, we rewrite the spinor S in an antisymmetrical tensor t . The relation between S and t is given by the next equations:

$$S_{11}=(t_{02}+it_{31})-i(t_{01}+it_{23}), \quad S_{12}=i(t_{03}+it_{12}),$$

$$S_{22} = (t_{02} + it_{31}) + i(t_{01} + it_{23}), \quad S_{11} = (t_{02} - it_{31}) - i(t_{01} - it_{23}), \quad (8)$$

$$S_{12} = i(t_{03} - it_{12}), \quad S_{21} = (t_{02} - it_{31}) + i(t_{01} - it_{23}).$$

Next we define a tensor f with the wave functions of a proton and of a neutron:

$$f_{0s} = i\Phi_P^* \beta u_s \Phi_N, \quad f_{8t} = i\Phi_P^* \beta u_s \alpha_t \Phi_N, \quad (9)$$

where α_s and β are the usual Dirac matrices. Coupling these f and t in the relativistic invariant form, we obtain the interaction Hamilton function:

$$H_{\text{int}} = \frac{1}{2} \left(\frac{g}{m_0 c} \right) \int (t_{4s} f^{4s}) dv, \quad (10)$$

where m_0 denotes the electron mass, and g is the coupling constant of this interaction. As far as we adopt the preceding assumption concerning with the interaction, we can not consider any other interaction form than that of Eq. (10).

Calculating the probability of β -disintegration caused by the interaction given in the equation (10) by the method of perturbation theory, we obtain the next results; when we put the mass of the neutrino zero,

$$p(\epsilon) d\epsilon = \frac{|\sigma|^2}{3\tau_0} (\epsilon_0 - \epsilon)^4 \epsilon \sqrt{\epsilon^2 - 1} d\epsilon, \quad \tau_0 = \frac{2\pi\hbar^7}{g^2 m_0^5 c^4}, \quad (11)$$

where ϵ , ϵ_0 and $|\sigma|$ are respectively the energy of the outgoing electron in the unit of $m_0 c^2$, the upper limit of ϵ , and the nuclear matrix element. The spectrum presented by the equation (11) is the $K-U$ type, and does not agree with the Fermi type that is ascertained by the experiments.

§ 3. Conclusion

The reason why the spectrum becomes the $K-U$ type is found in the following point: in order to make the interaction term gauge invariant, it is necessary to differentiate once the wave function of the neutrino as shown in Eq. (6). According to the same reason, if we assume the spin of the neutrino as $(2n+1)/2$, we obtain the following spectrum:

$$(\epsilon_0 - \epsilon)^{2n+2} \epsilon \sqrt{\epsilon^2 - 1} d\epsilon, \quad (12)$$

because, in this case, we must differentiate the wave function of the neutrino n times to derive the gauge invariant quantity. Therefore the spectrum becomes just the Fermi type only when $n=0$, that is, only when the spin of the neutrino is $1/2$.

The assumption concerning with the interaction adopted in this theory might seem to be too severe. If we do not take this assumption, we can consider the interactions which contain the wave function itself of the neutrino when $m \neq 0$. For example, when the spin of the neutrino is $3/2$, there are three types of

interactions with this property, i.e. vector-, tensor-, and pseudovector-types. According to the elaborate calculations by Mr. Nagasaka, we cannot obtain the Fermi type spectrum in every case. After all, the conclusion of this paper is not changed.

Appendix

Compared with the Van der Waerden's notation, our notation of upper and lower spinor suffices is inverted. In our notation, the relation between spinorial and vectorial representations is given by the following equations:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3 \end{pmatrix},$$

and

$$\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} A_0 + A_3 & A_1 - iA_2 \\ A_1 + iA_2 & A_0 - A_3 \end{pmatrix}.$$

The field equation (1) has four independent solutions. The orthogonal solutions A , B , C and D used in this calculation will be shown. Taking aside the plane-wave part $\exp i(\mathbf{k} \cdot \mathbf{r} - k_0 c t)$, the spin part of them are given by the following table.

	A	B	C	D	
a_{11}^1	$(k_0^2 - k_3^2)(k_1 + ik_3)$	$(k_1 + ik_2)^3$	$\kappa(2k_1^2 + 2k_2^2 + 3\kappa^2)$	0	a_{11}^1
a_{22}^2	0	$-(k_0 - k_3)(k_1^2 + k_2^2 + 3\kappa^2)$	0	0	a_{22}^2
b_{11}^1	0	0	$(k_0 - k_3)(k_1^2 + k_2^2 + 3\kappa^2)$	0	b_{11}^1
b_{22}^2	0	$-\kappa(2k_1^2 + 2k_2^2 + 3\kappa^2)$	$(k_1 - ik_2)^3$	$-(k_0^2 - k_3^2)(k_1 - ik_2)$	b_{22}^2
a_{11}^2	$\kappa^2(k_0 + k_3)$	$-(k_1 + ik_2)^2(k_0 + k_3)$	$-2\kappa(k_1 - ik_2)(k_0 + k_3)$	$\kappa(k_0 + k_3)(k_1 + ik_2)$	a_{11}^2
a_{22}^1	$(k_0 - k_3)^2(k_1 - ik_2)$	$(k_1 + ik_2)(k_0 - k_3)^2$	0	$\kappa(k_0 - k_3)^2$	a_{22}^1
b_{11}^2	$\kappa(k_0 - k_3)^2$	0	$(k_1 - ik_2)(k_0 - k_3)^2$	$-(k_0 - k_3)^2(k_1 + ik_2)$	b_{11}^2
b_{22}^1	$-\kappa(k_0 + k_3)(k_1 - ik_2)$	$-2\kappa(k_1 + ik_2)(k_0 + k_3)$	$(k_1 - ik_2)^2(k_0 + k_3)$	$\kappa^2(k_0 + k_3)$	b_{22}^1
ξ	$(k_0 - k_3)^2(k_0 + k_3)$	$(k_0 - k_3)(k_1 + ik_2)^2$	$\kappa(k_0 - k_3)(k_1 - ik_2)$	0	ξ
η	0	$-(k_1 + ik_2)(k_1^2 + k_2^2 + 2\kappa^2)$	$-\kappa(k_1 - ik_2)^2$	$\kappa(k_0^2 - k_3^2)$	η
ω	$\kappa(k_0^2 - k_3^2)$	$\kappa(k_1 + ik_2)^2$	$-(k_1 - ik_2)(k_1^2 + k_2^2 + 2\kappa^2)$	0	ω
θ	0	$\kappa(k_0 - k_3)(k_1 + ik_2)$	$-(k_0 - k_3)(k_1 - ik_2)^2$	$(k_0 - k_3)^2(k_0 + k_3)$	θ

Where k_1 , k_2 , k_3 and k_0 are the four components of the wave number vector, κ denotes mc/\hbar ,

$$a_{12}^1 = a_{21}^1 = \xi, \quad b_{12}^1 = b_{21}^1 = \omega,$$

$$a_{12}^2 = a_{21}^2 = \eta, \quad b_{12}^2 = b_{21}^2 = \theta.$$

The current vector $I_{\lambda\beta}^{\alpha}$ of this field is defined by the following equation:

$$I_{\lambda\beta}^{\alpha} = a_{\lambda\nu}^{\rho\kappa} a_{\beta\rho}^{\nu} + b_{\beta\nu}^{\rho\kappa} b_{\lambda\rho}^{\nu}.$$

Using this formula, we can carry out the normalisation integration I :

$$\begin{aligned}
 I &= \int I^0 dv, \\
 I^0 &= \frac{1}{2} (I_{11}^* + I_{22}^*) \\
 &= \frac{1}{2} (a_{11}^{1*} a_{11}^1 + a_{22}^{2*} a_{22}^2 + b_1^{11*} b_1^{11} + b_2^{22*} b_2^{22} \\
 &\quad + \xi^{*\xi} + \eta^{*\eta} + \omega^{*\omega} + \theta^{*\theta} \\
 &\quad + a_{11}^{2*} \xi + \xi^{*} a_{11}^2 + a_{22}^{1*} \eta + \eta^{*} a_{22}^1 \\
 &\quad + b_2^{11*} \omega + \omega^{*} b_2^{11} + b_1^{22*} \theta + \theta^{*} b_1^{22}).
 \end{aligned}$$

Inserting A , B , C or D in those equations, we have the following results.

$$\begin{aligned}
 I_A = I_D &= (k_0^2 - k_3^2) k_0 (k_0 - k_3) (k_1^2 + k_2^2 + 3\kappa^2), \\
 I_B = I_C &= 3\kappa^2 k_0 (k_0 - k_3) (k_1^2 + k_2^2 + 3\kappa^2).
 \end{aligned}$$

A Note on the Pressure Equation of the Two Fluid Model of Helium II

Tunemaru USUI

*Physics Department, Faculty of General Culture,
University of Tokyo*

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La premoekvacio de la dufluaĵa modelo de heliomo likva II ĉe ĝia dua movado estas derivata. Bazite de tiu ĉi rezultato, estas diskutataj la formuloj uzitaj de Pellam k. a. ĉe la analizo de iliaj eksperimentoj pri la Rayleigha disko kaj pri la pitot-tubo en la staranta ondo de dua sono.

§ 1. Introduction

Recently Pellam and his collaborator have performed two sorts of very interesting experiments concerning the stationary wave of second sound in liquid helium II. One is on the torque of the fluid pressure acting on a circular disc¹⁾ and the other, on the difference between the pressures at a loop and at a node of the wave²⁾. In view of the fact that the basic principles are quite the same as those of corresponding classical hydrodynamical devices except for the thermal character of the second sound, they appropriately called them thermal Rayleigh disc and thermal pitot tube, respectively. The observation on the latter seems to be primarily qualitative at present, but the former measurement is quantitative enough to be fitted for a detailed analysis. In fact, they have shown that their data fit rather well the expression which they derived on the basis of the two fluid model.

To the best of our knowledge, the stationary part of the acceleration involved in the Eulerian equation of motion of the two fluid model had not been given any direct experimental support till Pellam *et al.* carried out the above-mentioned measurements. It was introduced in order to answer the requirements which the general principles impose on the two fluid model. Therefore, it may be considered that experiments of such type as they have proposed make a criterion deciding the fate of the model.

Pellam *et al.* have, indeed, obtained a good quantitative agreement in the case of Rayleigh disc, as mentioned above, and also a general qualitative conformance to the behaviour expected in the case of pitot tube. However, their method of deriving the relevant theoretical formulae is of rather intuitive character, not based on the general equations of motion, and so it is not sufficiently convincing. For example, their derivation of the expressions for the torque acting on a Rayleigh disc does not directly take into account the character of the thermodynamical process involved in the second sound.

In view of these facts, it may be considered necessary to establish anew the pressure equation of the two fluid model and to see if their formulae are straightforward conclusions of the model or not. We shall try to make this aspect of the problem be settled down in the present note.

§ 2. Derivation of the pressure equation

We shall start from the momentum conservation law of the two fluid model:³⁾

$$\frac{\partial \rho_n \mathbf{v}_n}{\partial t} + \frac{\partial \rho_s \mathbf{v}_s}{\partial t} = -\rho_n (\mathbf{v}_n \cdot \text{grad}) \mathbf{v}_n - (\text{div } \rho_n \mathbf{v}_n) \mathbf{v}_n - \rho_s (\mathbf{v}_s \cdot \text{grad}) \mathbf{v}_s - (\text{div } \rho_s \mathbf{v}_s) \mathbf{v}_s \\ - \text{grad } p + \nabla \cdot \Pi_n,$$

where ρ_n , \mathbf{v}_n or ρ_s , \mathbf{v}_s are the density and the velocity of the normal or superfluid, respectively, and Π_n represents the viscosity tensor of the normal fluid. Instead of \mathbf{v}_n and \mathbf{v}_s , we can describe the field with the velocity of the centre of gravity:

$$\mathbf{v}_1 = (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) / \rho, \quad (2.1)$$

where ρ means the total density:

$$\rho = \rho_n + \rho_s, \quad (2.1')$$

and with the velocity of the normal fluid relative to the centre:

$$\mathbf{v}_2 = \mathbf{v}_n - \mathbf{v}_1. \quad (2.2)$$

Then the momentum conservation law becomes

$$\frac{\partial \rho \mathbf{v}_1}{\partial t} = -\rho (\mathbf{v}_1 \cdot \text{grad}) \mathbf{v}_1 - (\text{div } \rho \mathbf{v}_1) \mathbf{v}_1 + \nabla \cdot \Pi_1 \\ - \rho_n (\mathbf{v}_2 \cdot \text{grad}) \frac{\rho}{\rho_s} \mathbf{v}_2 - (\text{div } \rho_n \mathbf{v}_2) \frac{\rho}{\rho_s} \mathbf{v}_2 + \nabla \cdot \Pi_2 - \text{grad } p. \quad (2.3)$$

We shall assume henceforth that the centre of gravity is always at rest:

$$\mathbf{v}_1 = 0, \quad (2.4)$$

i.e., we confine ourselves on the phenomena where only the *second* motion can take place. In this case Eq. (2.3) reduces to

$$\text{grad } p + \rho_n (\mathbf{v}_2 \cdot \text{grad}) \frac{\rho}{\rho_s} \mathbf{v}_2 + (\text{div } \rho_n \mathbf{v}_2) \frac{\rho}{\rho_s} \mathbf{v}_2 = \nabla \cdot \Pi_2. \quad (2.5)$$

Carrying out some simple vectorial manipulations, we can transform Eq. (2.5) into the following, more convenient form:

$$\text{grad} \left(p + \frac{\rho \rho_n v_2^2}{\rho_s} \right) = \frac{\rho \rho_n}{\rho_s} [\mathbf{v}_2, [\nabla, \mathbf{v}_2]] + \left[[\mathbf{v}_2, \nabla], \frac{\rho \rho_n \mathbf{v}_2}{\rho_s} \right] \\ + (\lambda_n + 2\mu_n) \nabla (\nabla \cdot \mathbf{v}_2) - \mu_n [\nabla, [\nabla, \mathbf{v}_2]], \quad (2.5')$$

where $p \cdot \Pi_2$ has been explicitly written down with the viscosity coefficients λ_n and μ_n . Any further transformation into an integral form is in general impossible. So, we shall call Eq. (2.5') for the sake of convenience 'pressure equation'. It is to be noted that, while we are treating a general case, no time derivative does appear in this equation.

Now, let us apply the pressure equation to a one-dimensional laminar problem, e.g., the pitot tube experiment of Pellam. Considering the condition that v_2 is along the x -axis and a function of only x and t , one can write Eq. (2.5') in the following integral form:

$$p + \frac{\rho \rho_n v_2^2}{\rho_s} = (\lambda_n + 2\mu_n) \frac{\partial v_2}{\partial x} + \text{const.} \quad (2.6)$$

Since the time mean of $\partial v_2 / \partial x$ drops out in the case of stationary wave, this equation reduces to

$$\bar{p} + \overline{\frac{\rho \rho_n v_2^2}{\rho_s}} = \text{const.}, \quad (2.6')$$

which differs from the corresponding equation proposed by Pellam by a factor 2 in the dynamical pressure term. The discrepancy comes from the fact that he has presumably taken no account of the space variation of ρ_n and ρ_s .

§ 3. The Rayleigh disc torque

We shall next investigate the torque acting on a Rayleigh disc, set in the second sound field. From Eq. (2.4) and the mass conservation law: $\partial \rho / \partial t + \text{div } \rho \mathbf{v}_1 = 0$, we get

$$\frac{\partial \rho}{\partial t} = 0.$$

We shall assume here, in addition, that the density ρ is homogeneous everywhere. This assumption is justified by the fact that the velocity of the second sound wave, Eq. (3.4) below, derived on its basis, agrees with the experimental value.

If we neglect the viscous stress of normal fluid and furthermore the annihilation-production of both the fluids, we can write for the equation of the second motion⁴⁾,

$$\frac{D \mathbf{v}_2}{Dt} = -\text{grad}(\Pi + \theta), \quad (3.1)$$

where

$$\Pi = p / \rho$$

and

$$\theta = \int_{p=\text{const.}} (1 - \xi) \left(\frac{\partial s}{\partial \xi} \right)_p dT, \quad \text{with } \xi = \rho_n / \rho.$$

Once this conservative type of equation of motion is established, we can derive the so-called Thomson's theorem by the well-known process:

$$\frac{D}{Dt} \oint (\mathbf{v}_2 \cdot d\mathbf{r}) = 0.$$

Therefore we may be justified in assuming irrotational motion in the phenomenon of second sound:

$$[\mathbf{r}, \mathbf{v}_2] = 0,$$

or

$$\mathbf{v}_2 = -\text{grad } \Psi. \quad (3.2)$$

We assume as usual that Ψ and all the space or time derivatives of every quantities are small quantities of the first order. Then $\text{grad } p$ turns out to be of the second order. Retaining only small quantities of the first order in Eq. (3.1), and remembering the mass conservation law of the normal liquid, which reduces to

$$\frac{\partial \hat{\epsilon}}{\partial t} + \hat{\epsilon} \text{div } \mathbf{v}_2 = 0,$$

we can easily obtain the equation for the Ψ wave:

$$\Delta \Psi = \frac{1}{c_2^2} \frac{\partial^2 \Psi}{\partial t^2}, \quad (3.3)$$

where c_2 is the velocity of the second sound:

$$c_2^2 = \hat{\epsilon} (1 - \hat{\epsilon}) \left(\frac{\partial \hat{\epsilon}}{\partial \hat{\epsilon}} \right)_p \left(\frac{\partial T}{\partial \hat{\epsilon}} \right)_p. \quad (3.4)$$

For the monochromatic wave of circular frequency ω , the wave equation reduces to

$$\Delta \Psi = - \left(\frac{\omega}{c_2} \right)^2 \Psi. \quad (3.3')$$

Now we shall take up the pressure equation. Neglecting terms of orders higher than the second, we obtain from Eqs. (2.5'), (3.2) and (3.3')

$$\text{grad} \left(p + \frac{\rho \rho_n v_2^2}{\rho_s} \right) = \text{grad} \left(\frac{\rho \rho_n}{2 \rho_s} \left[\left(\frac{\omega}{c_2} \right)^2 \Psi^2 + v_2^2 \right] \right),$$

or

$$p = \text{const.} + \frac{1}{2} \frac{\rho \rho_n}{\rho_s} \left[\left(\frac{\omega}{c_2} \right)^2 \Psi^2 - v_2^2 \right]. \quad (3.5)$$

This result is formally the same as the corresponding one for ordinary sounds, except that the density is replaced with $\rho \rho_n / \rho_s$. Hence we can borrow the known result about the latter⁵⁾ and need only adapt it to the present case. When the angle of attack is $\pi/4$, the torque acting on a Rayleigh disc of radius a is given by the formula:

$$N = \frac{4}{3} \frac{\rho \rho_n}{\rho_s} a^3 \bar{v}_2^2 \left[1 + \frac{1}{5} \left(\frac{\omega a}{c_2} \right)^2 + \frac{19}{1050} \left(\frac{\omega a}{c_2} \right)^4 + \dots \right], \quad (3.6)$$

where \bar{v}_2^2 is the mean square velocity in the absence of the disc.

The first term coincides with the formula used by Pellam *et al.* The terms beyond the second represent the effect of finiteness of the ratio: (radius of the disc)/(wave length).

§ 4. Discussion

As stated in Section 1, Pellam's measurement on the thermal pitot tube is not yet sufficiently quantitative, but the general results were found to conform to the behaviour expected from Eq. (2.6'). A little examination shows that the pitot tube can be expected to be more sensitive at lower temperatures, and furthermore, the result is free from the uncertainty about the viscosity of normal fluid. It is, therefore, highly desirable to improve the precision of the measurement and to test rigorously the theoretical result Eq. (2.6).

As for the Rayleigh disc experiment, Pellam and Morse have derived their theoretical expression in the following manner. For sound waves in ordinary liquids, the torque N_{ordinary} , exerted on a Rayleigh disc of radius a at the angle of attack $\pi/4$, is given by

$$N_{\text{ordinary}} = \frac{4}{3} \rho a^3 \bar{v}^2,$$

where ρ is the density of the medium and \bar{v}^2 is its mean square velocity in the absence of the disc. Corresponding to the two fluid model of liquid He II, they have put for the torques exerted by the normal and superfluid

$$\text{and} \quad \left. \begin{aligned} N_n &= \frac{4}{3} \rho_n a^3 \bar{v}_n^2, \\ N_s &= \frac{4}{3} \rho_s a^3 \bar{v}_s^2, \end{aligned} \right\} \quad (4.1)$$

respectively. Because of the absence of mass motion, Eq. (2.4) holds and this, in turn, gives the relation (see Eq. (2.1))

$$\bar{v}_s = - \frac{\rho_n}{\rho_s} \bar{v}_n.$$

From these three equations they obtained for the resultant torque:

$$\begin{aligned} N &= N_n + N_s \\ &= \frac{4}{3} a^3 [\rho_n \bar{v}_n^2 + \rho_s \bar{v}_s^2] \\ &= \frac{4}{3} a^3 \frac{\rho \rho_n}{\rho_s} \bar{v}_2^2, \quad \text{the correct result.} \end{aligned} \quad (4.2)$$

However, when only the density of the normal fluid is in wave motion the phenomenon is quite another thing different from the second sound. Moreover, even when both the densities are fluctuating in accordance with the second sound, it is not clear that the separate torque could be given by the formulae (4.1) because of the presence of the force proportional to $\text{grad } T$. It may, therefore, be interesting to rederive the formula in a parallel manner and to see how the terms involving gradient of temperature drop out from the expression for the total torque.

In the same order of approximation as in Section 3, the equations of motion of both the fluids can be written as:⁴⁾

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \text{grad} \frac{\rho_n v_n^2}{2} + \frac{\rho_n}{\rho} \text{grad } p + \frac{\rho_n \rho_s}{\rho} \left(\frac{\partial s}{\partial \xi} \right)_p \text{grad } T = 0,$$

and

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \text{grad} \frac{\rho_s v_s^2}{2} + \frac{\rho_s}{\rho} \text{grad } p - \frac{\rho_n \rho_s}{\rho} \left(\frac{\partial s}{\partial \xi} \right)_p \text{grad } T = 0,$$

where the condition of irrotational motion Eq. (3.2) has been taken into account. As stated in Section 3, the total density ρ can be assumed constant. Therefore, the integrals

$$p_n = \int \frac{\rho_n}{\rho} dp, \quad p_s = \int \frac{\rho_s}{\rho} dp,$$

and

$$\theta = \int \frac{\rho_n \rho_s}{\rho} \left(\frac{\partial s}{\partial \xi} \right)_p dT$$

define definite functions. With these relations, the above set of equations of motion can be transformed into

$$\left. \begin{aligned} \rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \text{grad} \left[\frac{\rho_n v_n^2}{2} + p_n + \theta \right] &= 0, \\ \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \text{grad} \left[\frac{\rho_s v_s^2}{2} + p_s - \theta \right] &= 0, \end{aligned} \right\} \quad (4.3)$$

respectively.

In discussing the resultant torque $N = N_n + N_s$, we may simply add both the equations (4.3), keeping in mind the relation $p = p_n + p_s$, and Eq. (2.4):

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \text{grad} \left[\frac{1}{2} \frac{\rho_n \rho_s}{\rho_s} v_s^2 + p \right] = 0,$$

where the terms containing θ have dropped. In the conventional approximation

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_s \frac{\partial \mathbf{v}_s}{\partial t} = \left(\frac{\rho_n \rho_s}{\rho_s} \text{div } \mathbf{v}_s \right) \mathbf{v}_s = \frac{\rho_n \rho_s}{\rho_s} \Delta \Psi \text{grad } \Psi = - \text{grad} \frac{1}{2} \frac{\rho_n \rho_s}{\rho_s} \left(\frac{\omega}{c_s} \right)^2 \Psi^2.$$

Accordingly we finally obtain

$$\text{grad}\left(\phi + \frac{1}{2} \frac{\rho \rho_n}{\rho_s} \left[v_z^2 - \left(\frac{\omega}{c_2} \right)^2 \Psi^2 \right] \right) = 0,$$

the same result as Eq. (3.5).

Pellam *et al.* have compared their experimental data with the expression (4.2) and found a sufficiently good agreement. We shall here retain from a detailed analysis by using Eq. (3.6), but it may be safely concluded that the stationary part of acceleration, *i.e.*, $(\boldsymbol{v} \cdot \text{grad})\boldsymbol{v}$, has for the first time acquired its experimental support through the measurement of Rayleigh disc torque.

The present author wishes to thank Doctor J. R. Pellam for stimulating discussions about this problem held at the time of his stay in Japan.

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Letters to the Editor

On the Star Production by Cosmic-Ray Underground

S. Miura and S. Ogawa

*Institute of Theoretical Physics,
Nagoya University*

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Previously, we have analyzed the star production by the cosmic-ray underground.¹⁾ Then, we have intended that the charged star agent is π -meson produced by only γ -rays in equilibrium with the μ -meson component underground, since the additive interaction of μ -meson with matter should cause considerable modifications in the absorption spectrum of the cosmic-ray underground. This idea, however, is insufficient, because charged μ -mesons are able to produce π -meson by its own electromagnetic field in the collision with nucleons.²⁾ According to Williams-Weizsäcker method, the suggested γ - π cross section in (A) gives too large μ - π cross section (a factor ~ 10). Thus we must give some additions and modifications in our previous idea on the star production.

I) Charged star agents include not only π -meson but also μ -meson and in small fraction, protons. In the collision with nucleus by Coulomb interaction, μ -meson gives some fraction of its kinetic energy to nucleus, causing the nuclear evaporation. Recent experiment shows that high energy γ -ray makes nuclear evaporation with cross section 10^{-28} cm²/nucleon.³⁾ Thus μ -meson's cross section for star formation becomes, in Williams-Weizsäcker method,

$$\begin{aligned} &\sim (e^2/\hbar c) \log(E/\mu) \cdot 10^{-28} \text{ cm}^2/\text{nucleon} \\ &\sim 10^{-29} \text{ cm}^2/\text{nucleon}. \end{aligned}$$

The value is consistent with the experiment,

This process may belong to 1_p in Evans' notation.

II) The stars accompanied with several fast secondaries are thought to be more preferably caused by π -meson than by μ -meson. Because we cannot accept reasonably that such a weak interaction as electromagnetic effect of μ -meson may give multiple π -production. On the other hand, if it is confirmed that μ -meson itself produces π -mesons multiply,⁴⁾ it indicates an interesting feature of π -production,—suggesting “multiple” (not “plural”) production. Because, due to its weak interaction, μ -meson may not be able to collide several times with nucleons in “one” nucleus.

III) π -mesons and nucleon components can be produced in the following process.

a) μ -mesons produce π -meson by its own electromagnetic field in the collision with nucleon in nucleus. The cross section is,⁵⁾

$$\begin{aligned} \sigma &= (g^2/\hbar c)(e^2/\hbar c)^2(\hbar/\mu c)^2(M/\mu) \log(E/M) \\ &\simeq 0.7 \times 10^{-29} (g^2/\hbar c) \cdot \log(E/M); \\ &\text{cm}^2/\text{nucleon}. \end{aligned}$$

Taking into account that the average energy of μ -meson at the depth ~ 50 m H₂O underground is ~ 14 Bev, and that $(g^2/\hbar c) \sim 1$ from γ - π effect, σ becomes $\sim 3 \cdot 10^{-29}$ cm²/nucleon in consistent with the experiment. This effect responds to $2p$ and $3p$ (if recoil proton of high energy is included).

b) γ -rays in equilibrium with μ -meson component π -meson. If we take $\sigma_{\alpha-\pi} \simeq 10^{-28}$ cm²/nucleon, the contribution is of magnitude 10% compared with the interaction (a). This becomes clear from the fact that the value of $\sigma_{\alpha-\pi}$ is small by a factor 10 compared with $\sigma_{\alpha-\pi}$ taken in (A).

c) In the above process, recoil nucleons

may be ejected with considerable energy. Stars are also accompanied with such occasions.

IV) On stars are caused by neutrons emitted in the above process. These phenomena belong to relatively low energy process. In such energy region, charged particle loses its energy more by ionization than by nuclear collision. This implication is consistent with the fact that O_n is more frequent than O_p . Small fraction ($\sim 10\%$) of O_n may be caused by γ -ray.

O_p stars are caused by the capture of low energy π -meson and by the nuclear collision of the low energy protons.

Further the μ - π interaction mentioned in III, a), is accompanied with the corresponding energy loss of μ -meson. It would result in the considerable modifications of the depth-intensity spectrum of the μ -meson component underground. Wataghin⁶⁾ suggested that π -meson of high energy may diminish its nuclear interaction, from the analysis of the depth-intensity curve based on a definite spectrum of the μ -meson spectrum at sea level. His idea, however, is premature. Because the meson spectrum in high energy region at sea level is known not so precisely that the spectrum at sea level must be checked by another analysis, for instance, by the detailed size-frequency curve of the μ -meson bursts in high energy. On this point, some analysis may be reported in near future.

In conclusion, we should like to express our deep gratitude to Prof. S. Sakata, and we are much indebted to Mr. Hayakawa for his valuable discussions.

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Radiative Corrections to Anomalous Magnetic Moment of Nucleon in Pseudoscalar Meson Theory

K. Nakabayasi and I. Sato

Physical Institute, Faculty of Science,
Tôhoku University

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The anomalous magnetic moment of nucleon has been hitherto calculated in the lowest order by many authors,¹⁾ choosing in particular pseudoscalar meson which seems to be most promising from some evidences.²⁾ The result was that the moment ratio of neutron to proton moments was too large in absolute value compared with the experiment. In view of this discrepancy, we thus estimated the first-order radiative corrections using Feynman-Dyson method, in case of pseudoscalar coupling. (The coupling constants are denoted as $\beta_1 = \beta_2 = \beta$ (charged meson), β_3 (neutral meson to be added in symmetrical theory) and β_4 (purely neutral meson); they are equal to the conventional ones $f_\sigma^2/4\pi\hbar c$ in Heaviside units divided by 2π). The diagrams contributing to the moment in the fourth order are shown in Fig. 1. II and III are reducible to the second-order ones, II_0 and III_0 , responsible for the moment in the lowest order due to nucleon and meson currents, respectively. After renormalization, we obtain as the radiative corrections in nuclear magneton units

$$\mu_I = 0.472 \cdot \frac{1}{2} [(\beta_3 + \beta_4)^2 \tau_p + 4\beta(\beta_4 - \beta_3) \tau_n],$$

$$\mu_{II} = -0.242 \cdot$$

$$\frac{1}{2} [\{ (\beta_3 + \beta_4)^2 + 2\beta(\beta_4 - \beta_3) \} \tau_p + 2\beta(\beta_4 - \beta_3) \tau_n],$$

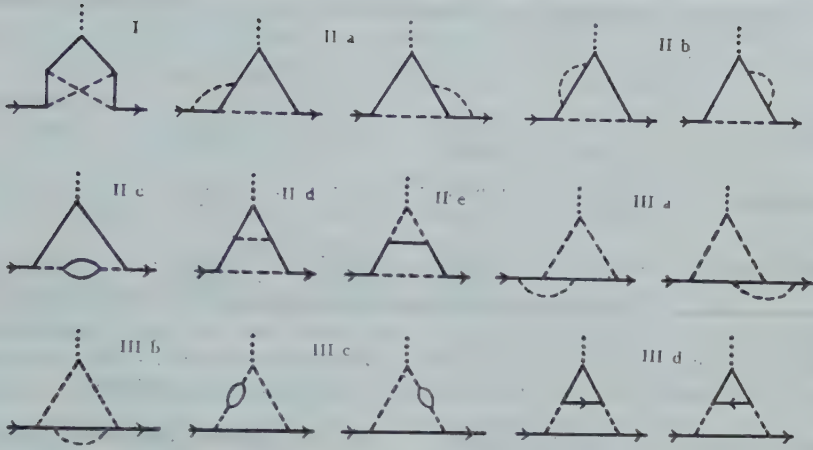


Fig. 1. Feynman-Dyson diagrams contributing to first-order radiative corrections for anomalous magnetic moment of nucleon. The full, broken and dotted lines denote nucleon, meson and photon lines, respectively.

$$\mu_{IIb} = -0.077.$$

$$\frac{1}{2}(2\beta + \beta_3 + \beta_4)[(\beta_3 + \beta_4)\tau_p + 2\beta\tau_n],$$

$$\mu_{IIc} = -0.128 \cdot [(\beta_3 + \beta_4)^2\tau_p + 2\beta^2\tau_n],$$

$$\mu_{IIa} = 0.626.$$

$$\frac{1}{2}[4\beta^2 + (\beta_3 + \beta_4)^2]\tau_p + 4\beta(\beta_3 + \beta_4)\tau_n],$$

$$\mu_{IIe} = -0.084 \cdot \frac{1}{2}\beta(2\beta - \beta_3 - \beta_4)(\tau_p - \tau_n),$$

$$\mu_{IIIa} = 0.089 \cdot \frac{1}{2}\beta(\beta_3 - \beta_4)(\tau_p - \tau_n),$$

$$\mu_{IIIb} = 0.213 \cdot \frac{1}{2}\beta(2\beta + \beta_3 + \beta_4)(\tau_p - \tau_n),$$

$$\mu_{IIIc} = 0.079 \cdot \beta^2(\tau_p - \tau_n),$$

$$\mu_{IIId} = -0.157\beta^2(\tau_p - \tau_n)$$

taking π -meson mass as $275m_e$, while the lowest-order moments are

$$\mu_{II0} = -0.237[(\beta_3 + \beta_4)\tau_p + 2\beta\tau_n],$$

$$\mu_{III0} = 0.348(\tau_p - \tau_n).$$

The anomalous moments up to the fourth order for various theories are tabulated in Table I. The radiative corrections are clearly in direction to remove the discrepancy existing in the lowest order, and this tendency accentuates itself muchly as neutral meson is mixed. These features peculiar to pseudo-scalar model are interpretable as follows: Since nucleon-meson coupling takes place through odd operator γ_5 , the nucleon acquires

large recoil whenever emits a virtual meson. This meson, too, gains large momentum and energy $\gtrsim Mc^2$, M being nucleon mass. This implies that the distribution of meson cloud around the nucleon is highly asymmetric or is associated with angular momentum of order \hbar . Angular momentum conservation now requires that the nucleon ought to undergo spin umklapping. This is the reason why nucleon contributions in the lowest order and from $II_{a,b,c}$ are not small and negative, where the nucleon interacts with the external electromagnetic field after emitting only one meson. In I and II_a , the nucleon emits however two mesons and recovers its spin direction, thus behaving normally so as to remedy the above unfavorable effect. And this radiative corrections turn out to be most predominant. On the other hand, the meson current contributions remain to be relatively small, owing to relativistic increase of meson inertia.

The values of coupling constant to reproduce the experimental moment-ratio and the resulting neutron moments are also included in Table I. The purely charged theory is to be rejected, as requires negatively large β . Thus the role of neutral mesons

Theory	μ_p		μ_n		β
	2nd order	4th order	2nd order	4th order	
Charged	0 +0.348 β	+0.271 β^2 -0.024 β^2	-0.474 β -0.348 β	-0.292 β^2 +0.024 β^2	~ -100
Symmetrical	-0.237 β +0.348 β	+0.303 β^2 +0.025 β^2	-0.474 β -0.348 β	+0.154 β^2 -0.025 β^2	4.09 ($\mu_n = -6.35$)
Symm. + neutral	-0.474 β +0.348 β	+0.210 β^2 +0.029 β^2	-0.474 β -0.348 β	+0.215 β^2 -0.029 β^2	2.14 ($\mu_n = -0.91$)

Table I. Anomalous moments calculated up to the fourth order, in nuclear magneton units, and values of the coupling constants to give experimental moment-ratio -1.07 . For μ_p and μ_n , the first (second) line corresponds to nucleon (meson) current contribution.

is decisively important; it is even possible to give the correct moments both for neutron and proton, adjusting β_1 with $\beta_3 = \beta$, that is, taking $\beta = 2.49$ and $\beta_4 = 1.37$.

This coupling constants correspond to the conventional one of order 10, being quite reasonable from other data.

Thus the radiative corrections are quite large, so higher order ones must be also taken into account, which shows however invalidity of the weak coupling approximation.

Full accounts will soon appear in Science Reports, Tōhoku University, first series, vol. 34, no. 4.

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On the Excited States of Even Nuclei

H. Horie,* M. Umezawa,* Y. Yamaguchi,**
and S. Yoshida*

Department of Physics, University of Tokyo*
and University of City Osaka**

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Recently many experiments have been performed for γ - γ and β - γ angular and direction-polarization correlations of various β -emitters, and considerable amounts of knowledge of their excited states have been accumulated. On the other hand, many authors¹⁾ have discussed the nuclear structure, especially from the standpoint of shell model. It is, however, regrettable that excited states of nuclei were scarcely analyzed but for some excited (odd) nuclei concerning β -decay or isomeric states. Therefore it seems worth while to examine the excited states of even nuclei on the basis of recent experimental results and Mayer's shell model.¹⁾ Of course, there are many excited states observed in various nuclear scatterings and reactions, on which detailed discussions will be presented in another paper.

First of all experimental results are summarized as Table I.

From this table, we see that there exists marked regularity. It seems therefore reasonable to consider that two like nucleons

Table I

	spin parity	spin parity	spin parity	spin	spin	spin	parity
2nd excited state	4	4	— —	—	—	4	
1st excited state	2 same parity	2 same parity	2 even	2	2	2	same parity
ground state	0	0	0 even	0	0	0	
nucleus	Ti ⁴⁶	Ni ⁶⁰	Ba ¹³⁴	Ca ⁴²	Pd ¹⁰⁶	Mg ^{24*}	
references	2) 3) 4)	2) 3) 4)	3) 4)	5)	4)	6)	

*) determined from life-time, conversion coefficient, and conversion ratio.

in the outmost *orbit* of these (even-even) nuclei couple so as to produce spin 2 and 4 in the first and second excited states, respectively (assuming that the "core" of nucleus gives neither spin nor magnetic moment). This is quite a contrast to the case of ground state, where two *like* nucleons in the same orbit couple so as to make spin zero.

Next, the first excited states of Sr^{86 7)8)9)}, Te^{124 10)}, and Sr^{88 4)} may be spin 1. These are not consistent with the model proposed above. Since these nuclei have *N* or *Z* between 30 and 80, where there are many isomers, we may consider that their first excited states are produced by the different configurations from the ones for ground states. It is expected that the same situation may be seen in higher excited states of all even nuclei.

These investigations suggest the following general picture for excited even nuclei. At first, we give the definite nuclear configuration for each (*even*) nucleus using Mayer's shell model, and then introduce an appropriate interaction *V* between the last two (like or unlike) nucleons inside the outmost orbit. Each excited state is regarded as a coupling mode of these two nucleons (under the definite configuration), which has a shifted energy level caused by perturbation *V*. We have considered here that "core" of nucleus does not essentially contribute to nuclear excitation. This optimistic assumption may not be so absurd, in so far as we

restrict ourselves within the energy regions of lower excitation. However this is not always the case at higher regions or "isomeric" region ($30 \leq N, Z \leq 80$), and as already discussed, states produced from different configuration must be taken into account. These arguments can be regarded as a natural generalization of nuclear shell structure theory to excited even nuclei.

The interaction *V* is undoubtedly connected with nuclear forces, while the nature of excited states are in the first approximation determined by *V*. Thus, if our model is correct, the nature of excited states will furnish some informations about nuclear forces. For instance if we tentatively assume that the interaction *V* between two like particles is short range repulsive interaction adopted by Mayer (eq. (2) of her paper), the above mentioned regularity about spin of excited states can readily be explained (though there may remain many other possibilities). What we can say under the assumption of Mayer's short range repulsive interaction is only that the potential used by Mayer leads to incorrect level spacings, indicating that her potential must be replaced by another more appropriate one.

The same model can be successfully applied to interpret various excited (even) nuclei accompanying complex β -decay.

Thus we think that the nuclear excited states of even nuclei could, to some extent, be interpreted too, upon the basis of current shell model.

The more detailed discussion of β -decay will be given in the later issue of this journal.

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On the Nuclear Potential in the Meson Theory

Y. Ataka

Department of Physics, Kyoto University

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In the present field theory, the deuteron problem can be solved by using the Schroedinger wave equation and the nuclear potential. Usually the nuclear potential in the meson theory is defined as the matrix element of the two nucleons' interaction Hamiltonian in the spatial representation calculated by the perturbation theory. In this case the nucleon recoil terms are dropped. As is well known, the meson theory of nuclear forces has suffered the r^{-3} difficulties of the potential functions. In this letter it will be shown that the static potential free from the

r^{-3} singularity will be gained except the tensor interaction of the spin 1 mesons.

Now when the two nucleons and the meson field are interacting, the state equation of this system is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = (H_0 + H_1 + H_M) \Psi, \quad (1)$$

where

$$H_0 = c\alpha^{(1)}(-i\hbar \text{grad}_1) + c\alpha^{(2)}(-i\hbar \text{grad}_2) + (\beta^{(1)} + \beta^{(2)})Mc^2, \quad (2)$$

H_M is the energy of the meson field and H_1 is the interaction Hamiltonian between them. Here taking the adiabatic approximation (that is, eliminating the meson wave functions from (1)), we get the Schroedinger wave equation of the two nucleons as follows:

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + H_2) \psi. \quad (3)$$

Van Hove¹⁾ gave H_2 in the momentum representation as

$$\begin{aligned} \langle \mathbf{p}_1, \mathbf{p}_2 | H_2 | \mathbf{p}_1', \mathbf{p}_2' \rangle &= (2\pi\hbar)^{-3} \\ &\delta(\mathbf{p}_1 - \mathbf{p}_1' + \mathbf{p}_2 - \mathbf{p}_2') \langle \mathbf{p}_1, \mathbf{p}_2 | O | \mathbf{p}_1', \mathbf{p}_2' \rangle \cdot \\ &\cdot \left[\left(\frac{\mathbf{p} - \mathbf{p}'}{\hbar} \right)^2 + \kappa^2 - \frac{1}{c^2} \left(\frac{E - E'}{\hbar} \right)^2 \right]^{-1}, \end{aligned} \quad (4)$$

here $E = c\sqrt{\mathbf{p}_1^2 + M^2c^2} = c\sqrt{\mathbf{p}_2^2 + M^2c^2}$,

$$E' = c\sqrt{\mathbf{p}_1'^2 + M^2c^2} = c\sqrt{\mathbf{p}_2'^2 + M^2c^2}$$

and for example in the neutral pseudoscalar theory

$$\begin{aligned} \langle \mathbf{p}_1, \mathbf{p}_2 | O | \mathbf{p}_1', \mathbf{p}_2' \rangle &= -f_3^2 \rho_2^{(1)} \rho_2^{(2)} \\ &+ i \left(\frac{f_2 f_3}{\kappa} \right) (\rho_1^{(1)} \rho_2^{(2)} + \rho_2^{(1)} \rho_1^{(2)}) \frac{E - E'}{\hbar c}. \end{aligned} \quad (5)$$

The equation (1) is not completely relativistic, because the retardation of the meson field cannot be considered. In order to drop the retardation terms in the above calculation, we take the limit $c \rightarrow \infty$, then nucleon recoil terms in (4) and (5) become automatically to vanish. Of course H_0 must be rewritten as

$$H_0 = -\frac{\hbar^2}{M}(\mathcal{A}_1^2 + \mathcal{A}_2^2). \quad (2')$$

In the next, we transform H_2 into the spatial representation such as

$$\begin{aligned} \langle \mathbf{r}_1, \mathbf{r}_2 | H_2 | \mathbf{r}_1', \mathbf{r}_2' \rangle &= \int \cdots \int \langle \mathbf{r}_1, \mathbf{r}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle d\mathbf{p}_1 d\mathbf{p}_2 \\ &\langle \mathbf{p}_1, \mathbf{p}_2 | H_2 | \mathbf{p}_1', \mathbf{p}_2' \rangle d\mathbf{p}_1' d\mathbf{p}_2' \langle \mathbf{p}_1', \mathbf{p}_2' | \mathbf{r}_1', \mathbf{r}_2' \rangle \end{aligned} \quad (6)$$

using the formulas²⁾

$$\begin{aligned} \langle \mathbf{p} | \rho_1 | \mathbf{p}' \rangle &= N_p N_{p'} \left\{ \frac{e(\boldsymbol{\sigma} \mathbf{p})}{E_p + Mc^2} + \frac{e(\boldsymbol{\sigma} \mathbf{p}')}{E_{p'} + Mc^2} \right\}, \\ \langle \mathbf{p} | \rho_2 | \mathbf{p}' \rangle &= N_p N_{p'} i \left\{ \frac{e(\boldsymbol{\sigma} \mathbf{p})}{E_p + Mc^2} - \frac{e(\boldsymbol{\sigma} \mathbf{p}')}{E_{p'} + Mc^2} \right\}, \\ \langle \mathbf{p} | \rho_3 | \mathbf{p}' \rangle &= N_p N_{p'} \\ &\left\{ 1 - \frac{c^2(\mathbf{p} \cdot \mathbf{p}') + ic^2(\boldsymbol{\sigma} \mathbf{p} \times \mathbf{p}')}{(E_p + Mc^2)(E_{p'} + Mc^2)} \right\}, \end{aligned} \quad (7)$$

where

$$N_p = \sqrt{\frac{E_p + Mc^2}{2E_p}}.$$

Here putting $\mathbf{p}_1 - \mathbf{p}_1' = \hbar \mathbf{k}$, equation (6) is

$$\begin{aligned} \langle \mathbf{r}_1, \mathbf{r}_2 | H_2 | \mathbf{r}_1', \mathbf{r}_2' \rangle &= (2\pi\hbar)^{-6} \int \cdots \int e^{\frac{i}{\hbar} \mathbf{p}_1(\mathbf{r}_1 - \mathbf{r}_1')} \\ &e^{\frac{i}{\hbar} \mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_2')} \cdot \\ &\cdot (-\hbar \mathbf{k} + \mathbf{p}_1, \hbar \mathbf{k} + \mathbf{p}_2 | \mathbf{O} | \mathbf{p}_1, \mathbf{p}_2) \\ &e^{i\mathbf{k}(\mathbf{r}_1' - \mathbf{r}_2')} (k^2 + x^2)^{-1} d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} (2\pi)^{-3}. \end{aligned} \quad (8)$$

Expanding the term $(-\hbar \mathbf{k} + \mathbf{p}_1, \hbar \mathbf{k} + \mathbf{p}_2 | \mathbf{O} | \mathbf{p}_1, \mathbf{p}_2)$ into the Taylor series with respect to \mathbf{p}_1 and \mathbf{p}_2 , the first approximation gives

$$\begin{aligned} \langle \mathbf{r}_1, \mathbf{r}_2 | H_2 | \mathbf{r}_1', \mathbf{r}_2' \rangle &= (2\pi\hbar)^{-6} \int \cdots \int e^{\frac{i}{\hbar} \mathbf{p}_1(\mathbf{r}_1 - \mathbf{r}_1')} \\ &e^{\frac{i}{\hbar} \mathbf{p}_2(\mathbf{r}_2 - \mathbf{r}_2')} \cdot \\ &\cdot (-\hbar \mathbf{k}, \hbar \mathbf{k} | \mathbf{O} | o, o) e^{i\mathbf{k}(\mathbf{r}_1' - \mathbf{r}_2')} (k^2 + x^2)^{-1} \\ &d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{k} (2\pi)^{-3} \\ &= V(r) \delta(\mathbf{r}_1 - \mathbf{r}_1') \delta(\mathbf{r}_2 - \mathbf{r}_2'). \end{aligned} \quad (9)$$

Therefore (3) is rewritten in the center of mass system as

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ -\frac{\hbar^2}{M} \Delta + V(r) \right\} \psi. \quad (10)$$

In the pseudoscalar theory $V(r)$ is easily calculated

$$\begin{aligned} V(r) &= -f_3^2 \int \frac{e^2(\boldsymbol{\sigma}^{(1)} \hbar \mathbf{k})(\boldsymbol{\sigma}^{(2)} \hbar \mathbf{k})}{2E_k(Mc^2 + E_k)} \frac{e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k} (2\pi)^{-3}}{k^2 + x^2} \\ &= f_3^2 (\boldsymbol{\sigma}^{(1)} \text{grad})(\boldsymbol{\sigma}^{(2)} \text{grad}) G(r). \end{aligned} \quad (11)$$

Here

$$\begin{aligned} G(r) &\equiv \frac{1}{4} \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k} (2\pi)^{-3}}{(k^2 + x_0^2)(k^2 + x^2)} \\ &= \frac{1}{4(x_0^2 - x^2)} \frac{e^{-\kappa r} - e^{-\kappa_0 r}}{4\pi r}, \end{aligned} \quad (12)$$

and therefore

$$\begin{aligned} V(r) &= \frac{f_3^2}{4\pi} \frac{x^2}{4(x_0^2 - x^2)} \left\{ \frac{1}{3} (\boldsymbol{\sigma}^{(1)} \boldsymbol{\sigma}^{(2)}) \right. \\ &\quad \left[\frac{e^{-\kappa r}}{r} - \left(\frac{x_0}{x} \right) \frac{e^{-\kappa_0 r}}{r} \right] \\ &\quad + S^{(12)}(r) \left[\frac{e^{-\kappa r}}{r} \left(\frac{1}{3} + \frac{1}{xr} + \frac{1}{x^2 r^2} \right) \right. \\ &\quad \left. \left. - \left(\frac{x_0}{x} \right)^2 \frac{e^{-\kappa_0 r}}{r} \left(\frac{1}{3} + \frac{1}{x_0 r} + \frac{1}{x_0^2 r^2} \right) \right] \right\}, \end{aligned} \quad (13)$$

which agrees with Kemmer's result in the limit $M \rightarrow \infty$. In the other models the same calculations are very easy and the similar results are gained except the tensor interactions of the spin 1 mesons.

At last, in the neutral scalar theory, the first and the next approximations of O are carried out and we get

$$V(r) = -f_1^2 \left\{ E(r) + \frac{\hbar}{2M} [(\boldsymbol{\sigma} \mathbf{L}) - i(\boldsymbol{\sigma} \mathbf{r} \mathbf{p})] \frac{F'(r)}{r} \right\} \quad (14)$$

where

$$E(r) \equiv \frac{e^{-\kappa r}}{4\pi r}$$

and

$$F(r) = \int \frac{e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k} (2\pi)^{-3}}{2E_k(k^2 + x^2)}.$$

Here we approximate

$$\frac{1}{2Mc^2} F(r) \equiv G(r), \quad (15)$$

then it is clear that

$$V(r) = -\frac{f_1^2}{4\pi} \frac{e^{-\kappa r}}{r} + \frac{f_1^2}{4\pi} \frac{\kappa^2}{4(\kappa_0^2 - \kappa^2)}$$

$$\left[\frac{e^{-\kappa r}}{r} \left(\frac{1}{\kappa r} + \frac{1}{\kappa^2 r^2} \right) - \left(\frac{\kappa_0}{\kappa} \right)^2 \frac{e^{-\kappa_0 r}}{r} \left(\frac{1}{\kappa_0 r} + \frac{1}{\kappa_0^2 r^2} \right) \right] \cdot$$

$$\cdot \left[\frac{(\sigma L)}{\hbar} - (r \cdot \text{grad}) \right]. \quad (16)$$

Thus the spin-orbit coupling proposed by many authors can be gained.

In conclusion the writer wishes to express his cordial thanks to Mr. H. Enatsu for his valuable suggestions and kind advices to this work.

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Production of Vector π -Mesons by High Energy Nucleon-Nucleon Collisions

K. Ida

Department of Physics, Tokyo University

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On the π -meson production by nucleon-nucleon collisions, several calculations have been performed according to Feynman-Dyson's method.⁽¹⁾⁽²⁾⁽³⁾ However, these are chiefly concerned with pseudoscalar π -meson, while with vector π -meson only the region near the threshold energy is studied. As for the model of π -meson, pseudoscalar type is now generally accepted, but in the present situation of the meson theory the possibility of the vector type can not be excluded entirely.⁽⁴⁾⁽⁵⁾ It seems that high energy vector π -meson production will give us some clue on this point.

In what follows angular and energy distribution of the emitted meson and total cross section for a single production is computed. We treat this process as the sixth order with respect to coupling constant f .

The effect of damping and multiple production is not considered.

The interaction Hamiltonian density is

$$H(x) = f \psi(x) \gamma_\mu \tau^\alpha \psi(x) \phi_\mu^\alpha(x), \quad (1)$$

$\psi(x)$: wave function for the nucleon,

$\phi(x)$: wave function for the meson,

τ^α : isotopic spin operator.

Here we adopt the symmetrically theory and vector coupling only. We shall calculate the transition probability from a state in which we have two nucleons with momentum p_1, p_2 to a state in which we have two nucleons with momentum q_1, q_2 and a meson with momentum k .

According to the Feynman-Dyson diagram (Fig. 1) the absolute square of the collision operator which corresponds to these transitions is as follows:

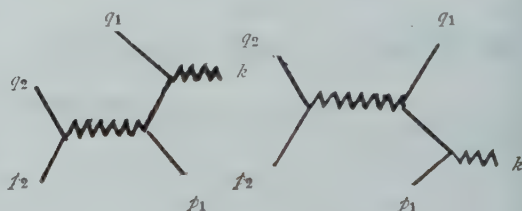


Fig. 1

$$|C|^2 = \left\{ \frac{1}{2(2\pi)^3} \right\} (2\pi)^8 \frac{1}{4} \sum_{\text{spin}} f^6$$

$$\times |(\bar{\psi}(q_2) \tau^\alpha \gamma_\lambda \psi(p_2)) (\bar{\psi}(q_1)$$

$$\times \{ \tau^3 \gamma_\mu (q_{1\lambda} \gamma_\lambda + k_\lambda \gamma_\lambda - m)^{-1} \tau^3 \gamma_\nu$$

$$+ \tau^3 \gamma_\nu (p_{1\lambda} \gamma_\lambda - k_\lambda \gamma_\lambda - m)^{-1} \tau^3 \gamma_\mu \} \psi(p_1)$$

$$\left\{ -\delta_{\lambda\nu} + \frac{(q_2 - p_2)_\lambda (q_2 - p_2)_\nu}{\mu^2} \right\}$$

$$\{ (q_2 - p_2)^2 - \mu^2 \}^{-1}$$

$$+ (q_1 q_2; p_1 p_2) - (q_1 q_2) - (p_1 p_2) |^2$$

$$\times \left(-\delta_{\mu\mu} + \frac{k_\mu k_\mu}{\mu^2} \right), \quad (2)$$

where m : nucleon mass, μ : meson mass.

$(q_1 q_2)$ or $(p_1 p_2)$ are expressions which can be obtained from the previous one by inter-

changing q_1 with q_2 or p_1 with p_2 respectively, and $(q_1 q_2; p_1 p_2)$ by interchanging q_1 with q_2 and p_1 with p_2 simultaneously. $\frac{1}{4} \sum_{spin}$ means to sum over the final spin states of the nucleons and averaging over the initial spin states. Every notation has its meaning just as defined by Feynman. We shall carry on our calculations in the center of gravity system. To fix the consideration we take the case of positively charged mesons in proton-neutron collisions. The other cases can be computed in a similar way. Further we consider the case of almost elastic scattering, that is, the energy of the emitted meson is much smaller than nucleon energy (see (7))

$$p_1 \sim p_2 \sim q_1 \sim q_2 \sim p \gg k. \quad (3)$$

As $p \gg m$ in this case, we shall calculate cross section in powers of m/p , leaving only the first non-vanishing term. Accordingly μ is also neglected compared with p . Under this approximation the second term which shows the polarization of the emitted meson is nearly zero, so in what follows we shall drop this term. The second term of the fifth line of expression (2) which shows the internal meson line vanishes also by the divergence theorem.

Now the calculation of the cross section is straightforward and can be performed as usual taking into consideration various simplifications mentioned in the above.

The total cross section becomes:

$$\sigma_{tot} = \left(\frac{f^2}{4\pi}\right)^2 \frac{9p^2}{2E_1 E_2 \mu^2} \log \frac{\bar{\epsilon}}{\mu} \left(\log \frac{4\bar{\epsilon}}{\mu} \right) \propto \frac{1}{\mu^2} \left(\log \frac{\bar{\epsilon}}{\mu} \right)^2, \quad (4)$$

$\bar{\epsilon}$: maximum energy of the emitted meson.

The differential cross section of the emitted meson is roughly proportional to

$$\frac{k^2 dk d(\cos \theta)}{\epsilon^2 - k^2 \cos^2 \theta} \quad (5)$$

and its angular distribution is nearly given by

$$k < \mu: \text{ spherically symmetry,} \\ \frac{\mu}{m} p > k > \mu \quad \bar{\theta} = \frac{\mu}{k} \text{ forward or backward,} \quad (6)$$

$\bar{\theta}$: the average broadening of the emitted meson.

The energy spectrum of the emitted meson is approximately

$$d\epsilon \propto \frac{\log \frac{\epsilon}{\mu}}{\epsilon}, \quad (7)$$

especially

$$\theta \sim 0^\circ; 180^\circ \quad \frac{k^2}{\mu^2} dk, \\ \theta \sim 90^\circ; \quad \frac{k^2}{\epsilon^2} dk,$$

The average deflection angle of nucleons after the collisions become

$$\bar{\alpha} \sim \frac{\mu}{p} \quad \epsilon < \frac{\mu}{m} E. \quad (8)$$

In conclusion, the present author expresses his sincerest thanks to Messrs. H. Fukuda and G. Takeda for their suggestion of this problem and valuable advice throughout this work.

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Remarks on the Production of Cosmic-Ray Mesons

Y. Yamaguchi

Department of Physics,
Osaka City University

March 19, 1951

In the previous paper¹⁾, we have discussed the production of cosmic-ray mesons, upon the basis of available informations at that time. It seems, however, necessary to add some supplementary arguments on this problem, referring to recent data obtained by the photographic technique. It is the purpose of this short note to satisfy this requirement.

It is evident that the phenomena caused by collision of a high energy nucleon with a nucleus are very much complicated owing to the fact of "plurimultiple" production of "shower particles", which seems to be confirmed by geomagnetic effect of stars with thin tracks²⁾ and so on.³⁾ In such a case the comparison of observed data with various theoretical predictions is too difficult to derive any definite conclusion about mesonic interaction. While so far only two examples^{4,5)} of single collision have been observed, they give direct informations about shower production without complications due to plurality, and thus they are very useful for our theoretical analysis. Though their statistics are poor, these data show the following important features of shower production:

- (1) Number of shower particles produced by a single collision is unexpectedly small. In *LFS*-shower⁴⁾ the primary particle with 3×10^{13} ev produced only 15 charged shower particles.
- (2) Shower particles are not distributed isotropically in the center of mass system, but are collimated in the directions of the incident particles. The shower in the laboratory system consists of two groups of shower particles

(i.e., dense and diffuse shower).

Not only these two single showers but also other showers, for example, *FN*-shower⁶⁾ or *R*-shower,⁷⁾ do not contradict these facts.

These features are beautifully explained by Fermi in his recent paper.⁸⁾ According to him the production of nucleon-pairs (at extremely high energy region) and the conservation law of the total angular momentum are essential to (1) and (2), respectively. In all of other theories these facts were not taken into account carefully.

Next let us give some remarks on the anisotropic distribution of shower particles. We may safely assume that an average energy of shower particles is much smaller than the energy of incident nucleon, because of the multiple production of shower particles. For marked anisotropic distribution, we can define the angles $\theta_{1/2}$ and θ_T , introduced by Bradt, Kaplon and Peters,⁷⁾ both for dense and diffuse shower in the laboratory system. Under the assumption just mentioned, we can express these angles in a very simple way: for example, if we adopt the angular distribution of shower particles

$$\cos^{2n} \theta \sin \theta d\theta d\varphi$$

in the center of mass system, the characteristic angles for a shower produced by a collision of two nucleons with energy γMc^2 (in the center of mass system) are roughly given as follows:

$$\theta_{1/2} \approx \frac{C_{1/2}}{\gamma}, \quad \theta_T \approx \frac{C_T}{\gamma} \quad \text{for dense shower,}$$

$$\theta_{1/2} \approx \frac{D_{1/2}}{\gamma}, \quad \theta_T \approx \frac{D_T}{\gamma} \quad \text{for diffuse shower,}$$

where the constants C and D are summarized in the following table (see next page).

If our assumption is correct, we can determine the parameter n and the incident energy γMc^2 . For instance, in *LFS*-shower

$$\begin{cases} n \sim 4, \\ \gamma Mc^2 \sim 120 \text{ Bev (i.e., } \sim 3 \times 10^{13} \text{ ev in the laboratory system)} \end{cases}$$

n	$C_{1/2}$ dense shower	$D_{1/2}$ diffuse shower	C_T dense shower	D_T diffuse shower
0	0.58	1.37	0.91	4.4
1	0.34	2.95	0.65	7.6
2	0.263	3.80	0.48	9.8
3	0.222	4.5	0.40	11.5
4	0.195	5.1	0.36	13.1

The half aperture θ_T^* of cone (in the center of mass system) containing 90% of shower particles is ~ 40 degrees. These values are completely in agreement with those stated in original report.⁴⁾

Finally it can be shown that the production rate of heavier mesons⁹⁾ is also consistent with Fermi's theory.

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On the Mass of Cohesive Meson and the Mass Difference of Nucleons

H. Enatsu and P. Y. Pac

Department of Physics, Kyoto University

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Pais¹⁾ and Sakata²⁾ have introduced independently the cohesive meson field in order

to remove the divergence difficulties of the electromagnetic self-energies of electron and proton. As to the mass difference of proton and neutron, it was found that the theoretical prediction has the right sign and order of magnitude, but Kawabe and Umezawa³⁾ have shown that one cannot obtain an adequate mass of cohesive meson to explain the observed mass difference, although the calculation was carried out in a relativistically invariant way.

We desire to point out that, when the method of evaluation is modified, the right value of mass difference can be given. The self-energies of nucleon have been computed to the second order of coupling constants by employing the formalism of relativistic quantum electrodynamics developed by Feynman and Dyson.⁴⁾ After the substitutions of momentum representations for the $D_F(x)$ and $S_F(x)$ functions are made, a variable transformation,

$$k_\mu \rightarrow k'_\mu + up_\mu \quad (1)$$

can be introduced as usual. In our treatments the so-called surface integral,⁵⁾ namely,

$$\int f(k) d^4k = \int \left(1 - up_\mu \frac{\partial}{\partial k_\mu}\right) f(k'_\mu + up_\mu) d^4k' \quad (2)$$

is considered.

One thus obtains the electromagnetic and cohesive mesonic masses of a proton, with the result

$$\delta m = \delta m_1 + \delta m_2 + \delta m_3, \quad (3)$$

$$\delta m_1 = m \left[\frac{3}{2\pi} \left(\frac{e^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} - \frac{3}{4\pi} \left(\frac{f^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} \right], \quad (4)$$

$$\delta m_2 = -m \left[\frac{1}{4\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{8\pi} \left(\frac{f^2}{4\pi\hbar c} \right) \left\{ 1 - \lambda^2 + (\lambda^4 - 6\lambda^2) \log \lambda + \frac{(-\lambda^5 + 8\lambda^3 - 16\lambda)}{\sqrt{4 - \lambda^2}} \tan^{-1} \left(\frac{\sqrt{4 - \lambda^2}}{\lambda} \right) \right\} \right], \quad (5)$$

$$\delta m_3 = -m \left[\frac{1}{8\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{16\pi} \left(\frac{f^2}{4\pi\hbar c} \right) \right], \quad (6)$$

where m : Proton mass ($=1837 m_e$),

α : C meson mass,

m : Electron mass,

$\left(\frac{f^2}{4\pi\hbar c} \right)$: Coupling constant of C meson, (7)

$$\left(\frac{e^2}{4\pi\hbar c} \right) = \frac{1}{137.3},$$

$$\lambda = \frac{\alpha}{m}, \quad M = \frac{mc}{\hbar},$$

$$K_0 = \sqrt{M^2 + K^2}, \quad K \rightarrow \infty.$$

Accordingly, δm_1 can be eliminated by the Pais-Sakata's relation

$$f^2 = 2e^2. \quad (8)$$

The second part δm_2 coincides exactly with the finite term of Kawabe and Umezawa.³⁾

The last term δm_3 is produced by the surface integral part, $\frac{\partial}{\partial k_\mu}$, of Eq. (2), and readily seen to be not negligible compared to δm_2 .

As the empirical value of the mass difference of proton and neutron is given by

$$\delta m = -2.47 m_e, \quad (9)$$

finally, we obtain the mass of cohesive meson:

$$\alpha = 110 m_e. \quad (10)$$

This is the expected order of magnitude.

Detailed account will be given in the later issue of this journal.

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On the Accuracy of the Molière Function, 1

J. Nishimura* and K. Kamata**

* Department of Physics, Kobe University

** Scientific Research Institute

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The accuracy of Molière functions¹⁾ have hitherto been discussed by various authors.²⁾³⁾ These arguments, however, have been done only in the qualitative way, and seems to be unsatisfactory.

In order to examine the accuracy in more detail, we compare his functions with ours⁴⁾ which are derived analytically under the approximation B.⁵⁾ We first evaluate the angular distribution function at the shower maximum⁶⁾ from the formulae (6) and (7) of the reference (4), and compare them with Molière's results.

(I) As shown in Fig. 1, his differential angular distribution function⁷⁾ of high energy particles agrees fairly well with ours for

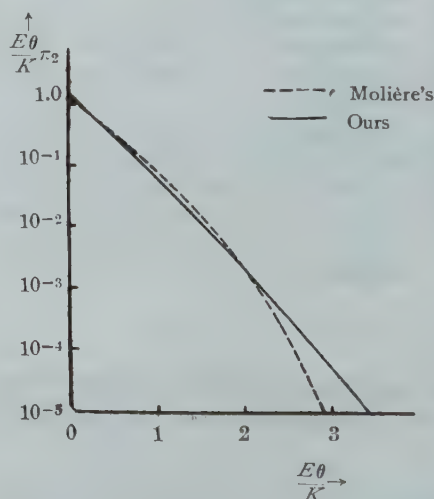


Fig. 1. Normalized differential angular distribution function of high energy particles at shower maximum. These functions are normalized as $\int_0^\infty \pi_2 2\pi \theta d\theta = \frac{K^2}{E^2}$. E : energy of electrons, K : 21 Mev.

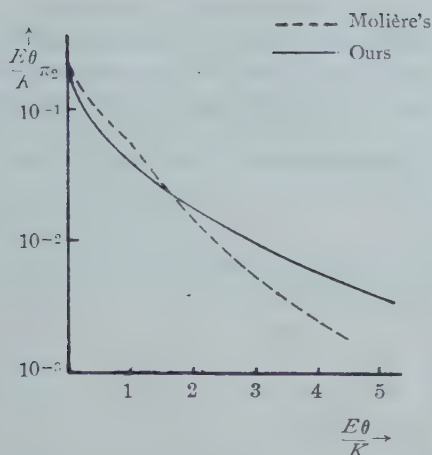


Fig. 2. Normalized integral angular distribution functions at shower maximum. They are normalized as $\int_0^\infty \pi_2 2\pi \left(\frac{\epsilon\theta}{K}\right) d\left(\frac{\epsilon\theta}{K}\right) = 1$. ϵ : critical energy of the material.

$\frac{E\theta}{K} < 2$, but not for $\frac{E\theta}{K} > 2$. Furthermore it becomes negative for $\frac{E\theta}{K} > 3$ as pointed out by Bethe and Eyges.⁶⁾ This inaccuracy is due to his rough numerical integration.[†]

(II) The integral* angular distribution is also shown in Fig. 2. His function is larger for $\frac{\epsilon\theta}{K} < 1.5$, and smaller for $\frac{\epsilon\theta}{K} > 1.5$ than ours. It should be noted that our distribution function does not contain the contribution of single scattering while Molière took account of this contribution, which may make his function larger than ours for $\frac{\epsilon\theta}{K} < 1.5$. The difference for $\frac{\epsilon\theta}{K} > 1.5$ seems to be due to the fact that the average number of particles having such energy is estimated too small in Arley approximation used by him.**

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△ In this paper, we limit ourselves to the shower initiated by a electron of very high energy.

□ Differential angular distribution function $\pi_2(E, \theta) 2\pi\theta d\theta dE$ represents the average number of electrons with energy between E and $E+dE$ lying the angle from the axis between θ and $\theta+d\theta$.

† More strictly, the interpolation formula (5) of reference (1) is not so accurate.

* Integral angular distribution function $\pi_2(\theta) 2\pi\theta d\theta$ is given by the formula, $\pi_2(\theta) = \int_0^\infty \pi(E, \theta) dE$.

** The possibility that this difference is due to the inaccuracy mentioned in (I), may be excluded, because the particles corresponding to $\frac{\epsilon\theta}{K} > 1.5$ have the energy less than the critical energy of the material.

A Possible Explanation for the Deviations of Nuclear Magnetic Moments from the Schmidt Lines

H. Miyazawa

Department of Physics, University of Tokyo

March 22, 1951

The magnetic moments of odd nuclei when plotted against nuclear spin, fall into two groups roughly paralleling the Schmidt lines. This fact constitutes a strong support to the nuclear shell model¹⁾ which assigns a definite stationary state to nuclear particle. No satisfactory explanation has yet been given, however, for the uniform deviations from the Schmidt lines. Several attempts^{2,3)} to obtain a better agreement by mixing different states have succeeded partly but are not thoroughly convincing.

The experimental fact that the deviations are in the opposite direction for nuclei with nucleon spin parallel or antiparallel to total

1) G. Molière, Cosmic Radiation, ed. by W. Heisenberg (Dover Publications, New York, 1946, chap. 3)

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spin, suggests that the deviation has its origin in spin rather than in orbital angular momentum. In this communication we shall consider the intrinsic magnetic moment of a nucleon bound in a nucleus. It is not equal to the value measured in free states, because the nucleon is greatly influenced by closely packed nuclear particles. Consider a Fermi gas composed of nucleons with momentum up to P . In this nucleus the virtual transitions of a nucleon to states with momentum below P are excluded according to the Pauli principle, and consequently for a nucleon with momentum zero, say, the virtual mesons with momentum less than P are lacking. This naturally results in the modification of the anomalous magnetic moment which is regarded as arising from the virtual meson cloud around the nucleon. It is generally believed that among the virtual mesons, those with momentum of the order of or below the meson rest mass π has a large contribution to the magnetic moment: for otherwise the recoil nucleon current would become much larger to destroy the approximate equality $\mu_p - 1 = |\mu_n|$. Now P is about 1.5π ; therefore a considerable decrease is expected in the anomalous magnetic moment.

This diminution can be calculated with a suitable meson model (pseudoscalar theory with cut off) adjusted so as to fit the magnetic moment of the free nucleon. For heavy nuclei the nucleus may be taken as a Fermi gas of infinitely large dimension. The spherical wave functions instead of plane waves are used. The result is, for the residual odd nucleon,

$$\Delta\mu = 0.68 - 0.23 \left\langle \frac{z}{r} (\sigma r) \right\rangle_{av} \text{ n.m.} \quad (1)$$

The second term is in general not large. One can thus conclude that the anomalous magnetic moment of the bound nucleon is about half of that of free nucleon. Although the above figures have been obtained for a special model of the meson, the conclusion

is nearly independent of the model if the model is adapted for free nucleon moment.

In order to see if the above mentioned change $\Delta\mu$ is sufficient to account for the

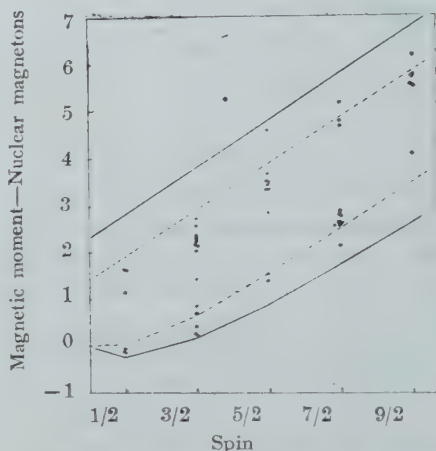


Fig. 1. Magnetic moments of odd proton nuclei. Only heavy nuclei ($A > 20$) are plotted. Full lines represent the Schmidt lines, dashed lines the present theory.

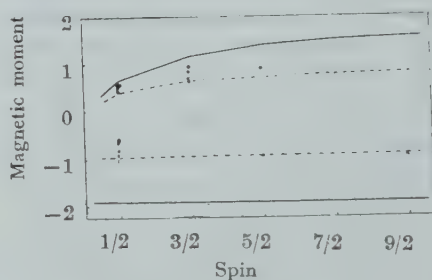


Fig. 2. Magnetic moments of odd neutron nuclei.

observed data, the magnetic moments of odd nuclei are calculated by the single particle model, with intrinsic moments

$$\mu_p = 2.79 - \Delta\mu,$$

$$\mu_n = -1.91 + \Delta\mu.$$

We see that even better agreement is achieved, if we put

$$\Delta\mu = 1 - 0.37 \langle \sigma r \rangle_{av},$$

the resulting curves being plotted in Figs. 1 and 2. This value of $\Delta\mu$ is larger than that of (1) by the factor of $3/2$, but in view of the crudeness of the model adopted, we may conclude that a possible explanations for the deviations of nuclear magnetic moments from the Schmidt lines are given in this way.

One can prove that the diminution of the anomalous magnetic moment calculated

here is equivalent to a part of the exchange magnetic moment, the so-called spin exchange moment.

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Connection between Particle Models and Field Theories, I

—The Case Spin 1/2—

Smio TANI

Department of Physics, Tokyo University

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If we neglect, in Feynman-Dyson's rule of calculation, the contributions due to all closed loops of fermion line, we may deal with the one-particle theory for the fermion. The transformation found by Pryce, by Foldy and Wouthuysen and also independently by the author enables us to get some intuitive explanation of fermion's behavior when it is free. In this paper we will show that also when the fermions are interacting with other fields, this transformation is rather suitable to get a physical interpretation of the results. For example, we are able to compute the effects of the Zitterbewegung. Although it is profitable to work in the one-particle theory, we must interpret the negative energy states in accordance with the positron theory. So we will discuss how far we should modify the one-particle theory in order to maintain the straight parallelism to the quantized field theory. A remark is added concerning with the interpretation of charge operators, i.e., Dirac operators ρ 's. Several results of the transformation are tabulated for the sake of reference in future.

§ 1. Introduction

In the relativistically invariant field theory we may often neglect the effect of vacuum fluctuation of a fermion field. This means to omit the effect of all closed loops of the fermion; in this case every fermion line is continuous and the number of fermion lines will be never changed. So we may start with the one-particle theory if we only pay attention to the treatment of the negative energy states. In the present paper we shall treat the case of spin 1/2. The cases of boson with spin 0 or 1 may be treated in an analogous way, which will be shown in the following paper.

Let us consider a single fermion interacting with some type of meson field. (Of course, the meson field should be quantized, for the mesons are emitted or absorbed.)

There is no difficulty in making a generalization to cases of two or more fermions. We start from the Hamiltonian formalism in which time plays a special role. We prefer this to the relativistically symmetrical formalism, because here we concern ourselves with the intuitive explanation of the results rather than the mathematical aspects of Lorentz-covariance.

The Hamiltonian of a free fermion H_0 is given by

$$H_0 = m\rho_3 + (\boldsymbol{p}\boldsymbol{\sigma})\rho_1. \quad (1)$$

In the usual matrix representation of ρ and σ , this Hamiltonian is diagonal when $\mathbf{p}=0$, and in this case the operator ρ_3 can be directly interpreted as the sign of the energy. In order to maintain this property also for the case of $\mathbf{p}\neq 0$, it is more convenient to eliminate the "perturbation" term $(\mathbf{p}\sigma)\rho_1$ and diagonalize H_0 by means of a transformation, which has been proposed by Pryce,¹⁾ by Foldy and Wouthuysen²⁾ and also independently by the author.³⁾ Then the Hamiltonian has the form*

$$U^{-1}H_0U = \sqrt{m^2 + \mathbf{p}^2} \cdot \rho_3, \quad (2)$$

where

$$U = \exp \left(-\frac{i}{2} \frac{(\mathbf{p}\sigma)}{p} \rho_2 \tan^{-1} \left(\frac{p}{m} \right) \right). \quad (3)$$

After the transformation the wave function is also reduced to a very simple form, (for pos. energy states)

$$\exp [i(\mathbf{p}\mathbf{X} - E_p t)] \times \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ((\mathbf{p}\sigma) = +p), \quad \times \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ((\mathbf{p}\sigma) = -p), \quad (4)$$

(for neg. energy states)

$$\exp [i(\mathbf{p}\mathbf{X} + E_p t)] \times \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ((\mathbf{p}\sigma) = +p), \quad \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ((\mathbf{p}\sigma) = -p),$$

here and later on making use of the abbreviation,

$$E_p = \sqrt{m^2 + \mathbf{p}^2}. \quad (5)$$

The transformation defined by (3) makes it possible to interpret the free fermion's behavior semi-classically; As will be discussed in the next paragraph, we can readily see the velocity dependence of operators ρ 's or σ 's in the operator equations in the new representation. Also we can separate out the Zitterbewegung amplitude from the coordinate operator, which behaves just in the same way as the position of a classical particle treated relativistically. In this paper we are interested mainly in the benefit of making this transformation, which seems not to have been fully appreciated by other authors; and we will show that by virtue of this transformation we are able to find a semi-classical model of the fermion, which is interacting with some type of meson field.

§ 2. Transformation U defined by Eq. (3)

At the beginning, one should notice to the fact that if one writes the operator

* In the following we employ the \mathbf{p} -representation rather than the \mathbf{X} -representation, because the momentum of a particle is more useful in the physical interpretation of the results than its position.

U given by (3) explicitly as a 4-4-matrix, then one will find that each column is nothing but one of four normalized orthogonal solutions of the usual Dirac equation respectively.

As far as we are concerned with the operators commutable with \mathbf{p} , we may look upon \mathbf{p} as a parameter. Let A denote an operator in the new representation transformed by means of U and let A^0 be its old and usual representation. Then, they are connected with each other by the relation,

$$A = U^{0-1} A^0 U^0, \quad (6)$$

where U^0 means the operator given by (3) in which spin operators ρ^0 's and σ^0 's are inserted. As the result of this transformation, the new representation A becomes dependent of the velocity of the particle also when the dynamical variable A^0 does not depend on it. On the other hand, the spin part of the wave function is velocity independent (see (4)), so the velocity dependence of A gives directly the information how the expectation values of various spin quantities depend on the velocity of the particle.

In the following we reproduce the results of the transformation for several dynamical quantities.

(i) *The ρ -operators*

$$U^{-1} \rho_1 U = \frac{m}{\sqrt{m^2 + \mathbf{p}^2}} \rho_1 + \frac{(\mathbf{p}\sigma)}{\sqrt{m^2 + \mathbf{p}^2}} \rho_3, \quad (\text{A, 1})$$

$$U^{-1} \rho_2 U = \rho_2, \quad (\text{A, 2})$$

$$U^{-1} \rho_3 U = -\frac{(\mathbf{p}\sigma)}{\sqrt{m^2 + \mathbf{p}^2}} \rho_1 + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}} \rho_3. \quad (\text{A, 3})$$

An interpretation of these equations for the charge operators will be investigated later. Here we give only the following remark. As has been discussed by Fock,⁴⁾ if we imagine a fictitious 3-dimensional ρ -space, we shall find this ρ -space rotating about a certain axis in the Heisenberg representation, the angular velocity of rotation being $2\sqrt{m^2 + \mathbf{p}^2}$. The axis of rotation in the usual representation of ρ 's lies in the ρ_1 - ρ_3 -plane and deviates from the ρ_3 -axis by the angle $\sin^{-1}((\mathbf{p}\sigma)/\sqrt{m^2 + \mathbf{p}^2})$. In our representation this axis is fixed and represented by ρ_3 ; but when \mathbf{p} grows greater ρ -space rotates also about the ρ_2 -axis. This fact is illustrated in Fig. 1.

(ii) *The spin operators*

The transformation for the spin vector is given by

$$U^{-1} \sigma U = \sigma_{||} + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}} \sigma_{\perp} + \frac{1}{\sqrt{m^2 + \mathbf{p}^2}} [\mathbf{p} \times \sigma] \rho_2, \quad (\text{B})$$

* The equations of motion for ρ 's show that their motion is of spherical character. This has a certain connection with the use of exclusion principle when quantized. This point will be discussed in Part II where we shall meet with the hyperbolic cases corresponding to symmetrical statistics.

where

$$\sigma_{||} = \frac{(\mathbf{p}\sigma)}{p^2} \mathbf{p}, \quad (7)$$

and

$$\sigma_{\perp} = \sigma - \sigma_{||} \quad (8)$$

denotes the longitudinal and transverse part of spin-vector respectively. The velocity dependence of the spin, which is shown in several textbooks⁵⁾ by computing its expectation value, is shown in (B) as an operator relation. The transverse part becomes smaller by the Lorentz factor $\sqrt{1-v^2} (= m/\sqrt{m^2+p^2})$, while the longitudinal part remains unchanged.

(iii) The position operator

The transformation for the position operator is given by

$$U^{-1} \mathbf{X} U = \mathbf{X} + \tilde{\mathbf{X}} + \hat{\mathbf{X}}, \quad (C)$$

where

$$\tilde{\mathbf{X}} = \frac{1}{2\sqrt{m^2+p^2}} \left(\sigma_{\perp} + \frac{m}{\sqrt{m^2+p^2}} \sigma_{||} \right) \rho_2, \quad (C, 1)$$

and

$$\hat{\mathbf{X}} = \frac{1}{2} \left(1 - \frac{m}{\sqrt{m^2+p^2}} \right) \frac{1}{p^2} [\mathbf{p} \times \sigma]. \quad (C, 2)$$

\mathbf{X} is the coordinate operator in the semi-classical sense; in other words, it is canonically conjugate to the momentum \mathbf{p} , and its time derivative (in the new representation) corresponds to the relativistic velocity of the particle; that is,

$$\dot{\mathbf{X}} = \frac{1}{i} [X, \sqrt{m^2+p^2} \rho_3] = \frac{\mathbf{p}}{\sqrt{m^2+p^2}} \rho_3. \quad (9)$$

$\tilde{\mathbf{X}}$ denotes the position shift due to Schroedinger's Zitterbewegung.⁶⁾ The longitudinal magnitude is $1/2m$ times the Lorentz factor. The transverse-longitudinal ratio is also $\sqrt{1-v^2}$. Its time derivative is equal to that part of $\rho_1 \sigma$ which gives rise to pair formation and destruction. (Cf. Eq. (D, 1) in (iv)). That is, we have

$$\dot{\tilde{\mathbf{X}}} = \rho_1 \left(\sigma_{\perp} + \frac{m}{\sqrt{m^2+p^2}} \sigma_{||} \right), \quad (10)$$

$\hat{\mathbf{X}}$ can be interpreted as a shift of the centre of motion. This term has the

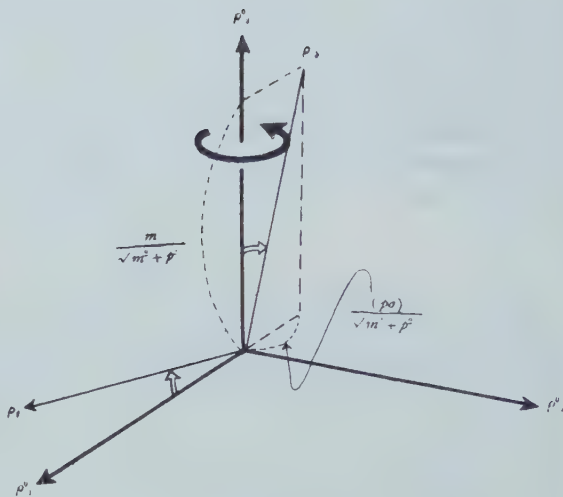


Fig. 1. The rotation of ρ -space in the Heisenberg representation.

The fixed axes are represented by ρ^0 's; the axis of rotation discussed by Fock is here represented by ρ_3^0 . As p becomes greater, ρ -space rotates about ρ_2 -axis.

effect to compensate the decrement of the transverse spin angular momentum and conserve the total angular momentum against an adiabatic change of the translational momentum. This point of view is justified as follows: The total angular momentum is a constant of motion and invariant against the transformation U . That is,

$$U^{-1}\left\{[\mathbf{X} \times \mathbf{p}] + \frac{1}{2}\boldsymbol{\sigma}\right\}U = [\mathbf{X} \times \mathbf{p}] + \frac{1}{2}\boldsymbol{\sigma}. \quad (11)$$

But the left hand side is

$$[\mathbf{X} \times \mathbf{p}] + [\tilde{\mathbf{X}} \times \mathbf{p}] + [\hat{\mathbf{X}} \times \mathbf{p}] + \frac{1}{2}\sigma_{||} + \frac{m}{2\sqrt{m^2 + \mathbf{p}^2}}\sigma_{\perp} + \frac{1}{2\sqrt{m^2 + \mathbf{p}^2}}\rho_2[\mathbf{p} \times \boldsymbol{\sigma}], \quad (12)$$

according to (B) and (C). Each term in (12) is velocity dependent; in other words, gets changes by the transformation U . So there should be some pairs of terms to compensate with each other and keep the whole expression invariant. The first of these pairs is the "orbital angular momentum associated with the Zitterbewegung" and the odd part* of the spin angular momentum. They are connected by

$$[\tilde{\mathbf{X}} \times \mathbf{p}] = -\frac{1}{2} \frac{1}{\sqrt{m^2 + \mathbf{p}^2}}[\mathbf{p} \times \boldsymbol{\sigma}]_{\rho_2}. \quad (13)$$

The second is the "increase of the orbital angular momentum by virtue of the shift $\hat{\mathbf{X}}$ " and the decrease of the transverse spin angular momentum which has been discussed in (ii). Because there is the relation,

$$[\hat{\mathbf{X}} \times \mathbf{p}] = \frac{1}{2}\left(1 - \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\right)\sigma_{\perp} \quad (14)$$

which justifies the interpretation of $\hat{\mathbf{X}}$ stated above. It should be remarked that its time derivative vanishes, i.e.,

$$\dot{\hat{\mathbf{X}}} = 0. \quad (15)$$

The term \mathbf{X} together with $\hat{\mathbf{X}}$ constitute the even part (i.e. the part which does not give rise to pair formation and destruction) of \mathbf{X} in the usual representation. This even part may be identified to the position of the centre of gravity in the relativistic mechanics of a classical particle. This point has been fully discussed by Pryce¹⁾ and Møller.**⁷⁾

* We use the term "odd part" for that part of any operator which contains either ρ_1 or ρ_2 , and "even part" for that containing neither of them.

** Møller has availed himself of the transformation of variables which may be defined by means of $A \rightarrow \tilde{A} = U^{-1}AU$,

where
$$U = \exp \left[\frac{i}{2} \tan^{-1} \left(\frac{\mathbf{p}}{m} \right) \cdot \rho_2 \left(1 - \frac{(\mathbf{p}\boldsymbol{\sigma})}{p} \right) \right],$$

and essential points in his results are the same as ours.

(iv) *Supplement*

$$U^{-1}\rho_1\sigma U = \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}}\rho_3 + \left(\sigma_{\perp} + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\sigma_{\parallel}\right)\rho_1, \quad (\text{D, 1})$$

$$U^{-1}\rho_2\sigma U = \frac{1}{\sqrt{m^2 + \mathbf{p}^2}}[\mathbf{p} \times \boldsymbol{\sigma}] + \left(\sigma_{\parallel} + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\sigma_{\perp}\right)\rho_2, \quad (\text{D, 2})$$

$$U^{-1}\rho_3\sigma U = \left(\sigma_{\perp} + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}}\sigma_{\parallel}\right)\rho_3 - \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}}\rho_1. \quad (\text{D, 3})$$

(D, 1) is the transformation for the operator $\rho_1\sigma$ which appears in the expression of the current in the case of electro-magnetic interaction.

By summing up Eqs. (D, 2) above and (C) in (ii), we see that the operators σ and $\rho_2\sigma$ together form a six-vector and their transformation character with respect to the Lorentz transformation is shown in the right hand side of these equations. For most of other quantities, e.g. the scalar density ρ_3 or the time-component density of the 4-vector I , our transformation equations show their Lorentz properties only when integrated over the whole space, but not the transformation properties of the densities themselves.

§ 3. The treatment of the interaction

In this paragraph we show that the transformation defined by (3) is much useful in treating the problem of field reaction.

In the ordinary representation of fermion operators, the interaction Hamiltonian H' of a fermion with meson field is of the form

$$H' = \sum_{\mathbf{k}} \sum_{\lambda} a_{\mathbf{k}\lambda}^* e^{-i(\mathbf{k}\mathbf{X})} \cdot \langle \text{interactor} \rangle + \text{hermitian conjugate}, \quad (16)$$

where $a_{\mathbf{k}\lambda}^*$ denotes the emission operator of the meson with momentum \mathbf{k} and polarization λ , and in the place of $\langle \text{interactor} \rangle$ one should put a suitable operator according to what type of meson and coupling is considered. (E.g. in the case of scalar meson and scalar coupling, put $g(2\pi/\epsilon_{\mathbf{k}})^{1/2}\rho_3$ in this place.) We denote the coupling constant by g and abbreviate as

$$\epsilon_{\mathbf{k}} = \sqrt{\mu^2 + \mathbf{k}^2}, \quad (17)$$

μ being the meson mass. Naturally the total Hamiltonian is given by

$$\mathfrak{H} = H_0 + H' + H_{\text{mes.}}, \quad (18)$$

where $H_{\text{mes.}}$ is the Hamiltonian of the isolated meson field which is given by

$$H_{\text{mes.}} = \sum_{\mathbf{k}} \sum_{\lambda} \epsilon_{\mathbf{k}} a_{\mathbf{k}\lambda}^* a_{\mathbf{k}\lambda}. \quad (19)$$

Now in order to treat the field reaction, let us go over into the interaction representation of the total system. Then the Hamiltonian function $(\mathfrak{H})_{\text{int.}}^*$ has

* $()_{\text{int.}}$ denotes the respective operator in the interaction representation.

the form

$$\begin{aligned}
 (\mathfrak{H})_{\text{int.}} = & \sum_{\mathbf{k}} \sum_{\lambda} a_{\mathbf{k}\lambda}^* i\epsilon_{\mathbf{k}} \exp [i m \rho_3 + (\mathbf{p}\boldsymbol{\sigma}) \rho_1] t] \cdot e^{-i(\mathbf{k}\mathbf{X})} \cdot \\
 & \langle \text{interactor} \rangle \cdot \exp [-i(m\rho_3 + (\mathbf{p}\boldsymbol{\sigma})\rho_1)t] + \text{herm. conj.} . \quad (20)
 \end{aligned}$$

In interaction representation, the operators ρ 's or σ 's behave in the same way as in the Heisenberg representation of free motion. Though the equations of motion for ρ 's and σ 's are integrated by Fock,⁵⁾ the answers look too much complicated to be written down as concrete operator relations. Moreover, the factor $e^{-i(\mathbf{k}\mathbf{X})}$ behaves much more unharmonically in time due to the existence of the Zitterbewegung. Thus, in the representation hitherto used, the time dependence of the right hand member of (20) is far from a simple exponential one; so we must make use of the familiar technique of projection operators, and write down the matrix elements of $(\mathfrak{H})_{\text{int.}}$ abandoning the exclusive use of the operator form. These troubles are caused by the fact that the operator $m\rho_3 + (\mathbf{p}\boldsymbol{\sigma})\rho_1$ is not yet diagonalized explicitly. But after the transformation U we have a simple exponential time factor, as will be revealed in the following.

If we perform the transformation U we have

$$\begin{aligned}
 U^{-1}(\mathfrak{H})_{\text{int.}}U = & \sum_{\mathbf{k}} \sum_{\lambda} a_{\mathbf{k}\lambda}^* \exp (i\epsilon_{\mathbf{k}}t) \exp (i\sqrt{m^2 + \mathbf{p}^2}\rho_3t) \cdot U^{-1} \cdot e^{-i(\mathbf{k}\mathbf{X})} \cdot U \times \\
 & U^{-1}\langle \text{interactor} \rangle U \cdot \exp (-i\sqrt{m^2 + \mathbf{p}^2}\rho_3t) + \text{herm. conj.} . \quad (21)
 \end{aligned}$$

At first sight, the right hand side of (21) may look rather complicated; however, it is arranged in a physically interesting way. In the new representation of the interactor, $U^{-1}\langle \text{interactor} \rangle U$, one can see the velocity dependence of the interactor, as was shown in the preceding paragraph. Let us consider the operator $e^{-i(\mathbf{k}\mathbf{X})}$ in the new representation and use the following notation

$$U^{-1}e^{-i(\mathbf{k}\mathbf{X})}U = e^{-i(\mathbf{k}, \mathbf{X} + \tilde{\mathbf{X}} + \hat{\mathbf{X}})} = e^{-i(\mathbf{k}\mathbf{X})} \cdot \langle \text{recoil} \rangle, \quad (22)$$

where the second statement is due to the results of (iii) of the preceding paragraph. The term $\langle \text{recoil} \rangle$ turns out to be unit operator either if the recoil momentum \mathbf{k} might vanish or if the fermion had no inner degree of freedom other than that of translation. We will analyze this term in the next paragraph. As will be shown there, we can consider $\langle \text{recoil} \rangle$ as expressing the disturbance of inner degree of freedom ρ and σ caused by the recoil. It consists of two types of expressions, one containing ρ_2 and the other free of ρ 's. Now we should classify the operator $\langle \text{recoil} \rangle \cdot U^{-1}\langle \text{interactor} \rangle U$ into two parts, one of which contains either ρ_1 or ρ_2 and the other does not contain. The term containing ρ_1 or ρ_2 will give rise to a pair formation or destruction. The distinction of the sign of energy is self-evident after the transformation, and only ρ_1 and ρ_2 causes the change of the sign of energy. Consequently one will never feel it necessary to use the projection operators. It is noticeable that the equations of

motion for ρ 's and σ 's are very simple in our representation. Only ρ_1 and ρ_2 show the dependence on time, which is given by

$$(\rho_1)_{\text{int.}} = \rho_1 \cdot \exp[-i2\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3] = \exp[i2\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3] \cdot \rho_1, \quad (23)$$

$$(\rho_2)_{\text{int.}} = \rho_2 \cdot \exp[-i2\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3] = \exp[i2\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3] \cdot \rho_2. \quad (23')$$

The operator of momentum transfer, $e^{-i(\mathbf{k}\mathbf{X})}$, also behaves in a simple manner, which was not the case hitherto; that is to say, we have

$$\begin{aligned} (e^{-i(\mathbf{k}\mathbf{X})})_{\text{int.}} &= e^{-i(\mathbf{k}\mathbf{X})} \cdot \exp[i(\sqrt{m^2 + (\mathbf{p} + \mathbf{k})^2} - \sqrt{m^2 + \mathbf{p}^2})t \cdot \rho_3] \\ &= \exp[i(\sqrt{m^2 + \mathbf{p}^2} - \sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2})t \cdot \rho_3] \cdot e^{-i(\mathbf{k}\mathbf{X})}. \end{aligned} \quad (24)$$

As the consequence of these simplification brought about by U , we get a single exponential time factor, after uniting the time factors of the term $(\langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U)_{\text{int.}}$ and of the term $(e^{-i(\mathbf{k}\mathbf{X})})_{\text{int.}}$. This procedure is equivalent to shift the factor $\exp[-i\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3]$ in (21) from the right to the left and $\exp[+i\sqrt{m^2 + \mathbf{p}^2}t \cdot \rho_3]$ from the left to the right till they are unified into a single exponential function, thereby taking account of the commutation relations for ρ 's and $e^{-i(\mathbf{k}\mathbf{X})}$.

Finally, we have the Hamiltonian in the interaction representation, which is of the form

$$\begin{aligned} U^{-1}(\mathfrak{H})_{\text{int.}}U &= \sum \sum \sum a_{\mathbf{k}\lambda}^* \exp[i(\sqrt{m^2 + \mathbf{p}^2}\rho_3 \pm \sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2}\rho_3 + \epsilon_{\mathbf{k}})t] \cdot \times \\ &\quad \cdot e^{-i(\mathbf{k}\mathbf{X})} \cdot \langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U + \text{herm.conj.} \end{aligned} \quad (25)$$

Here the double sign in the time factor should be settled to be positive when we deal with that part of $\langle \text{recoil} \rangle \cdot U^{-1} \langle \text{interactor} \rangle U$ containing ρ_1 or ρ_2 and to be negative for the other part. (The summation indices \pm denote the summation over both parts in Eq. (25).) In our new representation we find interaction operators of somewhat complicated appearance, but the physical meaning of these operators is clear and able to be read directly. Moreover, the interaction Hamiltonian shows a simple dependence on time in the interaction representation. Thus we can readily find a physical interpretation of field reaction on the fermion as far as we take the free motion as the basis of describing the effect of the meson field. If we desire, we can at once make a non-relativistic or extreme-relativistic approximation to any order in p/m or m/p .

In computing the perturbation effects caused by H' by means of the contact transformation method, we meet with the integral of the form,

$$\int U^{-1}(\mathfrak{H})_{\text{int.}}U dt. \quad (26)$$

Then, as is evident from (25), we have the following factors after integration

$$\frac{\exp[i(E_{\mathbf{p}\rho_3} \pm E_{\mathbf{p}-\mathbf{k}\rho_3} + \epsilon_{\mathbf{k}})t]}{i(E_{\mathbf{p}\rho_3} \pm E_{\mathbf{p}-\mathbf{k}\rho_3} + \epsilon_{\mathbf{k}})}. \quad (27)$$

Comparing this with the corresponding expression of the quantized field theory, we find that in our calculation the energy differences are as well exact for transitions accompanied by pair formation or destruction as for those not accompanied by them; this is due to the existence of the factor ρ_3 and the discrimination of the double sign in (25). Such correspondence to the quantized theory is automatically established also in any higher order processes, which should be computed from the following type of integrals.

$$\left(\frac{1}{i}\right)^n \int dt' \int dt'' \dots \int dt_n^{n-1} \mathfrak{H} \langle t' \rangle \cdot \mathfrak{H} \langle t'' \rangle \dots \mathfrak{H} \langle t_n \rangle. \quad (28)$$

This is due to the fact that the requirement of the momentum conservation in higher order processes are automatically fulfilled by virtue of the spatial exponential factor $e^{\pm i(\mathbf{k}\mathbf{X})}$. But if we want to maintain a perfect parallelism of our scheme to that of the quantized theory from which all closed loops are dropped out, we must deviate a little from the formal procedure: That is, instead of computing with

$$\left(\frac{1}{i}\right)^n \int dt' \int dt'' \dots \int dt_n^{n-1} U^{-1} \cdot (\mathfrak{H})_{\text{int.}} \langle t' \rangle \cdot (\mathfrak{H})_{\text{int.}} \langle t'' \rangle \cdot \dots \cdot (\mathfrak{H})_{\text{int.}} \langle t_n \rangle \cdot U, \quad (29)$$

which results from merely substituting $U^{-1}(\mathfrak{H})_{\text{int.}}U$ given by (25) into (28), we must employ the following expression

$$\begin{aligned} & \left(\frac{1}{i}\right)^n \int dt' \int dt'' \dots \int dt_n^{n-1} U^{-1} \cdot (\mathfrak{H})_{\text{int.}} \langle t' \rangle \cdot U \cdot \rho_3 \cdot U^{-1} (\mathfrak{H})_{\text{int.}} \langle t'' \rangle \cdot U \cdot \times \\ & \times \dots \times \rho_3 \cdot U^{-1} \cdot (\mathfrak{H})_{\text{int.}} \langle t_n \rangle \cdot U. \end{aligned} \quad (30)$$

This prescription involves nothing new, because it corresponds to the use of projection operator technique in the usual method of calculation and distinguishing the sign of energy in order to take only the holes of the negative energy sea as giving the true contributions. Let $+\Lambda(\mathbf{p})$, $-\Lambda(\mathbf{p})$ respectively denote the projection operator into the positive or negative energy state for a fermion with momentum \mathbf{p} . In our representation the operator which represents the operator $+\Lambda(\mathbf{p}) - \Lambda(\mathbf{p})$ is nothing but ρ_3 irrespective of \mathbf{p} . So by inserting ρ_3 our results will become in conformity with that of position theory.⁸⁾ In the Appendix I, we shall check these points more closely.

§ 4. Recoil of a spin 1/2 particle

As was already mentioned in the preceding paragraph, we meet in the interaction Hamiltonian with a factor of the type given by (22).

Here we give the term $\langle \text{recoil} \rangle$ explicitly as*

* N.B. $U^{-1} e^{-i(\mathbf{k}\mathbf{X})} U = e^{-i(\mathbf{k}\mathbf{X})} \cdot e^{+i(\mathbf{k}\mathbf{X})} U^{-1} e^{-i(\mathbf{k}\mathbf{X})} \cdot U = e^{-i(\mathbf{k}\mathbf{X})} U^{-1} (\mathbf{p} - \mathbf{k}) \cdot U(\mathbf{p})$.

$$\langle recoil \rangle = \exp \{ -i\Phi(\rho_2(\sigma z) + (\sigma \mathbf{r})) / \sqrt{z^2 + \mathbf{r}^2} \}, \quad (31)$$

where

$$z = (\sqrt{m^2 + \mathbf{p}^2} + m)\mathbf{k} + (\sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2} - \sqrt{m^2 + \mathbf{p}^2})\mathbf{p}, \quad (32)$$

$$\mathbf{r} = [\mathbf{k} \times \mathbf{p}], \quad (33)$$

and

$$\sin \Phi = \left\{ \frac{z^2 + \mathbf{r}^2}{4E_p(m + E_p)E_{p-k}(m + E_{p-k})} \right\}^{1/2} \quad (34)$$

In (34) the variable Φ vanishes in case of $\mathbf{k} = 0$. $\langle recoil \rangle$ would have been reduced to unit operator if the particle had no inner degree of freedom so that the charge operators ρ 's and the spin operators σ 's disappeared and the effect of the recoil was only to change the momentum from \mathbf{p} to $\mathbf{p} - \mathbf{k}$. Thus we are justified in interpreting the term $\langle recoil \rangle$ as due to the recoil disturbance of the inner degree of freedom. Expanding the exponential in (31) we have

$$\langle recoil \rangle = \cos \Phi + i \sin \Phi \cdot \frac{\rho_2(\sigma z) + (\sigma \mathbf{r})}{\sqrt{z^2 + \mathbf{r}^2}}. \quad (35)$$

The term containing z may be interpreted as the disturbance of the Zitterbewegung caused by the recoil, for the factor $\rho_2(\sigma z)$ has a similar construction as the Zitterbewegung in the free motion given by (C, 1). The term containing \mathbf{r} corresponds to the disturbance of the shift of the centre of motion given by (C, 2). In fact, such correspondence is justified in the non-relativistic limit. In this case the term $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$ is given by

$$\tilde{\mathbf{X}} \approx \frac{1}{2m} \rho_2 \sigma \quad (36)$$

and

$$\hat{\mathbf{X}} \approx \frac{1}{4m^2} [\mathbf{p} \times \sigma], \quad (37)$$

respectively to the first order of \mathbf{p}/m . Putting these expressions into (22) we have the expression

$$\begin{aligned} e^{-i(\mathbf{k}, \mathbf{X} + \tilde{\mathbf{X}} + \hat{\mathbf{X}})} &\approx \exp \left[-i((\mathbf{k}, \mathbf{X}) + \frac{1}{2m} \rho_2(\mathbf{k}, \sigma) + \frac{1}{4m^2} (\sigma \cdot [\mathbf{k} \times \mathbf{p}])) \right] \\ &= e^{-i(\mathbf{k}, \mathbf{X})} \left[-i \left(\frac{1}{2m} \rho_2(\sigma \mathbf{k}) + \frac{1}{4m^2} (\sigma \cdot [\mathbf{k} \times \mathbf{p}]) \right) \right], \end{aligned} \quad (38)$$

which checks our statement.

Φ is the safety measure how far we may neglect the disturbance of inner degrees of freedom due to the recoil. The behavior of Φ as a function of \mathbf{p}/m , \mathbf{k}/m , and θ (θ denoting the angle between \mathbf{p} and \mathbf{k}) is much complicated: and it is hard to make a general discussion about this behavior. Here we restrict

ourselves to the limiting case of $k \rightarrow \infty$. In this case (34) reduces to

$$\sin \Phi_{k \rightarrow \infty} = \frac{1}{2} \left\{ \frac{(1 + \sqrt{1 + (p/m)^2} - (p/m) \cos \theta)^2 + (p/m)^2 \sin^2 \theta}{(1 + \sqrt{1 + (p/m)^2}) \sqrt{1 + (p/m)^2}} \right\}^{1/2} \quad (39)$$

It should be remarked that it is only when $p \rightarrow \infty$ and $\theta = \pi$ that $\Phi_{k \rightarrow \infty}$ reaches the value $\pi/2$ and the recoil effects on the spin quantities are expected to change the character of the interactor entirely.

§ 5. Some examples of application

In our system of calculation, operators appear a little more complicated than those in the current use, but they readily furnish us with physical models and are suitable for a qualitative discussion. For a precise numerical evaluation the resort to the Feynman-Dyson's scheme is recommended. Some results of applying our method will be surveyed in the following.

(i) Compensation effect in the self-energy problem

As was explained at the end of § 3, we will insert ρ_3 's in the train of the transition operators in the transformation function by which the field reaction is taken into account. We might expect that in computing the 2nd order self-energy, the contributions due to pair formation in the intermediate state and those which do not come from intermediate pairs cancel each other. In higher order processes, it is not a simple matter to make a general discussion on the classification of compensating terms. But in each practical case, the computation is not so hard. As will be observed in Appendix II, there are two groups of interaction type, i.e. $\rho_3 - \rho_1$ group and $1 - \rho_2$ group. $\rho_3 - \rho_1$ group comprises those interaction type with ρ_3 for their even part and ρ_1 for their odd part. As long as the fermion undergoes a single type of interaction, the mixing of $\rho_3 - \rho_1$ type and $1 - \rho_2$ type in higher order processes does not occur. Then, for classification of compensating terms, one should compute the sign of a train like either

$$\rho_1 \bar{\rho}_3 \rho_1 \bar{\rho}_3 \rho_3 \dots \quad \text{and} \quad \rho_3 \bar{\rho}_3 \rho_1 \bar{\rho}_3 \rho_1 \dots \text{ etc.} \quad (40)$$

or

$$\rho_3 \bar{\rho}_3 \rho_3 \bar{\rho}_3 1 \dots \quad \text{and} \quad 1 \bar{\rho}_3 \rho_3 \bar{\rho}_3 \rho_2 \dots \text{ etc.} \quad (41)$$

(Here in order to distinguish the inserted ρ_3 from ρ 's contained in the original expression, we have put a bar upon them.)

Now in a process where the ultra-violet contribution plays a main role, the compensation effect will be stressed for that type of interaction which contains ρ_3 in the interactor of the usual representation, as compared to ρ_1 type; for, as the consequence of recoil by ultra-violet mesons, the fermion have a large virtual momentum, and ρ_3 goes over into ρ_1 . (Cf. § 2, Eqs. (A), the velocity dependence of ρ -operators). A similar reasoning shows us that 1-type interaction will be

much more compensated than ρ_2 -type. In fact, as is well known, the negative value of the second order self-energy in cases of neutral scalar and pseudovector mesons were found and investigated by several authors.⁹⁾¹⁰⁾

(ii) *Anomalous magnetic moment*

Let us consider a fermion of electric charge e in a homogeneous magnetic field of strength H and the direction \mathbf{e} (an unit vector). Then we have the interaction Hamiltonian of the fermion with this magnetic field

$$eU^{-1}\rho_1(\boldsymbol{\sigma}\mathbf{A})U, \quad (42)$$

with

$$\mathbf{A} = \frac{1}{2}H[\mathbf{e} \times \mathbf{X}]. \quad (43)$$

As the results of the transformation shown in § 2, we have

$$\frac{eH}{2} \left(\left\{ \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} \rho_3 + \left(\boldsymbol{\sigma}_\perp + \frac{m}{\sqrt{m^2 + \mathbf{p}^2}} \boldsymbol{\sigma}_\parallel \right) \rho_1 \right\} \cdot \{ [\mathbf{e} \times \mathbf{X}] + [\mathbf{e} \times \dot{\mathbf{X}}] + [\mathbf{e} \times \ddot{\mathbf{X}}] \} \right). \quad (44)$$

The second part of the first factor is proportional to $\dot{\mathbf{X}}$, i.e. the current associated with the Zitterbewegung. In the non-relativistic limit we can put, according to (36) and (37)

$$\tilde{\mathbf{X}} = \frac{1}{2m} \rho_2 \boldsymbol{\sigma}, \quad \hat{\mathbf{X}} = \frac{1}{4m^2} [\mathbf{p} \times \boldsymbol{\sigma}],$$

and have the reduced Hamiltonian

$$\frac{eH}{2} \left(\left\{ \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} \rho_2 + \rho_1 \boldsymbol{\sigma} \right\} \cdot \left\{ [\mathbf{e} \times \mathbf{X}] + \frac{1}{2m} [\mathbf{e} \times \boldsymbol{\sigma}] \rho_2 + \frac{1}{4m^2} [\mathbf{e} \times [\mathbf{p} \times \boldsymbol{\sigma}]] \right\} \right), \quad (45)$$

in which we find the interaction energy due to the normal magnetic moment,

$$\frac{eH}{4m} (\rho_1 \boldsymbol{\sigma} \cdot [\mathbf{e} \times \boldsymbol{\sigma}] \rho_2) = \frac{eH}{2m} \rho_3 (\boldsymbol{\sigma} \mathbf{e}) \quad (46)$$

Thus we recognize the position shift of Zitterbewegung and the current accompanied by it as the origin of the normal magnetic moment. If the fermion interacts with some kind of meson field, then the Zitterbewegung amplitude will be modified and we shall have

$$e^{-iS} \tilde{\mathbf{X}} e^{+iS} \quad (47)$$

in the place of $\tilde{\mathbf{X}}$ with no mesonic interaction. Here e^{+iS} denotes the transformation function to take the virtual meson cloud into account. How to construct this transformation function in its interaction representation was discussed in § 3. We may bring it back into its Schroedinger representation by merely dropping the exponential time factor. By actual calculation it turns out that this transformation function must modify the factor ρ_2 in the unperturbed Zitter-amplitude

in order that we might have a positive sign of anomalous magnetic moment. Because the modification of the σ factor (in the unperturbed amplitude) will always decrease the effective value of the spin component in a certain direction. These points were fully discussed by Welton¹¹⁾ and Koba.¹²⁾

In the transformation function the operators S has the same character as the interactor in the first approximation. From this fact, one may expect a positive sign of anomalous magnetic moment in the case of neutral scalar meson¹³⁾¹⁴⁾ or of neutral vector meson (photon), and a negative sign in the case of neutral pseudoscalar.¹⁵⁾ In the former cases the interactor have a factor ρ_3 or ρ_1 respectively; so these cases are expected to give rise to a modification of ρ_2 in the unperturbed amplitude, for this modification is determined to be great or small essentially according to the order of magnitude of $[\rho_2, S]$. On the other hand, in the latter case of pseudoscalar, the interactor is ρ_2 , which is obviously commutable with ρ_2 in the unperturbed Zitter-amplitude; then the modification of σ dominates and the result becomes negative.

By making use of the calculational technique developed in this paper, one can find Koba's model of the anomalous magnetic moment of an electron out of the usual quantum electrodynamical scheme: but we will not go into the details of this problem any further.

(iii) *Minute remarks on nuclear interactions*

At the end of § 4, we have noticed that in the expression

$$\langle \text{recoil} \rangle = \cos \Phi + i \sin \Phi \frac{\rho_2(\sigma \mathbf{Z}) + (\sigma \mathbf{r})}{\sqrt{\mathbf{Z}^2 + \mathbf{r}^2}}$$

$\cos \Phi$ vanishes only in the special case of $k \rightarrow \infty$, $p \rightarrow \infty$, $\theta = \pi$. Thus, generally speaking, it is rather small region of the integration domain where the type of the effective interactor becomes altered very much from the original expression in which no account is taken of the disturbance of ρ 's and σ 's. From this fact we may say that interaction types containing derivative of the meson field amplitude (in their usual representation) will yield higher singularity compared to those free from derivative. However, this cannot be considered as contradicting to the equivalence theorem¹⁶⁾ in nuclear force problem, because in the g^2 -order and Born approximation the even part of interactor of the vector-type coupling is proportional to the respective operator of the scalar-type coupling, as is verified from the results in Appendix II. The equivalence is guaranteed in such a problem as we retain only the lowest order contributions, but the deviation occurs in higher order contributions, especially when we take the intermediate pair creations into account.

At the end we add that the recoil disturbance of Zitterbewegung yields the tensor force without r^{-3} -singularity in the case of pseudoscalar meson and pseudoscalar coupling;¹⁷⁾ and that the recoil shift of centre $\hat{\mathbf{X}}$ yields the spin-orbit coupling in the case of scalar meson and scalar coupling, which was discussed by Dancoff.¹⁸⁾

§ 6. Interpretation of ρ -operators

We have met in § 2 the rotation of ρ -space. So here we add some points of interest in the physical interpretation of ρ 's, by the way.

The expectation value of ρ_3 , ρ_2 , 1 , and ρ_1 are scalar, pseudoscalar, 4-component of vector and of pseudovector respectively. These four operators constitute a closed set, in which every product of their linear combinations is again such a linear combination of them. The multiplication law in this closed set suggests us how these operators should be interpreted physically. However, we take a different way and for some while consider the relativistically symmetrical formalism, and then come back to the formalism initially adopted. This way might be longer but more smooth than the short-cut mentioned above.

Now, in the relativistically symmetrical formalism, the above mentioned density operators correspond to 1 , γ_{1234} , γ_4 , and γ_{123} respectively. They are generators of reflexions in the representation of the full Lorentz group. But we can interpret them as generators of reflexions of world line in Feynman diagram, instead of reflexions of coordinate axis. This will be clearly seen in the S -matrix of Feynman's theory. The S -matrix is constituted of a sum of terms like

$$\langle \text{Green function} \rangle \langle \text{interactor} \rangle^{(n)} \langle G.F. \rangle^{(n-1)} \langle G.F. \rangle^{(n-2)} \dots \dots \dots \langle \text{Intr.} \rangle^{(2)} \langle G.F. \rangle^{(1)} \langle \text{Intr.} \rangle \langle G.F. \rangle^* \quad (48)$$

Here the energy differences corresponding to some integral of D -functions are inessential for our purpose and not explicitly written down. In computing the transition probability of a certain process we shall multiply the S -matrix of the interested process with its hermitian conjugate from the left. Thus we shall have operators of the following type,

$$\begin{array}{ccccccc} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(1)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(2)} \dots \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n-1)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n)} \langle G.F. \rangle & \begin{array}{c} \text{Centre} \\ \downarrow \\ \times \end{array} \\ \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n-1)} \langle G.F. \rangle \dots \langle \text{Intr.} \rangle^{(2)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(1)} \langle G.F. \rangle & \end{array} \quad (49)$$

Let us take an example to fix our idea, and suppose that there stands a factor ρ_3 as a part of the $(n-1)$ -th interactor on each side of the position, "centre"; then we can make this $(n-1)$ -th interactor free of ρ_3 , putting two ρ_3 -factors into the centre, i.e.

$$\begin{array}{ccccccc} \dots \langle \text{Intr.} \rangle^{(n-1)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n)} \langle G.F. \rangle \rho_3 & \begin{array}{c} \text{Centre} \\ \downarrow \downarrow \\ \rho_3 \rho_3 \end{array} & \rho_3 \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n)} \langle G.F. \rangle \langle \text{Intr.} \rangle^{(n-1)} \dots \\ \text{free} & & \text{free} \\ \text{of } \rho_3 & & \text{of } \rho_3 \end{array} \quad (50)$$

* $G.F.$ stands for Green function and Intr. stands for interactor. The suffix attached to the interactor is showing its order in the train.

Of course, by virtue of $\rho_3^2 = 1$, the ρ_3 factors in the centre are cancelled. But they have left their trace in the remainder of (50): In view of the constitution of the Green's functions, $1/\sum_{\mu=1}^4 \gamma_{\mu} q_{\mu} - im$, where q_{μ} denotes the propagation 4-vector for respective states, it is evident that there is a change of sign $q_{\mu} \rightarrow -q_{\mu}$ ($\mu=1, 2, 3$) where ρ_3 has passed over.

By proceeding in this way, one can, in general, omit $\rho_1(\propto \gamma_{1234})$, $\rho_2(\propto \gamma_{123})$, $\rho_3(\propto \gamma_4)$ contained in interactors, but, on the other hand, making the following replacement, $q_{\mu} \rightarrow -q_{\mu}$,

$$\begin{array}{lll} \mu=1, 2, 3 & \text{in all factors of the train (48) lying on the left-side} & \\ & \text{of that interactor in which we omitted} & \rho_3, \\ \mu=4 & \text{"} & \rho_2, \\ \mu=1, 2, 3, 4 & \text{"} & \rho_1. \end{array} \quad (51)$$

The above statement in the relativistically symmetrical theory is a little obscured when we come back to the asymmetrical system. But we shall find the analogous conclusion here, too.

By application of ρ_3 the sign of energy does not change, while the sign of momentum will be reversed.

$$\begin{aligned} \rho_3^{-1} \rho_3 \rho_3 &= \rho_3, \\ \rho_3^{-1} U \rho_3 &= U^{-1}. \end{aligned} \quad (52)$$

By application of ρ_2 the sign of energy will be reversed, while the momentum remains unchanged.

$$\begin{aligned} \rho_2^{-1} \rho_3 \rho_2 &= -\rho_3, \\ \rho_2^{-1} U \rho_2 &= U. \end{aligned} \quad (53)$$

And by application of ρ_1 the sign of both energy and momentum will be reversed.

$$\begin{aligned} \rho_1^{-1} \rho_3 \rho_1 &= -\rho_3, \\ \rho_1^{-1} U \rho_1 &= U^{-1}. \end{aligned} \quad (53)$$

The apparent paradox, that we have obtained the same result in both cases of symmetrical formalism and asymmetrical one in spite of the different interpretation of the expectation values of operators, will be solved when we consider that it is the mutual commutation relations between these operators which underlies our discussion.

In this way we are justified in terming ρ 's as the charge operators: For the reflexions of world line in a Feynman diagram means physically an occurrence of a pair formation or destruction when the change of the sign of energy is contained in the reflexion. Thus three distinct types of reflexion attributed to each one of ρ 's correspond to different types of pair formation or destruction. In actual cases, these pair formation or destruction are accompanied with the simultaneous

change of momentum, and their exact representation in Feynman diagrams are difficult to be given ; but neglecting this fact we sum up the results schematically in Fig. 2.

In the sense stated above, the rotation of ρ -space which we met with in §2, can be interpreted as the change of the character of pair formation and destruction.

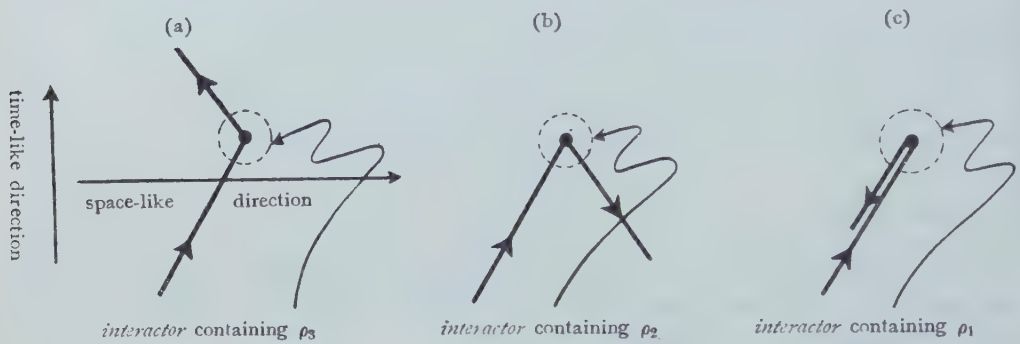


Fig. 2. Schematic representation of the effect of ρ 's contained in the interactors.
Here we omit the momentum transfer caused by the interaction.

§ 7. Summary and acknowledgement

We have shown in this paper, that in case when we can neglect the effect of the vacuum fluctuation of the spin 1/2 field, we can get semi-classical models of spin 1/2 particle which facilitate us much in qualitative understanding of the results. We should start from the one-particle theory for this fermion, and make use of the transformation proposed by Pryce, by Foldy and Wouthuysen, and also by the author. Then, we find semi-classical interpretation of the behavior when the particle moves without interaction. The results was summed up in §2. , and they furnish us with quite suitable basis for treating the interaction. The treatment of the interaction was described in §3. The result is in conformity with the positron theory, if we adopt the prescription stated there.

In concluding, I wish to express my cordial thanks to Prof. Tomonaga for his interest in this work and valuable discussions.

Appendix I

Here we discuss the correspondence between the one-particle theory and the quantized field theory.

In constructing the transformation function, we must compute integrals of following type, (given by (28), § 2)

$$\left(\frac{1}{i}\right)^n \int^t H\langle t'\rangle dt' \int^{t''} H\langle t''\rangle dt'' \int^{t'''} \dots \int^{t_{n-1}} H\langle t_n\rangle dt_n. \tag{a, 1}$$

In our system each $H\langle t \rangle$ is of the form, according to (25),

$$\begin{aligned} & \sum_{\mathbf{k}} \sum_{\lambda} e^{+i\mathbf{e}_{\mathbf{k}} \cdot \mathbf{r}} a_{\mathbf{k}\lambda}^* \{ e^{-i(\mathbf{k}\mathbf{X})} e^{i(E_{\mathbf{p}-\mathbf{k}} - E_{\mathbf{p}})\rho_3 t} \cdot \langle U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \rangle_{\text{even}} \\ & + e^{-i(\mathbf{k}\mathbf{X})} e^{i(E_{\mathbf{p}-\mathbf{k}} + E_{\mathbf{p}})\rho_3 t} \cdot \langle U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \rangle_{\text{odd}} \} \\ & + \text{hermitian conjugate.} \end{aligned} \quad (\text{a}, 2)$$

Here $\langle \rangle_{\text{even}}$, $\langle \rangle_{\text{odd}}$ denotes respectively the even and odd parts of the expressions inside the brackets. On the other hand, in the quantized theory, we should substitute the following expression for $H\langle t \rangle$ in place of the above expression into (a, 1)

$$\begin{aligned} & \sum_{\mathbf{k}} \sum_{\lambda} \sum_{\mu, \mu'=1,2} a_{\mathbf{k}\lambda}^* \{ b_{+\mu}^*(\mathbf{p}-\mathbf{k}) b_{+\mu'}(\mathbf{p}) e^{i(\mathbf{e}_{\mathbf{k}} + E_{\mathbf{p}-\mathbf{k}} - E_{\mathbf{p}})\rho_3 t} \phi_{+\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{+\mu'}(\mathbf{p}) \\ & + b_{-\mu}^*(-\mathbf{p}+\mathbf{k}) b_{-\mu'}^*(-\mathbf{p}) e^{i(\mathbf{e}_{\mathbf{k}} - E_{\mathbf{p}-\mathbf{k}} + E_{\mathbf{p}})\rho_3 t} \phi_{-\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{-\mu'}(\mathbf{p}) \\ & + b_{+\mu}^*(\mathbf{p}-\mathbf{k}) b_{-\mu'}^*(\mathbf{p}) e^{i(\mathbf{e}_{\mathbf{k}} + E_{\mathbf{p}-\mathbf{k}} + E_{\mathbf{p}})\rho_3 t} \phi_{+\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{-\mu'}(\mathbf{p}) \\ & + b_{-\mu}^*(-\mathbf{p}+\mathbf{k}) b_{+\mu'}(\mathbf{p}) e^{i(\mathbf{e}_{\mathbf{k}} - E_{\mathbf{p}-\mathbf{k}} - E_{\mathbf{p}})\rho_3 t} \phi_{-\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{+\mu'}(\mathbf{p}) \} \\ & + \text{hermitian conjugate,} \end{aligned} \quad (\text{a}, 3)$$

where b^* and b denote the creation and annihilation operators of fermions, and $\phi(\mathbf{p})$ and $\phi^*(\mathbf{p})$ are eigen-solutions of Dirac equation with momentum and its complex conjugate. μ is the spin index, and \pm denotes the sign of charge. By comparing (a, 2) with (a, 3), we shall find the term by term correspondence between two cases. For there is the following correspondence of the expressions in both cases.

$$\begin{aligned} \sum_{\mu, \mu'=1,2} \phi_{+\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{+\mu'}(\mathbf{p}) & \rightarrow \frac{1}{2} (1 + \rho_3) \cdot U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 + \rho_3), \\ \sum \phi_{-\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{-\mu'}(\mathbf{p}) & \rightarrow \frac{1}{2} (1 - \rho_3) \cdot U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 - \rho_3), \\ \sum \phi_{+\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{-\mu'}(\mathbf{p}) & \rightarrow \frac{1}{2} (1 + \rho_3) \cdot U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 - \rho_3), \\ \sum \phi_{-\mu}^*(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle \phi_{+\mu'}(\mathbf{p}) & \rightarrow \frac{1}{2} (1 - \rho_3) \cdot U^{-1}(\mathbf{p}-\mathbf{k}) \langle Intr. \rangle U(\mathbf{p}) \cdot \frac{1}{2} (1 + \rho_3). \end{aligned} \quad (\text{a}, 4)$$

Thus the correspondence of the transition elements of both theory is checked with respect to the elementary processes described by the interaction Hamiltonian: This statement is also true in general higher order processes, as is clear from the following reasoning. Here we neglect all vacuum effects, and the train of intermediate states in quantized theory which appear in (a, 1) is continuous, in other words, every particle generated in the preceding step must vanish in the next step. In the unquantized system we do not need to say about this continuity

of intermediate states, but we notice that here the distinction between process with and without pair formation is automatically established by virtue of ρ_3 in the exponential time factor before and after integration over time. The spatial exponential factor takes automatically the momentum conservation in higher order processes into account.

However, we must take care as regards to the sign of a train of terms, each of which is given by (a, 3). For positive energy states the creation operators stand on the left-side, while the annihilation operators on the right. For negative energy states the situation is just reversed. So, if a certain intermediate state is one of positive energy, we have a factor $b_+(\mathbf{p})b_+^*(\mathbf{p})$, which can be identified to unity in the one-particle theory. But for a negative intermediate state we shall have a factor $b_-^*(\mathbf{p})b_-(\mathbf{p}) = 1 - b_-(\mathbf{p})b_-^*(\mathbf{p})$, which should be put -1 ; because we should discard all effects due to the negative energy sea. This discrimination of sign of intermediate states can be established in our system of calculation by inserting ρ_3 into the middle of two transition operators of adjacent step.

Appendix II

We have purposely separated the term $\langle recoil \rangle$ in the discussion of § 3. The following results will be useful in calculations of any kind of interaction type. By suitable change of the notation, one will readily find the essential part of the term $\langle recoil \rangle U^{-1} \langle interactor \rangle U$ for any kind of interaction. Here we shall abbreviate as,

$$\mu_p = m + E_p = m + \sqrt{m^2 + \mathbf{p}^2},$$

$$\mu_q = m + E_q = m + \sqrt{m^2 + \mathbf{q}^2}.$$

$$2[E_q E_q \mu_p \mu_q]^{1/2} U^{-1}(\mathbf{q}) \cdot \mathbf{1} \cdot U(\mathbf{p}) \\ = \{ \mu_p \mu_q + (\mathbf{p} \cdot \mathbf{q}) + i(\mathbf{q} \times \mathbf{p}) \cdot \boldsymbol{\sigma} \} + i\rho_2 \{ (q\sigma)\mu_q - (p\sigma)\mu_q \}, \quad (\text{b, 1})$$

$$2[E_p E_q \mu_p \mu_q]^{1/2} U^1(\mathbf{q}) \cdot \rho_2 \cdot U(\mathbf{p}) \\ = \rho_2 \{ \mu_p \mu_q + (\mathbf{q} \cdot \mathbf{p}) + i([\mathbf{q} \times \mathbf{p}] \cdot \boldsymbol{\sigma}) \} + i \{ (q\sigma)\mu_p - (p\sigma)\mu_q \}, \quad (\text{b, 2})$$

$$2[E_p E_q \mu_p \mu_q]^{1/2} U^{-1}(\mathbf{q}) \cdot \rho_1 \cdot U(\mathbf{p}) \\ = \rho_1 \{ \mu_p \mu_q - (\mathbf{q} \cdot \mathbf{p}) - i([\mathbf{q} \times \mathbf{p}] \cdot \boldsymbol{\sigma}) \} + \rho_3 \{ (q\sigma)\mu_p + (p\sigma)\mu_q \}, \quad (\text{b, 3})$$

$$2[E_p E_q \mu_p \mu_q]^{1/2} \cdot U^{-1}(\mathbf{q}) \cdot \rho_3 \cdot U(\mathbf{p}) \\ = \rho_3 \{ \mu_p \mu_q - (\mathbf{q} \cdot \mathbf{p}) - i([\mathbf{q} \times \mathbf{p}] \cdot \boldsymbol{\sigma}) \} - \rho_1 \{ (\mathbf{q} \cdot \boldsymbol{\sigma})\mu_p + (\mathbf{p} \cdot \boldsymbol{\sigma})\mu_q \}, \quad (\text{b, 4})$$

$$2[E_p E_q \mu_p \mu_q]^{1/2} \cdot U^{-1}(\mathbf{q}) \cdot \boldsymbol{\sigma} \cdot U(\mathbf{p}) \\ = \{ \mu_p \mu_q \boldsymbol{\sigma} + (\mathbf{p}\boldsymbol{\sigma})\mathbf{q} + (q\boldsymbol{\sigma})\mathbf{p} - (\mathbf{p} \cdot \mathbf{q})\boldsymbol{\sigma} - i[\mathbf{q} \times \mathbf{p}] \} \\ + i\rho_2 \{ \mu_p \mathbf{q} - \mu_q \mathbf{p} - i\mu_q [\mathbf{p} \times \boldsymbol{\sigma}] - i\mu_p [\mathbf{q} \times \boldsymbol{\sigma}] \}, \quad (\text{b, 5})$$

$$\begin{aligned}
 & 2[E_p E_q \mu_p \mu_q]^{1/2} \cdot U^{-1}(q) \cdot \rho_2 \sigma \cdot U(p) \\
 &= \rho_2 \{ \mu_p \mu_q \sigma + (p\sigma)q + (q\sigma)p - (pq)\sigma - i[q \times p] \} \\
 & \quad + i \{ \mu_p q - \mu_q p - i\mu_p [q \times \sigma] - i\mu_q [p \times \sigma] \}, \quad (b, 6)
 \end{aligned}$$

$$\begin{aligned}
 & 2[E_p E_q \mu_p \mu_q]^{1/2} \cdot U^{-1}(q) \cdot \rho_1 \sigma \cdot U(p) \\
 &= \rho_1 \{ \mu_p \mu_q \sigma - (p\sigma)q - (q\sigma)p + (pq)\sigma + i[q \times p] \} \\
 & \quad + \rho_3 \{ \mu_p q + \mu_q p - i\mu_p [q \times \sigma] + i\mu_q [p \times \sigma] \}, \quad (b, 7)
 \end{aligned}$$

$$\begin{aligned}
 & 2[E_p E_q \mu_p \mu_q]^{1/2} \cdot U^{-1}(q) \cdot \rho_3 \sigma \cdot U(p) \\
 &= \rho_3 \{ \mu_p \mu_q \sigma - (p\sigma)q - (q\sigma)p + (pq)\sigma + i[q \times p] \} \\
 & \quad - \rho_1 \{ \mu_p q + \mu_q p - i\mu_p [q \times \sigma] + i\mu_q [p \times \sigma] \}, \quad (b, 8)
 \end{aligned}$$

where

$$U(p) = \exp \left\{ -\frac{i}{2} \rho_2 \frac{(p\sigma)}{|p|} \tan^{-1} \left(\frac{|p|}{m} \right) \right\} = (2\mu_p \cdot E_p)^{-1/2} \{ \mu_p - i\rho_2 (p\sigma) \},$$

$$U^{-1}(q) = \exp \left\{ +\frac{i}{2} \rho_2 \frac{(q\sigma)}{|q|} \tan^{-1} \left(\frac{|q|}{m} \right) \right\} = (2\mu_q \cdot E_q)^{-1/2} \{ \mu_q + i\rho_2 (q\sigma) \}.$$

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On the Matrix Elements of the Beta Decay¹⁾Mituo TAKETANI, *Tokyo*

Seitaro NAKAMURA, Ken-ichi ONO and Minoru UMEZAWA

Department of Physics, University of Tokyo

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f values of the β decay for forbidden transitions are classified according to the experimental spectra. They are found to fall into distinct groups, if the correct formula for the forbidden probabilities of Fermi theory worked out by Nakamura, Shima and Kobayasi are adopted. The possible selection rules, and the conditions for the uniqueness of the matrix elements adopted, are also discussed. From the results, interaction types of the Fermi theory could be limited to some extent.

§ 1. β -ray spectra in the case of the forbidden transitions

Recently the analysis of β -ray spectra has made great advances, undoubtedly due to the good use of the thinnest sources and the improved availability of many radio-active isotopes. While we have reasons to believe that the Fermi theory of β -decay is well founded in the case of allowed transitions, both with regard to the spectra and the selection rules, the check of the theory seems not to be finished in the cases of the forbidden transitions. In addition to the encouraging fact that Konopinski and Uhlenbeck²⁾ have once succeeded in reproducing the so-called $K-U$ type spectrum with a specific second forbidden formula of the Fermi theory, the discovery of shapes corresponding to their predictions on the basis of some 1st forbidden formulae speaks strongly in favor of the Fermi theory. Actually, as was first confirmed by L.M. Langer and H.C. Price³⁾ with the spectrum of Y^{91} , the forbidden correction factor, the so-called ' a ' type, of Konopinski and Uhlenbeck's theory finds its direct counterpart in nature, not the linear combination of more than two correction factors such as in the case of RaE .

Nevertheless, it can not be overlooked that beta emitters have been discovered which have allowed spectra and forbidden life times. Such instances have recently been worked out with much more reliability. Konopinski and Uhlenbeck were the first to point out that the allowed shape of Na^{24} might correspond to the beta decay exclusively induced by the matrix elements of $\{a$ or $\{\gamma_5$, which happen to be predominant among various matrix elements of the first forbidden formula. This interpretation can be extended to the cases of allowed spectra with the first forbidden life times. On the other hand, since the second forbidden formulas have no matrix elements corresponding to the allowed shape, it may be plausible, for the cases of beta emitters which have longer life times, to attempt

to derive the required spectra, i.e., the allowed shape, by suitably adjusting more than two matrix elements of the second forbidden transitions predicted from any of the five Fermi interactions. Unfortunately, this attempt was found to be hopeless in the case of P^{32} . If we try to explain P^{32} by means of the selection rule $\Delta J = \pm 2$ in the tensor interaction of the second forbidden transition, there is no alternative but to adopt the following correction factor in formula (30c) of Konopinski's¹¹⁾ article for β -ray spectra:

$$\begin{aligned} C_{2T} \sim & \sum_{ij} |A_{ij}|^2 3a \\ & + \sum_{ij} |T_{ij}|^2 \frac{1}{12} [3D_+ - C] \\ & - \left[\sum_{ij} T_{ij} A_{ij}^* + \text{comp. conj.} \right] \frac{1}{2} E \end{aligned}$$

where a , C , D_+ , E , are the complicated functions of W_0 , W , Z defined by (31), (29) in Konopinski's²⁾ article.

Generally speaking, owing to the rapid decrease in $3D_+ - C$ near the low energy end, we cannot expect the allowed shape by combining the adjustable correction factors involved in the tensor interactions. We meet with the same difficulties in the other interactions and also in the cases of other elements.

In order to settle these difficulties in the Fermi theory, some alternative such as the combination of the Fermi theory and the Yukawa theory, or the more speculative participation of the mu-meson in the beta decay of the nucleus have been suggested by one of the authors. At the present stage however, it

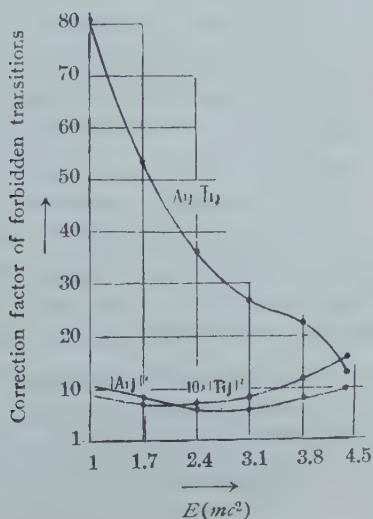


Fig. 1 shows values of a , $3D_+ - C$, $1/2E$ by substituting the values of W_0 , Z for the β -decay of P^{32} plotted versus W . We cannot cancel out the steep decrease of the correction factor corresponding to $|T_{12}|^2$, by linearly combining the other two, without giving an unreasonably large value to $|A_{12}|/|T_{12}|$.

may be necessary to arrange an obvious perspective of the data concerning the spectra, matrix elements, and the selection rules, before making a next step. Possibilities could not be passed over that the present difficulties will be overcome by developing the Fermi theory in the detail.

Accordingly, we have analysed the data available to us in the following way: first, a classification of the shapes of the beta ray spectra and the possible assignments of the matrix elements was made, secondly, the absolute values of these matrix elements or the specific life times, i.e., $\bar{f}t$ values, were computed by the exact formula of the probabilities predicted by the Fermi theory in the case of the forbidden transitions. The formulas were mainly worked out by Nakamura, Shima and Kobayasi.⁴⁾ (See Appendix 2). We shall denote these specific life times by $\bar{f}t$, distinguishing them from those evaluated conventionally from the formula of the allowed probability. The differences between the values of f and \bar{f} may amount to as much as 10^3 , since these differences are very marked in the cases of low maximum energy.

§ 2. Matrix elements

1. Uniqueness of the matrix elements

The above-mentioned classification of the data concerning the spectra and $\bar{f}t$ values enables us to discuss the matrix elements adopted in each group.

In the first place, it must be answered whether the selected matrix elements could be enough to represent the actual transformation of the nucleus without ambiguity. The existing analysis was already sufficient to claim that the assigned matrix elements are unique and sufficient except in the case of the second group. In this case, in order to reproduce the allowed shape from the forbidden formula, the question what matrix elements should really contribute in the transition is very difficult to answer. According to our analysis, these spectra could not be expressed by any linear combinations of available matrix elements.

In the following we will assume that they are due to the single matrix, $\int \alpha$ or $\int \gamma_5$, and we first explain the reasons why the values $\bar{f}t$ vary between 0.85×10^6 and 5.2×10^7 , the interval of which covers as much as one to sixty. (See Table 1).

A peculiarity of the nuclear wave functions might be mentioned for their explanation. If we confine our attention to the spin and the parity of nuclei, it may be natural to assume that they are partly due to $\int \alpha$ and partly to $\int \gamma_5$, the former takes part in the beta decay with the smaller $\bar{f}t$ values, the latter with the larger. This assumption is almost consistent with the consideration of the relative values of $\int \alpha$ and $\int \gamma_5$. (See Section II, (1)) The main difficulty lies in the fact that the formula includes other matrix elements which have the same selection rules as those of $\int \alpha$ or $\int \gamma_5$, thus destroying the uniqueness of the latters. (Thorough analysis of them appears in Feenberg's article.¹⁾)

A solution of this difficulty may be to analyse the conventional selection

Table I. Beta decay in the case of the forbidden transitions

Elements	$W_0(\text{mc}^2)$	$t(\text{sec})$	Types of Spectra	$f t$	$\bar{f} t$ corrected	Selection Rules and possible Matrix Elements
1st Group						
Y^{90}	5.29	2.2×10^5	a 6)	1.8×10^8	3×10^7	$\Delta J = \pm 2$, yes B_{ij}
K_r^{85}	2.17	2.96×10^8	a		4.1×10^7	
A^{39}	2.108	4.5×10^3	a ✓		4.5×10^7	
Cl^{38}	10.43	2.3×10^3	a 7)	4.0×10^7	5×10^7	
S_r^{89}	3.88	4.7×10^6	a 8)	3.8×10^8	6.2×10^7	
S_r^{90}	2.05	$8. \times 10^8$	a 6)	1.5×10^9	$7. \times 10^7$	
K_r^{85}	2.17	2.96×10^8	a		4.1×10^7	
Y^{91}	4.01	5.3×10^6	a 9)	5.4×10^8	8.7×10^7	
K^{42}	8.06	4.46×10^4	a	1.1×10^3	1.32×10^8	
2nd Group						
Ca^{45}	1.5	1.3×10^7	1 10)	3.0×10^5	"	$\int a$
Au^{198}	2.88	2.3×10^5	1 9)	8.5×10^5	"	
Na^{24}	3.74	5.3×10^4	1 11)	9.6×10^5	"	
P_m^{147}	1.44	1.2×10^3	1 12)	1.8×10^6	"	
V^{48}	2.42	1.4×10^6	1 13)	1.8×10^6	"	
S_m^{151}	1.15	6.3×10^8	1	2.0×10^6	"	
P_r^{143}	2.83	1.2×10^6	1 14)	3.9×10^6	"	
Z_r^{97}	4.75	6.1×10^4	1	3.5×10^6	"	
Re^{186}	3.1	3.2×10^6	1 15)	1.86×10^7	"	$\int \gamma_5$
Na^{22}	2.12	8.2×10^7	1 13)	3.9×10^7	"	
P^{32}	4.37	1.2×10^6	1 9)	4.96×10^7	"	
T_m^{170}	2.94	1.2×10^7	1 16)	5.2×10^7	"	
Cu^{62}	6.7	3.3×10^4	1	2×10^7	"	
T_b^{160}	2.7	6.5×10^7	1	$1.7 \sim 3.4 \times 10^7$	"	
Ag^{110}	2.25 (2.04)	1.9×10^7	1	1×10^7 (6×10^6)	"	
3rd Group						$\Delta J = \pm 2$, no
RaE	3.3	4.3×10^5	linear comb. 9)	9.8×10^7	1.3×10^8	A_{ij}, T_{ij}
4th Group						$\Delta J = \pm 3$, no
Be^{10}	2.08	10^{13}	c 17)	6×10^{12}	6×10^9	S_{ijk}
5th Group						A_{ij}, T_{ij}
Cl^{36}	2.4	3×10^{13}	linear comb. if c 19)	4×10^{13} 4×10^{13}	5×10^8 2×10^{11}	S_{ijk}

Note 1. It was customarily pointed out that the empirical transition classification breaks down in the 1st group, because of very large $f t$ values. But this opinion seems to be very doubtful, since they used the allowed formula for f function in the case of forbidden transitions. If we improve it by using the correct formula of forbidden transitions, which will be indicated by the notation \bar{f} , the results of $\bar{f} t$ in the 1st group can be classified in a distinct group of the reasonable order (Compare $f t$ given by the Table III of Feenberg and Hammack³), with our $\bar{f} t$ in the 1st group of our table I).

Note 2. The column "Types of Spectra". Type a , or c , means the β -ray spectra with experimental correction factors denoted respectively by the functions mentioned below:

$$a = \frac{1}{36} [(W_0 - W)^2 + W^2 - 1],$$

$$c = \frac{1}{90} [3(W^2 - 1)^2 + 3(W_0 - W)^4 + 10(W^2 - 1)(W_0 - W)^2].$$

1 means allowed type spectra. Linear comb. means that the type of spectra can be reproduced by the suitable linear combination of the respective matrix elements, in C_{2T} in this case.

rules more thoroughly. If the spin-orbit couplings could be assumed to be very weak,¹⁸⁾ as in the case of the deuteron problem where the D -state probability in the ground state of the deuteron is about 4%, we have the following relations.

$$J = I + L$$

where

$$I = \sum \text{Nucleon Spin},$$

$$L = \sum \text{Nucleon Orbital Moment}.$$

If we could further assume that, for the stationary state of any nucleus, practically no exchange of angular momentum between I and L take place, that is, each separately is the constant of motion, the selection rules for the beta decay not

M	A	
$\int a$	$\Delta I = 0, L = 0 \longleftrightarrow 0$	Yes
$\int \gamma_5$	$I = 0 \longleftrightarrow 0, \Delta L = 0 \longleftrightarrow 0$	Yes
B_{ij}	$\Delta J = \pm 2$	Yes
S_{ijk}	$\Delta J = \pm 3$	No

only for J but also those for I and L hold independently. For instance, $\int \gamma_5$ not only forbids the change in J values but also requires I and L to be conserved, while the term $\int \sigma \cdot r$ allows the change in I and L by ± 1 although it forbids the change in J . (See Table II) We have therefore the following

table II for the matrix elements participating in the first and the second forbidden transition.

According to this table II, we can uniquely conclude: All terms except M vanishes if the conditions A are fulfilled:

Since the selection rules for the above terms are thus determined, we can definitely know the validity of our predictions in comparing this result with the informations about the actual nuclear transitions. Most crucial test of our theory will be gained in the spin states of the second group. If it will not be established that, for instance, the beta decay of P^{32} is followed by the selection rule $\Delta I = 0 \longleftrightarrow 0, \Delta L = 0 \longleftrightarrow 0$, our consideration should be withdrawn.

II. Relative values of the specific life times and the corresponding matrix elements

It was shown in the table I that the values ft evaluated in the first group are greater than those in the second group by the factor ~ 10 . On the other hand, we know that B_{ij} participates in the beta decay in the first group and $\int a$

Table II. Selection rules in the case of very weak spin-orbit coupling
The first forbidden matrix elements: (See Appendix 1)

Matrix Element	$\int \gamma_5$	$\int \mathbf{r}$	$\int \mathbf{a}$	$\int \boldsymbol{\sigma} \cdot \mathbf{r}$	$\int \boldsymbol{\sigma} \times \mathbf{r}$	B_{ij}
I	0	0	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)
L	0	0, ± 1 (no 0-0)	0	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)
J	0	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0	0, ± 1 (no 0-0)	0, $\pm 1, \pm 2$ (no 0-0, 1/2-1/2, 0-1)
Parity Change	Yes	Yes	Yes	Yes	Yes	Yes

The second forbidden matrix elements:

Matrix Element	R_{ij}	$R_{ij}\gamma_5$	A_{ij}	$\int \mathbf{a} \times \mathbf{r}$	$\int \mathbf{a} \cdot \mathbf{r}$	T_{ij}	S_{ijkl}
AI	0		0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)
AL	0, $\pm 1, \pm 2$ (no 0-0, 0-1)		0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, ± 1 (no 0-0)	0, $\pm 1, \pm 2$ (no 0-0, 0-1)	0, $\pm 1, \pm 2$ (no 0-0, 0-1)
AJ	0, $\pm 1, \pm 2$ (no 0-0, 0-1)		0, $\pm 1, \pm 2$ (no 0-0, 0-1)	0, ± 1 (no 0-0)	0	0, $\pm 1, \pm 2$ (no 0-0, 0-1)	0, $\pm 1, \pm 2, \pm 3$ (no 0-0, 0-1, 0-2)
Parity Change	No	Yes	No	No	No	No	No

in the second group. That is to say, B_{ij} is the first order term with regard to both $\boldsymbol{\sigma}$ and \mathbf{r} , while $\int \mathbf{a}$ is of the first order in \mathbf{a} and 0-th order in \mathbf{r} . With regard to the fourth group, the condition $A_{ij}/T_{ij} = -5.8$ was needed for reproducing the measured spectrum of RaE. Since $A_{ij} = \int a_i x_j + a_j x_i - \frac{2}{3} \delta_{ij} \mathbf{a} \cdot \mathbf{r}$ is of the first order both in \mathbf{a} and in \mathbf{r} , $T_{ij} = \{x_i (\boldsymbol{\sigma} \times \mathbf{r})_j + x_j (\boldsymbol{\sigma} \times \mathbf{r})_i\}$ of the first order in $\boldsymbol{\sigma}$ and of the second order in \mathbf{r} , we can roughly estimate the ratio of various matrix elements as follows:

$$|\int \mathbf{a}| : |B_{ij}| \doteq |A_{ij}| : |T_{ij}| = 5.8 : 1. \quad (1)$$

This ratio agrees well with that of the values $\bar{f}t$ in the first group ($\sim 10^7$) to that of the second group ($\sim 10^6$).

Next, it will be mentioned that there is a difference in the values $\bar{f}t$ between the second and the third groups by the factor of about 10. It is easily shown that the following relation holds if we assume the free particle model for the nucleon:

$$|\int \mathbf{a}| : |\int \gamma_5| \doteq (P_1 + P_2) : (P_1 - P_2)$$

where P_1 and P_2 represent the magnitude of the momenta before and after the

beta decay. That is to say,

$$(P_1 + P_2)^2 \sim \text{Kinetic energy of the nucleon inside the nucleus} \\ (\text{a few } 10 \text{ Mev.}).$$

$$(P_1 - P_2)^2 \sim \text{The energy emitted inside the nucleus} \\ (\text{a few MeV.}).$$

Therefore we have the following order-estimation :

$$|\int \alpha|^2 : |\int \gamma_5|^2 = (P_1 + P_2)^2 : (P_1 - P_2)^2 = 10 : 1. \quad (2)$$

As for the second forbidden transition, the magnitudes of the values $\bar{f}t$ in the 3rd and the 4th groups may not be unreasonable compared with those of the first forbidden groups, if we take into account the above considerations about the order of the matrix elements, A_{ij} , T_{ij} and S_{ijk} .

The analysis of the beta decay of Cl^{36} was precisely studied by the Columbia University Group¹⁹⁾, and the spin of Cl^{36} was measured²⁰⁾ to be 2 and that of A^{36} is reasonably assumed to be 0. C. S. Wu and L. Feldman have shown that the measured spectrum can be reproduced by the vector or tensor interactions with $A_{ij} : T_{ij} = \sqrt{18} : 1$. It is very interesting that if we assign the beta decay of Cl^{36} to the matrix elements S_{ijk} , $\bar{f}t$ becomes unreasonably small as shown in Table I, while we obtain $\bar{f}t$ of a reasonable order in the case of the linear combination theory mentioned above.

§ 3. Discussions

We shall determine the necessary conditions for the interaction types of the Fermi theory, in the light of the results obtained above. To derive the required matrix elements in each group, the following interaction types should be needed :

Table III.

		Matrix	Interaction	$\bar{f}t$
Group 1		B_{ij}	tensor or pseudovector	$10^7 \sim 10^8$
Group 2	(i)	$\int a$	tensor or vector	$10^5 \sim 10^6$
	(ii)	$\int \gamma_5$	pseudoscalar or pseudovector	$10^6 \sim 10^7$
Group 3, 5		A_{ij}, T_{ij}	tensor or vector	$10^8 \sim 10^9$
Group 4		S_{ijk}	tensor or pseudovector	$10^9 \sim 10^{10}$

It is evident that a single type of interaction can not explain the present data consistently. Therefore, it is not surprising to assume that besides the tensor interaction usually adopted for the allowed transition, the pseudoscalar or the

pseudovector interaction should be accepted simultaneously. It is interesting to note that each ft value of group 2, which have the allowed spectrum, never exceed that of the group 1, which have the first forbidden 'a' type spectrum. In view of this fact, the above assumption seems to be able to answer the problem of (forbidden life time and allowed shape.)

Appendix 1. For convenience's sake the explicit forms of the matrix elements appearing in the β -decay of the forbidden transitions will be quoted below from Konopinski's article

$$B_{ij} = |\{ \sigma_i x_j + \sigma_j x_i - \frac{2}{3} \delta_{ij} \sigma \cdot r \}|, \quad R_{ij} = |\{ x_i x_j - \frac{1}{3} \delta_{ij} r^2 \}|,$$

$$A_{ij} = |\{ \sigma_i x_j + \sigma_j x_i - \frac{2}{3} \delta_{ij} \sigma \cdot r \}|, \quad T_{ij} = |\{ x_i (\sigma \times r)_j + x_j (\sigma \times r)_i \}|,$$

$$S_{ijk} = |\{ \sigma_i x_j x_k - \frac{2}{5} \delta_{ij} \sigma_k - \frac{4}{5} \delta_{ik} \sigma_j \sigma \cdot r \}|.$$

Appendix 2. The life time for forbidden transitions when $F(Z, W) = 1$ was calculated by Nakamura, Shima and Kobayasi,⁴⁾ by using Konopinski's formula. Some data necessary for the analysis in Table I. are quoted because of its difficulty of access.

$$\frac{1}{\tau_{1T}} = \frac{G^2}{\pi^2} \sum_{ij} |B_{ij}|^2 I_8(W_0^6) + \text{another terms}, \quad (4)$$

$$\begin{aligned} \frac{1}{\tau_{2T}} = \frac{G^2}{\pi^2} & \left[\sum |S_{ijk}|^2 \left\{ \frac{1}{72} I_{15}(W_0^8) \right. \right. \\ & + \{ \sum |T_{ij}|^2 \} \frac{1}{12} \left\{ I_{17}(W_0^8) \pm \frac{aZ}{30\rho} I_{12}(W_0^7) + \frac{a^2 Z^2}{16\rho^2} I_9(W_0^6) \right\} \\ & + \{ \sum |A_{ij}|^2 \} I_8(W_0^6) \\ & \left. \left. - \{ \sum T_{ij} A_{ij}^* + \sum T_{ij}^* A_{ij} \} \frac{1}{2} \left\{ I_{11}(W_0^7) \pm \frac{aZ}{24\rho} I_{10}(W_0^6) \right\} \right] \quad (5) \end{aligned}$$

$$I_8(W_0^6) = \left(\frac{W_0^6}{2 \cdot 5 \cdot 7 \cdot 9} - \frac{59 W_0^4}{5 \cdot 7 \cdot 9 \cdot 16} - \frac{407 W_0^2}{5 \cdot 7 \cdot 9 \cdot 32} - \frac{1}{2 \cdot 5 \cdot 7 \cdot 9} \right) R + \left(\frac{W_0^3}{3 \cdot 8} + \frac{W_0}{3 \cdot 32} \right) L,$$

$$I_9(W_0^6) = \left(\frac{W_0^6}{3 \cdot 7} - \frac{179 W_0^4}{3 \cdot 4 \cdot 5 \cdot 7} - \frac{1889 W_0^2}{3 \cdot 5 \cdot 7 \cdot 8} - \frac{26}{3 \cdot 5 \cdot 7} \right) R + \left(2 W_0^3 + \frac{7 W_0}{8} \right) L,$$

$$I_{10}(W_0^6) = \left(\frac{W_0^6}{5 \cdot 7} - \frac{33 W_0^4}{4 \cdot 5 \cdot 7} - \frac{901 W_0^2}{3 \cdot 5 \cdot 7 \cdot 8} - \frac{2}{3 \cdot 7} \right) R + \left(W_0^3 + \frac{3 W_0}{8} \right) L,$$

$$I_{11}(W_0^7) = \left(\frac{W_0^7}{4 \cdot 5 \cdot 7 \cdot 9} - \frac{W_0^5}{2 \cdot 7 \cdot 9} - \frac{247 W_0^3}{5 \cdot 7 \cdot 9 \cdot 16} - \frac{2 W_0}{5 \cdot 7 \cdot 9} \right) R + \left(\frac{W_0^4}{3 \cdot 8} + \frac{W_0^2}{3 \cdot 16} \right) L,$$

$$\begin{aligned}
 I_{12}(W_0^7) &= \left(\frac{W_0^7}{4.7} - \frac{67W_0^6}{7.8} - \frac{21611W_0^3}{7.32} - \frac{1175W_0}{7.64} \right) R + \left(\frac{15W_0^4}{2} + \frac{45W_0^3}{8} + \frac{15}{64} \right) L, \\
 I_{13}(W_0^7) &= \left(\frac{W_0^{7.8}}{2.7.81} - \frac{103W_0^6}{4.5.7.81} - \frac{1139W_0^4}{4.7.25.27} - \frac{134W_0^2}{7.25.81} \right) R + \left(\frac{W_0^5}{4.5} + \frac{W_0^3}{4.9} \right) L, \\
 I_{17}(W_0^7) &= \left(\frac{4W_0^7}{5.7.81} - \frac{11W_0^6}{4.5.27} - \frac{4831W_0^4}{7.8.25.27} - \frac{398W_0^2}{7.25.81} + \frac{2}{25.81} \right) R \\
 &\quad + \left(\frac{W_0^5}{2.5} - \frac{W_0^3}{8.9} \right) L. \\
 R &= (W_0^2 - 1)^{1/2}, \quad L = \log \{ W_0 + (W_0^2 - 1)^{1/2} \}. \quad (6)
 \end{aligned}$$

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Note on Schwinger's Variational Method*

Tosio KATO

Department of Physics, University of Tokyo

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It is shown that one of the variational methods of Schwinger applied to the one-dimensional wave equation gives a lower bound of the absolute value of the scattering phase provided the potential is semi-definite and too strong, a condition satisfied for potentials proposed for the neutron-proton scattering. Further a simple modification of Schwinger's iteration procedure is proposed, which converges not only mathematically but also more rapidly than the original one. An upper bound of the absolute value of the phase is also shown to be obtained if the iteration method is carried out to the second approximation. Numerical examples are given in the case of a square-well potential, and very accurate upper and lower bounds of the phases are obtained.

§ 1. Introduction

Recently many attempts¹⁾ have been made to extend the variational methods to problems related to eigenfunctions belonging to the continuous spectrum. In particular, the scattering phase has been calculated successfully by these methods.¹⁾ They are similar to the usual Ritz variational method for computing eigenvalues of Hermitian operators, the main difference lying in the following points:

1. The stationary value is unique (so long as we are concerned with simple problems containing only one phase);

2. The stationary value is in general neither maximum nor minimum. Thus we do not know in general whether the value of the phase given by these methods is an upper or a lower bound to the correct one.

But there is an exception to this general rule. *One of the methods of Schwinger²⁾ is capable of giving either an upper or a lower bound to the correct phase under rather general conditions.* In the present note we consider for simplicity the phase of S -scattering by a short-range potential, and in the first part we shall show that the Schwinger method gives a lower bound of the *absolute value* of the phase provided (1) the potential is semi-definite (everywhere non-negative or non-positive) and (2) it is "weak" in the sense that the correct phase is less than π in absolute value. These conditions are fulfilled for all potentials so far proposed for the neutron-proton scattering. This is a remarkable property of the Schwinger method, for it seems that other methods hitherto proposed do not give an upper or a lower bound of the phase even under the restrictions just stated.

* Though the manuscript of this paper was received last summer, its publication has been delayed to this number through the carelessness in business affairs.

It will be noted that the same result also holds in more general cases (higher angular momentum, proton-proton scattering).

Schwinger has also given a systematic method for successively improving the trial wave function.²⁾ If his method is applied to the case where the above conditions are satisfied, it should give a monotonic sequence of values of the phase. We have attempted to prove this and the convergence of the sequence to the correct value, but unfortunately we have not been successful. Instead we are lead to a modification of the method which not only proved to be convergent ultimately to the correct value but also converges more rapidly than the original method of Schwinger. When u_1, u_2 are respectively the first and second approximations of the wave function determined by the Schwinger iteration method, our modification consists in replacing u_2 by the best function u_2' which is expressible as a linear combination of u_1 and u_2 . It is important to note that there is no need of additional calculation in obtaining u_2' and the corresponding value of the phase, *so that u_2' can still be regarded as a second, and not a third, approximation*. Nevertheless a simple example of the square-well potential shows that the improvement is remarkable and u_2' gives a very accurate lower bound of the phase.

We shall also show that the Schwinger method is able to give an estimate of the *upper bound* of the absolute value of the phase too, provided the iteration method is carried out to the second approximation. It may be remarked that the method of obtaining this opposite bound is analogous to the estimation of lower bounds of eigenvalues of Hermitian operators.³⁾

A natural question which presents itself is whether there are other methods for estimating upper or lower bounds of the phase. The answer is affirmative, and it can be shown that we can derive formulae more general and accurate than Schwinger's, although it seems that his method is distinguished among them by its simplicity. In a subsequent paper^{3a)} we want to treat the problem from a wider point of view.

§ 2. Extremum property of the stationary value

We consider the radial wave equation for the S -scattering problem:

$$d^2u/dr^2 + k^2u + W(r)u = 0, \quad (1)$$

where k is the wave number and $W(r)$ is the potential multiplied by $-2m/\hbar^2$ (m =reduced mass). We assume that $|W(r)|$ is integrable at $r \rightarrow \infty$. The phase δ is given by $u(r) \sim \sin(kr + \delta)$, $r \rightarrow \infty$, where $u(r)$ is the solution of (1) satisfying the boundary condition $u(0)=0$. As is well known the phase δ is positive (negative) if $W(r) \geq 0$ (≤ 0) everywhere. More generally δ is a monotonically increasing function of $W(r)$.

According to the method of Schwinger,²⁾ the quantity $k \cot \delta$ is characterized as the unique stationary value of the functional

$$J[u] = \frac{\int_0^\infty W u^2 dr - \int_0^\infty W(r) u(r) dr \int_0^\infty G(r, r') W(r') u(r') dr'}{\left[k^{-1} \int_0^\infty W u \sin kr dr \right]^2}, \quad (2)$$

and the function which makes (2) stationary is the correct wave function. Here $G(r, r')$ is the Green function given by

$$G(r, r') = k^{-1} \sin kr_> \cos kr_<, \quad r_< = \text{Min}(r, r'), \quad r_> = \text{Max}(r, r'), \quad (3)$$

and satisfies the equation

$$\{ (d^2/dr^2) + k^2 \} G(r, r') = -\delta(r - r'). \quad (4)$$

To examine the extremum property of $J[u]$, let us first assume $W(r) \geq 0$ and set

$$W^{\frac{1}{2}} u = v, \quad k^{-1} W^{\frac{1}{2}} \sin kr = f. \quad (5)$$

Then (2) can be written as

$$J[u] = \frac{(v, v) - (v, Hv)}{(v, f)^2} = \frac{(v, Av)}{(v, f)^2} \quad (A = 1 - H), \quad (6)$$

where $(f, g) \equiv \int f^* g dr$ is the usual scalar product* and H is an integral operator with the kernel

$$H(r, r') = W(r)^{\frac{1}{2}} G(r, r') W(r')^{\frac{1}{2}}. \quad (7)$$

H and A are Hermitian operators.

Functionals of the abstract form (6) appear frequently in the variational methods in different fields of application, e.g. in the variational treatments of the collision problem by other authors¹⁾ as well as in the problem of the asymptotic neutron densities by Marshak and Davison.²⁾ So it would be worth while to derive in general terms a necessary and sufficient condition that the stationary value of the functional (6) be the *minimum*.

First we note that we may normalize v by

$$(v, f) = 1 \quad (8)$$

without loss of generality, for (6) is homogeneous in v . Then we have by (6) and (8)

$$J[u] = (v, Av), \quad \delta J = 2(\delta v, Av), \quad \delta^2 J = (\delta v, A\delta v), \quad (9)$$

$$(\delta v, f) = 0. \quad (10)$$

It follows from (9) and (10) that $\delta J = 0$ if

$$Av = \text{const} \cdot f \quad \text{or} \quad v = \text{const} \cdot A^{-1}f, \quad (11)$$

* In what follows we consider only real functions so that $(f, g) = (g, f)$.

and this gives the correct solution. The corresponding stationary value of J is

$$J_0 = J[A^{-1}f] = 1/(A^{-1}f, f). \quad (12)$$

The case where A^{-1} does not exist requires special treatment, but there is no essential difficulty and we may omit it.

Now we can state the lemma: *In order that the stationary value (12) of $J[u]$ be the minimum, it is necessary and sufficient that one of the following equivalent conditions are satisfied.**

(A) the quadratic form

$$(\varphi, A\varphi) + c(\varphi, f)^2 \quad (13)$$

is positive definite for some fixed constant c ;

(B) if we denote by P the integral operator** with the kernel

$$P(r, r') = f(r)f(r'), \quad (14)$$

all eigenvalues of the Hermitian operator $A + cP$ are positive for some fixed constant c .

The equivalence of (A) and (B) is clear, since (13) can be rewritten as $(\varphi, (A + cP)\varphi)$. Hence we have only to prove the lemma for (A). If (A) is satisfied, we have

$$(\varphi, A\varphi) > 0 \quad \text{if} \quad (\varphi, f) = 0 \quad \text{and} \quad \varphi \neq 0, \quad (15)$$

whence follows by (9) and (10) that $\delta^2 J > 0$ for $\delta v \neq 0$, and the stationary value (12) is the absolute minimum, showing that (A) is sufficient. Next suppose that (12) is a minimum of J at least in the neighborhood of the correct wave function v given by (11). Then we must have $\delta^2 J > 0$ at least for sufficiently small $\delta v \neq 0$. But as δv is restricted only by the condition (10), the relation (15) must hold. Now take an arbitrary φ and set

$$\varphi' = \varphi - (\varphi, f)(v, f)^{-1}v$$

with v given by (11). Then we have $(\varphi', f) = 0$ and hence by (15) $(\varphi', A\varphi') > 0$ provided $\varphi' \neq 0$, i.e., φ is not a constant multiple of v . On substituting φ' into this inequality and noting (11) and (12), we immediately obtain

$$(A\varphi, \varphi) - J_0(\varphi, f)^2 > 0.$$

Since this is true if φ is not a constant multiple of v , (A) holds if we take c such that $c > -J_0$, since $(v, f) = \text{const } (A^{-1}f, f) \neq 0$. Thus (A) is shown to be necessary.

* Similar condition has been derived also by Davison, reference 4). Roughly speaking, his condition is equivalent to our (B) with the special value $c=0$. Consequently, his condition is sufficient but not necessary. In fact, there are examples where J has the minimum and yet (B) is not satisfied with $c=0$ (e.g. the case $\pi/2 < \delta < \pi$ in our problem).

** In abstract terms, we should define P as a constant multiple of the projection operator on the one-dimensional subspace spanned by f .

Returning to our problem, we shall next show that the condition (B) is satisfied if the correct phase δ is less than π . For this purpose, suppose that $A+cP$ had an eigenvalue ≤ 0 or, what comes to the same thing, that $H-cP$ had an eigenvalue $\lambda \geq 1$ for every choice of c . If φ is the corresponding eigenfunction, we should have*

$$\lambda \varphi = (H - cP)\varphi. \quad (16)$$

Noting (7), (14), (5) and setting $\varphi = W^{\frac{1}{2}}\psi$, it follows** that

$$\lambda \psi(r) = \int_0^\infty G(r, r') W(r') \psi(r') dr' - ck^{-2} \left(\int_0^\infty W\psi \sin kr dr \right) \sin kr. \quad (17)$$

Operating with $\lambda^{-1}(d^2/dr^2 + k^2)$ on both sides, we obtain by (4)

$$d^2\psi/dr^2 + k^2\psi + \lambda^{-1}W(r)\psi = 0. \quad (18)$$

Thus ψ is a solution of the wave equation with the "potential" $\lambda^{-1}W$ instead of W . Also it follows from (17) and (3) that ψ satisfies the boundary condition $\psi(0)=0$ and that for $r \rightarrow \infty$

$$\begin{aligned} \psi(r) &\sim \lambda^{-1} \left(\int_0^\infty W\psi \sin kr dr \right) (k^{-1} \cos kr - ck^{-2} \sin kr) \\ &= \text{const.} \sin(kr + \theta), \quad \tan \theta = -c^{-1}k. \end{aligned} \quad (19)$$

If we take c very large, $\tan \theta$ is negative and very small. But θ must be positive because $\lambda^{-1}W \geq 0$ in the wave equation (18). Hence θ must be at least very near to π , and we should be able to make $\theta > \delta$ by taking c sufficiently large, for δ is less than π by hypothesis. But this is a contradiction, for the phase belonging to the wave equation (18) cannot be larger than δ since the "potential" $\lambda^{-1}W$ is not stronger than W if $\lambda \geq 1$.

Thus we have obtained the result that the stationary value of J is the absolute minimum provided $W(r) \geq 0$ and $\delta < \pi$, and the Schwinger method gives an upper bound of $k \cot \delta$, or a lower bound of δ , in such a case. These conditions are fulfilled in the case of neutron-proton scattering.

Quite in the same way we can show that the method gives a lower bound of $k \cot \delta$, or an upper bound of δ (lower bound of $|\delta|$) provided $W(r) \leq 0$ and $\delta > -\pi$.

It will be remarked that these are the only cases where the Schwinger method gives certainly an upper or a lower bound of $k \cot \delta$. The proof will be omitted here, though it is not difficult.

These results can immediately be extended to more general case with higher

* Since we have assumed that $|W(r)|$ is integrable, the operator $H-cP$ belongs to Hilbert-Schmidt type and has a pure point spectrum.⁵⁾

** In deriving (17) we have divided (16) by $W(r)^{\frac{1}{2}}$. Therefore (17) is at first correct only for such r with $W(r) > 0$. But as $\lambda \neq 0$ we can define $\psi(r)$ by (17) itself for all r without encountering any contradiction, and (17) is extended to all values of r .

angular momentum and containing Coulomb force. We have only to replace $G(r, r')$ by the Green function appropriate to the wave equation containing the centrifugal force and the Coulomb potential, $k^{-1} \sin kr$ by the corresponding regular solution of the wave equation, and δ by the *relative* phase shift. Then we have only to take as W the short range potential as above.

§ 3. Improvement of the successive approximation

Schwinger⁽⁶⁾ has given a method for systematically improving the trial function $u(r)$. When $u_1(r)$ is the starting trial wave function (first approximation), his method consists in taking

$$u_2(r) = \int_0^\infty G^*(r, r') W(r') u_1(r') dr' \quad (20)$$

with

$$G^*(r, r') = k \cot \delta \cdot k^{-1} \sin kr \cdot k^{-1} \sin kr' + G(r, r') \quad (21)$$

as the second approximation, where $k \cot \delta$ should be replaced by its first approximation $J[u_1]$.

Again let us assume $W(r) \geq 0$ and set $v_1 = W^{\frac{1}{2}} u_1$, $v_2 = W^{\frac{1}{2}} u_2$. Then we have by (20), (21), (6), (5) and (7)

$$v_2 = (v_1, Av_1)(v_1, f)^{-1} f + (1 - A)v_1. \quad (22)$$

We observe that

$$(v_1, v_2) = (v_1, v_1) \quad (23)$$

as is immediately seen by taking the scalar product of (22) with v_1 .

Now we consider the problem to make $J[u_2']$ stationary when u_2' is restricted to linear combinations of u_1 and u_2 . It is convenient to write the corresponding $v_2' = W^{\frac{1}{2}} u_2'$ in the form

$$W^{\frac{1}{2}} u_2' = v_2' = a_1(v_1, f)^{-1} v_1 + a_2(v_2, f)^{-1} v_2. \quad (24)$$

Then we have by (6)

$$J[u_2'] = \frac{a_1^2 J_1 + 2a_1 a_2 J_{12} + a_2^2 J_2}{(a_1 + a_2)^2}, \quad (25)$$

where we have set $J_1 = J[u_1]$, $J_2 = J[u_2]$ and

$$J_{12} = J[u_1, u_2] \equiv (Av_1, v_2) / (v_1, f)(v_2, f). \quad (26)$$

If we regard (25) as a function of the two-dimensional vector (a_1, a_2) , the result of (11) and (12) shows that the stationary value occurs for

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \text{const.} \begin{pmatrix} J_1 & J_{12} \\ J_{12} & J_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{const.} \begin{pmatrix} J_2 - J_{12} \\ J_1 - J_{12} \end{pmatrix}, \quad (27)$$

and that the stationary value is

$$\begin{aligned} J'_2 = J[u'_2] &= \left\{ (1, 1) \begin{pmatrix} J_1 & J_{12} \\ J_{12} & J_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}^{-1} \\ &= (J_1 J_2 - J_{12}^2) / (J_1 - 2J_{12} + J_2). \end{aligned} \quad (28)$$

Hence

$$\begin{aligned} J_1 - J'_2 &= (J_1 - J_{12})^2 / (J_1 - 2J_{12} + J_2), \\ J_2 - J'_2 &= (J_2 - J_{12})^2 / (J_1 - 2J_{12} + J_2). \end{aligned} \quad (29)$$

When the stationary value of $J[u]$ is the minimum (the case $\delta < \pi$), the quantities (29) should be positive, for u'_2 is a better approximation than both u_1 and u_2 . In fact the denominator of (29) is positive in this case; this can be shown if we note that it can be rewritten as

$$(z, Az) \quad \text{with} \quad z = (v_1, f)^{-1}v_1 - (v_2, f)^{-1}v_2 \quad (30)$$

and that $(z, f) = 0$ (see (15)).

Hitherto u_1, u_2 (and the corresponding v_1, v_2) have been arbitrary in these equations. Now we take as v_2 the Schwinger second approximation (22). Then J_1, J_2 are respectively the first and second approximations of $k \cot \delta$. To obtain u'_2 and J'_2 by (27) and (28), we have only to calculate J_{12} . We have by (22)

$$Av_1 = v_1 - v_2 + (v_1, Av_1)(v_1, f)^{-1}f,$$

and hence by (26), (23) and (6)

$$J_{12} = J_1 - p, \quad (31)$$

with

$$p = \{ (v_2, v_2) - (v_1, v_1) \} / (v_1, f)(v_2, f). \quad (32)$$

It follows from (29) that

$$\begin{aligned} J_1 - J'_2 &= p^2 / (2p - J_1 + J_2), \\ J_2 - J'_2 &= (p - J_1 + J_2)^2 / (2p - J_1 + J_2). \end{aligned} \quad (33)$$

The denominator in (33) is positive according to what is shown above, provided the stationary value of $J[u]$ is the minimum. Also (27) becomes

$$a_1 : a_2 = (p - J_1 + J_2) : p. \quad (34)$$

Thus u'_2 and $J[u'_2]$ is immediately obtained if p is known. But p is easily calculated by (32), where the necessary quantities are given in the original notations by

$$\begin{aligned} (v_1, v_1) &= \int_0^\infty W u_1^2 dr, & (v_2, v_2) &= \int_0^\infty W u_2^2 dr, \\ (v_1, f) &= k^{-1} \int_0^\infty W u_1 \sin kr dr, & (v_2, f) &= k^{-1} \int_0^\infty W u_2 \sin kr dr, \end{aligned} \quad (35)$$

and these are in any case calculated in the course of computing $J[u_1]$ and $J[u_2]$. In this way we see that *there is no need of additional calculation if we take u'_2 as*

the second approximation instead of u_2 , while the resulting value of $k \cot \delta$ is improved considerably as we shall illustrate by the following example.

Let us consider the case of the square-well potential

$$W(r) = \frac{\pi^2}{4b^2} (0 \leq r \leq b), \quad = 0 (r > b),$$

so that the resonance occurs at zero energy. The exact value of $k \cot \delta$ is

$$k \cot \delta \equiv J_0 = \frac{1}{2} b k^2 + 0.0326727 \ b^3 k^4 + O(k^6). \quad (36)$$

To apply the variational method, we take as the first approximation

$$u_1(r) = \sin \frac{\pi}{2b} r \quad (0 \leq r \leq b), \quad = 1 (r \geq b),$$

which is the correct wave function for $k=0$. If we calculate the values of $J[u]$ for u_1 and the Schwinger second approximation u_2 , we obtain

$$\left. \begin{aligned} J[u_1] - J_0 &= 0.0033030 \ b^3 k^4 + \dots, \\ J[u_2] - J_0 &= 0.0000353 \ b^3 k^4 + \dots \end{aligned} \right\} \quad (37)$$

On the other hand, the modified second approximation u_2' gives

$$J[u_2'] - J_0 = 0.0000033 \ b^3 k^4 + \dots. \quad (38)$$

Thus $J[u_2']$ is more than ten times better than $J[u_2]$. In this example $J[u_2]$ is already so accurate that practically further improvement would be unnecessary, but in other problems the improvement may be expected to be of practical importance. By the way $J[u_1]$, $J[u_2]$ and $J[u_2']$ are all *larger than* J_0 and we have a verification of the general result of § 2; in fact, the conditions $W' \geq 0$ and $\delta < \pi$ are satisfied.

Hitherto we have assumed $W' \geq 0$, but the expressions (33), (34) and (35) are significant without this assumption. Although in the general case we cannot expect to show rigorously that u_2' always gives better result than u_1 and u_2 , it is very plausible that this is usually the case, for it is easily seen that u_2' makes $J[u]$ stationary in the general case too when u is restricted to linear combinations of u_1 and u_2 .

§ 4. Upper and lower bounds of the phase

In this section we shall show that also an *opposite* bound of $k \cot \delta$ can be obtained in the case discussed in § 2, provided the second approximation of the Schwinger iteration method is calculated. Then we shall show in Appendix that the modified iteration method of § 3 is certainly convergent to the correct result.

From (22) we have

$$w \equiv v_2 - v_1 = (v_1, Av_1)(v_1, f)^{-1} f - Av_1. \quad (39)$$

Also we note that

$$(w, w) = (v_2, v_2) - (v_1, v_1) \quad (40)$$

by virtue of (23). It follows from (39) that

$$(A^{-1}w, w) = (v_1, Av_1)^2 (v_1, f)^{-2} (A^{-1}f, f) - (v_1, Av_1). \quad (41)$$

Substituting (6) and (12) into (41) we can easily derive

$$J[u_1] - J_0 = \frac{(A^{-1}w, w)}{(v_1, f)^2} \left\{ 1 + \frac{(A^{-1}w, w)}{(v_1, f)^2 J[u_1]} \right\}^{-1}. \quad (42)$$

If we replace A by $A + cP$ in (42), $J[u]$ and J_0 are increased by the constant c (see § 2. P is given by (14)), and hence the left side of (42) is unchanged, while $J[u_1]$ on the right becomes infinite when $c \rightarrow +\infty$. If we assume $W \geq 0$ and $\delta < \pi$ as in § 2, the operator $A + cP$ is positive definite for sufficiently large c . Let $a > 0$ be the limit of the smallest eigenvalue of $A + cP$ when $c \rightarrow +\infty$ (since P is positive semi-definite, all eigenvalues of $A + cP$ are non-decreasing functions of c). In other words, $1 - a$ is the limit of the largest eigenvalue of $H - cP$. Then the inverse of the positive-definite operator $A + cP$ has the largest eigenvalue a^{-1} for $c \rightarrow +\infty$. If we further note that w and v_2 are independent of c even if we replace A by $A + cP$ as is directly seen from (39), we have

$$0 < \lim_{c \rightarrow +\infty} ((A + cP)^{-1}w, w) \leq a^{-1}(w, w) = a^{-1}\{(v_2, v_2) - (v_1, v_1)\}.$$

On replacing A by $A + cP$ in (42) and making $c \rightarrow +\infty$, we obtain therefore

$$0 \leq J[u_1] - J_0 \leq a^{-1}(v_1, f)^{-2}\{(v_2, v_2) - (v_1, v_1)\} \quad (a > 0). \quad (43)$$

Since $J[u_1]$, (v_1, f) , (v_1, v_1) and (v_2, v_2) are known by (35), (43) gives a lower bound of J_0 provided we have only a rough estimate of a . It will be noted that $1 - a$ is the largest value of λ for which the phase belonging to the wave equation (18) is just equal to π . If δ is not very near π , a will not be very small compared with unity (cf. the example below). Also it will be noted that, for the purpose of applying (43), we have only to calculate (v_2, v_2) and $J[u_1]$ is not necessary.

As an example, let us apply (43) to the case of the square-well potential discussed in § 3. The result is

$$J[u_1] - J_0 \leq a^{-1}(0.0029744 \, b^3 k^4 + \dots).$$

If we adopt the exact value of a : $a = 0.87779 + O(k^2)$, we obtain

$$J_0 \geq \frac{1}{2} b k^2 + 0.0325872 \, b^3 k^4 + \dots \quad (44)$$

This gives a lower bound of J_0 , the exact value of which is given by (36). The coefficient of $b^3 k^4$ in (44) is smaller than the exact one only by an amount 0.0000855. Thus the accuracy of the estimate of the lower bound is much better than the corresponding upper bound $J[u_1]$ and comparable with that of $J[u_2]$ (see

(37)), and may be regarded as sufficient for practical purposes.

In the same way we can estimate an upper bound of $k \cot \delta$ in the case where $W \leq 0$ and $\delta > -\pi$.

Appendix

Convergence of the modified iteration method

Again we assume $W \geq 0$, $\delta < \pi$ (the case $W \leq 0$, $\delta > -\pi$ can be treated in the same way). By (33), (32), (30) and (40), we have

$$J[u_1] - J[u_2'] = (v, w)^2 / (v_1, f)^2 (v_2, f)^2 (z, Az). \quad (45)$$

As in §4 the left side is unchanged if we replace A by $A + cP$. The same is true for z . Let β be the limit for $c \rightarrow -\infty$ of the largest eigenvalue of $A + cP$ (which is a non-decreasing function of c , see §4). We have shown in §3 that $(z, f) = 0$ and $(z, Az) > 0$. Hence $Pz = 0$ and

$$0 < (z, Az) = \lim_{c \rightarrow -\infty} (z, (A + cP)z) \leq \beta(z, z). \quad (46)$$

In particular it follows that $\beta > 0$, and we have by (45) and (46)

$$J[u_1] - J[u_2'] \geq \beta^{-1} (v, w)^2 / (v_1, f)^2 (v_2, f)^2 (z, z). \quad (47)$$

The denominator on the right becomes

$$\begin{aligned} & (v_2, f)^2 (v_1, v_1) - 2(v_1, f)(v_2, f)(v_1, v_2) + (v_1, f)^2 (v_2, v_2) \\ &= \{(v_1, f) - (v_2, f)\}^2 (v_1, v_1) + (v_1, f)^2 \{(v_2, v_2) - (v_1, v_1)\} \\ &= (w, f)^2 (v_1, v_1) + (v_1, f)^2 (v, w) \\ &\leq 2(f, f)(v_1, v_1)(w, w) \end{aligned}$$

by (30), (23) and the Schwarz inequality, and we obtain from (47)

$$\begin{aligned} J[u_1] - J[u_2'] &\geq (2\beta)^{-1} (f, f)^{-1} (v_1, v_1)^{-1} (w, w) \\ &= (2\beta)^{-1} \|f\|^{-2} \|v_1\|^{-2} \|Av_1 - (v_1, Av_1)(v_1, f)^{-1} f\|^2, \end{aligned} \quad (48)$$

where we have written $(w, w) = \|w\|^2$ etc. and substituted (39).

Now suppose that we have calculated a sequence $J[u_n']$, $n = 1, 2, \dots$, by the modified iteration method given in §3, where $u_1' = u_1$ and u_n' is determined from u_{n-1}' just as u_2' was determined from u_1 . Then $J[u_n']$ form a non-increasing sequence as we have shown before. Since the sequence is bounded below by $J[u_n'] \geq J_0$, it is in any case convergent to a certain limit and we have

$$0 \leq J[u_n'] - J[u_{n+1}'] \rightarrow 0, \quad n \rightarrow \infty. \quad (49)$$

If we apply (48) to u_n' in place of u_1 and note (49), we have $(v_n' = W^{\frac{1}{2}} u_n')$

$$\|v_n'\|^{-1} \{Av_n' - (v_n', Av_n')(v_n', f)^{-1} f\} \rightarrow 0, \quad n \rightarrow \infty, \quad (50)$$

in the sense of *mean convergence* with respect to the norm $\|\cdot\|$. Except for the case $\delta = \pi/2$, A has no eigenvalue equal to zero, so that A^{-1} is a bounded operator⁷⁾ and we can operate with A^{-1} on (50) (the exceptional case $\delta = \pi/2$ offers no real difficulty, we have only to replace A by $A + cP$ and hence J by $J + c$ for some fixed c). Thus we have

$$\|v_n'\|^{-1} \{v_n' - (v_n', Av_n')(v_n', f)^{-1} A^{-1} f\} \rightarrow 0. \quad (51)$$

Since the norm of $\|v_n'\|^{-1} v_n'$ is equal to unity, it follows that

$$\|v_n'\|^{-1} |(v_n', f)| \cdot |J[u_n']| = \|v_n'\|^{-1} |(v_n', Av_n')| \cdot |(v_n', f)|^{-1} \rightarrow \|A^{-1} f\|^{-1}. \quad (52)$$

Hence the sequence $\|v_n'\| \cdot |(v_n', f)|^{-1}$ has a limit, for this is the case with $J[u_n']$. If we take the scalar product of (51) with f , we have

$$\|v_n'\|^{-1} \{(v_n', f) - (v_n', Av_n')(v_n', f)^{-1} (A^{-1} f, f)\} \rightarrow 0. \quad (53)$$

Since $\|v_n'\| \cdot |(v_n', f)|^{-1}$ was shown to have a finite limit, we can multiply both sides of (53) by $\|v_n'\|$

$\cdot (v_n', f)^{-1}$, and noting (6) and (12), we obtain

$$1 - J_0^{-1} J[u_n'] \rightarrow 0 \quad \text{or} \quad J[u_n'] \rightarrow J_0. \quad (54)$$

Thus the sequence $J[u_n']$ converges to the correct value J_0 .

On the other hand, multiplying (51) by $\|v_n'\| (v_n', f)^{-1}$, we have

$$(v_n', f)^{-1} v_n' - J[u_n'] A^{-1} f \rightarrow 0$$

or, by (54),

$$(v_n', f)^{-1} v_n' \rightarrow J_0 \cdot A^{-1} f$$

in the sense of mean convergence. Thus u_n' is also convergent to the correct wave function at least in the sense of mean convergence with the weight function $W(r)$, provided it is properly normalized.

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Statistics of Kagomé Lattice

Itiro SYÔZI

Department of Physics, Osaka University

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The transition temperature of the kagomé lattice with $Z=4$ is obtained and compared with that of the square lattice.

After the work of Onsager,¹⁾ who solved exactly the problem of Ising model for the case of plane square lattice, the same problems for the honeycomb and triangular lattice were treated by several authors.²⁾ Other than these three types of lattices, there is left a lattice, called in Japanese kagomé (woven bomboo pattern), which consists exclusively of equivalent lattice points and equivalent bonds. Since the number of nearest neighbors of a lattice point is as many as in the square lattice, namely four, it is interesting to verify the natural conjecture that the curie point, in general, is determined solely by the relation $\text{ch} 2H = \sec \pi/Z$ established by Onsager for the three types of lattices.

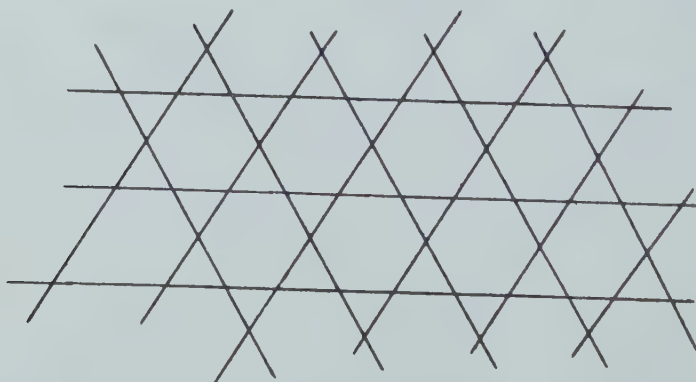


Fig. 1. Kagomé Lattice

Let us start from a variant of the honeycomb lattice, which has an extra spin on the middle point of every side as well as on every vertex (say decorated honeycomb lattice). Let its interaction parameter be L . By summing at first over the spin variables with respect to the vertices in the partition function of this lattice, we arrive at the partition function of the kagomé lattice (Star-triangle transformation), with an interaction parameter K ; in fine

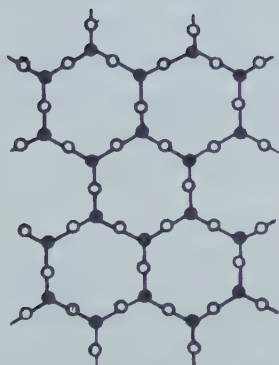


Fig. 2. Decorated Honeycomb Lattice

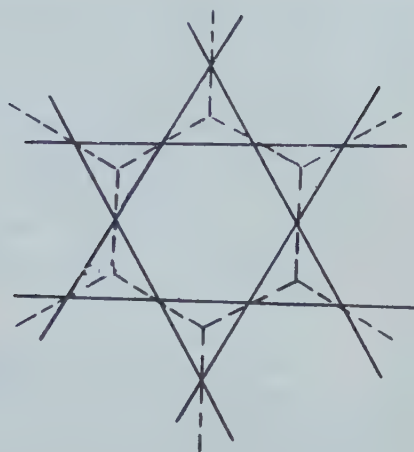


Fig. 3. Star-Triangle Transformation

$$\sum_{\mu_0} e^{L\mu_0(\mu_1+\mu_2+\mu_3)} = 2(\text{ch}^3 L \text{ch} 3L)^{1/4} e^{K(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)},$$

$$e^{4K} = 2\text{ch} 2L - 1.$$

Then there holds the relation between the partition function of the decorated honeycomb lattice $f(L)$ and that of the kagomé lattice $f_k(K)$

$$f(L) = 2^N (\text{ch}^3 L \cdot \text{ch} 3L)^{N/4} f_k(K),$$

where N is the number of vertices.

On the other hand, the decorated honeycomb lattice can be transformed to the ordinary honeycomb lattice, by summing at first over the spins on the side, with interaction parameter H ; in fine,

$$\sum_{\mu_0} e^{L\mu_0(\mu_1+\mu_2)} = 2(\text{ch} 2L)^{1/2} e^{H\mu_1\mu_2}$$

where

$$e^{2H} = \text{ch} 2L.$$

Then the partition function of the decorated honeycomb lattice is connected to that of the honeycomb lattice $f_h(H)$ by

$$f(L) = 2^{3N/2} (\text{ch} 2L)^{3N/4} f_h(H).$$

Thus we have

$$f_k(K) = \left(\frac{2\text{ch} 2K}{\text{ch} H} \right)^{N/2} f_h(H)$$

where

$$e^{4K} = 2e^{2H} - 1.$$

By means of this relation, we can deduce the thermodynamic properties of the kagomé lattice from that of the honeycomb lattice. In the ferromagnetic case, the specific heat becomes logarithmically infinite at the temperature $e^{4K} = 3 + 2\sqrt{3}$, corresponding to $\text{ch} 2H = 2$ of the honeycomb lattice, which is a little below than

that of the square lattice $e^{\mathcal{U}} = 3 + 2\sqrt{2}$.

But in the anti-ferromagnetic case, there is no phase change. Thus the transition temperature of a plane lattice is not determined solely by its nearest neighbor number Z , and the former formula appears to be rather of fortuitous origin. The method here employed to reduce a new lattice to the known lattice may be applied in a number of ways. But it leads, in general, to rather complicated lattices of meagre physical interest.

The author wishes to express his thanks to Professor K. Husimi who brought the kagomé lattice to his attention.

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Note added in proof: Mr. T. Utiyama has developed a general method valid for the lattice obtained by replacing all the black squares of chess-board by an "elementary figure" ($\equiv E.F.$) which is supposed to be a rectangle with $\nu (\geq 0)$ vertical cross-pieces. In $E.F.$, the "vertical, upper and lower horizontal interactions" are denoted, from left to right, by β ; $\beta_{2\mu+1}$, $\beta^{2\mu}$ and $\beta_{2\mu}$ ($\mu \equiv 0, \dots, \nu$), where some β 's may be 0 or $+\infty$. For the partition function etc., he has given explicit formulas depending on four parameters S_3 , S^1 , etc. which can be calculated by inspection of the $E.F.$ For example,

$$S_3 = \sum_{\nu} \text{ch}(\beta - \beta_1 - \dots - \beta_{2\nu+1}) \cdot \text{sh}(\beta^{0*} + \beta_0^*) \text{ch}(\beta^{2*} + \beta_2^*) \dots \text{ch}(\beta^{2\nu*} + \beta_{2\nu}^*), \quad (2^\nu \text{ terms});$$

here, we write 1-st, $\text{ch}(\beta - \beta_1 - \dots - \beta_{2\nu+1})$ (\equiv the "leading factor" $\equiv L.F.$), 2-nd, all the $\text{ch}(\beta \pm \beta_1 \pm \dots \pm \beta_{2\nu-1} - \beta_{2\nu+1})$ obtained from the $L.F.$ by alternations of signs before $\beta_1, \dots, \beta_{2\nu-1}$, and 3-rd, the factor with "*" -argument" of each term which is $\text{ch}(\beta^{2\mu*} + \beta_{2\mu}^*)$ or $\text{sh}(\beta^{2\mu*} + \beta_{2\mu}^*)$ according as the signs before $\beta_{2\mu-1}$ and $\beta_{2\mu+1}$ are equal or not. S^1 is written down by the same rule, with the $L.F.$: $\text{sh}(\beta + \beta_1 + \dots + \beta_{2\nu+1})$ and with "*" -argument": $(\beta^{2\mu*} - \beta_{2\mu}^*)$, therefore is simply equal to its $L.F.$, if $\beta^{2\mu} = \beta_{2\mu}$.

The Curie temperature is characterized by the Eq (C): $|S'| = |S_3|$, where if some β 's are $+\infty$, we equate the coefficients of the highest terms in $\exp(+\infty)$. For example, we have for the Kagomé lattice with $\beta^{2\mu} = \beta_{2\mu}$, $\beta_3 = \beta$, $\beta_1 = +\infty$: $S_3 \equiv \text{ch}(\beta - \beta_1 - \beta_3) \text{sh} 2\beta_2^* + \text{ch}(\beta + \beta_1 - \beta_3) \text{ch} 2\beta_0^* \text{sh} 2\beta_2^* \equiv (\exp(\beta_1)/2) \cdot (2\text{ch} 2\beta/\text{sh} 2\beta)$, $S' \equiv \text{sh}(\beta + \beta_1 + \beta_3) \equiv (\exp(\beta_1)/2) \cdot \exp(2\beta)$, therefore, from (C), $\exp(4\beta) = 3 + 2\sqrt{3}$.

In the same manner, we have from the Eq. (C) for the case $\nu=0$ with four different positive interactions, the relation: $gd^2\beta + g'd^2\beta^0 + g''d^2\beta_0 + g''d^2\beta^1 = \pi$, which is characteristic for $\nu=0$.

On the Positron Theory of Vacuum

Yasuhisa KATAYAMA

Department of Physics, Kyoto University

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In our previous consideration, the so-called ambiguities of quantum field theory can be removed by the certain general conditions of integrals. These procedures are equivalent with the subtraction theory which subtract the ambiguous terms from the matrix element in question. In this paper, we again examine these subtractions from the slightly different view point and develop this consideration along the positron theoretical subtraction and indicate the gauge difficulties can be removed by some technical procedure treating with the density matrix of the vacuum. To clarify it, we utilize the solution of density matrix in the constant external electromagnetic field obtained by Fock's proper time method and separate the non gauge terms from it by this procedure and then arrive the throughout gauge invariant result. For the case of the density matrix in the arbitrary external field, we need the slight modifications, but there does not appear the essential changes of the subtraction terms.

§ 1. Introductory remarks

It is believed that in the region of quantum electrodynamics the extremely beautiful theories of Tomonaga-Schwinger-Feynman have swept away every difficulty except the divergences themselves. Certainly the problems of divergent difficulties are not the own subject of these theories, but the characters and meanings of these difficulties and the relations of these ones to the other results have been more clarified by these theories. Mathematically the divergent difficulties are closely connected with the singular functions in the commutation relations. That is to say, the divergences in the quantum field theory are partly due to these singular functions.

While the same singular functions play the very important roles in the finite regions and one obtains the satisfactory results which are compatible with the experiments. Accordingly, the behaviors of these functions in the finite regions would remain unchanged in the future theory.

However, the behaviors of singular functions are not so decisively separated that we can pick up only the favourable parts. Even in the finite results there exists the one which is due to these parts and the other not due to them. The ambiguity problems are those examples which are due to the unfavourable parts of singularities. Therefore the ambiguity problems are not considered separately from the divergent difficulties. But the only reason why we have to pursue these problems is due to the fact that the existences of the formal requirements concerning with these appearances are independent on the divergences. In order to

harmonize the above two situations, we are obliged to use some unsatisfactory techniques¹⁾ in the courses of calculations, though each of which is not reasonably acceptable in the strict sense.

In spite of these circumstances, if we will forcibly draw a line between the ambiguity problems and the divergence difficulties, the situations are as follows:²⁾ The former come from the infinite remote boundary values in the momentum space, that is to say, in the coordinate space they come from the behaviors of

$$\lim_{x \rightarrow 0} \prod_{i=1}^n x_{\mu_i} S^{(1)}(x) \quad \text{or} \quad \lim_{x \rightarrow 0} \prod_{i=1}^n x_{\mu_i} S_F(x), \quad (n=1, 2, 3, \dots), \quad (1)$$

while the latter due to

$$\lim_{x \rightarrow 0} S^{(1)}(x) \quad \text{or} \quad \lim_{x \rightarrow 0} S_F(x). \quad (2)$$

Therefore if we are able to remove the divergence difficulties throughout in the theory, the singular functions become regular

$$\lim_{x \rightarrow 0} [S^{(1)}(x)]_{\text{reg.}} = \text{finite} \quad (2')$$

and then

$$\lim_{x \rightarrow 0} \prod_{i=1}^n x_{\mu_i} [S^{(1)}(x)]_{\text{reg.}} = 0, \quad (1')$$

which indicate that there exist no ambiguous results.

In our previous work,³⁾ we proposed the method holding the identity (1') apart from the singular behaviors of functions (2) and utilized the conditions*

$$\int (dq) \left(\frac{1}{2} q^2 + L \right) \delta'(q^2 + L) = 0 \quad (3)$$

and

$$\int (dq) L \delta''(q^2 + L) = 0, \quad (4)$$

which are deduced from

$$\lim_{x \rightarrow 0} x_{\mu} S^{(1)}(x) = 0 \quad (3')$$

and

$$\lim_{x \rightarrow 0} x_{\mu} x_{\nu} S^{(1)}(x) = 0. \quad (4')$$

This procedure follows from above-mentioned point of view.

However, even though this method succeeded in a few examples and the reason why these proposed conditions are necessitated is explained by the general discussions analysed by Fukuda and Kinoshita,⁴⁾ what degrees we are obliged to

* The third condition proposed by Fukuda and Kinoshita can be deduced from $\lim_{x \rightarrow 0} x_{\lambda} x_{\mu} x_{\nu} S^{(1)}(x) = 0$ and the second condition. This condition is referred to (4'').

utilize these conditions is yet questionable. Therefore the results which are removed ambiguous terms in virtue of this method, are not more reliable than the original one. We hardly say that the results does not miss out the physically meaningful terms by wrong applications of these conditions. Expressing the other way, these conditions are so strong that the results remain the whole physical terms, as criticized by Koba et al.⁵⁾

Accordingly, in this paper we try to examine the other reliable subtraction procedure maintaining the above mentioned point of view, and indicate the previously proposed method is slightly stronger than the one developed in the following.

For this purpose, we treat the density matrix of vacuum in the external electromagnetic field defined by⁶⁾

$$\rho(x', x'') = \langle [\phi(x'), \bar{\psi}(x'')] \rangle_0 = -S^{(1)}(x', x''), \quad (5)$$

where the function $S^{(1)}(x', x'')$ is the modified $S^{(1)}(x' - x'')$ function in the external field and $\phi(x')$, $\bar{\psi}(x')$ are the electron wave functions in Heisenberg representation concerning only with the external fields and not the virtual fields. The cases including virtual fields and the more complicated cases are not considered in this paper. Therefore we only consider the closed loop type processes in Feynman-Dyson diagram.

Firstly, we deduce the solution of density matrix in the constant external field for preparations in section 2. Then we separate the non-gauge terms and obtain the gauge invariant unambiguous results after these terms are subtracted. In section 4, we consider the case of arbitrary external field and separate the subtracting terms in this case.

§ 2. Solution of density matrix in the constant external field

Defining the density matrix $\rho(x', x'') \equiv -S^{(1)}(x', x'')$ as equation (5), the equations which the density matrix holds are as follows:

$$\left[\gamma_\mu \left(\frac{\partial}{\partial x'_\mu} - ieA_\mu(x') \right) + m \right] S^{(1)}(x', x'') = 0, \quad (6)$$

or

$$\left(\frac{\partial}{\partial x''_\mu} + ieA_\mu(x'') \right) S^{(1)}(x', x'') \gamma_\mu - m S^{(1)}(x', x'') = 0. \quad (6')$$

To solve these equations, we reduce these first-order equations to the second-order ones by

$$S^{(1)}(x', x'') = \left[\gamma_\mu \left(\frac{\partial}{\partial x'_\mu} - ieA_\mu(x') \right) - m \right] A^{(1)}(x', x'') \quad (7)$$

and then we have

$$\left[\left(\frac{\partial}{\partial x'_\mu} - ieA_\mu(x') \right)^2 - m^2 + \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}(x') \right] \Delta^{(1)}(x', x'') = 0, \quad (8)$$

where $\sigma_{\mu\nu}$ is Pauli's spin matrix, $\sigma_{\mu\nu} = \frac{\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu}{2i}$.

According to Fock's proper time method,⁷⁾ we replaced this equation by five dimensional equation

$$i \frac{\partial R}{\partial u} + \left[\left(\frac{\partial}{\partial x'_\mu} - ieA_\mu(x') \right)^2 - m^2 + \frac{e}{2} \sigma_{\mu\nu} F_{\mu\nu}(x') \right] R = 0, \quad (9)$$

where

$$\Delta^{(1)}(x', x'') = \int_{-\infty}^{+\infty} du \varepsilon(u) R, \quad \varepsilon(u) = \frac{u}{|u|}, \quad (10)$$

and separate (9) by

$$R = e^{is} f \quad (11)$$

into two equations

$$\frac{\partial S}{\partial u} + \left[\left(\frac{\partial S}{\partial x'_\mu} - eA_\mu(x') \right)^2 + m^2 \right] = 0, \quad (12a)$$

and

$$\frac{\partial f}{\partial u} + \left[\square'^2 S - \frac{ic}{2} \sigma_{\mu\nu} F_{\mu\nu}(x') - e \frac{\partial A_\mu(x')}{\partial x'_\mu} \right] f = i \square'^2 f. \quad (12b)$$

(12a) and (12b) are the fundamental equations of Fock.

Next, replacing generating function S by

$$-m^2 u + e \int_0^1 da A_\mu(x_1) z_\mu + S', \quad (13)$$

we have

$$\frac{\partial S'}{\partial u} + \left(\frac{\partial S'}{\partial x'_\mu} + e \int_0^1 da a F_{\mu\nu}(x_1) z_\nu \right)^2 = 0 \quad (14a)$$

and

$$\begin{aligned} \frac{\partial f}{\partial u} + \left[\square'^2 S' + e \int_0^1 da a^2 \frac{\partial F_{\mu\nu}(x_1)}{\partial x_{1\mu}} z_\nu - \frac{ic}{2} \sigma_{\mu\nu} \left(2 \int_0^1 da a F_{\mu\nu}(x_1) \right. \right. \\ \left. \left. + \int_0^1 da a^2 \frac{\partial F_{\mu\nu}(x_1)}{\partial x_{1\lambda}} z_\lambda \right) \right] f = i \square'^2 f, \end{aligned} \quad (14b)$$

where

$$x_1 = ax' + (1-a)x'' = x + z \left(a + \frac{1}{2} \right)$$

and

$$z = x' - x'', \quad x = \frac{x' + x''}{2}.$$

It is remarkable that the solutions of above equations have no non-gauge terms and only include the apparently gauge invariant terms.

Here we assume the external electromagnetic fields are constant, i.e., $F_{\mu\nu} = \text{const.}$ and then the fundamental equations in question become very simpler forms,

$$\frac{\partial S'}{\partial u} + \left(\frac{\partial S'}{\partial x'_\mu} + \frac{e}{2} F_{\mu\nu} x'_\nu \right)^2 = 0, \quad (15a)$$

$$\frac{\partial f}{\partial u} + \left[\square'^2 S' - \frac{ie}{2} \sigma_{\mu\nu} F_{\mu\nu} \right] f = 0. \quad (15b)$$

The assumption of constant field in the above equations is equivalent to the assumption that we neglect the higher order derivatives of the arbitrary external field. And then if we take the effects of the higher order derivatives into considerations, only slight modifications are necessitated for this solution.

For the solution of equation (15a), we assume the form

$$S' = (z, g(F^2, u), z) \equiv g_\mu [g(F^2, u)]_{\mu\nu} x'_\nu \quad (16)$$

and obtain

$$g(F^2, u) = \frac{1}{4} (eF) \coth (eFu) \quad (16')$$

or

$$S' = \frac{1}{4} (z, eF \coth (eFu), z) \quad (16'')$$

from the equation

$$\frac{dg}{du} + 4g^2 - \frac{e^2}{4} F^2 = 0,$$

where we read

$$[F^n]_{\mu\nu} = \underbrace{F_{\mu\alpha} F_{\alpha\beta} \cdots F_{\lambda\nu}}_n$$

for the sake of simplicity. Substituting the above solution into (15b) and using

$$\square'^2 S' = \frac{3}{2} \frac{1}{u} + \frac{1}{2} S_p [(eF) \coth (eFu)], \quad S_p[1] = 1, \quad (17)$$

we have the solution

$$\begin{aligned} f &= f_0 \exp \left[-\frac{1}{2} S_p \left\{ \log n^4 + \log \frac{\sinh(eFu)}{(eFu)} \right\} + \frac{ie}{2} (\sigma F) u \right] \\ &= \frac{f_0}{u^2} \exp \left[-\frac{1}{2} S_p \left\{ \log \frac{\sinh(eFu)}{(eFu)} \right\} + \frac{ie}{2} (\sigma F) u \right] \end{aligned} \quad (18)$$

and

$$(\sigma F) \equiv \sigma_{\mu\nu} F_{\mu\nu},$$

f_0 being the constant which is determined that the solution $\mathcal{A}^{(1)}(x', x'')$ must tend to $\mathcal{A}^{(1)}(x' - x'')$ when the charge e becomes zero.

$$f_0 = \frac{i}{16\pi^2}. \quad (19)$$

Summarizing the previous results, we have

$$\begin{aligned} S^{(1)}(x', x'') &= e^{ieA_\mu(x)z_\mu} \left[\gamma_\mu \left(\frac{\partial}{\partial x'_\mu} + \frac{ie}{2} F_{\mu\nu} z_\nu \right) - m \right] \\ &\times \frac{i}{16\pi^2} \int d\eta \frac{\epsilon(\eta)}{\eta^2} \exp \left[-im^2\eta + \frac{ie}{2} (\sigma F \eta) + \frac{i}{4} (z, eF \coth(eF\eta), z) \right. \\ &\quad \left. - \frac{1}{2} S_p \log \frac{\sinh(eF\eta)}{(eF\eta)} \right]. \end{aligned} \quad (20)$$

To transform this result into momentum space, we use the free vacuum electron density matrix

$$-\frac{i}{16\pi^2} \int d\eta \frac{\epsilon(\eta)}{\eta^2} e^{-im^2\eta + \frac{i}{4} \frac{z_\mu^2}{\eta}} = \mathcal{A}^{(1)}(z) = \frac{1}{(2\pi)^3} \int (d\hat{p}) e^{i\hat{p}z} \delta(\hat{p}^2 + z) \quad (21)$$

and $z = m^2$ and $\frac{\partial}{\partial x} \equiv \partial_x$. Finally we have the solution of density matrix in the constant field as follows:

$$\begin{aligned} S^{(1)}(x', x'') &= e^{ieA_\mu(x)z_\mu} \frac{1}{(2\pi)^3} \int (d\hat{p}) e^{i\hat{p}z} \left[i\hat{\gamma}\hat{p} - m + \frac{ie}{2} (\gamma F z) \right. \\ &\quad \left. + \frac{1}{2\partial_x} \{ \gamma, (eF \partial_x) \cot(eF \partial_x), z \} \right] \\ &\times \exp \left[-\frac{e}{2} (\sigma F) \partial_x + \frac{1}{4\partial_x} \{ eF \partial_x (\cot(eF \partial_x) - 1), z \} \right. \\ &\quad \left. - \frac{1}{2} S_p \log \frac{\sin(eF \partial_x)}{(eF \partial_x)} \right] \delta(\hat{p}^2 + z). \end{aligned} \quad (22)$$

In the above equations (20) and (22), the reason why we don't write $A_\mu(x) \rightarrow \frac{1}{2} x_\lambda F_{\lambda\mu}$ is that in the arbitrary field the same original form $A_\mu(x)$ appears and these terms are only the origin of the non-gauge results and we dare indicate these circumstances more precisely.

§ 3. Separation of non-gauge terms and subtraction of ambiguous terms

From the solution of density matrix (22), we can expand it in power series

$$S^{(1)}(x', x'') = \sum_{n=0}^{\infty} \sum_{r=0}^n \frac{1}{2^{n-r} r! (n-r)!} \left[\frac{\partial^n S^{(1)}(x', x'')}{\prod_{s=1}^r \partial(eA_{\mu_s}) \prod_{s=r+1}^n \partial(eF_{\mu_s \sigma_s})} \right] \prod_{s=1}^r (eA_{\mu_s}) \prod_{s=r+1}^n (eF_{\mu_s \sigma_s}), \quad (23)$$

where 2^{n-r} factor of denominators is due to the doubled summations of $F_{\rho\sigma}$. The above equation also can be rewritten in virtue of the character of exponential functions of A_μ ,

$$e^{ieA_\mu(x)z_\mu} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left[\frac{\partial^n e^{-ieA_\mu(x)z_\mu} S^{(1)}(x', x'')}{\prod_{s=1}^n \delta(eF_{\rho_s\sigma_s})} \right]_{e=0} \prod_{s=1}^n (eF_{\rho_s\sigma_s}). \quad (24)$$

Noticing that the non-gauge terms if they would occur, are due to the first factor of equation (24), we separate it into the two parts, one of which is free from the non-gauge terms and other perhaps the non-gauge one, i.e.

$$S_G^{(1)}(x', x'') = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left[\frac{\partial^n e^{-ieA_\mu(x)z_\mu} S^{(1)}(x', x'')}{\prod_{s=1}^n \delta(eF_{\rho_s\sigma_s})} \right]_{e=0} \prod_{s=1}^n (eF_{\rho_s\sigma_s}) \quad (25)$$

and

$$S_N^{(1)}(x', x'') = (e^{ieA_\mu(x)z_\mu} - 1) \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left[\frac{\partial^n e^{-ieA_\mu(x)z_\mu} S^{(1)}(x', x'')}{\prod_{s=1}^n \delta(eF_{\rho_s\sigma_s})} \right]_{e=0} \prod_{s=1}^n (eF_{\rho_s\sigma_s}). \quad (26)$$

It is noticeable that in the case in which the non-diagonal $S^{(1)}(x', x'')$'s play a role as the case of transitions of real electron the above separation is not correct and the more complicated separation of (26) is necessary, and therefore the above separation is only meaningful in the case of vacuum electron problem in which only the diagonal $S^{(1)}(x, x)$'s are treated.

Bearing the above remarks in mind, we can evaluate the each terms of (26). Expanding also in power series

$$S_N^{(1)}(x', x'') = I_N + (II_N + II_{N'}) + (III_N + III_{N'} + III_{N''}) + \dots, \quad (27)$$

we only consider the diagonal $S_N^{(1)}(x, x)$ as follows:

$$\begin{aligned} \lim_{z \rightarrow 0} I_N &= \lim_{z \rightarrow 0} [ieA_\mu z_\mu S^{(1)}(z)] = \frac{iv}{(2\pi)^3} (\gamma A) \int (d\rho) \left(\frac{1}{2} \rho^2 + x \right) \delta'(\rho^2 + x), \\ \lim_{z \rightarrow 0} II_N &= \lim_{z \rightarrow 0} \left[\frac{1}{2!} (ieA_\mu z_\mu)^2 S^{(1)}(z) \right] = \frac{c^2 m}{2(2\pi)^3} A^2 \int (d\rho) x \delta''(\rho^2 + x), \\ \lim_{z \rightarrow 0} II_{N'} &= \lim_{z \rightarrow 0} \left[\frac{1}{2} (ieA_\mu z_\mu) \left(\frac{\partial e^{-ieA_\mu z_\mu} S^{(1)}(x', x'')}{\partial (eF_{\rho\sigma})} \right)_{e=0} eF_{\rho\sigma} \right] \\ &= \frac{-e^2}{2(2\pi)^3} \left[\frac{i}{2} (\gamma A) (\sigma F) + (\gamma F A) \right] \int (d\rho) x \delta''(\rho^2 + x), \quad (28) \\ \lim_{z \rightarrow 0} III_N &= \lim_{z \rightarrow 0} \left[\frac{1}{3!} (ieA_\mu z_\mu)^3 S^{(1)}(z) \right] = -\frac{ic^3}{6(2\pi)^3} (\gamma \cdot l) A^2 \int (d\rho) x^2 \delta'''(\rho^2 + x), \\ \lim_{z \rightarrow 0} III_{N'} &= \lim_{z \rightarrow 0} III_{N''} = 0 \end{aligned}$$

and the higher than third order terms are always zero in this limit. According

to these results, there only exist the four non-gauge terms

$$S_N^{(1)}(x, x) = \lim \{I_N + II_N + III_N + IIII_N\}.$$

This fact is simply explained in the general discussions: that is, the reasons why these terms appear are that the singular $S^{(1)}(z)$ function have third order singularity in a point $z=0$ and then

$$\lim_{z \rightarrow 0} \prod_{i=1}^n z_{\mu_i} S^{(1)}(z) \neq 0, \quad \text{for } n=1, 2 \text{ and } 3 \quad (29.1)$$

and

$$\lim_{z \rightarrow 0} z_{\mu} \frac{\partial S^{(1)}(z)}{\partial x} \neq 0. \quad (29.2)$$

The first three inequalities raise three non-gauge terms I_N , II_N and III_N , and the second one non-gauge term $IIII_N$.

From these circumstances, if we assume *a-priori* following procedure independently of the analytic behaviors of $f(x)$

$$\lim_{x \rightarrow 0} x^n f(x) \rightarrow 0 \quad n=1, 2, 3, \dots, \quad (30)$$

the above unsatisfactory terms would vanish. In our previous works, we utilized three conditions of momentum space integration form (3), (4) and (4'') instead of (30). To illustrate these circumstances, we can calculate the gauge invariant terms (25).

The gauge invariant terms are evaluated in the similar way for the diagonal elements as follows:

$$S_G^{(1)}(x', x'') = I_G + II_G + \dots \quad (31)$$

and

$$\begin{aligned} \lim_{z \rightarrow 0} I_G &= \lim_{z \rightarrow 0} \left[\frac{1}{2} \left(\frac{\delta e^{-ieA_{\mu} z_{\mu}} S^{(1)}(x', x'')}{\delta (cF_{\rho\sigma})} \right)_{\epsilon=0} (cF_{\rho\sigma}) \right] = \frac{em}{2(2\pi)^3} (\sigma F) \int (dp) \delta'(p^2 + z), \\ \lim_{z \rightarrow 0} II_G &= \lim_{z \rightarrow 0} \left[\frac{1}{8} \left(\frac{\delta^2 e^{-ieA_{\mu} z_{\mu}} S^{(1)}(x', x'')}{\delta (eF_{\rho\sigma}) \delta (eF_{\alpha\beta})} \right)_{\epsilon=0} (eF_{\rho\sigma}) (eF_{\alpha\beta}) \right] \\ &= -\frac{e^2 m}{4(2\pi)^3} \left(\frac{1}{2} (\sigma F)^2 + \frac{1}{3} S_{\nu}(FF) \right) \int (dp) \delta''(p^2 + z). \quad (32) \end{aligned}$$

In these results, if we utilize the conditions (3) and (4) devising slight modifications, both terms are dropped, while using the condition (30), they remain and have non-zero values. And the second equation destroys the equivalence theorem in γ -decay contrary to the previous paper in which the same theorem holds. That is, the previous conditions are more powerful, though a little dangerous.* In this paper, we only subtract the certain terms with the weaker condition (30).

* For example, if we would remove the difficulties of equivalence theorem in the matrix II_G , the first term of the scalar type γ decay being the gauge invariant also is dropped in this procedure.

Finally, after the subtraction procedure (30), we obtain for the diagonal element

$$\begin{aligned} S_G^{(1)}(x, x) &= -\frac{m}{(2\pi)^3} \int (d\rho) \exp \left[-\frac{e}{2} (\sigma F) \partial_\alpha - \frac{1}{2} S_p \log \frac{\sin(eF \partial_\alpha)}{(eF \partial_\alpha)} \right] \delta(\rho^2 + x) \\ &= -\frac{m}{(2\pi)^3} \int (d\rho) S_p \left(\frac{\sin(eF \partial_\alpha)}{(eF \partial_\alpha)} \right)^{-\frac{1}{2}} \delta(\rho^2 + x - \frac{e}{2} (\sigma F)). \end{aligned} \quad (33)$$

From this result, we have

$$S_p[\gamma S_G^{(1)}(x, x)] = 0 \quad (34)$$

and conclude that there does not exist any effect of vacuum polarization under the influences of the constant external field.⁸⁾

To demonstrate the correctness of above result, we will apply it to the processes of γ -decay and compare with the many authors' calculation results. From (33), we have

$$-\frac{m}{(2\pi)^3} \int (d\rho) \exp \left[-\frac{e}{2} (\sigma F) \partial_\alpha + \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n-1}}{n(2n)!} S_p (eF \partial_\alpha)^{2n} \right] \delta(\rho^2 + x), \quad (35)$$

where B_{2n-1} 's are Bernoulli's numbers

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n}-1)\pi^{2n}} \left[1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \dots \right]. \quad (36)$$

The exponential function can be expanded in power series of e ,

$$\exp \equiv 1 - \frac{e}{2} (\sigma F) \partial_\alpha + \frac{e^2}{2} \left\{ \frac{1}{4} (\sigma F)^2 + B_1 S_p F^2 \right\} \partial_\alpha^2 - \frac{e^3}{4} \left\{ \frac{1}{12} (\sigma F)^3 + B_1 (\sigma F) S_p F^2 \right\} \partial_\alpha^3 + \dots$$

Then (35) becomes

$$\begin{aligned} & -\frac{m}{(2\pi)^3} \int (d\rho) \delta(\rho^2 + x) + \frac{em}{2(2\pi)^3} (\sigma F) \int (d\rho) \delta'(\rho^2 + x) \\ & - \frac{e^2 m}{2(2\pi)^3} \left(\frac{1}{4} (\sigma F)^2 + B_1 S_p F^2 \right) \int (d\rho) \delta''(\rho^2 + x) \\ & + \frac{e^3 m}{4(2\pi)^3} \left(\frac{1}{12} (\sigma F)^3 + B_1 (\sigma F) S_p F^2 \right) \int (d\rho) \delta'''(\rho^2 + x) + \dots \end{aligned} \quad (37)$$

From this result we can evaluate the matrix elements of the processes in question. And we have

A) γ -decay of scalar meson with scalar coupling⁹⁾

$$\begin{aligned} f\phi\langle\bar{\psi}\psi\rangle_0 &= \frac{f}{2} \phi S_p [S_G^{(1)}(x, x)] \\ &= \frac{e^2 f m}{8(2\pi)^3} \phi \left\{ \frac{1}{2} S_p (\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) F_{\mu\nu} F_{\rho\sigma} + \frac{1}{3} S_p (1) F_{\mu\nu}^2 \right\} \int (d\rho) \delta''(\rho^2 + x) \\ &= -\frac{e^2 f}{24\pi^2 m} \phi F_{\mu\nu}^2. \end{aligned} \quad (38)$$

B) γ -decay of pseudoscalar meson with pseudoscalar coupling

$$\begin{aligned} \tilde{f}\phi\langle\psi\gamma_5\psi\rangle_0 &= \frac{\tilde{f}}{2} \phi S_p [\gamma_5 S_G^{(1)}(x, x)] \\ &= \frac{e^2 \tilde{f} m}{16(2\pi)^3} \phi S_p (\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) F_{\mu\nu} F_{\rho\sigma} \int (d\rho) \delta''(\rho^2 + x) \\ &= \frac{e^2 \tilde{f}}{4\pi^2 m} \phi (F_{23} F_{14} + F_{31} F_{24} + F_{12} F_{34}). \end{aligned} \quad (39)$$

This term destroys the equivalence theorem as discussed above.

C) γ -decay of vector meson with tensor coupling¹⁰⁾

$$\begin{aligned}
 \frac{i g}{2\mu} U_{st} \langle \psi \gamma_s \gamma_t \psi \rangle &= \frac{i g}{4\mu} U_{st} S_{\mu} [\gamma_s \gamma_t S_{\alpha\beta}^{(1)}(x, x)] \\
 &= -\frac{e^3 g m}{2^5 3 (2\pi)^3} U_{st} \left\{ S_{\mu} (\gamma_s \gamma_t \gamma_{\mu} \gamma_{\nu}) F_{\mu\nu} F_{\rho\sigma}^2 + \frac{1}{2} S_{\mu} (\gamma_s \gamma_t \gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\sigma} \gamma_{\alpha} \gamma_{\beta}) F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} \right. \\
 &\quad \left. \times \int (d\ell) \delta''(\ell^2 + x) \right\} \\
 &= \frac{e^3 g}{24\pi^2 \mu m^3} U_{st} \{ F_{0\nu} (F_{12} F_{34} + F_{23} F_{14} + F_{31} F_{24}) \varepsilon_{0\nu st} \}.
 \end{aligned} \tag{40}$$

These results agree the ones of other authors^{9), 10)} entirely.

§ 4. Extension to the solution of density matrix in the arbitrary external field

From the above subtraction procedure in the constant external field, we will next analyse the effects of arbitrary external field on this procedure. However, as it is very complicate to obtain the rigorous solution of density matrix, we only consider the qualitative behaviors of the effect of field.

In this case, it is very convenient to represent the field quantities in momentum space, say

$$(A_{\mu} A_{\nu} \cdots F_{\rho\sigma} \cdots F_{\alpha\beta}) \rightarrow \int \cdots \int (dk_1) \cdots (dk_n) e^{i \sum_{i=1}^n k_i x} (A_{\mu}(k_1) A_{\nu}(k_2) \cdots F_{\rho\sigma}(k_r) \cdots F_{\alpha\beta}(k_n)).$$

Then the effects differing from ones of the constant field are represented by some k 's dependent factors in the equations. One group of these effects appears in the δ -function of density matrix (22) which are estimated by

$$\delta(p^2 + x) \rightarrow \delta[p^2 + x + \sum_{i,j} d_{ij} k_i k_j],$$

where d_{ij} are the k_i 's independent some functions of other parameter a_1, \cdots, a_n by which the density is integrated from 0 to 1 finally. Next effects appear in the displacement effects of the momentums of vacuum electrons except one in δ -function. These effects are represented by

$$p \rightarrow p + \sum_i' c_i k_i,$$

where c_i 's are the some functions of a_i 's and slightly different forms in each momentum p in the expression. \sum' is the summation of only the k_i 's of field strengths $F_{\rho\sigma}$ multiplied. This reason is due to the fact that the behaviors of A_{μ} type factors appear in this solution in quite similar way as the constant field case, as stated in the section 2.

For example, let the certain term of the solution in the constant field be

$$M = \int (d\rho) f(\rho) (A \cdots, F \cdots F) \delta(p^2 + x), \tag{41}$$

then we can estimate the same term in the arbitrary field by

$$\bar{M} = \prod_{i=1}^n \int_0^1 da_i \int (dk_i) e^{ik_i x} \cdot \int (dp) f(p + \sum_{i=1}^n c_i k_i) \delta(p^2 + x + \sum_{i,j} d_{ij} k_i k_j) \quad (41')$$

$$\times (A(k_1) \dots, F(k_r) \dots F(k_n)).$$

We can also treat the non-diagonal density replacing by more general form

$$\bar{M}' = \prod_{i=1}^n \int_0^1 da_i \int (dk_i) e^{ik_i (x - \frac{x}{2} + za_1 \dots a_n)} \int (dp) e^{ipz} f(p + \sum_{i=1}^n c_i k_i) \quad (41'')$$

$$\times \delta(p^2 + x + \sum_{i,j} d_{ij} k_i k_j) (A(k_1) \dots, F(k_r) \dots F(k_n)).$$

Using these crude estimations, we also examine the preceding subtraction procedure.

Firstly, we estimate the subtraction terms \bar{I}_N , \bar{II}_N and \bar{III}_N , in which only A_μ -type factors appear, and then only δ -function are modified. However, after the integration of momentum of vacuum is performed, this modified effects vanish perfectly. That is, in these terms, there exist any change essentially.

$$\lim_{z \rightarrow 0} \bar{I}_N = \lim_{z \rightarrow 0} I_N, \quad \lim_{z \rightarrow 0} \bar{II}_N = \lim_{z \rightarrow 0} II_N \quad \text{and} \quad \lim_{z \rightarrow 0} \bar{III}_N = \lim_{z \rightarrow 0} III_N.$$

To examine the next subtraction terms \bar{II}_N and \bar{III}_N , we modify I_G as

$$\begin{aligned} \bar{I}_G = & -\frac{e}{(2\pi)^3} \int_0^1 da \int (dp) e^{ipz} \int (dq) e^{iq(x - \frac{x}{2} + za)} [\gamma, F(q), p + c_1 q] \\ & + \frac{1}{2} (i\gamma(p + c_2 q) - m)(\sigma F(q)) \delta'(p^2 + x + dq^2) \\ = & -\frac{e}{(2\pi)^3} \int_0^1 da \int (dp) e^{ipz} \int (dq) e^{iqz} e^{iq(x - \frac{x}{2} + za)} [\gamma F p] + \frac{1}{2} (i\gamma p - m)(\sigma F) \delta'(p^2 + x + dq^2) \\ & -\frac{e}{(2\pi)^3} \int_0^1 da \int (dp) e^{ipz} \int (dq) e^{iq(X - \frac{x}{2} + za)} [c_1 (\gamma F(q) q) + \frac{c_2}{2} (i\gamma q)(\sigma F(q))] \delta'(p^2 + x + dq^2). \end{aligned} \quad (42)$$

The term of first line is identical with the I_G except the argument of δ -function and then the terms of \bar{II}_N and \bar{III}_N constructed from it is identical with II_N and III_N . The essential changes, if they would appear, are due to the effects of the term of second line. But after some calculations, we have

$$\begin{aligned} \lim_{z \rightarrow 0} \bar{II}_N = & \lim_{z \rightarrow 0} II_N + \frac{2c_2^2}{(2\pi)^3} \int_0^1 da_1 da_2 \int (dp) \int (dk_1) (dk_2) e^{i(k_1 + k_2)X} \\ & \delta''(p^2 + x + d_1 k_1^2 + d_2 k_2^2 + d_3 k_1 k_2) \\ & \times (p A(k_1)) \left[c_1' (\gamma F(k_2) k_2) + \frac{c_2'}{2} (i\gamma k_2) (\sigma F(k_2)) \right] \\ = & \lim_{z \rightarrow 0} II_N, \\ \lim_{z \rightarrow 0} \bar{III}_N = & \lim_{z \rightarrow 0} III_N - \frac{2c_2^3}{(2\pi)^3} \prod_{i=1}^3 \int_0^1 da_i \int (dk_i) e^{ik_i X} \int (dp) \left(1 + \frac{1}{2} p^2 \partial_x \right) \delta''(p^2 + x + \sum_{i,j} d_{ij} k_i k_j) \end{aligned}$$

$$\times (A(k_1)A(k_2)) \left[c_1''(\gamma F(k_3)k_3) + \frac{c_2''}{2}(i\gamma k_3)(\sigma F(k_3)) \right] \\ = \lim_{\varepsilon \rightarrow 0} III_{N'}.$$

Accordingly in these terms the effects of derivatives give no modifications. For the term $III_{N'}$, we are also the similar consideration as above.

Consequently, we can conclude that the effects of high derivatives of field strength give no contributions to the subtraction terms and then one can only subtract the same terms in the constant field even in the arbitrary field.

However, if we subtract all divergent terms according to the positron theoretical method, there exists one subtraction term which differs from ones in the case of constant field, i.e.

$$- \frac{e}{(2\pi)^3} \int_0^1 da \int (dp) \int (dq) e^{iqx} \left[c_1(\gamma F(q)q) + \frac{c_2}{2}(i\gamma q)(\sigma F(q)) \right] \delta'(p^2 + x). \quad (43)$$

This term is nothing else the divergent charge renormalization term.

After all the solution of density matrix in the arbitrary field is

$$\bar{S}_G^{(0)}(x, x) = - \frac{m}{(2\pi)^3} \int (dp) S_p \left[\frac{\sin(e \int da \int dk F \partial_x)}{(e \int da \int dk F \partial_x)} \right]^{-\frac{1}{2}} e^{-\frac{e}{2} (\sigma \int da \int dk F) \partial_x} \delta(p^2 + x - \sum d_{ij} k_i k_j) \\ + (\text{contributions of effects of second group, } p \rightarrow p + \sum c k) \\ + \frac{e}{(2\pi)^3} \int_0^1 da \int (dp) \int (dq) e^{iqx} \left[c_1(\gamma F q) + \frac{c_2}{2}(i\gamma q)(\sigma F) \right] \delta'(p^2 + x). \quad (44)$$

§ 5. Conclusions

In this paper, we proposed a more satisfactory subtraction procedure than the previous ones. This procedure is a little weaker than the latter and then a part of formal requirements, i.e., the equivalence theorem does not hold in this procedure contrary to the previous ones. As far as the gauge invariance is concerned, this procedure is reasonable. However, if we will apply any subtraction procedure to the more complicated processes than treated here, our condition (30) has the very inconvenient form against the conditions (3), (4) and (4''). For this reason the subtraction procedure here proposed must be extended more convenient form.

By the way, we only indicate that there exists a more satisfactory subtraction procedure at the background of the strong and apparently curious conditions in our previous paper.

In conclusion the author wishes to express his sincere gratitude to Prof. M. Kobayasi for his kind guidance and encouragement.

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On Gauge Invariance and Equivalence Theorems

Ziro Koba, Nobumichi Mugibayashi and Shinzô Nakai

Department of Physics, Faculty of Science, Osaka University

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The gauge invariance theorem and the equivalence theorems are studied in the Feynman-Dyson formalism. Our results may be of considerable use in reducing complicated calculations. When the Dyson diagram contains a closed loop, certain ambiguities appear as usual. From the standpoint that surface integrals in the momentum space should be estimated accurately, however, they are easily removed by a simple prescription. As an application of our method, the two gamma-decay of a neutral meson is reexamined and entirely consistent results are obtained.

§ 1. Introduction and summary

Recent developments in the quantum electrodynamics mainly due to Tomonaga,¹⁾ Schwinger,²⁾ Feynman³⁾ and Dyson⁴⁾ have enabled us to calculate radiative corrections in a completely covariant manner and to obtain finite results by eliminating divergences through mass and charge renormalization. There remain, however, still obscurities concerning the gauge invariance, validity of the equivalence theorems, and though several authors⁽⁵⁻⁹⁾ have made remarkable contributions with respect to these points, one can hardly regard them as ultimately settled.

We have investigated these problems from a practical point of view, *i.e.*, in such a form as to be immediately useful for carrying out the covariant perturbation. First we consider in what form the gauge invariance theorem and the equivalence theorems are generally guaranteed in the Feynman-Dyson formalism, and obtain certain rules of dividing complicated diagrams into suitable classes, which method proves very convenient in the higher order calculation.

Next we consider the well-known ambiguities, which are due to charged or nucleonic closed loops in the Dyson diagram. One finds the inconsistency that in the momentum representation even the result obtained by a *formal* calculation does not always satisfy the requirements of the gauge invariance or the equivalence, though they are of course automatically fulfilled in the coordinate representation. We trace the origin of this fact and ascertain that it is nothing but what gives rise to the ambiguous terms in the *practical* calculation (*i.e.* the non-gauge-invariant or inequivalent terms which have been already studied by many authors and also those newly introduced ones in forms of surface integrals). Thus we have got a simple prescription: *erase the terms that yield the above-mentioned formal discrepancy, and the ambiguities in the practical calculation will be removed simultaneously.*

As an example we apply our method of calculation to the case of the two-gamma-decay of a neutral meson and arrive at the following consistent results.

- i) Matrix elements for scalar meson with scalar coupling and for pseudoscalar meson with pseudoscalar coupling become finite and gauge invariant.
- ii) The equivalence theorem for pseudoscalar meson is valid in this process.
- iii) Transitions in other cases are forbidden in agreement with Furry's theorem and/or Yang's selection rule.

Finally we discuss the relation of our method to other treatments briefly.

§ 2. Gauge invariance and equivalence theorems in the Feynman-Dyson formalism

As is well known, gauge invariance and equivalence theorems are all alike mathematically in the sense that they are concerned with the behaviour of a term with space-time derivatives. It is therefore to be expected that these problems may be treated in parallel. Indeed this analogy is especially conspicuous when the equivalence theorems for neutral meson are considered.

A. Gauge invariance theorem

In the quantum electrodynamics, the gauge invariance of the Feynman-Dyson formalism has been proved by Feynman himself in his second paper cited above.* We shall here extend his procedure to the more general case of nucleon-meson system interacting with electromagnetic field.

For a certain process we have many Dyson diagrams, the number of which increases very rapidly in general when the order of the process become high, and their treatment comes to be more and more difficult. Therefore it is desirable to have some rules to divide those complicated diagrams into suitable classes. We shall see that the requirement of the gauge invariance in fact affords us such rules.

It is sufficient to confine our considerations to real photons because we can discuss the gauge invariance for virtual photons in the same way before taking the vacuum expectation value of the P -bracket. Accordingly we may regard the Dyson diagrams of the nucleon-meson system which interacts with photon field as consisting essentially of some charged open lines in connection with several real photon lines.

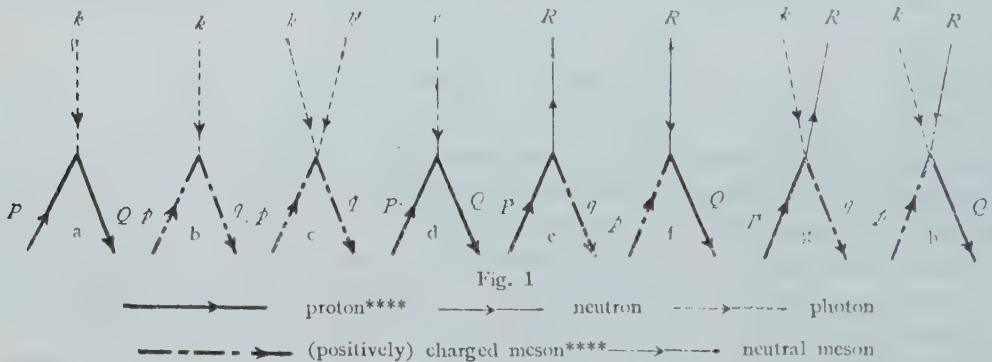
Now we attend to a particular photon and remove its representative line from the diagrams; then several diagrams are reduced to the same incomplete form, which we call "skeleton" and denote by \tilde{D}_g , in distinction from the original ones, D . Conversely we may construct a class of diagrams \tilde{D}_g belonging to D from one skeleton \tilde{D}_g by attaching that photon line to all charged lines and all (possible) vertices, and thus we can classify D into a number of \tilde{D}_g 's by

* Compare ref. 3, see especially section 8 of the latter.

starting from individual \bar{D}_θ 's. As will be seen in the following, each \bar{D}_θ forms a gauge invariant class, that is, the sum of the matrix elements corresponding to \bar{D}_θ is invariant under any gauge transformation with respect to that photon, and therefore our classification of diagrams may be called a gauge-invariant one.* Considering such a class as a whole,** one can not only evaluate the matrix elements fairly but also classify the nature of the problematic terms appearing when charged lines have closed loops.***

In order to avoid unnecessary complication, we discuss only the essential part of the proof of gauge invariance in the nucleon-meson-photon system. We may assume namely that the skeleton is a charged open line connecting with one or two neutral lines at each vertex, without destroying the generality. First we decompose the skeleton into elementary parts \tilde{E}_θ ; that is to say we examine each possible type of vertex separately.

Here we restrict ourselves to the scalar meson field—other cases can be treated in a similar way—and enumerate all possible \tilde{E}_θ 's (Fig. 1).



Whether the coupling in d, e and f is scalar or vector will be indicated by suffix 1 or 2 respectively.

In these diagrams neutral particles are all assumed to be actual, their energy-momenta satisfying certain relations,***** whereas P , Q , p and q are in general

* If the charge flow consists of two (or more) independent branches, \bar{D}_θ is divided into two (or more) subclasses. (This is quite natural, since the gauge invariance is closely related to the conservation law of charge-current.)

Further, some modification of our rule is required when one carries out the so-called "renormalization" procedure. (See § 3 of the work by Koba, Kotani and Nakai.¹⁰⁾)

** In the processes where two (or more) real photons are involved, a class which is gauge invariant with respect to one photon is not necessarily invariant with regard to the gauge of the other. In such a case one has to group the diagrams into certain superclasses, each of which being gauge invariant with regard to all photons.

*** An illustrative example calculated by our method has been given by Kotani and some of us.¹⁰⁾

**** Although the sense of the arrow in charged lines is of little importance, it is convenient to fix it, e.g., to the direction of the positive charge flow as in the present paper.

***** This assumption is inessential to our argument.

energy-momentum vectors for virtual states; when they do correspond to actual states, however, they satisfy the following equations of motion:

$$(i\gamma P + m)\psi(P) = 0, \quad \bar{\psi}(Q)(i\gamma Q + m) = 0, \quad (P, Q, p \text{ or } q \text{ being actual.}) \quad (2.1)$$

$$(p^2 + \mu^2)\phi(p) = 0, \quad \phi^*(q)(q^2 + \mu^2) = 0.$$

Neutral bosons in the above figures are described as being absorbed; the case in which one of them is emitted may be obtained by reversing the sign of its energy-momentum.

Writing down the essential parts of the matrices corresponding to these diagrams, one gets in the momentum representation*:

$$\frac{i\gamma Q - m}{Q^2 + m^2} A_\mu(k) \gamma_\mu \frac{i\gamma P - m}{P^2 + m^2}, \quad (2.2a) \quad \frac{1}{q^2 + \mu^2} iA_\mu(k)(q+p)_\mu \frac{1}{p^2 + \mu^2}, \quad (2.2b)$$

$$\frac{1}{q^2 + \mu^2} 2A_\mu(k) A_\nu(k') \delta_{\mu\nu} \frac{1}{p^2 + \mu^2}, \quad (2.2c)$$

$$\frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(r) \tau_3' \frac{i\gamma P - m}{P^2 + m^2}, \quad (\text{scalar coupling}) \quad (2.2d_1)**$$

$$\frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(r) \tau_3' (i\gamma r) \frac{i\gamma P - m}{P^2 + m^2}, \quad (\text{vector coupling}) \quad (2.2d_2)**$$

$$\frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ \frac{i\gamma P - m}{P^2 + m^2}, \quad (\text{scalar coupling}) \quad (2.2e_1)$$

$$-\frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ (i\gamma q) \frac{i\gamma P - m}{P^2 + m^2}, \quad (\text{vector coupling}) \quad (2.2e_2)$$

$$\frac{i\gamma Q - m}{Q^2 + m^2} \tau_- \psi(R) \frac{1}{p^2 + \mu^2}, \quad (\text{scalar coupling}) \quad (2.2f_1)$$

$$\frac{i\gamma Q - m}{Q^2 + m^2} (i\gamma p) \tau_- \psi(R) \frac{1}{p^2 + \mu^2}, \quad (\text{vector coupling}) \quad (2.2f_2)$$

$$-\frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ A_\mu(k) \gamma_\mu \frac{i\gamma P - m}{P^2 + m^2}, \quad (\text{vector coupling}) \quad (2.2g)$$

$$\frac{i\gamma Q - m}{Q^2 + m^2} A_\mu(k) \gamma_\mu \tau_- \psi(R) \frac{1}{p^2 + \mu^2}. \quad (\text{vector coupling}) \quad (2.2h)$$

When one has to do with external charged lines the propagation function in (2.2) should be replaced by the corresponding wave function:

$$\frac{i\gamma P - m}{P^2 + m^2} \rightarrow \psi(P), \quad \frac{i\gamma Q - m}{Q^2 + m^2} \rightarrow \bar{\psi}(Q),$$

$$\frac{1}{p^2 + \mu^2} \rightarrow \phi(p), \quad \frac{1}{q^2 + \mu^2} \rightarrow \phi^*(q),$$

if P, Q, p or q is actual. (2.3)

* For the interaction Hamiltonian used in this section, see Appendix 1.

** $\tau_3' = \tau_3$ (symmetrical theory), or $\tau_3' = 1$ (neutral theory).

An arbitrary skeleton \tilde{D}_g can be constructed from these elementary \tilde{E}_g 's by connecting same kind of charged lines in succession, and correspondingly the matrix element for \tilde{D}_g may be expressed in terms of (2.2) by identifying the propagation functions with one another.

Now performing the gauge transformation

$$A_\mu(k) \rightarrow A_\mu(k) + ik_\mu A(k) \quad (2.4)$$

in (2.2a) and (2.2b) and picking out the contribution from the second term of (2.4) (we shall denote this procedure by $A_\mu(k) \rightarrow ik_\mu$), we obtain* by the help of the conservation law of energy-momentum and the equations of motion:

$$\frac{i\gamma Q - m}{Q^2 + m^2} \eta_P - \eta_Q \frac{i\gamma P - m}{P^2 + m^2}, \quad (2.5a)$$

$$\frac{1}{q^2 + \mu^2} \eta_m - \eta_q \frac{1}{p^2 + \mu^2}, \quad (2.5b)$$

where $\eta_P = 1$ if P is virtual and $\eta_P = 0$ if P is actual. It is very important that apart from η both (2.5a) and (2.5b) are the difference of two terms with same functional forms but with arguments different from each other by the energy-momentum k of the photon under consideration.

We examine the gauge invariance of the elementary skeletons by combining (2.2) and (2.5). There are now two cases to be considered separately, according to whether it is possible or not to insert the photon line in question, say k_μ into the vertex.

1. *The cases where an insertion of k_μ into vertex is not possible (cases a, c, d₁, d₂, e₁, f₁, g and h)*

For example, let us construct two diagrams from c_1 by inserting k_μ into charged line and evaluate the corresponding matrices according to (2.2a), (2.2c₁), (2.2f₁) and (2.2b) and by the use of the Lorentz condition for A_μ . Then we get

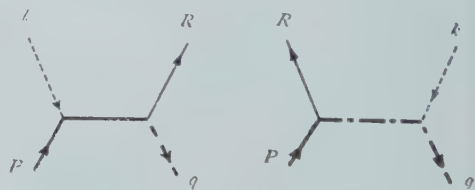


Fig. 2

$$\begin{aligned} & -\frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ \frac{i\gamma(P+k) - m}{(P+k)^2 + m^2} A_\mu(k) \gamma_\mu \frac{i\gamma P - m}{P^2 + m^2} \\ & + \frac{1}{q^2 + \mu^2} 2iA_\mu(k) q_\mu \frac{1}{(q-k)^2 + \mu^2} \bar{\psi}(R) \tau_+ \frac{i\gamma P - m}{P^2 + m^2}. \end{aligned} \quad (2.6)$$

By $A_\mu(k) \rightarrow ik_\mu$:

$$\begin{aligned} & \frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ \left\{ \frac{i\gamma(P+k) - m}{(P+k)^2 + m^2} \eta_P - \eta_{P+k} \frac{i\gamma P - m}{P^2 + m^2} \right\} \\ & + \left\{ \frac{1}{q^2 + \mu^2} \eta_{q-k} - \eta_q \frac{1}{(q-k)^2 + \mu^2} \right\} \bar{\psi}(R) \tau_+ \frac{i\gamma P - m}{P^2 + m^2}. \end{aligned}$$

* Scalar function $A(k)$ is omitted, for it appears always as a common factor.

But $\eta_{P+k} = \eta_{q-k} = 1$ since both $P+k$ and $q-k$ are virtual, so that this becomes

$$= \frac{1}{q^2 + \mu^2} \bar{\psi}(R) \tau_+ \frac{i\gamma(P+k) - m}{(P+k)^2 + m^2} \eta_P - \eta_q \frac{1}{(q-k)^2 + \mu^2} \bar{\psi}(R) \tau_+ \frac{i\gamma P - m}{P^2 + m^2}, \quad (2.7)$$

the two terms of which have again same functional form with arguments differing from each other by k .

2. The cases where an insertion of k_μ into vertex is possible
(cases b , c_2 and f_2)

For example, we consider following three diagrams constructed by inserting k_μ into two charged meson lines and a vertex,

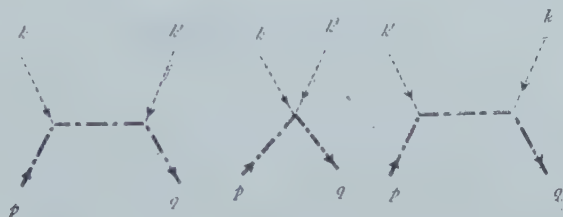


Fig. 3

and evaluate the corresponding matrices in terms of (2.2b) and (2.2c), making use of the Lorentz conditions for A_μ , A_ν , too.

$$\frac{1}{q^2 + \mu^2} 2iA_\nu(k') q_\nu \frac{1}{(p+k)^2 + \mu^2} 2iA_\mu(k) p_\mu \frac{1}{p^2 + \mu^2} \quad (2.8)$$

$$+ \frac{1}{q^2 + \mu^2} 2A_\mu(k) A_\nu(k') \delta_{\mu\nu} \frac{1}{p^2 + \mu^2} + \frac{1}{q^2 + \mu^2} 2iA_\mu(k) q_\mu \frac{1}{(q-k)^2 + \mu^2} 2iA_\nu(k') p_\nu \frac{1}{p^2 + \mu^2}.$$

By $A_\mu(k) \rightarrow ik_\mu$, we obtain a result of the same type as (2.5) or (2.7),

$$\frac{1}{q^2 + \mu^2} 2iA_\nu(k') q_\nu \frac{1}{(p+k)^2 + \mu^2} \eta_P - \eta_q \frac{1}{(q-k)^2 + \mu^2} 2iA_\nu(k') p_\nu \frac{1}{p^2 + \mu^2}. \quad (2.9)$$

Thus, from whatever elementary skeleton we may start, we obtain a result similar to (2.7) or (2.9) after $A_\mu(k) \rightarrow ik_\mu$ by taking into account all possible combinations with regard to k_μ . Proceeding with these procedures up to the initial and final charged segments of the skeleton, therefore, we reach the following conclusion.

Considering all possible connection of the skeleton with the particular photon k_μ , one gets in general an expression of the form

$$F(t) \eta_{initial} - \eta_{final} F(t-k), \quad (2.10)$$

after $A_\mu(k) \rightarrow ik_\mu$. When the charged line is open, $\eta_{initial} = \eta_{final} = 0$ since initial and final charged segments are both external, so that (2.10) vanishes and therefore in this case the gauge invariance is unambiguously guaranteed.

However, when the charged line forms a closed loop, initial and final segments coincide and are of course internal; accordingly $\eta_{initial} = \eta_{final} = 1$, and moreover the expression (2.10) should be integrated with respect to a certain intermediate energy-momentum. Namely, we have then

$$\int F(t) dt - \int F(t-k) dt. \quad (2.11)$$

This does not always give zero because of non-vanishing effects of the variable translation in a divergent integral, and therefore we can say nothing definite about the gauge invariance in such a case. We shall discuss this problem in detail in the next section.

B. Equivalence theorems

These theorems have been already investigated by many authors from their respective points of view.¹⁰ We now study them in the Feynman-Dyson formalism and examine to what extent (including higher order processes) they are applicable.

In spite of the before-mentioned analogy between gauge invariance and equivalence theorems, one should not lose sight of their essential difference: The former is a requirement which any formalism must always satisfy, while the latter is, so to speak, an expedience which can be utilized in certain limited cases. One should also notice that photon is neutral while meson may have charge. In fact the adequacy of equivalence theorems for charged meson is restricted to a very narrow region, as will be seen later.

Replacing the photon by the neutral meson and the charged line by the nucleonic line, we can apply the arguments used in the proof of gauge invariance theorem to the cases of equivalence. For that purpose we introduce the similar notions as the equivalent class \bar{D}_e , the skeleton \bar{D}_e etc.*

1. Neutral scalar meson

In the case of the equivalence theorems we have only one diagram for elementary skeleton \tilde{E}_e 's: The corresponding matrix is given by

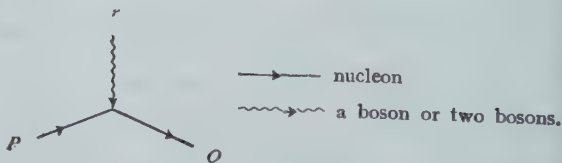


Fig. 4

$$\frac{i\gamma Q - m}{Q^2 + m^2} \chi(r) \xi \frac{i\gamma P - m}{P^2 + m^2}, \quad (2.12)$$

* It is natural to define the equivalent class as a group of those diagrams belonging to \mathcal{D} which are obtained by sliding the end of the meson line in question along the open line of nucleon. This definition coincides with the analogous one in the proof of gauge invariance, when the meson is neutral. The charged meson line, however, cannot be left out from the nucleonic line without destroying the conservation of charge. Therefore the notions as the skeleton lose their significance in the charged meson theory.

$\chi(r)$ being a wave function (or wave functions) of a boson (or bosons) r , and ξ the coupling between r and nucleon fields. For Fig. 4 and (2.12) one has of course the understanding as (2.1) and (2.3) respectively.

Now we consider the interaction between a neutral meson and a nucleon,

$$\frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' \frac{i\gamma P - m}{P^2 + m^2} \quad (\text{scalar coupling}). \quad (\text{Fig. 5}) \quad (2.13)$$

Changing the coupling to a vector one, one obtains from (2.13),

$$\frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' (i\gamma l) \frac{i\gamma P - m}{P^2 + m^2}.$$

Using the energy-momentum conservation $P + l = Q$ and taking into account the understanding as (2.1) and (2.3), this becomes

$$\begin{aligned} &= \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' (i\gamma Q + m - i\gamma P - m) \frac{i\gamma P - m}{P^2 + m^2} \\ &= \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' \gamma_P - \eta_Q \phi^0(l) \tau_3' \frac{i\gamma P - m}{P^2 + m^2}. \end{aligned} \quad (2.14)$$

Next we examine \tilde{E}_e by combining (2.13) and (2.14):

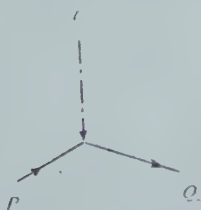


Fig. 5



Fig. 6

Matrices for vector coupling are

$$\begin{aligned} &\frac{i\gamma Q - m}{Q^2 + m^2} \chi(r) \xi \frac{i\gamma(P+l) - m}{(P+l)^2 + m^2} \phi^0(l) \tau_3' (i\gamma l) \frac{i\gamma P - m}{P^2 + m^2} \\ &+ \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' (i\gamma l) \frac{i\gamma(Q-l) - m}{(Q-l)^2 + m^2} \chi(r) \xi \frac{i\gamma P - m}{P^2 + m^2} \\ &= \frac{i\gamma Q - m}{Q^2 + m^2} \chi(r) \xi \frac{i\gamma(P+l) - m}{(P+l)^2 + m^2} \phi^0(l) \tau_3' \gamma_P - \eta_Q \phi^0(l) \tau_3' \frac{i\gamma(Q-l) - m}{(Q-l)^2 + m^2} \chi(r) \xi \frac{i\gamma P - m}{P^2 + m^2} \\ &+ \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \chi(r) (\tau_3' \xi - \xi \tau_3') \frac{i\gamma P - m}{P^2 + m^2}. \end{aligned} \quad (2.15)$$

(2.15) is of the same type as (2.7) or (2.9) except the third term. Hence, if

$$\tau_3' \xi - \xi \tau_3' = 0 \quad (2.16)$$

is valid for all i (i being the index of the vertices), just the same argument as in the gauge invariance theorem leads to the confirmation that the vector coupling in neutral scalar meson theory is of an illusory character, except in the case of closed nucleonic loop.

2. Neutral pseudoscalar meson

In this case the corresponding expression to (2.14) is

$$\begin{aligned}
 & \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' (i\gamma l) \gamma_5 \frac{i\gamma P - m}{P^2 + m^2} \\
 &= \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' (i\gamma Q + m - i\gamma P + m - 2m) \gamma_5 \frac{i\gamma P - m}{P^2 + m^2} \\
 &= -2m \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' \gamma_5 \frac{i\gamma P - m}{P^2 + m^2} \\
 &\quad - \frac{i\gamma Q - m}{Q^2 + m^2} \phi^0(l) \tau_3' \gamma_5 \eta_F - \eta_Q \phi^0(l) \tau_3' \gamma_5 \frac{i\gamma P - m}{P^2 + m^2}, \quad (2.17)
 \end{aligned}$$

the first term bringing out the equivalence of the pseudovector coupling to the case of pseudoscalar coupling. The remaining terms have an analogous form to (2.14) and one can easily derive the condition for the validity of the equivalence between both couplings of pseudoscalar meson theory except in the closed nucleonic loop:

$$\tau_3' \gamma_5 \xi_i + \xi_i \tau_3' \gamma_5 = 0, \quad \text{for all } \xi_i. \quad (2.18)$$

3. Charged scalar or pseudoscalar meson

In this case we cannot proceed to the second step in the above reasoning owing to the conservation of charge. It is in general impossible to combine an elementary skeleton with the charged meson line under consideration. Therefore, the equivalence theorems for charged meson are valid only in the following three exceptional cases:

- (i) Both nucleonic lines forming a vertex with a charged meson line are external. In such a case $\eta = 0$ in (2.15) and (2.17). (E.g. second order nuclear force.)
- (ii) The contact interaction plays the rôle of cancelling a redundant term as in §2-A-2. (E.g. second order $\gamma - \pi^\pm$ process.)
- (iii) All bosons except the charged meson under consideration are neutral meson, in which the equivalent class can possess its meaning.

4. Vector and pseudovector meson

Applying the above procedures to the tensor coupling of the vector (pseudovector) meson field, we obtain, besides a non-vanishing (vanishing) equivalent term to the vector (pseudovector) coupling, an additional term which represents

an interaction including the first order space-time derivative of nucleon wave function. In the case of neutral vector (pseudovector) meson, such an additional term alone appears if the following condition analogous to (2.16) or (2.18) is satisfied:

$$\begin{aligned} \text{neutral vector meson:} \quad & \tau_3' \gamma_\rho \hat{\xi}_4 + \hat{\xi}_4 \tau_3' \gamma_\rho = 0, \\ \text{neutral pseudovector meson:} \quad & \tau_3' \gamma_\rho \hat{\xi}_4 - \hat{\xi}_4 \tau_3' \gamma_\rho = 0, \quad \text{for all } \hat{\xi}_4. \end{aligned} \quad (2.19)$$

§ 3. Ambiguities due to a closed loop

Among the difficulties of the present quantum field theory there are those contradictions that the final result obtained in our covariant formalism does not always satisfy the requirements of the gauge invariance or equivalence, though they are of course formally fulfilled at the beginning of the calculation. These so-called "ambiguities of the current field theory" appear when the corresponding Dyson diagram contains a closed loop. In order to remove such ambiguities many authors have proposed various prescriptions.⁵⁽⁷⁻¹¹⁾

In this section we shall take the case of a closed charged line and examine the validity of the gauge invariance both in coordinate representation and in momentum representation. Comparing these two different verifications we can find a natural means to remove the apparent contradiction. The same argument can be applied to the case of the equivalence theorems.

We consider $n+1$ real neutral bosons interacting with $n+1$ virtual fermion lines and assume one of these bosons to be a photon. A gauge invariant class \bar{D}_θ for this process contains n diagrams, in every one of which the relative position of the boson except that photon is the same. Its representative diagram is given by Fig. 7.

The corresponding matrix is,* apart from a common numerical coefficient,

$$M(x_n, x_{n-1}) = \int dx_0 \cdots dx_n \{ S_F(x_0 - x_{n-1}) \gamma_\mu S_F(x_p - x_0) \Gamma'_p \cdots \Gamma'_{n-1} \} A_\mu(x_0) \phi_1(x_1) \cdots \phi_n(x_n), \quad (3.1)$$

in coordinate space, and

$$\begin{aligned} M(k_p, k_{p-1}) &= (-2i)^{n+1} \int dt_0 \cdots dt_n \left\{ \frac{i\gamma t_0 - m}{t_0^2 + m^2} \gamma_\mu \frac{i\gamma t_p - m}{t_p^2 + m^2} \Gamma_p \cdots \Gamma_{p-1} \right\} \\ &\cdot A_\mu(k_0) \phi_1(k_1) \cdots \phi_n(k_n) \delta(t_p + k_0 - t_0) \delta(t_{n+1} + k_p - t_p) \cdots \delta(t_0 + k_{p-1} - t_{p-1}) \end{aligned} \quad (3.1')$$

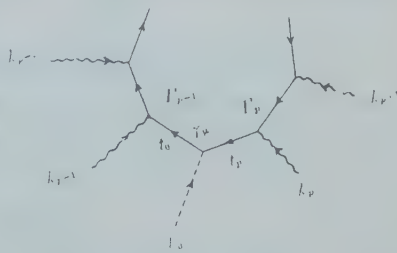


Fig. 7

* For simplicity's sake we shall omit here and in what follows in this section the operation S_{fur} which must be taken with respect to the fermion closed loop, because it is not essential to our argument,

in momentum space, where ϕ_p is the potential of the p -th boson field or its space-time derivative and I'_p the coupling between the boson k_p and the nucleon field, and p is assumed to have a period n , i.e., $p+n \equiv p$. The representations in coordinate and momentum space are connected with the Fourier transformations of $S_F(x)$, $A_\mu(x_0)$ and $\phi_p(x_p)$:

$$S_F(x) = \frac{-i}{8\pi^4} \int \frac{i\gamma t - m}{t^2 + m^2} e^{-itx} dt, \\ A_\mu(x_0) = A_\mu(k_0) e^{ik_0 x_0}, \quad \phi_p(x_p) = \phi_p(k_p) e^{ik_p x_p}. \quad (3.2)$$

Performing the gauge transformation $A_\mu(x_0) \rightarrow \partial A(x_0)/\partial x_{0\mu}$, (3.1) changes into

$$M'(x_p, x_{p-1}) = (-2i) \int dx_0 \cdots dx_n \{ \partial(x_p - x_0) S_F(x_0 - x_{p-1}) - \partial(x_0 - x_{p-1}) S_F(x_p - x_0) \} \cdot \\ \cdot \Gamma_p \cdots \Gamma_{p-1} A(x_0) \phi_1(x_1) \cdots \phi_p(x_p) \cdots \phi_n(x_n). \quad (3.3)$$

Here we have carried out a partial integration* and used the equation

$$\left(\gamma \frac{\partial}{\partial x} - m \right) S_F(x) = -2i \delta(x).$$

Correspondingly, $A_\mu(k_0) \rightarrow ik_{0\mu} A(k_0)$ transforms (3.1') into

$$M'(k_p, k_{p-1}) = (-2i)^{n+1} \int dt_0 \cdots dt_n \left\{ \frac{i\gamma t_0 - m}{t_0^2 + m^2} - \frac{i\gamma t_p - m}{t_p^2 + m^2} \right\} \Gamma_p \cdots \Gamma_{p-1} \cdot \\ \cdot A(k_0) \phi_1(k_1) \cdots \phi_n(k_n) \delta(t_p + k_0 - t_0) \delta(t_{p-1} + k_p - t_p) \cdots \delta(t_0 + k_{p-1} - t_{p-1}). \quad (3.3')$$

Up to this stage the two representations are connected uniquely by (3.2).

Now, integrating (3.3) with respect to x_0 and summing up from $p=1$ to $p=n$, we obtain

$$\sum_{p=1}^n M'(x_p, x_{p-1}) = 0. \quad (3.4)$$

Thus the gauge invariance is guaranteed if all diagrams belonging to the same \bar{D}_g are taken into account. Integrating (3.3'), however, with respect to t_0, t_2, \dots, t_n and summing up over all p , we obtain (writing t instead of t_1),

$$\sum_{p=1}^n M'(k_p, k_{p-1}) = (-2i)^{n+1} \int dt \{ F(t) - F(t - k_0) \} A(k_0) \phi_1(k_1) \cdots \phi_n(k_n), \quad (3.4')$$

where

$$F(t) = \prod_{j=1}^n \Gamma_j \frac{i\gamma(t - \sum_{l=1}^j k_l) - m}{(t - \sum_{l=1}^j k_l)^2 + m^2}$$

(This gives an example for (2.11).) (3.4') does not always vanish because¹²⁾

* We drop the surface integrals in the coordinate space. This corresponds to $k\delta(k)=0$ in momentum space.

$$\int dt \{ F(t) - F(t - k_0) \} = \int dt \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l!} \left(k_0 \cdot \frac{\partial}{\partial t} \right)^l F(t) \quad (3.5)$$

gives zero only if $\int dt F(t)$ is convergent or logarithmically divergent, but becomes finite and non-vanishing when $\int dt F(t)$ is linearly or quadratically divergent and diverges when $\int dt F(t)$ contains higher divergency. (See Appendix 3.) Thus we see that the gauge invariance is not always valid in momentum space, while it is guaranteed in coordinate space.

As is evident from above considerations this discrepancy originates in changing the order of the integrations in the divergent integral including singular functions. To make it clearer, we shall examine the most simple expression obtained by inserting (3.2) into (3.3) with $p=1$.

$$\begin{aligned} M'(1, n) = & \left(\frac{-i}{8\pi^4} \right)^{n+1} \int dx_0 \cdots dx_n dt_0 \cdots dt_n \left(\frac{i\gamma t_0 - m}{t_0^2 + m^2} - \frac{i\gamma t_1 - m}{t_1^2 + m^2} \right) l_1' \cdot \\ & \cdot \prod_{j=2}^n \left\{ \frac{i\gamma t_j - m}{t_j^2 + m^2} l_j' e^{-i\mu_j(t_j - t_{j-1})} e^{ik_j x_j} \right\} e^{-i\mu_0(x_0 - x_n)} e^{-i\mu_1(x_1 - x_0)} \cdot \\ & \cdot e^{ik_0 x_0} e^{ik_1 x_1} A(k_0) \phi_1(k_1) \cdots \phi_n(k_n). \end{aligned} \quad (3.6)$$

Integrating two terms of (3.6) separately, that is, integrating the first (second) term in the order $t_1(t_0)$, x_0 , one obtains,

$$\begin{aligned} M'(1, n) = & (-2i) \left(\frac{-i}{8\pi^4} \right)^n \int dx_1 \cdots dx_n dt_2 \cdots dt_n \\ & \left\{ \int dt_0 \frac{i\gamma t_0 - m}{t_0^2 + m^2} e^{-i\mu_0(x_1 - x_n)} e^{ik_0 x_1} - \int dt_1 \frac{i\gamma t_1 - m}{t_1^2 + m^2} e^{-i\mu_1(x_1 - x_n)} e^{ik_0 x_n} \right\} e^{ik_1 x_1} \Gamma_1 \cdot \\ & \cdot \prod_{j=2}^n \left\{ \frac{i\gamma t_j - m}{t_j^2 + m^2} l_j' e^{-i\mu_j(t_j - t_{j-1})} e^{ik_j x_j} \right\} A(k_0) \phi_1(k_1) \cdots \phi^n(k^n). \end{aligned} \quad (3.7)$$

Here the integration variable t_0 in the first term and t_1 in the second are completely independent of each other so that we may identify them. Then we can reach (3.4) by integrating (3.7) over all t 's and taking into account other members of the same \bar{D}_g . (3.7) will be still correct even if one integrates (3.6) inversely first with respect to x_0 , next t_1 in the first term and t_0 in the second. This time, however, t_0 and t_1 are not independent but are connected by the conservation law arising from the integration over x_0 , so that the different result from above will be obtained. Indeed, this is the case of (3.4').

After the above analysis we can now set up a method to remove the discrepancy between the two representations and to recover the gauge invariance in the practical calculation. For that purpose it is sufficient to notice that such a factor as (3.4'), which contradicts with the requirement of gauge invariance if (3.5) does not vanish, will necessarily appear in the actual evaluation of the matrix element as a non-gauge-invariant term. Hence such a term should be

ignored in order that one may obtain a consistent result. This rather crude procedure might be justified by the inference that in the future correct theory the gauge invariance will hold also in the momentum representation. Thus we have got the following prescription:

*Those non-gauge-invariant terms should be dropped which coincide, by the procedure $A_\mu(k) \rightarrow ik_\mu$, with (3.5) which remains in the formal proof of gauge invariance in momentum space.**

§ 4. Two-gamma-decay of neutral meson

This theme of "the decay of a neutral meson into two photons through virtual proton field" has been already analyzed by many authors. And as it reveals the so-called ambiguities most explicitly, it is regarded as a stumbling stone of the new formalism. In spite of several attempts to remove ambiguities,¹³⁾ it seems that the problem is still unsettled. We shall now reexamine it from our above-mentioned point of view.

There are two diagrams which represent this process: This pair of diagrams makes not only a gauge invariant class for the photon k and l but also an equivalent class for the neutral meson p .

The essential part of the matrix element is given in general by

$$M = I_{\mu\nu} A_\mu(k) A_\nu(l) \Phi(p),$$

$$I_{\mu\nu} = \int dt S p \left\{ \Gamma \frac{i\gamma t - m}{t^2 + m^2} \gamma_\mu \frac{i\gamma(t+k) - m}{(t+k)^2 + m^2} \gamma_\nu \frac{i\gamma(t+k+l) - m}{(t+k+l)^2 + m^2} \right\} + (k \leftrightarrow l, \mu \leftrightarrow \nu), \quad (4.1)$$

where $\Phi(p)$ means the neutral meson wave function, Γ the coupling of the neutral meson with the proton field and $(k \leftrightarrow l, \mu \leftrightarrow \nu)$ represents the expression obtained by interchanging k and l , μ and ν in the foregoing term.

A formal proof of gauge invariance is afforded by the substitution $A_\mu(k) \rightarrow ik_\mu$:**

$$M' = I'_\nu A(k) A_\nu(l) \Phi(p),$$

$$I'_\nu = i I_{\mu\nu} k_\mu$$

$$= \int dt S p \left\{ \Gamma \frac{i\gamma(t+k) - m}{(t+k)^2 + m^2} \gamma_\nu \frac{i\gamma(t+k+l) - m}{(t+k+l)^2 + m^2} \right\} - \int dt S p \left\{ \Gamma \frac{i\gamma t - m}{t^2 + m^2} \gamma_\nu \frac{i\gamma(t+l) - m}{(t+l)^2 + m^2} \right\}. \quad (4.2)$$

* This prescription is not trivial, because the non-gauge-invariant part in the matrix element is by no means unique, since any gauge-invariant expression could be added to it, and the term to be struck out remains in general ambiguous.

** As (4.1) is symmetric with respect to A_μ and A_ν , it is sufficient to examine the gauge invariance for one of them, the other can be deduced by symmetrization. (See, e.g., (4.9).)

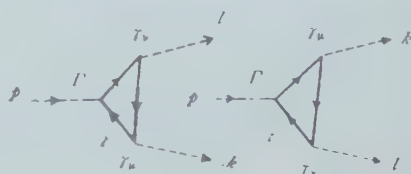


Fig. 8

1. *Scalar (scalar)**; $\Phi = \phi^0$, $\Gamma = 1$.

$$I'_\nu (S.s.) = -8im \int k_\lambda \frac{\partial}{\partial t_\lambda} \frac{t_\nu}{(t^2 + m^2)^2} dt$$

$$= 4\pi^2 m k_\nu, \quad (4.3)$$

$$I_{\mu\nu} (S.s.) = 8m \{ k_\nu l_\mu - (kl) \delta_{\mu\nu} \} \int dt \int_0^1 2x dx \int_0^1 dy \frac{1 - 4xy(1-x)}{(t^2 + L)^3}$$

$$- 4i\pi^2 m \delta_{\mu\nu}, \quad (4.4)$$

where $L = 2(kl)xy(1-x) + m^2$, x and y are Feynman's integration parameters. In (4.4) the first term is gauge invariant whereas the second is not. However, the latter coincides with (4.3) when $A_\mu(k) \rightarrow i k_\mu$ and so should be dropped according to our foregoing arguments.

2. *Pseudoscalar (pseudoscalar)*; $\Phi = \phi^0$, $\Gamma = \gamma_5$.

$$I'_\nu (Ps.ps.) = 0, \quad (4.5)$$

$$I_{\mu\nu} (Ps.ps.) = - \int dt \int_0^1 2x dx \int_0^1 dy \frac{8mk_\rho l_\sigma \epsilon(\mu\nu\rho\sigma)}{(t^2 + L)^3}, \quad (4.6)$$

with

$$\epsilon(\mu\nu\rho\sigma) = \begin{cases} 1 & \text{when } (\mu\nu\rho\sigma) \text{ is an even permutation of } (1234), \\ -1 & \text{,, odd ,,} \\ 0 & \text{otherwise.} \end{cases}$$

(4.6) is evidently gauge invariant.

3. *Pseudovector (pseudovector)*; $\Phi = \phi_\alpha^0$, $\Gamma = \gamma_5 \gamma_\alpha$, $(p_\alpha \phi_\alpha^0 = 0)$.

$$I'_\nu (Pv.pv.) = 4 \int dt k_\lambda \frac{\partial}{\partial t_\lambda} \frac{t_\rho l_\sigma}{(t^2 + m^2)^2} \epsilon(\omega\rho\sigma)$$

$$= 2i\pi^2 k_\rho l_\sigma \epsilon(\omega\rho\sigma), \quad (4.7)$$

$$I_{\mu\nu} (Pv.pv.) = -i \int dt \int_0^1 2x dx \int_0^1 dy \frac{1}{\{(t + kx + lxy)^2 + L\}^3} \cdot$$

$$\{ Sp(\gamma_5 \gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\sigma \gamma_\rho \gamma_\tau) t_\rho (t+k)_\sigma (t+k+l)_\tau + m^2 Sp(\gamma_5 \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\rho) (t+l)_\rho \}$$

$$+ (k \leftrightarrow l, \mu \leftrightarrow \nu).$$

After taking *Spur* by the help of (A.7) and (A.8), performing the variable translation $t \rightarrow t - kx - lxy$ in the first term and adding the contribution from the second,

$$= -\frac{4}{3} i \int dt \int_0^1 2x dx \int_0^1 dy \frac{1}{(t^2 + L)^3} [2k_\rho l_\sigma p_\alpha \epsilon(\mu\nu\rho\sigma) \{ x(1-x)(1+2y) + xy(1-xy) \}$$

* *Scalar (scalar)* or *(S.s.)* stands for *scalar meson with scalar coupling*, and so forth.

$$\begin{aligned}
& + 2(k_\rho I_\sigma \{k_\nu \varepsilon(\mu \mu \rho \sigma) - l_\mu \varepsilon(\mu \nu \rho \sigma)\} + (kl)(k-l)_\rho \varepsilon(\mu \mu \nu \rho)) \{x(1-x)(1-3y) + x^2 y(1-y)\} \\
& + 3(k-l)_\rho \left\{ \frac{t^2}{4} (1-3x+3y) + 2(kl)x^2 y(1-x)(1-y) + L(1+x-xy) \right\} \varepsilon(\mu \mu \nu \rho) \\
& + 4i \int dt \int_0^1 2x dx \int_0^1 dy (k-l)_\lambda x(1-y) \frac{\partial}{\partial t_\lambda} \frac{t^2 t_\rho}{(t^2 + L)^3} \varepsilon(\mu \mu \nu \rho). \quad (4.8)
\end{aligned}$$

But the second line of (4.8) may be replaced by $-p_\alpha \varepsilon(\mu \nu \rho \sigma)$ because

$$\begin{aligned}
& A_\mu A_\nu \phi_\alpha^0 [k_\rho I_\sigma \{k_\nu \varepsilon(\mu \mu \rho \sigma) - l_\mu \varepsilon(\mu \nu \rho \sigma)\} + (kl)(k-l)_\rho \varepsilon(\mu \mu \nu \rho)] \\
& = -A_\mu A_\nu \phi_\alpha^0 p_\alpha \varepsilon(\mu \nu \rho \sigma),
\end{aligned}$$

in virtue of the identity in the determinant theory,

$$(af)|bcde| + (bf)|cdea| + (cf)|deab| + (df)|eabc| + (ef)|abcd| \equiv 0,$$

where a, \dots, f are 4-vectors, (af) is the scalar product, $|bcde|$ the determinant $b_\mu c_\nu d_\rho e_\sigma \varepsilon(\mu \nu \rho \sigma)$ etc.

Moreover the apparent divergent term proportional to t^2 in the third line of (4.8) is in reality finite, for it changes, by the partial integrations of parameters, into

$$\int dt \int_0^1 2x dx \int_0^1 dy \frac{t^2(1-3x+3xy)}{(t^2 + L)^3} = - \int dt \int_0^1 2x dx \int_0^1 dy \frac{8(kl)x^2(1-x)y(1-y)}{(t^2 + L)^3},$$

according to (A.3), so that it cancels the next term.

Consequently $I_{\mu\nu}(Pv, pv.)$ is reduced to

$$\begin{aligned}
I_{\mu\nu}(Pv, pv.) & = -16ip_\alpha k_\rho l_\sigma \varepsilon(\mu \nu \rho \sigma) \int dt \int_0^1 2x dx \int_0^1 dy \frac{xy(1-y)}{(t^2 + L)^3} \\
& + 2\pi^2(k-l)_\rho \varepsilon(\mu \mu \nu \rho). \quad (4.9)
\end{aligned}$$

Of the numerical coefficient 2 in the second term, which is not gauge invariant for A_μ or A_ν , $8/3$ comes from the term proportional to L in the third line of (4.8) and $-2/3$ originates in the surface integral on the occasion of the variable translation, i.e., in the last term of (4.8). This non-gauge-invariant term should be disregarded since it coincides with (4.7) after $A_\mu(k) \rightarrow ik_\mu$. The first term of (4.9) is indeed gauge invariant, but vanishes because of the auxiliary condition for pseudovector meson field $p_\alpha \phi_\alpha^0 = 0$. Therefore the transition for $Pv. (pv.)$ is forbidden in agreement with Yang's selection rule.¹⁴⁾

4. *Pseudoscalar (pseudovector)*; $\Phi = \phi^0$, $\Gamma = ip_\alpha \gamma_5 \gamma_\alpha$.

$$I'_\nu(Ps, pv.) = 0 \quad (4.10)$$

$I_{\mu\nu}$ in this case may be evaluated in two different ways, that is, directly by utilizing the result of $(Pv, pv.)$ and far more easily by the help of equivalence theorems.

First $I_{\mu\nu}(Ps, pv.)$ will be obtained by multiplying (4.9) by ip_α . Then the second term too becomes gauge invariant and one gets

$$I_{\mu\nu}(Ps, pv.) = -2m \int dt \int_0^1 2x dx \int_0^1 dy \frac{8mk_\rho l_\sigma \epsilon(\mu\rho\sigma)}{(t^2+L)^3} + 4i\pi^2 k_\rho l_\sigma \epsilon(\mu\nu\rho\sigma). \quad (4.11)$$

The first term is equivalent to $I_{\mu\nu}(Ps, ps.)$ but the second is not. However, as the condition (2.18) is fulfilled here, such an inequivalent term ought to have arisen from the proton closed loop. Indeed, after the separation of the equivalent term by the procedure described in § 2-B, we obtain the inequivalent term:

$$\begin{aligned} I_{\mu\nu}(Ps, pv.)_{\text{inequiv.}} &= \int dt S p \left\{ \gamma_5 \frac{i\gamma t - m}{t^2 + m^2} \gamma_\mu \frac{i\gamma(t+k) - m}{(t+k)^2 + m^2} \gamma_\nu \right\} + (k \leftrightarrow l, \mu \leftrightarrow \nu) \\ &\quad - \int dt S p \left\{ \gamma_5 \frac{i\gamma(t+k) - m}{(t+k)^2 + m^2} \gamma_\nu \frac{i\gamma(t+k+l) - m}{(t+k+l)^2 + m^2} \gamma_\mu \right\} - (k \leftrightarrow l, \mu \leftrightarrow \nu) \\ &= 2 \int dt \left\{ k_\lambda \frac{\partial}{\partial t_\lambda} \frac{t_\rho k_\sigma}{(t^2 + m^2)} + (2k+l)_\lambda \frac{\partial}{\partial t_\lambda} \frac{t_\rho l_\sigma}{(t^2 + m^2)^2} \right\} \epsilon(\mu\nu\rho\sigma) + (k \leftrightarrow l, \mu \leftrightarrow \nu) \\ &= 4i\pi^2 k_\rho l_\sigma \epsilon(\mu\nu\rho\sigma), \end{aligned} \quad (4.12)$$

which agrees with the second term of (4.11), so that this term should be ignored. Thus the equivalence theorem for pseudoscalar meson in this process is restored.

As is seen from this example, this technique of applying the equivalence theorem makes the calculation remarkably easier in comparison with the straightforward procedure.

5. Other cases.

In $S(v.)$ and $V(v.)$ there appear only non-gauge-invariant terms in $I_{\mu\nu}$, but one may drop them again by the reason that they coincide with I'_ν after $A_\mu(k) \rightarrow ik_\mu$. In $V(t.)$ and $Pv.(t.)$ $I_{\mu\nu}$ vanishes exactly. Hence our prescription does not contradict with Furry's theorem.¹⁵⁾

§ 5. Concluding remarks

We have shown above that, provided the effects of the variable translation in a divergent integral are estimated accurately, we can get satisfactory and consistent results for the two-gamma-decay of the neutral meson in complete agreement with various requirements (gauge invariance theorem, equivalence theorems, Furry's theorem and Yang's selection rule) by the help of only one simple rule that the non-gauge invariant or inequivalent terms which correspond to the remaining surface integrals in the formal proof of these theorems should be struck off. Our procedure is *necessary and sufficient* to remove those internal inconsistencies related to closed loops of charged or nucleonic lines in all cases. Of course this does not yet imply that we can always obtain an unambiguous result, since it may happen that two expressions different from one another by a gauge-invariant term appear equally justified even by our prescription. But in practice such cases are rather *artificial*,* and the requirements of gauge invariance

* Imagine two non-gauge-invariant terms X_μ and Y_μ . If $(Xk) = (Yk)$, we could either discard X_μ itself, or transform it into $(X_\mu - Y_\mu) + Y_\mu$ and discard Y_μ , retaining gauge invariant $(X_\mu - Y_\mu)$.

and equivalence are in most cases sufficient to determine the matrix element uniquely. We hope, in any case, that our rule can eliminate a good deal, if not all, of the ambiguities which have hindered us from carrying out the estimation of the transition probabilities.

At first one might be inclined to give up the standpoint adopted by us and to permit a free translation of the variable even in a divergent integral, taking no notice of its effect. Indeed by this procedure the discrepancy in the formal proof disappears, and most of the non-gauge-invariant or inequivalent terms are also struck out. However, there are other non-gauge-invariant or inequivalent terms presenting themselves *not* in a form of a surface integral as stated with regard to (4.9), and these do remain and spoil the whole result. They represent one of the difficulties which have bothered us from the early stage of the covariant formulation.

The methods which have been hitherto proposed to get rid of this ambiguity may be classified roughly into two groups, the one being the so-called regulator method (Pauli-Villars),⁽⁶⁾ and the other the assignation of appropriate values to some typical integrals of singular functions including convergent ones (Katayama, Fukuda-Kinoshita).⁽⁹⁾ As was pointed out by many authors the regulator method, when applied to the two-gamma-decay of neutral meson, contradicts with the equivalence theorems since it takes away some terms of physical significance together with the non-gauge-invariant terms.* In the latter method the circumstances are similar, *i.e.*, we fear that certain important terms might drop as the mathematically incorrect identities are employed there. Thus both procedures seem either inconsistent or too severe. On the contrary, our prescription may certainly be insufficient to turn away all kinds of ambiguities, but it will not commit a fault of eliminating a physically significant term.

In concluding we wish to express our sincere thanks to Messrs. H. Fukuda and T. Kinoshita for their useful advices and kindly having shown us their manuscript before publication, to Mr. Y. Katayama for valuable discussions, and especially to Mr. T. Kotani for stimulating debates and constant collaboration. One of us (Z.K.) is much obliged to Yukawa-Fellowship of Osaka University for the financial aid.

Appendix

1. Interaction Hamiltonian in interaction representation ($\hbar=c=1$.)

(Only those used in § 2, *i.e.*, the interactions between nucleon, scalar or pseudoscalar meson and photon, are mentioned.)

$$H^{nr} = -i\bar{\psi}\gamma_{\mu}\tau_F\psi A_{\mu},$$

* Recently Peaslee treated the infinite integrals by dropping the surface terms in momentum space, and showed that his procedure was equivalent in some cases to the regulator method.⁽⁷⁾

$$\begin{aligned}
H^{mr} &= ieA_\mu(\phi^* \cdot \partial\phi/\partial x_\mu - \partial\phi^*/\partial x_\mu \cdot \phi) + e^2 A_\mu^2 \phi^* \phi, \\
H^{nm} &= if\bar{\psi}\gamma_5' \tau_+ \psi \phi^* + if\bar{\psi}\gamma_5' \tau_- \psi \phi + if^0\bar{\psi}\gamma_5' \tau_3 \psi \phi^0 \\
&\quad + ig'\bar{\psi}\gamma_5' \gamma_\alpha \tau_+ \psi \cdot \partial\phi^*/\partial x_\alpha + ig'\bar{\psi}\gamma_5' \gamma_\alpha \tau_- \psi \cdot \partial\phi/\partial x_\alpha + ig^0\bar{\psi}\gamma_5' \gamma_\alpha \tau_3 \psi \cdot \partial\phi^0/\partial x_\alpha, \\
H^{nmr} &= -eg'\bar{\psi}\gamma_5' \gamma_\mu \tau_+ \psi A_\mu \phi^* + eg'\bar{\psi}\gamma_5' \gamma_\mu \tau_- \psi A_\mu \phi.
\end{aligned}$$

where $\gamma_5' = -i$, $g' = ig$ and $g^{0'} = ig^0$ for scalar meson $\gamma_5' = \gamma_5$, $g' = g$ and $g^{0'} = g^0$ for pseudoscalar meson; $\tau_3' = 1$ for charged plus pure neutral meson theory and $\tau_3' = \tau_3$ for symmetrical meson theory. The normal-dependent terms are omitted since they are not necessary for the evaluation of the matrix element.¹⁶⁾

2. Integration formulas (l, m, n etc.: zero or positive integer.)

$$\int dt \frac{t^{2m} \prod_{i=1}^{2l} t_{a_i}}{(t^2 + L)^n} = \frac{1}{2^l (l+1)!} \int dt \frac{t^{2(m+l)}}{(t^2 + L)^n} A_l(a_1 \cdots a_{2l}), \quad (\text{A. 1})$$

where $dt = dt_1 dt_2 \cdots dt_{2l}$, $t^{2m} = (t \cdot t)^m$, L is independent of t , and $A_l(a_1 \cdots a_{2l})$ is obtained by summing up products of l Kronecker δ 's over all possible pairs of $2l$ suffices a_1, \dots, a_{2l} , so that it consists of $(2l)!/2^l \cdot l!$ terms, e.g.,

$$A_2(\mu\nu\rho\sigma) = \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}.$$

$$\int dt \frac{t^{2m}}{(t^2 + L)^n} = \frac{(m+1)! (n-m-3)!}{(n-1)!} \frac{i\pi^2}{L^{n-m-2}},$$

only when $n-m \geq 3$. (A. 2)

A useful formula utilizing the partial integration of a parameter is

$$\int dt \int_0^1 dx \frac{(l+1)x^l - (l-l'+1)x^{l-l'}}{(t^2 + L)^n} t^{2m} = - \int dt \int_0^1 dx \frac{(m+1)x^{l-l'+1}(1-x^{l'}) \cdot \partial L / \partial x}{(t^2 + L)^n} t^{2(m-1)},$$

where $n-m \geq 2$, $l \geq l' \geq 1$. (A. 3)

It is frequently employed to decrease the order of the integral with regard to t .

To combine two integrals having different denominator from each other, use the following formula:

$$\begin{aligned}
\frac{1}{(t^2 + A)^n} &= \int_0^1 \frac{dx}{\{t^2 + A - (A-B)x\}^n} - n \int_0^1 dx \frac{(A-B)(1-x)}{\{t^2 + A - (A-B)x\}^{n+1}}, \\
\frac{1}{(t^2 + B)^n} &= \int_0^1 \frac{dx}{\{t^2 + A - (A-B)x\}^n} + n \int_0^1 dx \frac{(A-B)x}{\{t^2 + A - (A-B)x\}^{n+1}}, \quad (\text{A. 4})
\end{aligned}$$

in which both A and B are independent of t and x . As their applications we have

$$\begin{aligned}
\int dt \left\{ \frac{1}{(t^2 + A)^n} - \frac{1}{(t^2 + B)^n} \right\} t^{2m} &= -n \int dt \int_0^1 dx \frac{(A-B)t^{2m}}{\{t^2 + A - (A-B)x\}^{n+1}}, \text{ for all } m, n; \\
\int dt \frac{t^{2m}}{(t^2 + A)^n} &= \frac{n}{n-m-2} \int dt \int_0^1 dx \frac{B - (n-m-3)(A-B)(1-x)}{\{t^2 + A - (A-B)x\}^{n+1}} t^{2m}, \quad n-m \geq 3.
\end{aligned}$$

3. Estimation of surface integrals

Using (A. 1) and (A. 2) one finds :

$$\begin{aligned} \int dt \frac{\partial f(t)}{\partial t_p} &\equiv \int dt \frac{\partial}{\partial t_p} \frac{t^{2m} \prod_{i=1}^{2l-1} t_{\alpha_i}}{(t^2 + L)^n} \\ &= \int dt \left\{ \frac{l+m+1-n}{2^{l-1}(l+1)!} \frac{t^{2(l+m+1)}}{(t^2 + L)^n} + \frac{nL}{2^{l-1}(l+1)!} \frac{t^{2(l+m+1)}}{(t^2 + L)^{n+1}} \right\} \Delta_l(\rho a_1 \cdots a_{2l-1}) \quad (\text{A. 5}) \\ &= \begin{cases} 0 & \text{if } n > l+m+1, \text{ i.e., } \int f(t) dt \text{ converges or} \\ & \text{logarithmically diverges,} \\ \frac{i\pi^2}{2^{l-1}(l+1)!} \cdot \Delta_l(\rho a_1 \cdots a_{2l-1}) & \text{if } n = l+m+1, \text{ i.e., } \int f(t) dt \text{ is linearly} \\ & \text{divergent,} \\ \infty & \text{if } n < l+m+1. \end{cases} \end{aligned}$$

By iterating (A. 5) it is shown that

$$\begin{aligned} \int dt \frac{\partial^2 g(t)}{\partial t_p \partial t_\sigma} &\equiv \int dt \frac{\partial}{\partial t_p} \frac{\partial}{\partial t_\sigma} \frac{t^{2m} \prod_{i=1}^{2l} t_{\alpha_i}}{(t^2 + L)^n} \\ &= \begin{cases} 0 & \text{if } n > l+m+1, \\ \frac{i\pi^2}{2^{l-1}(l+2)!} \Delta_{l+1}(\rho \sigma a_1 \cdots a_{2l}) - \frac{i\pi^2}{2^{l-1}(l+1)!} \Delta_l(a_1 \cdots a_{2l}) \delta_{\rho\sigma} & \text{if } n = l+m+1, \text{ i.e., } \int g(t) dt \text{ is} \\ & \text{quadratically divergent,} \\ \infty & \text{if } n < l+m+1. \end{cases} \quad (\text{A. 6}) \end{aligned}$$

From (A. 5) and (A. 6), and taking into account the fact that the integral of an odd function over the whole region vanishes, one can conclude that the effect of the variable translation (the surface integral) in convergent or logarithmically divergent integral does not appear while it becomes finite in linearly or quadratically divergent integral and involves divergence in more highly divergent integral.

4. Calculation of Spurs

$$Sp(\gamma_s \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4\epsilon(\mu\nu\rho\sigma), \quad (\text{A. 7})$$

$$\begin{aligned} &Sp(\gamma_s \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau \gamma_\omega) \\ &= \frac{8}{3} \{ \delta_{\mu\nu} \epsilon(\rho\sigma\tau\omega) - \delta_{\nu\rho} \epsilon(\sigma\tau\omega\mu) + \delta_{\rho\sigma} \epsilon(\tau\omega\mu\nu) \\ &\quad - \delta_{\sigma\tau} \epsilon(\omega\mu\nu\rho) + \delta_{\tau\omega} \epsilon(\mu\nu\rho\sigma) - \delta_{\omega\mu} \epsilon(\nu\rho\sigma\tau) \} \\ &- \frac{4}{3} \{ \delta_{\mu\rho} \epsilon(\nu\sigma\tau\omega) - \delta_{\nu\sigma} \epsilon(\rho\tau\omega\mu) + \delta_{\rho\tau} \epsilon(\sigma\omega\mu\nu) \\ &\quad - \delta_{\sigma\omega} \epsilon(\tau\mu\nu\rho) + \delta_{\tau\mu} \epsilon(\omega\nu\rho\sigma) - \delta_{\omega\nu} \epsilon(\mu\rho\sigma\tau) \}. \end{aligned} \quad (\text{A. 8})$$

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Theory of Antiferromagnetism and Antiferromagnetic Resonance Absorption, I

Takeo NAGAMIYA

Department of Physics, Osaka University

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It will be pointed out that the anisotropy energy is of primary importance in the theory of antiferromagnetism. The formula for the susceptibility $\chi = (\chi_{||} + 2\chi_{\perp})/3$, which follows in van Vleck's theory from a rather arbitrary assumption that the parallel and perpendicular domains are distributed with a ratio 1:2, can only be justified by inclusion of anisotropy energy. Formula for the dependency of the susceptibility upon external magnetic field will be derived and it will be shown that the field-dependent part varies proportional to the square of the intensity of the magnetic field, and inversely proportional to the anisotropy constant.

§ 1. Introduction

Antiferromagnetism, originally proposed by Néel¹⁾ and formulated by van Vleck²⁾ and others, has recently been studied from various sides.³⁾⁻⁷⁾ It is supposed that the crystal lattice is divided into two sublattices, each of which being occupied by + and - spins, respectively, at the absolute zero of temperature, and an order-disorder transition is supposed to occur at the Curie point. This picture cannot of course be an exact one, as actually shown by Slater⁸⁾ for the case of one-dimensional assembly of spins of magnitude 1/2 with a negative exchange integral. The picture is exact only for the case of classical spins of infinite magnitude which interact with each other according to the usual exchange law. It may, however, be perhaps a good approximation for typical antiferromagnetics such as MnO and Cr₂O₃ where the magnetic ions have spins of magnitude greater than 1/2. In van Vleck's theory, which may be regarded as standard, it is assumed that each \pm spin is subjected to a molecular magnetic field arising from the surrounding \mp spins and to the external magnetic field, and that the effective field acting upon the spin \mathbf{S}^{\pm} is given by

$$\mathbf{H}_{eff}^{\pm} = \mathbf{H} + 2Jzg^{-1}\beta^{-1}\bar{\mathbf{S}}^{\mp}, \quad (1)$$

where J is the negative exchange integral, z the number of nearest neighbors, $g=2$ the Landé factor, $\beta = \hbar e / 4\pi mc$ the Bohr magneton, H the external magnetic field, and $\bar{\mathbf{S}}^{\mp}$ the statistical average of \mathbf{S}^{\pm} at a given temperature.

Now $\bar{\mathbf{S}}^{\pm}$ and \mathbf{H}_{eff}^{\pm} must have the same direction in order that the theory is free from inner contradiction. We therefore put

$$g\beta\bar{\mathbf{S}}^{\pm} = \chi^{\pm}\mathbf{H}_{eff}^{\pm} \quad (2)$$

On solving (1), (2), we have

$$g\beta\bar{S}^{\mp} = \frac{\chi^{\pm}(1+2Jzg^{-2}\beta^{-2}\chi^{\mp})}{1-(2Jzg^{-2}\beta^{-2})^2\chi^{+}\chi^{-}}\mathbf{H} \quad (3)$$

so that \bar{S}^{\mp} are parallel to the external field, except for the case that

$$\chi^{+}=\chi^{-}=1/(-2Jzg^{-2}\beta^{-2}), \quad (4)$$

where \bar{S}^{\pm} are perpendicular to the external field for vanishing field strength. For the latter case we have

$$\mathbf{M} = \frac{1}{2}Ng\beta(\bar{S}^{+} + \bar{S}^{-}) = \frac{Ng^2\beta^2}{-4Jz}\mathbf{H}. \quad (5)$$

\mathbf{M} is the intensity of magnetization per unit volume and N is the number of atoms per unit volume. (5) is the formula derived by van Vleck for the perpendicular case.

Van Vleck assumed that there are parallel and perpendicular domains with a ratio 1:2 and gave the susceptibility formula $\chi=(\chi_{||}+2\chi_{\perp})/3$. This assumption cannot be justified, however, since the difference in free energy per unit volume of the two types of domains, namely $\frac{1}{2}(\chi_{||}-\chi_{\perp})H^2$, is positive so that only perpendicular domains must be realized under thermal equilibrium. It then follows that the susceptibility is constant below the Curie point, unlike observations.

The anisotropy energy of antiferromagnetics is of the same order of magnitude as that of ferromagnetics as referred to by Néel.⁹ In ferromagnetics, the energy to the external magnetic field $-\mathbf{MH}$ can be greater than the anisotropy energy, and the magnetization \mathbf{M} can easily be turned to the direction of the external field. In antiferromagnetics, however, the energy to the external field $-\frac{1}{2}\chi H^2$ is so small that each spontaneous magnetization of the two sublattices can hardly deviate from the direction of easy direction for an ordinary field strength. Theory of the susceptibility of antiferromagnetics must thus be based on a consideration which takes into account the anisotropy energy from the beginning.

It is also hardly possible to account for the anomalous resonance absorption observed for Cr_2O_3 on the basis of existing theories, but it will be shown that it follows naturally from a theory which takes into account the anisotropy energy. A theory of the susceptibility of antiferromagnetics and its resonance absorption based on the mentioned idea will be developed in the following lines.

§ 2. The susceptibility at absolute zero

We shall first confine ourselves to the case of absolute zero. The magnetic moments of the \pm sublattices are $\mathbf{M}^{\pm}=\frac{1}{2}Ng\beta\mathbf{S}^{\pm}$; their direction cosines will be denoted as $a^{\pm}, \beta^{\pm}, \gamma^{\pm}$. Thus

$$\mathbf{M}^{\pm}=\frac{1}{2}Ng\beta\mathbf{S}^{\pm}=\mathbf{M}_0(a^{\pm}, \beta^{\pm}, \gamma^{\pm}), \quad \mathbf{M}_0=|\mathbf{M}^{\pm}|. \quad (6)$$

We assume that there is one axis of easy magnetization or there are several of

them which are mutually equivalent, and that the anisotropy energy has rotational symmetry about any one of these axes for small angle of deviation from it for M^+ or M^- . Since the magnetization vectors can not deviate significantly from a certain one of the axes of easy magnetization for a given magnetic domain, we shall select this axis as the z -axis and fix the x - and y -axes perpendicular to it. The anisotropy energy is then expressible for small angle of deviation in the form

$$F = \frac{1}{2} K [(a^{+2} + \beta^{+2}) + (a^{-2} + \beta^{-2})]. \quad (7)$$

The effective magnetic field acting upon M^\pm due to this energy is

$$- \frac{K}{M_0} (a^\pm, \beta^\pm, 0). \quad (8)$$

This must be added to the right hand side of (1) to obtain the total effective field, and the latter must have the same direction as M^\pm , so that we have

$$H - AM^\mp - \frac{K}{M_0} (a^\pm, \beta^\pm, 0) = M^\pm / \chi^\pm, \quad (9)$$

where

$$A = -4j_z g^{-2} \beta^{-2} / N. \quad (10)$$

We shall assume that the external magnetic field is in the yz -plane and denote its direction cosines by 0, β_H , γ_H , and solve (9). Eliminating χ^\pm we have

$$\frac{H\beta_H - AM_0\beta^\mp - (K/M_0)\beta^\pm}{H\gamma_H - AM_0\gamma^\mp} = \frac{\beta^\pm}{\gamma^\pm}. \quad (11)$$

Since the angle of deviation was assumed to be small, we put $\gamma^\pm = \pm 1$ and have

$$\beta^\pm = \frac{\mp H\gamma_H + K/M_0}{2AK - (H\gamma_H)^2 + (K/M_0)^2} H\beta_H. \quad (12)$$

It may generally be assumed that the molecular magnetic field is far greater than the external and effective anisotropy magnetic fields so that the second and the third terms in the denominator of (12) may be neglected compared with the first. We then have

$$\beta^\pm = \frac{H\beta_H}{2AM_0} \mp \frac{H^2\gamma_H\beta_H}{2AK}. \quad (13)$$

Therefore the total magnetization is

$$M = M^+ + M^- = (0, H\beta_H/A, 0),$$

and the susceptibilities parallel and perpendicular to the axis of magnetization are

$$\chi_{||} = 0, \quad \chi_{\perp} = \frac{1}{A}. \quad (14)$$

In powder specimens the axis are oriented at random so that its susceptibility is

$$\chi = \frac{1}{3}(\chi_{\parallel} + 2\chi_{\perp}) = \frac{2}{3A}. \quad (15)$$

Formulas (14) and (15) are in agreement with the result obtained by van Vleck, but there is an essential difference in the meaning of parallel and perpendicular. In our case they refer to the axis of easy magnetization whereas in van Vleck's theory they imply the direction of the spontaneous magnetizations of the sublattices with respect to the external field.

A higher approximation to the solution of (11) for strong field can be obtained by putting $\gamma^{\mp} = \pm(1 - \frac{1}{2}\beta^{\pm 2})$. The result is as follows:

$$\frac{1}{2}(\beta^{+} + \beta^{-}) = -\frac{H\beta_n}{2AM_0} \left(1 + \frac{(H\gamma_n)^2}{2AK} + O(H^4) \right), \quad (16)$$

$$\frac{1}{2}(\beta^{+} - \beta^{-}) = -\frac{H^2\beta_n\gamma_n}{2AK} \left(1 - \frac{(H\beta_n)^2}{2AK} + \frac{(H\gamma_n)^2}{2AK} + O(H^4) \right), \quad (16a)$$

$$\begin{aligned} \frac{1}{2}(\gamma^{+} + \gamma^{-}) &= \frac{1}{2}(\sqrt{1 - \beta^{+2}} + \sqrt{1 - \beta^{-2}}) = -\frac{(\beta^{+} + \beta^{-})(\beta^{+} - \beta^{-})}{4} \\ &= -\frac{H\beta_n}{2AM_0} \cdot \frac{H^2\beta_n\gamma_n}{2AK} (1 + O(H^2)). \end{aligned} \quad (17)$$

In deriving these formulas it was assumed that H is very small compared with the molecular magnetic field AM and moderately small compared with $(AM_0 \cdot K/M_0)^{\frac{1}{2}}$ or $(AK)^{\frac{1}{2}}$. The second term in the bracket of (16) comes from the denominator of (12). The magnetization parallel to the applied field is

$$\begin{aligned} M_0[(\beta^{+} + \beta^{-})\beta_n + (\gamma^{+} + \gamma^{-})\gamma_n] \\ = \frac{H}{A} \left(\beta_n^2 + \frac{H^2\beta_n^2\gamma_n^2}{AK} + O(H^4) \right) = \chi H \end{aligned} \quad (18)$$

where χ is the corresponding susceptibility. The susceptibility of a powdered specimen is therefore

$$\chi = \frac{1}{A} \left(\frac{2}{3} + \frac{2}{15} \frac{H^2}{AK} + O(H^4) \right) \quad (19)$$

It is observed that the susceptibility increases with increasing applied field strength. In the case of $\text{MnO}^{(9)}$ it is found that χ is about 58×10^{-6} and 67×10^{-6} for $H=7000$ Oe and 24000 Oe at absolute zero. Comparing this result with (19) we find $AK=6.6 \times 10^8$ Oe². On the other hand, $AM_0^2=(3/2)NkT_0$, as we shall see below (or according to van Vleck). Putting $2M_0/N=5$ Bohr magnetons $=4.6 \times 10^{-20}$ and $T_0=120^\circ\text{K}$, we find $AM_0=1.1 \times 10^6$ Oe, and therefore $K/M_0=6 \times 10^2$ Oe. The last value is of the same order of magnitude as that found for ordinary ferromagnetic materials. However, since H^2 is of the same order of magnitude as AK in this example, our formula (19) is not a good approximation and the last figure must be considered as representing only its order of magnitude.

§ 3. Susceptibility at an arbitrary temperature

The anisotropy energy may be considered as very small compared with NkT when the temperature is not extremely low, say for $T > 1^\circ\text{K}$. The statistical distribution of spins is therefore determined mainly by the molecular magnetic field. The specific heat, the Curie temperature, and the susceptibility above the Curie point are given almost exactly by usual theories. We shall show that the susceptibility below the Curie point also agree with that given by van Vleck for a powdered specimen if "parallel" and "perpendicular" are referred to the magnetic axis.

In this section we shall specialize our model in the following way. The spins will be assumed classical, and each of them will be assumed to rotate in a molecular anisotropy field. Let each spin magnetic moment in $+$ or $-$ sublattice be denoted as $\mathbf{m}^\pm = g\beta\mathbf{S}^\pm$ and the anisotropy energy as $f(\mathbf{m}^\pm)$. The potential energy for each spin is then

$$V^\pm = -\mathbf{m}^\pm \mathbf{H}_{eff}^\pm + f(\mathbf{m}^\pm), \quad \mathbf{H}_{eff}^\pm = \mathbf{H} - a\mathbf{m}^\mp, \quad a = N/2. \quad (20)$$

Since it is assumed that $f(\mathbf{m}^\pm) \ll kT$, the distribution is

$$\exp(-V^\pm/kT) = \exp(\mathbf{m}^\pm \mathbf{H}_{eff}^\pm/kT) (1 - f(\mathbf{m}^\pm)/kT),$$

and the mean atomic magnetic moment is

$$\bar{\mathbf{m}}^\pm = \frac{\int \mathbf{m}^\pm \exp(\mathbf{m}^\pm \mathbf{H}_{eff}^\pm/kT) (1 - f(\mathbf{m}^\pm)/kT) d\omega^\pm}{\int \exp(\mathbf{m}^\pm \mathbf{H}_{eff}^\pm/kT) (1 - f(\mathbf{m}^\pm)/kT) d\omega^\pm}, \quad (21)$$

where $d\omega^\pm$ is the element of the solid angle for \mathbf{m}^\pm .

For the following we shall drop the suffix *eff* and confine our attention to $+$ spin. The usual anisotropy energy per atom is the thermal mean of $f(\mathbf{m}^+)$ and is given by

$$\bar{f}(\mathbf{H}^+) = \frac{\int \exp(\mathbf{m}^+ \mathbf{H}^+/kT) f(\mathbf{m}^+) d\omega^+}{\int \exp(\mathbf{m}^+ \mathbf{H}^+/kT) d\omega^+}. \quad (22)$$

As before, we shall assume that this quantity has rotational symmetry about each of the axes of easy magnetization for small angle of deviation for the atomic magnetic moment, so that it is expressible in the form

$$\bar{f}(\mathbf{H}^+) = \alpha(a^2 + \beta^2). \quad (23)$$

Here a^+ and β^+ are the x - and y -component of the direction cosines of \mathbf{H}^+ , the magnetic axis being chosen as the z -axis, and α is a constant which depends on $|\mathbf{H}^+|$. (23) corresponds to (7) by putting $N\alpha = K$. Now

$$\begin{aligned} & \int \mathbf{m}^+ \exp(\mathbf{m}^+ \mathbf{H}^+/kT) f(\mathbf{m}^+)/kT \cdot d\omega^+ \\ &= \text{grad}_{\mathbf{H}^+} \int \exp(\mathbf{m}^+ \mathbf{H}^+/kT) f(\mathbf{m}^+) d\omega^+ \end{aligned}$$

$$\begin{aligned}
 &= \text{grad}_{\mathbf{H}^+} [\bar{f}(\mathbf{H}^+) \cdot \int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+] \\
 &= \text{grad}_{\mathbf{H}^+} \bar{f}(\mathbf{H}^+) \cdot \int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+ \\
 &\quad + \bar{f}(\mathbf{H}^+) / kT \cdot \int \mathbf{m}^+ \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+,
 \end{aligned}$$

where

$$\text{grad}_{\mathbf{H}^+} = \left(\frac{\partial}{\partial H_x^+}, \frac{\partial}{\partial H_y^+}, \frac{\partial}{\partial H_z^+} \right).$$

Thus (21) becomes, by use of (22),

$$\begin{aligned}
 \overline{\mathbf{m}^+} &= \frac{\int \mathbf{m}^+ \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+ \cdot (1 - \bar{f}(\mathbf{H}^+) / kT)}{\int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+ \cdot (1 - \bar{f}(\mathbf{H}^+) / kT)} \\
 &\quad - \frac{\text{grad}_{\mathbf{H}^+} \bar{f}(\mathbf{H}^+) \cdot \int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+}{\int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+ \cdot (1 - \bar{f}(\mathbf{H}^+) / kT)} \\
 &= \frac{\int \mathbf{m}^+ \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+}{\int \exp (\mathbf{m}^+ \mathbf{H}^+ / kT) d\omega^+} - \frac{\text{grad}_{\mathbf{H}^+} \bar{f}(\mathbf{H}^+)}{1 - \bar{f}(\mathbf{H}^+) / kT}.
 \end{aligned}$$

The first term is expressible by Langevin function and the denominator in the second term can be replaced by 1. We have

$$\overline{\mathbf{m}^+} = \frac{\mathbf{H}^+}{|\mathbf{H}^+|} mL(m|\mathbf{H}^+|/kT) - \text{grad}_{\mathbf{H}^+} \bar{f}(\mathbf{H}^+). \quad (24)$$

It follows from (23) that

$$\begin{aligned}
 \frac{\partial \bar{f}}{\partial H_x^+} &= \frac{\partial x}{\partial |\mathbf{H}^+|} (\alpha^{+2} + \beta^{+2}) \alpha^+ + \frac{2x}{|\mathbf{H}^+|} [\alpha^+ (1 - \alpha^{+2}) - \alpha^+ \beta^{+2}], \\
 \frac{\partial \bar{f}}{\partial H_y^+} &= \frac{\partial x}{\partial |\mathbf{H}^+|} (\alpha^{+2} + \beta^{+2}) \beta^+ + \frac{2x}{|\mathbf{H}^+|} [-\alpha^{+2} \beta^+ + \beta^+ (1 - \beta^{+2})], \\
 \frac{\partial \bar{f}}{\partial H_z^+} &= \frac{\partial x}{\partial |\mathbf{H}^+|} (\alpha^{+2} + \beta^{+2}) \gamma^+ + \frac{2x}{|\mathbf{H}^+|} [-\alpha^{+2} \gamma^+ - \beta^{+2} \gamma^+].
 \end{aligned}$$

Thus for small α^+ and β^+

$$\text{grad}_{\mathbf{H}^+} \bar{f}(\mathbf{H}^+) = \left(\frac{2x}{|\mathbf{H}^+|} \alpha^+, \frac{2x}{|\mathbf{H}^+|} \beta^+, 0 \right). \quad (25)$$

We assume that the applied field is in the yz -plane so that $\alpha^+ = 0$. Then (24) can be written

$$\begin{cases} \overline{m_y^+} |\mathbf{H}^+| = H_y^+ mL(m|\mathbf{H}^+|/kT) - 2x\beta^+, \\ \overline{m_z^+} |\mathbf{H}^+| = H_z^+ mL(m|\mathbf{H}^+|/kT). \end{cases} \quad (26)$$

Similar equations hold for $-$ spins.

In solving these equations we first put $\mathbf{H}=0$. Then we have

$$\bar{m} = mL(am\bar{m}/kT), \quad \bar{m}_y^\pm = 0, \quad \bar{m}_z^\pm = \pm \bar{m}, \quad (27)$$

which is the usual equation to determine the mean magnetic moment \bar{m} . The Curie point is given by $3kT_c = am^2$. Next we take $\mathbf{H} \neq 0$ and put

$$\bar{m}_z^\pm = \pm \bar{m} + \Delta \bar{m}_z^\pm \quad (28)$$

and assume that $a\bar{m}_y^\pm$, $a\Delta \bar{m}_z^\pm$, H_y , H_z are all small compared with am . We also assume that x/\bar{m} is likewise small. After some computations the following results can be obtained. For small field strength

$$\bar{m}_y^\pm = \frac{H_y}{2a} \mp \frac{\bar{m}}{4ax} \frac{1-l}{1+l} H_y H_z, \quad \Delta \bar{m}_z^+ = \Delta \bar{m}_z^- = \frac{l}{1+l} \frac{H_z}{a}, \quad (29)$$

where

$$l = \frac{am^2}{kT} L'(am\bar{m}/kT) = \frac{3T_c}{T} \left(1 - \frac{2T}{3T_c} - \left(\frac{\bar{m}}{m} \right)^2 \right). \quad (30)$$

Thus the perpendicular and parallel susceptibilities are

$$\chi_\perp = \frac{N}{2a} = \frac{Nm^2}{6kT_c}, \quad (31)$$

$$\chi_\parallel = \frac{N}{a} \frac{1}{1+l^{-1}} = \frac{Nm^2}{3kT_c} \left[1 + \frac{T}{3T_c} \left(1 - \frac{2T}{3T_c} - \left(\frac{\bar{m}}{m} \right)^2 \right)^{-1} \right]^{-1}, \quad (32)$$

in complete agreement with van Vleck's formula. For somewhat higher field strength which is still very small compared with am and is moderately small compared with \sqrt{ax} , we have

$$\frac{1}{2} (m_y^+ + m_y^-) = \frac{H_y}{2a} \left[1 + \left(\frac{1-l}{1+l} \right)^2 \frac{H_z^2}{4ax} \right], \quad (33)$$

$$\frac{1}{2} (\Delta \bar{m}_z^+ + \Delta \bar{m}_z^-) = \frac{H_z}{a} \left[\frac{l}{1+l} + \left(\frac{1-l}{1+l} \right)^2 \frac{H_y^2}{8ax} \right]. \quad (34)$$

In this case, the magnetic moment per atom parallel to the applied field is

$$\frac{1}{2} (\bar{m}_y^+ + \bar{m}_y^-) \beta_H + \frac{1}{2} (\bar{m}_z^+ + \bar{m}_z^-) \gamma_H = \frac{H}{2a} \left[\beta_H^2 + \frac{2l}{1+l} \gamma_H^2 + \left(\frac{1-l}{1+l} \right)^2 \frac{H^2 \beta_H^2 \gamma_H^2}{2ax} \right], \quad (35)$$

and the corresponding susceptibility for a powder specimen is

$$\chi = \frac{N}{2a} \left[\frac{2}{3} + \frac{2l}{1+l} \cdot \frac{1}{3} + \left(\frac{1-l}{1+l} \right)^2 \frac{H^2}{15ax} \right]. \quad (36)$$

Equations (29)–(36) correspond to (14)–(19) of the preceding section, since $a = NA/2$ and $x = K/N$. For $T=0$ the formulas reduce to the latter, because

$l=kT/am^2$ for small T . However, l would vanish more rapidly if our treatment were quantum-mechanical. By use of (31) and (32), the coefficients containing l in the equations above are expressible by χ_{\perp} and χ_{\parallel} ; namely, $2l/(1+l)=\chi_{\parallel}/\chi_{\perp}$, $(1-l)/(1+l)=1-\chi_{\parallel}/\chi_{\perp}$. Our approximation is valid only when \bar{m}_y^{\pm}/\bar{m} is small compared with 1; from (29) this implies that

$$(1-\chi_{\parallel}/\chi_{\perp})H^2/8ax \ll 1. \quad (37)$$

§ 4. Conclusion

The treatment above shows that the results obtained by van Vleck are essentially correct except for the meaning of parallel and perpendicular. These referred in van Vleck's theory to the applied magnetic field, whereas they referred in the present theory to one of the axes of easy magnetization along which the spontaneous magnetizations of the two sublattices are directed in the absence of external field and from which they are difficult to avert on applying a magnetic field. Formula (36) makes it possible to deduce the anisotropy constant at any given temperature by measuring the dependency of the susceptibility on field strength, so far as $(1-\chi_{\parallel}/\chi_{\perp})H^2/8ax$ is small compared with 1.

In conclusion the writer wishes to acknowledge his hearty thanks to Mr. Kei Yosida for his stimulating discussions on this problem. The present work was aided by the Grant for Fundamental Research delivered from the Ministry of Education.

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Note added in proof: Prof. Néel kindly awakened to the writer an attention to a paper, L. Néel, *Ann. de Phys.* 11e Serie, Tome 5, 1936, t. 232, that the underlying idea of the present paper is already given there. Prof. Néel has, however, derived the formula of the susceptibility only for the absolute zero of temperature, whereas the writer gives it for arbitrary temperatures. Prof. Néel has furthermore derived a sudden change of the direction of the magnetizations of the sublattices for a strong external field, an extension of which for arbitrary temperatures, especially in the case of MnF_2 , will be given by a forthcoming paper by K. Yosida.

Theory of Antiferromagnetism and Antiferromagnetic Resonance Absorption, II

Takeo NAGAMIYA

Department of Physics, Osaka University

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Resonance absorption of microwaves by antiferromagnetic substances will be discussed by extending Kittel's theory of ferromagnetic resonance absorption. Assuming the model proposed in Part I, it follows that there appears a broad resonance absorption peak below the Curie point for a powder specimen, qualitatively similar to that observed by Trownson and others for Cr_2O_3 , although the observed ordinary paramagnetic resonance cannot be predicted, probably due to an oversimplified assumption for the equation of motion.

§ 1. Introduction

When an antiferromagnetic substance is subjected to a strong static magnetic field and a weak oscillating magnetic field the magnetization vectors of the two sublattices, \mathbf{M}^+ and \mathbf{M}^- , will oscillate about their static values, and in some cases resonance absorption will occur. It was shown in the preceding paper that if the anisotropy energy is taken into consideration these vectors are directed almost parallel to an axis of easy magnetization, besides being mutually almost antiparallel, even for a strong applied field. We shall now develop a theory of resonance absorption in such an antiferromagnetic substance by extending Kittel's theory of ferromagnetic resonance absorption.

§ 2. Equation of motion

Kittel's theory starts from the equation of motion describing the relation between the time variation of the angular momentum vector, i.e. the magnetization vector divided by $\gamma = hc/4\pi mc$, and the torque acting upon the latter vector. In our case, the torque acting on \mathbf{M}^\pm results from the effective field acting on it which consists of the external applied field, the Weiss molecular magnetic field, and the anisotropy magnetic field. Assuming that the anisotropy energy per unit volume is of the form

$$F = \frac{1}{2} K [(a^{+2} + \beta^{+2}) + (a^{-2} + \beta^{-2})] \quad (1)$$

for small a^\pm and β^\pm , where a^\pm , β^\pm , γ^\pm are direction cosines of \mathbf{M}^\pm , the axis of actually occurring magnetizations of the sublattices being chosen as the z -axis,

and K is a constant which depends generally on temperature, the effective magnetic field due to the anisotropy energy has the following x, y, z components:

$$-\frac{K}{M_0} (\alpha^\pm, \beta^\pm, 0) \quad (2)$$

(see (8) and (25) of Part I, and notice that $2\alpha \cdot N/2 = K$), where M_0 is the value of $|\mathbf{M}^\pm|$ and can be approximated by its value for vanishing field strength. In terms of the x, y, z components of \mathbf{M}^\pm (2) is expressible as

$$-\frac{K}{M_0^2} (M_x^\pm, M_y^\pm, 0). \quad (3)$$

The total effective field is thus

$$\mathbf{H} - A\mathbf{M}^\mp - \frac{K}{M_0^2} (M_x^\pm, M_y^\pm, 0), \quad A = -4Jzg^{-2}\beta^{-2}/N. \quad (4)$$

The equation of motion is therefore

$$\begin{aligned} \frac{1}{\gamma} \frac{d\mathbf{M}^\pm}{dt} &= \mathbf{M}^\pm \times \left(\mathbf{H} - A\mathbf{M}^\mp - \frac{K}{M_0^2} (M_x^\pm, M_y^\pm, 0) \right) \\ &= \mathbf{M}^\pm \times (\mathbf{H} - A\mathbf{M}^\mp) + \frac{K}{M_0^2} (M_x^\pm M_y^\pm, -M_x^\pm M_x^\pm, 0). \end{aligned}$$

Since \mathbf{M}^+ and \mathbf{M}^- are nearly parallel and antiparallel to the z -axis, respectively, M_x^\pm appearing in this equation can be replaced by $\pm M_0$, and it becomes

$$\frac{1}{\gamma} \frac{d\mathbf{M}^\pm}{dt} = \mathbf{M}^\pm \times (\mathbf{H} - A\mathbf{M}^\mp) + \frac{K}{M_0} (\pm M_y^\pm, \mp M_x^\pm, 0). \quad (5)$$

We now define

$$\mathbf{M} = \mathbf{M}^+ + \mathbf{M}^-, \quad \mathbf{M}' = \mathbf{M}^+ - \mathbf{M}^-. \quad (6)$$

Then, adding and subtracting the equation (5) for $+$ and $-$ spin magnetizations, we have

$$\begin{aligned} \frac{1}{\gamma} \frac{d\mathbf{M}}{dt} &= \mathbf{M} \times \mathbf{H} + \frac{K}{M_0} (M_y', -M_x', 0), \\ \frac{1}{\gamma} \frac{d\mathbf{M}'}{dt} &= \mathbf{M}' \times (\mathbf{H} - A\mathbf{M}) + \frac{K}{M_0} (M_y, -M_x, 0). \end{aligned} \quad (7)$$

It can be shown that the static solution of this equation is quite consistent with the results obtained in Part I, (29) and (33).

Since our interest is in the oscillation of the magnetization vectors, we shall denote by $\delta\mathbf{H}$, $\delta\mathbf{M}$, $\delta\mathbf{M}'$ the deviations of \mathbf{H} , \mathbf{M} , \mathbf{M}' from their static values, and ask for their time variations. It follows by taking the variation of (6) that

$$\begin{aligned}\frac{1}{\gamma} \frac{d\delta\mathbf{M}}{dt} &= \delta\mathbf{M} \times \mathbf{H} + \mathbf{M} \times \delta\mathbf{H} + \frac{K}{M_0} (\delta M_y', -\delta M_x', 0), \\ \frac{1}{\gamma} \frac{d\delta\mathbf{M}'}{dt} &= \delta\mathbf{M}' \times (\mathbf{H} - A\mathbf{M}) + \mathbf{M}' \times (\delta\mathbf{H} - A\delta\mathbf{M}) + \frac{K}{M_0} (\delta M_y, -\delta M_x, 0).\end{aligned}\quad (8)$$

In the second of these equations \mathbf{M}' can be replaced by $(M_x', M_y', 2M_0)$ and the last anisotropy term can be neglected compared with the third term. Moreover, since $A=1/\chi_\perp$ (see (14) of Part I), $\mathbf{H}-A\mathbf{M}$ has the following components:

$$\mathbf{H}-A\mathbf{M} = (0, 0, a H_z), \quad a=1-\chi_\parallel/\chi_\perp. \quad (9)$$

Therefore, assuming that $\delta\mathbf{M}$, $\delta\mathbf{M}'$, $\delta\mathbf{H}$ are all proportional to $e^{i\omega t}$, (8) can be written as

$$\begin{aligned}\frac{i\omega}{\gamma} \delta M_x &= H_z \delta M_y - H_y \delta M_z - M_z \delta H_y + M_y \delta H_z + \frac{K}{M_0} \delta M_y', \\ \frac{i\omega}{\gamma} \delta M_y &= -H_z \delta M_x + H_x \delta M_z + M_z \delta H_x - M_x \delta H_z - \frac{K}{M_0} \delta M_x', \\ \frac{i\omega}{\gamma} \delta M_z &= H_y \delta M_x - H_x \delta M_y - M_y \delta H_x + M_x \delta H_y, \\ \frac{i\omega}{\gamma} \delta M_x' &= a H_z \delta M_y' - 2M_0 (\delta H_y - A \delta M_y) + M_y' (\delta H_z - A \delta M_z), \\ \frac{i\omega}{\gamma} \delta M_y' &= -a H_z \delta M_x' + 2M_0 (\delta H_x - A \delta M_x) - M_x' (\delta H_z - A \delta M_z), \\ \frac{i\omega}{\gamma} \delta M_z' &= M_x' (\delta H_y - A \delta M_y) - M_y' (\delta H_x - A \delta M_x).\end{aligned}\quad (11)$$

§ 3. The resonance condition

The resonance condition is furnished by the vanishing of the determinant of the coefficients of δM_x , δM_y , δM_z , $\delta M_x'$, $\delta M_y'$, $\delta M_z'$ in (11). Using $M_y' = -aM_0 H_y H_z' / AK$, $M_x' = -aM_0 H_x H_z' / AK$ (see (29) of the preceding paper), it gives rise to

$$\begin{aligned}& [\omega^2 \cos^2 \theta + a(1+a) \sin^2 \theta \cos^2 \theta] H^4 \\ & - [(1+a^2+2aG^2) \cos^2 \theta + (1-G^2) \sin^2 \theta] H^2 + (1-G^2)^2 = 0,\end{aligned}\quad (12)$$

where $\cos \theta = H_z'/H$, H is written for $H/(\omega/\gamma)$, and

$$G = (2AK)^{\frac{1}{2}} / (\omega/\gamma). \quad (13)$$

In order that the theory of the preceding paper is applicable it must be assumed that $2G^2 \gg aH^2$, so that the terms containing a in (12) must be considered as

small compared with the remaining terms. The solution of (12) neglecting a is

$$H^2 = \frac{(1 - G^2)^2}{1 - G^2 \sin^2 \theta}, \tag{14}$$

and the solution corrected for the dominant terms containing a is

$$\begin{aligned} H^2 &= \frac{(1 - G^2)^2}{1 - G^2 \sin^2 \theta} + \frac{a \cos^2 \theta H^2}{1 - G^2 \sin^2 \theta} (\sin^2 \theta \cdot H^2 - (a + 2G^2)) \\ &= \frac{(1 - G^2)^2}{1 - G^2 \sin^2 \theta} \left[1 - \frac{a \cos^2 \theta}{(1 - G^2 \sin^2 \theta)^2} ((a + 2G^2) \right. \\ &\quad \left. - (1 - (2 - a)G^2 + 3G^4) \sin^2 \theta) \right]. \end{aligned} \tag{15}$$

We shall confine ourselves to (14). The resonance value of H ranges from $(1 - G^2)$ to $(1 - G^2)^{\frac{1}{2}}$ corresponding to $\theta = 0$ and $\pi/2$ when $G < 1$, or from $(G^2 - 1)$ to ∞ corresponding to $\theta = 0$ and $\theta = \sin^{-1} G^{-1}$ when $G > 1$. Referring to the anisotropy constant in ferromagnetic substances, it may be assumed that G vanishes at the Curie point and increases with decreasing temperature. The

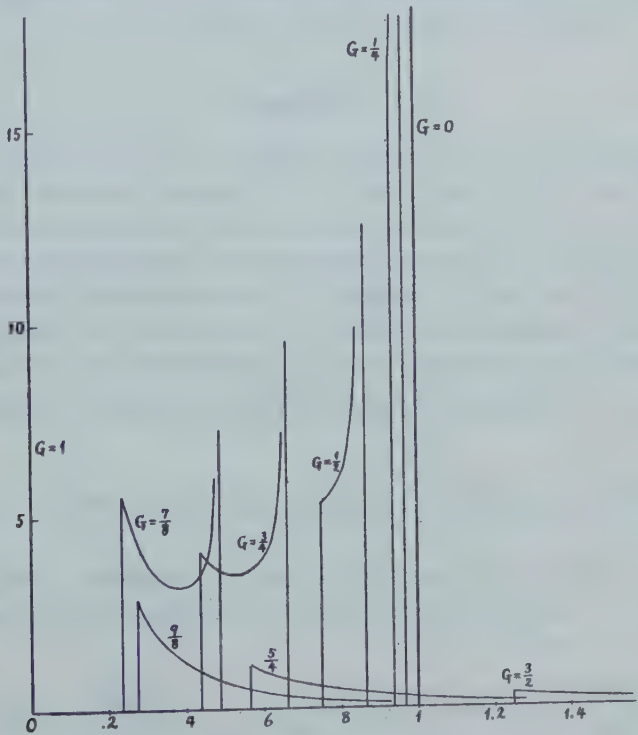


Fig. 1. Curves of $\sin \theta \cdot d\theta/dH$ vs. H for various values of G , representing qualitatively the resonance absorption curves by a powdered specimen; G increases with decreasing temperature.

resonance may therefore occur for any orientation of the crystal near the Curie point and for a limited range of orientation below a certain temperature for which $G=1$. For a powder specimen a rough idea of the resonance absorption curve may be obtained by plotting $\sin \theta \cdot d\theta/dH$, namely the number of small crystals resonating in the unit interval of H , as a function of H . From (14) we have

$$\begin{aligned} G \cos \theta &= |1 - G^2| \left(\frac{1}{H^2} - \frac{1}{1 - G^2} \right)^{\frac{1}{2}}, \\ G \sin \theta \frac{d\theta}{dH} &= |1 - G^2| \frac{1}{H^3} \left(\frac{1}{H^2} - \frac{1}{1 - G^2} \right)^{-\frac{1}{2}}. \end{aligned} \quad (16)$$

Fig. 1 shows the curves of $\sin \theta \cdot d\theta/dH$ versus H for several values of G . These absorption curves resembles those observed for Cr_2O_3 by Trounson and others¹⁾ qualitatively in that they have a finite breadth and shift to lower values of the field strength with decreasing temperature, disappearing at a certain temperature. However, these investigators observed besides such a peak the ordinary paramagnetic resonance peak (at $H=1$) below the Curie point. Also, Okamura and his collaborators²⁾ observed for MnS , MnO , and MnSe the ordinary paramagnetic resonance peak below the Curie point and did not find any other. The present theory is unable to account for these facts.

§ 4. Discussion

It is improbable that the non-occurrence of the ordinary paramagnetic resonance peak below the Curie point in the present theory is the result of the inadequacy of our model of antiferromagnetism, because, as mentioned in Part I, the dropping of the anisotropy energy has as its consequence the constancy of the susceptibility below the Curie point. It seems rather to be due to an oversimplification of our equation of motion. If we assume the spins to be classical and assume them to be subjected each to a molecular anisotropy field with a potential energy of the form $\frac{1}{2} k_0 (u_i^2 + \beta_i^2)$, the equation of motion for the spin i is

$$\frac{1}{\gamma} \frac{d\mathbf{m}_i}{dt} = \mathbf{m}_i \times \left(\mathbf{H} - \frac{a}{z} \sum_j \mathbf{m}_j \right) + k_0 \left(\frac{m_{iz}m_{iy}}{m^2}, \frac{m_{iz}m_{ix}}{m^2}, 0 \right),$$

where $a = -2Jzg^{-2}\beta^{-2}$, z the number of nearest neighbours, and the summation over j is to be extended over the nearest neighbours of i . Summing up this equation over i we have

$$\frac{1}{\gamma} \frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{H} + k_0 \left(\sum_i \frac{m_{iz}m_{iy}}{m^2}, -\sum_i \frac{m_{iz}m_{ix}}{m^2}, 0 \right),$$

where $\mathbf{M} = \sum_i \mathbf{m}_i$. The summation over i can be divided into the summation over the $+$ sites and the summation over the $-$ sites, and if we could replace

the components of m_i by their mean value in a certain sense for each site, we have the first of the equations (7). The second equation follows from a further simplifying assumption of the molecular magnetic field. These equations are exact at absolute zero.

If we could put the summation over i in the equation above simply equal to zero, we have the ordinary paramagnetic resonance only and no other at measurable field strength. It seems therefore that this resonance follows from the fluctuating terms in the summation and should vanish at absolute zero. Indeed, the measurements both by Trownson and others and by Okamura and others show that the paramagnetic resonance drops off with decreasing temperature and the corresponding absorption amount seems to tend to zero for $T \rightarrow 0$. The elucidation of this point must, however, await a more detailed theory.

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Note on the Magnetic Properties of the FeS_n System

Kei YOSIDA

Department of Physics, Osaka University

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The ferromagnetic properties of the FeS_n system are interpreted as an imperfectly compensated antiferromagnetism or ferrimagnetism as called by L. Néel. A special case of $n=1$, namely the substance $\text{FeS}_{1.00}$ is regarded as an antiferromagnetic material. In FeS_n , Fe^{3+} ions appear when $n>1$, and its ferromagnetism is explained by assuming that these Fe^{3+} ions are preferably situated on only one of the two sublattices which correspond to $+$ and $-$ spin orientations respectively.

§ 1. Introduction

The magnetic properties of the FeS_n system are very complicated, as shown by the experimental results given by Juza and Blitz¹⁾, Ziegler²⁾, Haraldsen³⁾, Miyahara⁴⁾ and others⁵⁾. According to their results, the stoichiometric FeS (50 percent S) is paramagnetic, and the paramagnetic susceptibility shows two knicks α and β at temperatures of 145°C and 325°C respectively. As n increases from 1 in FeS_n , the α point shifts to the lower temperature side, but the β point is fixed at 325°C , and a new knick γ begins to appear between α and β . At the

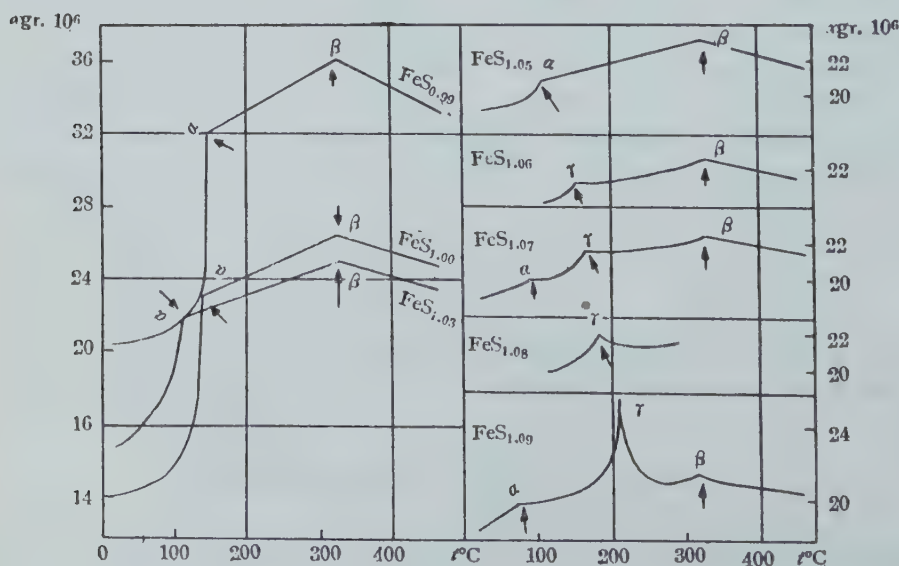


Fig. 1. Temperature-susceptibility curve of FeS_n in its paramagnetic region.

beginning, this knick γ is situated in the neighborhood of α , but then gradually shifts to β and at the same time grows to a peak with increasing n . (see Fig. 1)

At room temperature, the susceptibility χ increases abruptly when n is increased from $n=1.10$ and it becomes dependent upon the applied magnetic field. This means that FeS_n becomes ferromagnetic from this value of n . But the ferromagnetic properties of this material disappear when the temperature is lowered below 200°C . When n exceeds a certain value greater than 1.10, χ begins to decrease because of the transition from a monophasic to a two-phase region. The value of n for this transition depends upon the temperature, and at room temperature it is about 1.10, but increases with an increasing temperature, for example, to 1.11 at 300°C and 1.20 at 330°C . On account of the existence of such a narrow monophasic region, the magnetic properties of the FeS_n system are much more complicated than those of the CrS_n system⁽⁶⁾ which shows similar properties but has a comparatively wider monophasic region. In Fig. 2 is shown, as an example, the temperature dependency of the susceptibility of the ferromagnetic $\text{CrS}_{1.17}$.

The crystal structure of FeS as analyzed by Hägg and Sucksdorff⁽⁷⁾ is of NiAs type where S ions are arranged on a hexagonal closest packed lattice, the interstices of which are occupied by the Fe ions. No change was found in the crystal structure at α , β and γ transitions.

Pyrrhotite, known as a natural ferromagnetic material, has the composition of sulphur that corresponds to the maximum susceptibility. This ferromagnetic material shows remarkable features much different from other metallic ferromagnetics Fe, Co and Ni. For instance, its saturation intensity of magnetization is very small and amounts only to 23 gauss per gram. Its anisotropy energy is very large and it is very difficult to magnetize to the c-axis, contrary to the case of Co. In this direction it shows a paramagnetic behavior and the ferromagnetism appears only in the plane perpendicular to the c-axis. The gyromagnetic ratio in this plane is 0.63, unlike the usual value of 2.

In order to interpret this abnormal value of the gyromagnetic ratio, D. R. Inglis⁽⁸⁾ proposed the idea that the Fe ions in FeS_n are in the state of $L=2$ and $S=1/2$ and the L -vector is coupled with the S -vector in the opposite direction. This state is, however, consistent neither with free state of the Fe^{2+} ions nor with that of the Fe^{3+} ions contained in FeS_n .

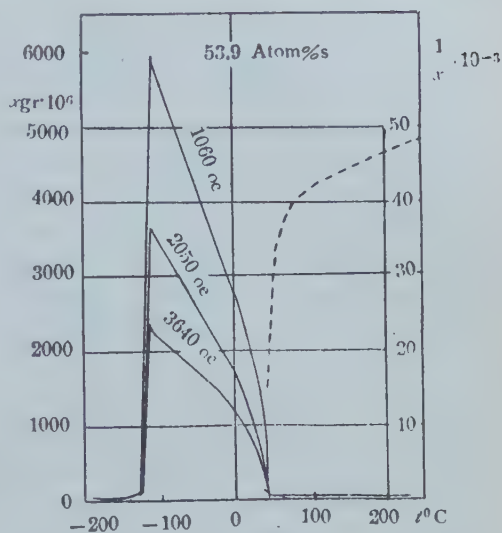


Fig. 2. Temperature-susceptibility curve of $\text{CrS}_{1.17}$

No comprehensive theoretical interpretation of the above mentioned complicated but interesting properties of the FeS_x system has yet been given. Recently, M. Shimizu⁹⁾ has theoretically derived the susceptibility-temperature curve of FeS , containing 50 atomic percent sulphur. His theory is based on the idea that two knicks in the χ - T curve arise from two kinds of antiferromagnetic arrangements of $+$ and $-$ spins, the orientations of spins in one arrangement being parallel to the c -axis and those in the other arrangement perpendicular to it. He showed that the ordered arrangements of them are thermally destroyed at two different Curie temperatures. Although his theory does not explain all the experimental results, it may probably be certain that at least one of the two knicks arises from antiferromagnetism.

In the present paper, we shall start from the idea that FeS with 50 percent sulphur is an antiferromagnetic material. When the percentage of sulphur is increased above 50 atomic percent, Fe^{3+} ions appear, mixing with Fe^{2+} ions, in order that the newly added sulphur atoms become doubly ionized. If these Fe^{3+} ions are assumed to tend to occupy only one sublattice of the two, each of which is respectively occupied by $+$ spins and $-$ spins alone, the magnetic moments of the two sublattices will no longer compensate with each other, and ferromagnetism will appear. Interpreting the ferromagnetism of pyrrhotite in such a way, it will be shown that other features of the FeS_x system can be explained at least qualitatively.

§ 2. Paramagnetic region

According to Van Vleck¹⁰⁾, the antiferromagnetic susceptibility above its Curie temperature T_c is given by the formula

$$\chi = \frac{N\mu_B^2 g^2 S(S+1)}{3k(T+T_c)}, \quad (1)$$

where N is the number of magnetic ions, S is the spin quantum number of an ion and μ_B , k and g are Bohr magneton, Boltzmann constant and Landé g -factor respectively.

Since in $\text{FeS}_{1.00}$ Fe atoms are doubly ionized and are in the state 5D , the magnitude of S is 2. Substituting $g=2$, and $T=T_c=325+273$ into Eq. (1), we obtain $\chi_{gr.}=28.7 \times 10^{-6}$ for the susceptibility per gram at the β point. This value is in rather good agreement with the value of 27.0×10^{-6} , obtained experimentally by Haraldsen at the same temperature. Also, the behavior of the experimental χ - T curve near the β point is similar to that of antiferromagnetics. On the other hand, R. Ueda and his coworkers¹¹⁾ observed a prominent peak in the specific heat-temperature curve at about 325°C . These facts seem to indicate that the β point corresponds to the Curie temperature of antiferromagnetics. Thus we may say that the spins in this substance are arranged in an antiferromagnetic configuration, and consequently the crystal lattice is resolved into two sublattices, each of

which being occupied by only one kind of spins. The directions of the spins on these two sublattices are opposite to each other, and therefore the magnetic moment of the whole crystal is zero in the absence of magnetic field, the substance being paramagnetic.

Generally, in semiconductors or ionic crystals, the magnetic ions interact with each other through the agency of intermediary non-magnetic ions, and this interaction, which is called "superexchange," does not depend only upon the distance between them, but also upon the relative positions of the magnetic ions with respect to the non-magnetic ions¹²⁾. This circumstance makes it difficult to infer the space-lattice of superstructure of FeS due to the antiferromagnetic spin orientations. The determination of this structure will perhaps be enabled by experiments of neutron diffractions.

Recently Prof. T. Nagamiya¹³⁾ pointed out that the anisotropy energy plays a very important rôle in the antiferromagnetism, contrary to the case of ferromagnetism. In the latter case, the energy $-MH$ of the saturation intensity of magnetization M to the applied magnetic field H is so large in comparison with the anisotropy energy that a comparatively weak field will suffice to orient the spins parallel to the applied field. On the other hand, in an antiferromagnetic medium, the energy of magnetization $-1/2\chi H^2$ is very small on account of its low susceptibility, so that even a very strong field will not suffice to make the spins deviate from the direction of easy magnetization to align parallel to the applied field. Now let us assume that, in the case of FeS_{1.00}, the direction of easy magnetization is in the plane perpendicular to the *c*-axis below the Curie temperature. Then it will be very difficult to orient the spins to the *c*-axis.

As to the α -transition at 145°C, the circumstances are more complicated and at present it is impossible to clarify what kinds of transitions do occur at this point. According to the x-rays analysis made by R. Ueda and others¹⁴⁾, the *c*-axis abruptly contracts at about this α point with increasing temperature, contrary to the behavior of the *a*-axis which shows only a gradual expansion. They also show that this temperature shifts to the lower temperature side as the percentage of sulphur is increased. The shift of this temperature seems to be intimately connected with the shift of the α -transition temperature with increasing percentage of sulphur. The cause of this phenomenon is, however, obscure to the writer, and it may only be suggested that this transition is related with the change of anisotropy, analogous to that of magnetite at a low temperature.

§ 3. Ferromagnetic region

In the preceding paragraph, it was mentioned that FeS_{*n*} becomes ferromagnetic when *n* exceeds 1.10 at room temperature; a natural pyrrhotite corresponds to the value of *n* ranging over 1.15~1.20. Its saturation intensity of magnetization is about 23 gauss per gram, which amounts to only about a tenth of that of iron

or cobalt. In pyrrhotite having the above mentioned range of values of n , the number of the Fe^{3+} ions amounts to 30~40 percent of all the Fe ions.

If we assume that all the Fe^{3+} ions occupy only one sublattice, then the magnetic moment of one Bohr magneton per Fe^{3+} ion survives, because the spin moment of Fe^{3+} ion is $5\mu_B$ and does not compensate that of Fe^{2+} ion on the other sublattice $4\mu_B$. Therefore, the intensity of magnetization per gram, σ , is given by

$$\sigma = \frac{N}{A} \cdot \mu_B \cdot 2x, \quad 2x < 0.5, \quad (2)$$

where N is the Avogadro's number, A is the molecular weight of FeS_{1+x} , and $2x$ is the ratio of the number of Fe^{3+} ions to that of all the Fe ions. Now if we put $A = 55.84 + (1+x)32.06$ and $x = 0.2$, we obtain $\sigma = 23.6$ from Eq. (2), which is in very good agreement with the experimental saturation value of magnetization of pyrrhotite.

In this way, the ferromagnetism of pyrrhotite is accounted for by an imperfect compensation of two antiparallel magnetic moments of unequal magnitude on two different sublattices. It may be called "ferrimagnetism" as it is named by L. Néel¹⁰.

In Fig. 2 is illustrated the temperature variation of the reciprocal susceptibility above the Curie point of $\text{CrS}_{1.17}$, which has properties analogous with pyrrhotite. Both this curve and the susceptibility curve of pyrrhotite represent the ferrimagnetic feature and obey the following relation given by Néel:

$$\frac{1}{\chi} = \frac{T}{C} + \frac{1}{\chi_0} - \frac{\sigma}{T - \theta}, \quad (3)$$

where C , χ_0 , θ and σ are certain constants.

Now, let us consider the relation between σ and x . The relation should be expressible as Eq. (2) up to $x = 0.25$, where the number of Fe^{3+} ions is equal to that to Fe^{2+} ions. Above this value, the relation must be replaced by

$$\sigma = \frac{N}{A} \cdot \mu_B (1 - 2x). \quad (4)$$

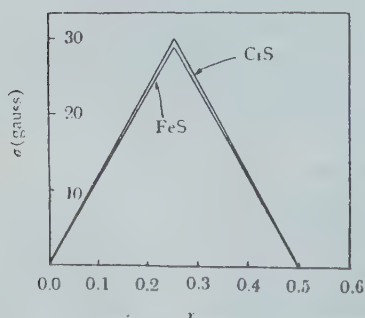


Fig. 3. σ - x curves of FeS_{1+x} and CrS_{1+x} .

Fig. 3 illustrates the σ - x curve calculated from Eqs. (2) and (4) in the cases of FeS_{1+x} and CrS_{1+x} . σ takes a maximum value at $x = 0.25$, and decreases symmetrically on both sides of this value of x . It disappears at $x = 0$ and $x = 0.5$ where all the Fe ions are Fe^{2+} and Fe^{3+} , respectively.

In actuality, σ increases with increasing sulphur in the FeS_n system, but since it separates into two phases, it is impossible to observe the maximum value. On the other

hand, in the CrS_n system, which has a comparatively wide region of the mono-phase, the maximum of σ can be observed at about $x = 0.25$.

In both FeS_n and CrS_n systems σ does not increase continuously from $x=0$, as shown by Fig. 3, but these materials become ferromagnetic only when x exceeds a certain value, for instance $x=0.10$ for FeS_n . This is because the Fe^{3+} ions or the Cr^{3+} ions do not all enter the single sublattice. This circumstance will be considered in the following paragraph.

As to the magnetic anisotropy of pyrrhotite, since in this substance ferromagnetism occurs because the Fe^{3+} ions take part in the antiferromagnetism of the Fe^{2+} ions, the direction of easy magnetization should coincide with the preferred direction of the spins of $\text{FeS}_{1.00}$. The latter was assumed to lie in the plane perpendicular to the c-axis. Therefore the direction of easy magnetization of pyrrhotite exists in this plane. In order to magnetize to the c-axis, the spins of all the Fe^{2+} ions as well as those of the Fe^{3+} ions must point towards the c-axis. The energy of the magnetization to the external field is then $-MH$, where M is the number of the Fe^{3+} ions multiplied by μ_B , whereas the anisotropy energy to balance this energy is the anisotropy energy per ion multiplied by the number of all the Fe ions, including Fe^{2+} and Fe^{3+} . The latter will be much larger than the former, and therefore it is expected that it is scarcely possible to magnetize the substance towards the c-axis. It is in striking contrast to the usual metallic ferromagnetics.

As the result of the preceding considerations it is expected that pyrrhotite shows a paramagnetic behavior in the direction of the c-axis, and that its susceptibility-temperature curve shows a kink at about 325°C , as in the case of $\text{FeS}_{1.00}$. There is, however, no experimental evidence to affirm this kink, so far as the writer is aware of.

In the introduction we referred to Inglis' explanation of the abnormal value of the gyromagnetic ratio of pyrrhotite in the plane of easy magnetization. This value is, here, interpreted in the following way. If we assume that, below the antiferromagnetic Curie point (that is in the ferromagnetic temperature range), the orbital angular momentum is coupled with the spin angular momentum to give the expression of the coupling energy $\lambda(LS)$, where λ is negative because in the Fe^{2+} ions the d -shell is more than half-filled, the vectors L and S of Fe^{2+} ions will be parallel to each other in the lowest energy state. The magnetic moment of the Fe^{2+} ion then becomes $6\mu_B$ instead of $4\mu_B$ as was assumed in the preceding paragraph, and the angular momentum $4\hbar$. No change is, however, necessary for the preceding discussions, since the difference of the magnetic moments of Fe^{2+} and Fe^{3+} ions remains still one Bohr magneton.

Now let $2Nx$ be the number of the Fe^{3+} ions and $2N(1-x)$ be that of the Fe^{2+} ions. Then the total magnetic moment and the total angular momentum are given by

$$[6 \times N - \{5 \times 2Nx + 6 \times N(1-2x)\}] \mu_B = 2Nx\mu_B,$$

and

$$\left[4 \times N - \left\{ \frac{5}{2} \times 2Nx + 4 \times N(1-2x) \right\} \right] \hbar = 3Nx\hbar$$

respectively. The ratio of the former to the latter is the gyromagnetic ratio g , and is given by

$$g = \frac{2Nx}{3Nx} = 0.67.$$

This value is in good accord with the experimental value of 0.63.

If the above consideration be correct, the g -factors of Fe^{2+} and Fe^{3+} ions are 2 and 1.5 respectively. Thus, in the Fe^{2+} ion, the orbital angular momentum is considered not to be quenched by the crystalline electrostatic field and so it is expected that in the magnetic microwave resonance experiments, the spectroscopic splitting factor of the Fe^{2+} ion will become considerably larger than the value of 2, like the case of cobalt Tutton salts.

§ 4. Para-ferro transition region

In the measurements of both the FeS_n system and the CrS_n system carried out by Haraldsen, ferromagnetism does not occur until n reaches a certain critical value. This shows that the existing Fe^{3+} ions do not gather into one sublattice unless the number of Fe^{3+} ions becomes greater than a certain value. We shall consider this situation in the following manner.

Now suppose that the crystal can be resolved into two sublattices and consider the state that the one sublattice is occupied only by $+$ spins and the other only by $-$ spins, that is, a perfect antiferromagnetic ordered state. Let $2N$ be the number of all the Fe ions, $2Nx$ be that of Fe^{3+} ions, and $2Nx\rho$ be the number of the Fe^{3+} ions on the one sublattice. Then the number of the Fe^{2+} ions on this sublattice is $N - 2Nx\rho$ and those of the Fe^{3+} ions and Fe^{2+} ions on the other sublattice are $2Nx(1-\rho)$ and $N - 2Nx(1-\rho)$ respectively.

The entropy S for given values of x and ρ , is given, in an usual manner, by

$$S = k \log \left(\frac{N}{2Nx(1-\rho)} \right) \cdot \left(\frac{N}{2Nx\rho} \right). \quad (5)$$

It is, however, very difficult to calculate the energy which may consist of electrostatic and magnetic parts. This is, at any rate, a function of x and ρ , and remains the same if one replaces the number of the Fe^{3+} ions on one sublattice, $2Nx\rho$, with that of the same ions on the other sublattice, $2Nx(1-\rho)$. Therefore, introducing one parameter a , we assume this energy in the following form;

$$E(x, \rho) = -ax^2(1-2\rho)^2. \quad (6)$$

When a magnetic field H is applied to the direction perpendicular to the c -axis, the following energy E_H will be added, apart from the energy arising from the antiferromagnetic background;

$$E_H = -2N\mu_R H \left[5x(1-\rho) + 4 \left\{ \frac{1}{2} - x(1-\rho) \right\} - 5x\rho - 4 \left(\frac{1}{2} - x\rho \right) \right] = -2N\mu_R H(1-2\rho)x. \quad (7)$$

Now if we write σ for $1-2\rho$, the state of $\sigma=0$ corresponds to the equal distribution of Fe^{3+} ions among two sublattices and that of $\sigma=1$ to the distribution that all the Fe^{3+} ions occupy only one sublattice. Using this σ , the free energy per ion is obtained from Eqs. (5), (6) and (7) as follows:

$$\begin{aligned} f = & -ax^2\sigma^2 - \mu_R H x \sigma + kT \left[\frac{x(1+\sigma)}{2} \log \frac{x(1+\sigma)}{2} \right. \\ & + \frac{1}{2} \left\{ 1 - x(1+\sigma) \right\} \log \frac{1}{2} \left\{ 1 - x(1+\sigma) \right\} \\ & + \frac{x(1-\sigma)}{2} \log \frac{x(1-\sigma)}{2} \\ & \left. + \frac{1}{2} \left\{ 1 - x(1-\sigma) \right\} \log \frac{1}{2} \left\{ 1 - x(1-\sigma) \right\} \right]. \quad (8) \end{aligned}$$

For a given value of x , the equilibrium value of σ is determined by $\partial f / \partial \sigma = 0$. After simple calculations this yields

$$\tanh \frac{2ax^2\sigma + \mu_R H}{xkT} = \frac{\sigma}{1 - x(1-\sigma^2)}. \quad (9)$$

For $H=0$, the condition for Eq. (8) to have a solution other than $\sigma=0$, that is, the condition for ferromagnetism to arise, is

$$\frac{2a}{kT} > \frac{1}{x(1-x)}. \quad (10)$$

From this equation, the Curie temperature θ is determined for a given value of x , as

$$\theta = \frac{2a}{k} x(1-x). \quad (11)$$

If θ determined by Eq. (11) is higher than the antiferromagnetic Curie temperature T_c which probably lies at about 325°C , the ferromagnetism will disappear at the latter temperature T_c .

Haraldsen measured the Curie temperature of FeS_n for various values of n

in the ferromagnetic region. Using these experimental values, $2a/k$ can be evaluated from Eq. (11). These results are shown by the following table:

n	1.11	1.12	1.14	1.15	1.20
θ	573°K	583°K	593°K	598°K	603°K
$2a/k$	3340°K	3200°K	2940°K	2850°K	2500°K

Noting that the precision of measurements is not so good, because the Curie points of various FeS_n lie in the neighborhood of the temperature of transition from a monophase to a two-phase region, and that our calculations are simplified by neglecting the temperature effect of the antiferromagnetic background, the constancy of the obtained values of $2a/k$ may be said as satisfactory. It is likely that the deviation from constancy for large n is caused by this simplified approximation. In particular, the observed Curie temperature 603°K for $n=1.20$ may correspond to the antiferromagnetic Curie point T_c , rather than to θ .

Furthermore, Haraldsen has measured the Curie temperature of CrS_n for $n=1.17, 1.20, 1.25$ and 1.33 . These values are all in the neighborhood of 35°C. In the CrS_n system, the ferromagnetism appears from $n=1.14$ at room temperature. Substituting $n=1.14$, and $T=293^\circ\text{K}$ into Eq. (10), $2a/k$ is calculated to 1450°K. Using this value of $2a/k$ in Eq. (10), we can calculate the Curie temperature θ for the above four cases. Each Curie temperature obtained in this manner is about 35°C or higher. Hence, the Curie points in these cases seem to arise from antiferromagnetic transition.

From the standpoint of this paragraph, the behavior of the $\sigma-x$ curve given in the preceding paragraph must be modified in the following way. In the case of FeS_n , we may assume that $2a/k$ is about 3000°K. Then at room temperature it follows from computations that this substance is paramagnetic when $n \leq 1.10$

or $n \geq 1.40$ and ferromagnetic when $1.40 > n > 1.10$, and that the maximum value of σ is attained at $n=1.25$. In Fig. 4, the $\sigma-x$ ($n=1+x$) curves are illustrated schematically. In this figure, the dashed lines are the case that all the Fe^{3+} ions occupy only one sublattice, that is, the case of absolute zero. The outer solid curve corresponds to room temperature, and the inner ones to higher temperatures.

According to the present theory, the ferromagnetism should appear at a sufficiently low

temperature for an arbitrary value of x , although it is actually not the case. This situation may probably be connected with complicated circumstances in the lower temperature region where the second transition corresponding to the α -transition in $\text{FeS}_{1.00}$ should be expected for FeS_{1+x} .

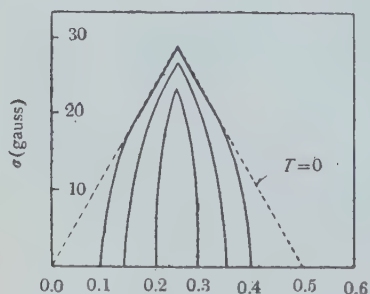


Fig. 4. $\sigma-x$ curve of FeS_{1+x}

§ 5. Conclusion

In this paper, we proposed the idea that the magnetic behavior of the FeS_n system is essentially based on the antiferromagnetism of FeS_{1.00} and that the ferromagnetism of pyrrhotite arises from the situation that the Fe³⁺ ions which appear with an excess of sulphur, prefer one of the two sublattices so that the magnetic moments of the ions on the two sublattices compensate with each other only imperfectly. The computational results based on this idea seems to favor the experimental facts qualitatively, although several assumptions used there remain still to be explained. These assumptions should be, in future, confirmed by rigorous computations based on more precise experimental knowledge, especially of the superstructure due to antiferromagnetism. Experiments on the variation of the saturation intensity of magnetization with foreign atoms, such as Cu, Zn, Ni etc., would also throw light on this problem. Similar considerations as for pyrrhotite were given for chromium sulphide.

In conclusion, the writer would like to express his cordial thanks to Prof. T. Nagamiya for his continual advices in this problem.

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Note On Many Fermion Problems

Toshiyuki NISHIYAMA

Department of Physics, Osaka University

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Non-equilibrium properties of the statistical operator are formulated in the scheme of second quantization. Some formulae of the so-called quantum hydrodynamics of Landau and those of Born and Green are examined from our point of view. Further we study Tomonaga's theory of one dimensional sound quanta introducing spin degeneracies, and we make some discussions about the three dimensional problem.

§ 1. Introduction and summary

In the preceding papers¹⁾ we have given a formulation of the density operator of N fermion ensemble in the scheme of second quantization, by which the equilibrium properties of the ensemble have been studied. Our present objective is the non-equilibrium properties of the assembly of many fermions. The mathematical difficulties concerning the many body problem have been proved to be so tremendous that one has been obliged to give up straightforward methods. At this stage of the problem several attempts have been made. Recently Tomonaga²⁾ has proposed an idea to describe many fermion problems by means of the density operator. This formulation enables us to replace the fermion field by the field of sound quanta, a boson field, just as in the neutrino theory of light, and to obtain, so far as we know, the most complete solution of the many body problem although it is restricted to the one dimensional problem. In this article we work out in two directions. The first aims to ask for the formulation of the kinetic theory in the scheme of second quantization and the second is directed to give, independently, the theory of sound quanta. In § 2 we consider the kinetic theory and the hydrodynamical equations such as the equation of motion and the equation of energy change are obtained. The formulae given by Born and Green³⁾ in the configuration space or by Kirkwood⁴⁾ in the classical theory and further those of the theory of quantum hydrodynamics of Landau⁵⁾ are examined from our point of view. In § 3 the commutation relations concerning the Fourier coefficients of the density and current operators are established and we prepare for the description in the momentum space. In § 4 the one dimensional problem of the theory of sound quanta is considered. We take spin degeneracies into consideration and obtain the solution of the eigenvalue problem. In § 5 we give some remarks on the dynamical correlation

function and in the last section the three dimensional problem of sound quanta is discussed. There we recall the neutrino theory of light and Pryce's work on this theme. He said, "It is to be hoped that really beautiful mathematical theory which has been developed in the course of its three years of life may eventually find application somewhere in physics", although the neutrino theory of light must be abandoned.

§ 2. Hydrodynamical equations

The temporal development of a field quantity $A(x)$, a functional of the fermion wave functions $\varphi^*(x)$ and $\varphi(x)$, is described by the equation:

$$\frac{\partial A(x)}{\partial t} = \frac{i}{\hbar} [\bar{H}(x'), A(x)]. \quad (2.1)$$

Here the bracket means $\int \{H(x')A(x) - A(x)H(x')\} dx'$ and $H(x)$ is the Hamiltonian density. We consider the following two field quantities, density and current:

$$\rho(x) = \varphi^*(x)\varphi(x), \quad (2.2. a)$$

$$j(x) = \frac{\hbar}{mi} [\nabla] \rho(x), \quad (2.2 b)$$

where

$$[\nabla] \rho(x) = \frac{1}{2} \{ \varphi^*(x) \nabla \varphi(x) - \nabla \varphi^*(x) \varphi(x) \}.$$

They are subjected to the following commutation relations:

$$[\rho(x), \rho(x')] = 0, \quad (2.3. a)$$

$$[j(x), \rho(x')] = \frac{\hbar}{2mi} \{ \varphi^*(x) \varphi(x') \nabla \delta(x-x') + \varphi^*(x') \varphi(x) \nabla \delta(x-x') - \nabla \varphi^*(x) \varphi(x') \delta(x-x') - \varphi^*(x') \nabla \varphi(x) \delta(x-x') \}, \quad (2.3. b)$$

or

$$\langle [j(x), \rho(x')] \rangle_n = \frac{\hbar}{mi} \langle \rho(x) \rangle_n \nabla \delta(x-x'), \quad (2.3. b')$$

where $\langle A \rangle_n$ means the representation of A in the n -th subspace¹⁾ and

$$\langle \rho(x) \rangle_n = \sum_{\alpha=1}^n \delta(x-x_\alpha), \quad \langle j(x) \rangle_n = \frac{\hbar}{mi} \sum_{\alpha=1}^n \{ \nabla \delta(x-x_\alpha) - \delta(x-x_\alpha) \nabla \},$$

we introduce further the energy density which is defined by

$$E(x) = \frac{m}{2} \left(\frac{\hbar}{mi} \right)^2 [\nabla] \cdot [\nabla] \rho(x) + V(x) \rho(x), \quad (2.4)$$

where

$$V(x) = \frac{1}{2} \int \{ G(x', x) \rho(x') - G(x, x') \delta(x - x') \} dx'.$$

We assume the Hamiltonian density as follows:

$$\mathbf{H}(x) = \frac{\hbar^2}{2m} \frac{\partial \varphi^*(x)}{\partial x} \frac{\partial \varphi(x)}{\partial x} + \frac{1}{2} \int \varphi^*(x) \varphi^*(x') G(x, x') \varphi(x') \varphi(x) dx. \quad (2.5)$$

One gets the mass conservation law from (1.1),

$$m \frac{\partial \rho(x)}{\partial t} + \nabla \cdot m j(x) = 0, \quad (2.6)$$

which can be more directly derived from the Lagrangian density of the form

$$\mathbf{L}(x) = \varphi^*(x) i \hbar \varphi(x) - \frac{\hbar^2}{2m} \frac{\partial \varphi^*(x)}{\partial x} \frac{\partial \varphi(x)}{\partial x} - \frac{1}{2} \int \varphi^*(x) \varphi^*(x') G(x, x') \varphi(x') \varphi(x) dx'.$$

One finds further from (1.1) the equation of motion and the change of energy density of the form (see § 7),

$$\frac{\partial j(x)}{\partial t} + \nabla \cdot \mathbf{T}(x) = 0, \quad \mathbf{T}(x) = \left(\frac{\hbar}{mi} \right)^2 [\nabla][\nabla] \rho(x), \quad (2.7)$$

$$\frac{\partial E(x)}{\partial t} + \nabla \cdot h(x) = 0, \quad h(x) = \frac{m}{2} \left(\frac{\hbar}{mi} \right)^3 [\nabla][\nabla] \cdot [\nabla] \rho(x), \quad (2.8)$$

where the interaction energy is omitted (see Appendix. 2). If we assume the current operator expressible in the following form by a suitable operator $V(x)$ and its Hermite conjugate $V^*(x)$,

$$j(x) = \{ \rho(x), V(x) \} = \frac{1}{2} \rho(x) V(x) + \frac{1}{2} V^*(x) \rho(x), \quad (2.9)$$

we get the following equation corresponding to Euler's hydrodynamical equation:

$$\begin{aligned} \left\{ \rho(x), \frac{\partial V(x)}{\partial t} \right\} + \nabla \cdot \mathbf{P}(x) + \frac{1}{2} \nabla V^*(x) \cdot \{ \rho(x), V(x) \} + \frac{1}{2} \{ \rho(x), V(x) \} \cdot \overline{\nabla V(x)} \\ - \{ \rho(x), V(x) \} \times \omega(x) + \omega^*(x) \times \{ \rho(x), V(x) \} = 0, \end{aligned} \quad (2.10)$$

where

$$\omega(x) = \frac{1}{2} \nabla \times V(x), \quad \omega^*(x) = \frac{1}{2} \nabla \times V^*(x),$$

are vorticities, and $\mathbf{P}(x)$ is the pressure dyadic defined by

$$\mathbf{P}(x) = \mathbf{T}(x) - \frac{1}{2} \{ \rho(x), V(x) \} V(x) - \frac{1}{2} \overline{V^*(x) \{ \rho(x), V(x) \}}. \quad (2.11)$$

The dyadic notations used here are those devised by Chapman & Cowling.⁹

§ 3. Description in momentum space

One obtains the commutation relations and the formulae concerning the Fourier coefficients of quantities when the field quantities are developed in the Fourier series.

The Fourier coefficients of the wave function are given by

$$\begin{aligned}\varphi^*(x) &= \frac{1}{\sqrt{v}} \sum_k a_k^* \exp(-ikx), \quad \varphi(x) = \frac{1}{\sqrt{v}} \sum_k a_k \exp(ikx), \\ [a_k, a_l]_+ &= [a_k^*, a_l^*]_+ = 0, \quad [a_k, a_l^*]_+ = \delta_{k,l},\end{aligned}\tag{3.1}$$

and the coefficients of the hydrodynamical quantities are expressed by

$$\text{density:} \quad \rho_k = \sum_l a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}, \tag{3.2. a}$$

$$\text{current:} \quad j_k = \frac{\hbar}{m} \sum_l l a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}, \tag{3.2. b}$$

$$\text{stress dyadic:} \quad T_k = \frac{\hbar^2}{m^2} \sum_l l l a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}. \tag{3.2. c}$$

Hereby one finds the following commutation relations

$$\begin{aligned}[\rho_k, \rho_l] &= 0, \quad [\rho_k, j_l] = \frac{\hbar}{m} k \rho_{k+l}, \\ [j_k^{(x)}, j_l^{(y)}] &= \frac{\hbar}{m} (k^{(y)} j_{k+l}^{(x)} - l^{(x)} j_{k+l}^{(y)}),\end{aligned}\tag{3.3}$$

(x): x -component of vectors

and the temporal development of the density operator is expressed by

$$\rho_k(t) = \rho_k(o) - ik \cdot j_k(o) t - \frac{1}{2} k \cdot T_k(o) \cdot k t^2 + \dots \tag{3.4}$$

Taking the interaction energy of the form

$$H_{\text{int}} = \frac{1}{2} \sum_k G_k \rho_k \rho_{-k} - \frac{1}{2} G(o) \rho_o, \tag{3.5}$$

$$G_k = G_{-k} = \frac{1}{v^2} \iint G(x, x') \exp ik(x-x') dx' dx, \tag{3.6}$$

into consideration, the stress tensor becomes

$$T_k = \left(\frac{\hbar}{m}\right)^2 \sum_l l l a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}} - \frac{1}{m} \sum_l G_l \rho_{l+k} \rho_{-l} \mathbf{I}_{kl}, \quad k \mathbf{I}_{kl} = l. \tag{3.7}$$

§ 4. The one dimensional problem

S. Tomonaga solved the eigenvalue problem concerning the one dimensional many fermion assembly under the following premises:

- (1) No fermion is in the region of the wave number $n > 5n_0/4$, $\rho_0 = 2n_0 = N$,
- (2) No hole is in the region of the wave number $n < 3n_0/4$,
- (3) The wave length of the sound wave is larger than $4L/N$.

We introduce the spin degeneracies in the conventional manner and then consider the eigenvalue problem and the correlation of particles in the one dimensional box of length L . We have proposed in the preceding paper to introduce canonical variables

$$\rho_k^+ = \frac{1}{\sqrt{2}}(\rho_{k,+} + \rho_{k,-}), \quad \rho_k^- = \frac{1}{\sqrt{2}}(\rho_{k,+} - \rho_{k,-}), \quad (4.1)$$

$$\pi_k^+ = \frac{1}{\sqrt{2}}(\pi_{k,+} + \pi_{k,-}), \quad \pi_k^- = \frac{1}{\sqrt{2}}(\pi_{k,+} - \pi_{k,-}),$$

$$[\rho_k^w, \pi_l^{w'}] = i\hbar \delta_{k,l} \delta_{w,w'}, \quad [\rho_k^w, \rho_l^{w'}] = [\pi_k^w, \pi_l^{w'}] = 0, \quad w = +, -$$

where

$$\rho_k^w = \sum a_{l-\frac{k}{2},w}^* a_{l+\frac{k}{2},w}, \quad (4.2)$$

$$\pi_k^w = \frac{\pi i}{kL} \left(\sum_{l>0} a_{l-\frac{k}{2},w}^* a_{l+\frac{k}{2},w} - \sum_{l<0} a_{l-\frac{k}{2},w}^* a_{l+\frac{k}{2},w} \right).$$

The Hamiltonian of the assembly becomes

$$H = \frac{\hbar^2}{2m} \sum_k k^2 (N_{k,+} + N_{k,-}) + \sum G_k \rho_k^+ \rho_{-k}^+ - \frac{1}{2} G(0) N, \quad (4.3)$$

where

$$G_{-k} = G_k = \frac{1}{L^2} \iint G(x, x') e^{ik(x-x')} dx dx', \quad N = N_+ + N_-. \quad (4.4)$$

The Lagrangian density of sound quanta takes the form

$$L(x) = \frac{1}{2} \sum_\lambda \left\{ \left| \dot{S}_\lambda(x) \right|^2 - c_0^2 \left| \frac{\partial S_\lambda(x)}{\partial x} \right|^2 - c_0^2 \mu_\lambda^2 |S_\lambda(x)|^2 \right\}, \quad (4.5)$$

$$\lambda = +, -, \quad c_0^2 = \frac{\hbar^2 k_0^2}{m^2}, \quad \mu_+ = \frac{1}{c_0} \sqrt{\frac{N}{m} G_k |k|^2}, \quad \mu_- = 0$$

Thereby one obtains the momentum and the energy

$$G = \frac{1}{2} \sum_\lambda \int \left\{ \dot{S}_\lambda(x) \frac{\partial S_\lambda(x)}{\partial x} + \frac{\partial S_\lambda(x)}{\partial x} \dot{S}_\lambda(x) \right\} dx, \quad (4.6)$$

$$H^s = \hbar \sum_{k\lambda} \omega_k^\lambda M_k^\lambda, \quad (4.7)$$

where

$$S_{\lambda}(x) = \frac{1}{L} \sum_k \sqrt{\frac{N}{2m}} \frac{1}{c_k^{\lambda}} \pi_k \exp(-ikx), \quad c^- = c_0, \quad (4.8)$$

$$\delta_k^{\lambda*} = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\frac{N|k|}{2mc_k^{\lambda}}} \pi_{-k}^{\lambda} + i \sqrt{\frac{2mc_k^{\lambda}}{N|k|}} \rho_k^{\lambda} \right), \quad (4.9. a)$$

$$\delta_k^{\lambda} = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{\frac{N|k|}{2mc_k^{\lambda}}} \pi_k^{\lambda} - i \sqrt{\frac{2mc_k^{\lambda}}{N|k|}} \rho_{-k}^{\lambda} \right), \quad (4.9 b)$$

$$\omega_k^- = |k|c_0, \quad \omega_k^+ = |k|c_k^+, \quad c_k^+ = \left(\frac{\hbar^2 k_0^2}{m^2} + \frac{N}{m} G_k \right)^{\frac{1}{2}}. \quad (4.10)$$

One finds that H^S takes the alternative form

$$H^S = H + \frac{\hbar}{2} \sum_k (\omega_k^- - \omega_k^+) - H_0 - \frac{1}{2} G_0 N^2 + \frac{1}{2} G(o) N, \quad (4.11)$$

under Tomonaga's premise. One readily obtains the conservation law

$$\dot{\rho}_k^+ = \frac{k_0 L}{\pi m} k^2 \pi_{-k}^+, \quad \dot{\rho}_k^- = \frac{k_0 L}{\pi m} k^2 \pi_{-k}^-, \quad (4.12. a)$$

and the equations of motion

$$\dot{\pi}_{k_i}^+ = -2 \frac{m}{N} c_k^{+2} \rho_{-k}^+, \quad \dot{\pi}_{k_i}^- = -2 \frac{m}{N} c_0^2 \rho_{-k}^-. \quad (4.12. b)$$

It is interesting that the last equation is independent of the interaction energy.

§ 5. The correlation function

We define the correlation function of the N fermion assembly by making use of the reduced density operator of two explicit coordinates in the form (σ : spin variable, v : volume of box)

$$\begin{aligned} C(\xi) &= \frac{1}{N(N-1)} \sum_{\sigma, \sigma'} \int \langle \varphi^*(x+\xi, \sigma) \varphi^*(x, \sigma') \varphi(x, \sigma') \varphi(x+\xi, \sigma) \rangle_{AV} dx \\ &= \frac{1}{N(N-1)v} \sum_{k, l} \sum_q \sum_{\lambda, \lambda'} \langle a_{k, \lambda}^* a_{l+q, \lambda'}^* a_{l, \lambda'} a_{k+q, \lambda} \rangle_{AV} \exp(iq\xi), \end{aligned} \quad (5.1)$$

or alternatively

$$\begin{aligned} C(\xi) &= \frac{1}{N(N-1)} \int \langle \{ \rho(x+\xi) \rho(x) \} - \delta(\xi) \} \rho(x) \rangle_{AV} dx \\ &= \frac{1}{N(N-1)v} \sum_{k, l} \sum_q \sum_{\lambda, \lambda'} \langle a_{k, \lambda}^* a_{k+q, \lambda} a_{l+q, \lambda'}^* a_{l, \lambda'} - N_{k, \lambda} \delta_{k, l} \delta_{\lambda, \lambda'} \rangle_{AV} \exp(iq\xi). \end{aligned} \quad (5.2)$$

We divide this function into the three parts. The first is the Coulomb term

$$C_c(\xi) = \frac{1}{N(N-1)v} \sum_{k,l} \sum_{\lambda,\lambda'} \langle N_{k,\lambda} N_{l,\lambda'} \rangle_{AV},$$

the second is the exchange term

$$C_{ex}(\xi) = \frac{-1}{N(N-1)v} \sum_k \sum_q \sum_\lambda \langle N_{k,\lambda} N_{k+q,\lambda} \rangle_{AV} \exp(iq\xi),$$

and the last is the so-called collision term

$$C_{col}(\xi) = \frac{1}{N(N-1)v} \sum_{k \neq l} \sum_{q \neq 0} \sum_{\lambda, \lambda'} \langle a_{k,\lambda}^* a_{l+q,\lambda'}^\dagger a_{l,\lambda'} a_{k-q,\lambda} \rangle_{AV} \exp(iq\xi).$$

This term contributes considerably, if the interaction energy has appreciable value, although it is neglected in the scheme of the Hartree-Fock approximation. The major task of our theory consists in the evaluation of this term, therefore we shall especially pay attention for it.

At first we calculate the second term in the right side of (5.2):

$$\begin{aligned} I &= \partial(\xi) \int \rho(x) dx = \frac{1}{v} \sum_k \sum_\lambda N_{k,\lambda} \sum_q \exp(iq\xi) \\ &= \frac{1}{v} \sum_{k,q} \sum_\lambda \{ N_{k,\lambda} N_{k+q,\lambda} \exp(iq\xi) + N_{k,\lambda} (1 - N_{k+q,\lambda}) \exp(iq\xi) \}. \end{aligned} \quad (5.3)$$

For the complete degeneracy, namely at the ground state and at the limit of the vanishing interaction energy, the first term in the right side gives the exchange term and the second

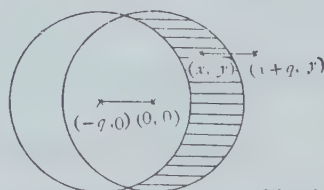


Fig. 1

$$\begin{aligned} x^2 + y^2 + z^2 &< k_0^2 \\ (x+q)^2 + y^2 + z^2 &> k_0^2 \\ x, y\text{-plane} \end{aligned}$$

term gives non vanishing value only if k -state is occupied and $k+q$ -state is unoccupied. We write

$$\langle I \rangle_{AV} = -C_{ex}(\xi) + \frac{1}{v} \sum_q f(q) \exp(iq\xi), \quad (5.4)$$

where $f(q)$ is given by (see fig. 1)

$$\begin{aligned} f(q) &= \frac{v}{8\pi^3} \pi \left(k_0^2 q - \frac{1}{12} |q|^3 \right), \quad |q| \leq 2k_0, \\ &= \frac{v}{8\pi^3} \frac{4}{3} \pi k_0^3 = N, \quad |q| > 2k_0. \end{aligned} \quad (5.5)$$

Next we consider the one dimensional problem. The correlation function is a function of ρ_k^+ only and given by

$$C(\hat{\xi}) = \frac{1}{N(N-1)L} \sum (2\rho_q^+ \rho_{-q}^+ - \sum_{k,\lambda} N_{k,\lambda})_{AV} \exp(iq\hat{\xi}). \quad (5.6)$$

From (4.9) one gets

$$\rho_q^+ \rho_{-q}^+ = \frac{|q|L}{2\pi} \frac{c_0}{c_q} (M_b^+ + M_{-q}^+ + 1). \quad (5.7)$$

If we are only interested in the ground state, we find

$$\begin{aligned} C(\hat{\xi})_q &= \frac{1}{N(N-1)L} \sum_q \left(\frac{|q|L}{2\pi} \frac{c_0}{c_q} - f(q) \right) \exp(iq\hat{\xi}) + C_{ex}(\hat{\xi}) \\ &= \frac{1}{N(N-1)L} \sum_q \frac{|q|L}{\pi} \left(\frac{c_0}{c_q} - 1 \right) \cos q\hat{\xi} - \frac{1}{2} \left(\frac{\sin k_0 \hat{\xi}}{k_0 \hat{\xi}} \right)^2, \end{aligned} \quad (5.8)$$

which splits up into two parts:

$$C_p(\hat{\xi}) = \frac{1}{L} \left\{ \frac{1}{2} - \frac{L}{N} \delta(\hat{\xi}) + \frac{1}{N^2} \sum_q \left| \frac{Lq}{2\pi} \right| \left(\sqrt{\frac{T_q}{T_q + 4U_q}} + 1 \right) \cos q\hat{\xi} \right\}, \quad (5.8'a)$$

$$C_a(\hat{\xi}) = \frac{1}{L} \left\{ \frac{1}{2} + \frac{1}{N^2} \sum_q \left| \frac{Lq}{2\pi} \right| \left(\sqrt{\frac{T_q}{T_q + 4U_q}} - 1 \right) \cos q\hat{\xi} \right\} \quad (5.8'b)$$

corresponding to parallel spins and antiparallel spins respectively. Here Tomonaga's notations

$$T_q = \frac{\hbar^2}{m} |q|k_0, \quad U_q = \left| \frac{Lq}{2\pi} \right| G_q, \quad (5.9)$$

are adopted. From these formulae we can infer that the three dimensional correlation function contains terms of the form

$$-\int_0^{2k_0} \frac{dq}{q} \exp(iq\hat{\xi}) = -8\pi \frac{\sin^2 k_0 \hat{\xi}}{\hat{\xi}^2}. \quad (5.10)$$

Heisenberg⁸⁾ suggested that any two antiparallel spins do not approach to each other, though this estimation would meet with great mathematical difficulties. From our point of view, his suggestion seems to be reasonable and fit into our interpretation.

We shall add some remarks on the energy change caused by the dynamical correlation of particles. We consider that fermions move in a prescribed uniform positive charge distribution. The ground state energy is divided into three parts:

H_0 : for the complete degeneracy,

$$E_0 = \frac{\hbar}{2} \sum_{|k| < 2k_0} |k| C_0 \left(\sqrt{\frac{T_k + 4U_k}{T_k}} - 2 \right) + \frac{\hbar}{2} \sum_{|k| > k_0} 2k_0 C_0 \left(\sqrt{\frac{T_k + 4U_k}{T_k}} - 2 \right),$$

for the zero point energy and

$$V_0 = -\frac{1}{2} G(o) N, \quad (5.11)$$

which turns out at the vanishing interaction energy into the form,

$$V_0 = -\frac{1}{2} \sum_q \sum_k \sum_\lambda G_q N_{k,\lambda} N_{k+q,\lambda} - \sum_{|q| < 2k_0} G_q \left| \frac{Lq}{2\pi} \right| - \sum_{|q| > 2k_0} G_q \frac{Lk_0}{\pi} \quad (5.11')$$

that cancels the first order term in the zero point energy. We have arranged the series of the zero point energy so that the terms concerning the dynamical correlation function vanish for the vanishing interaction energy.

We have dealt with the operators ρ_k and π_k , but the operators

$$\rho^+(k) = \sum_{l>0} a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}, \quad \rho^-(k) = \sum_{l<0} a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}$$

are the more fundamental ones. One reason lies in the fact that one cannot from the commutation relation (4.1) derive the commutation relation given by Tomonaga,

$$[\rho^+(k), \rho^+(l)] = [k] \delta_{k,-l}, \quad [\rho^-(k), \rho^-(l)] = -[k] \delta_{k,-l},$$

$$[\rho^\pm(k), \rho^\mp(l)] = 0, \quad [k] = \frac{L}{2\pi} k,$$

from which, however, one can obtain (4.1). If one remembers the neutrino theory of light and the fact that $\rho^+(k)$ represents the progressive wave and $\rho^-(k)$ represents the regressive wave, one readily think of that relationship between the regressive waves representing negative energy states and positons or holes.

§ 6. The three dimensional problem

When one intends to apply the theory to the three dimensional problem, one meets with the difficulties that the density operator is no more governed by the wave equation and it is hardly possible to compose the canonical variables of the density and the current operators under such simple assumptions as the case of one dimensional problem.⁹⁾ Here we consider how we could construct the Hamiltonian in the quadratic form by introducing suitable operators in place of the density operator. At first the following operators are introduced

$$\tau_k = \sum_{l//k} a_{l-\frac{k}{2}}^* a_{l+\frac{k}{2}}, \quad l = |l|n, \quad k = |k|n, \quad \text{with unite vector } n, \quad (6.1)$$

$$[a_l, \tau_k] = a_{l+k} \delta_{n,n'}, \quad [\tau_k, a_l^*] = a_{l-k}^* \delta_{n,n'}, \quad k = |k|n, \quad l = |l|n',$$

which induces various translations along the radial directions in the momentum space. One meets with such operators in the neutrino theory of light. The canonical conjugate variable to τ_k is given by

$$\sigma_k = \frac{\pi i}{|k|L} \hbar \left(\sum_{l>0} a_{l+\frac{k}{2}}^* a_{l-\frac{k}{2}} - \sum_{l<0} a_{l+\frac{k}{2}}^* a_{l-\frac{k}{2}} \right), \quad l//k, \quad (6.2)$$

which is constructed according to the principle of the approximation for the one dimensional case. Here we ask for the Hamiltonian in the quadratic form of the canonical variables τ_k and σ_k . If the interaction energy is absent, it is possible to compose the Hamiltonian in such a form by adding terms

$$\frac{2\pi}{L} \frac{\hbar^2 k_0}{m} \sum_{r \neq r'} \sum_{r''}^{\infty} a^* \left(r - \frac{|k|}{2} \right)_n a \left(r + \frac{|k|}{2} \right)_n a^* \left(r' + \frac{|k|}{2} \right)_n a \left(r' - \frac{|k|}{2} \right)_n, \quad k = |k|n, \quad (6.3)$$

which vanishes under our approximation, to the Hamiltonian of the form

$$K_0^S = \frac{\hbar^2 k_0}{m} \sum_{|k| < k_0} |k| (N_k - 1) + \frac{\hbar^2 k_0}{m} \sum_{|k| > k_0} |k| N_k. \quad (6.4)$$

Indeed, one is led to the Hamiltonian of the familiar form

$$K_0^S = \frac{1}{2} \frac{m}{N_0} C_0^2 \sum_k \tau_k \tau_{-k} + \frac{1}{2} \frac{N_0}{m} \sum_k |k|^2 \sigma_k \sigma_{-k} - \frac{\hbar}{2} \sum_k |k| C_0, \quad C_0 = \frac{\hbar k_0}{m} = \frac{\pi \hbar}{L m} N_0. \quad (6.5)$$

Before the interaction energy is introduced, we ask for the structure of the k -th component of the density operator. We have

$$\rho_k = \sum_{x,y} \sqrt{\frac{N_{xy}}{m}} \rho_{xy} = \sum_{x,y,z} a^*(x, y, z) a(x, y, z+k), \quad (6.6)$$

where ρ_{xy} is given by

$$\rho_{xy}(k) = \sqrt{\frac{m}{N_{xy}}} \sum_z a^* \left(x, y, z - \frac{k}{2} \right) a \left(x, y, z + \frac{k}{2} \right),$$

$$N_{xy} = \frac{L}{\pi} \sqrt{k_0^2 - x^2 - y^2}, \quad k = (c, c, |k|). \quad (6.7)$$

The canonical conjugate variables are defined by

$$\pi_{xy}(k) = \frac{\pi i}{k L} \sqrt{\frac{N_{xy}}{m}} \hbar \left\{ \sum_{z>0} a^* \left(x, y, z + \frac{k}{2} \right) a \left(x, y, z - \frac{k}{2} \right) - \sum_{z<0} a^* \left(x, y, z + \frac{k}{2} \right) a \left(x, y, z - \frac{k}{2} \right) \right\},$$

$$[\rho_{xy}(k), \pi_{x'y'}(k)] = i \hbar \delta_{x,x'} \delta_{y,y'}, \quad (6.8)$$

and the other commutators vanish. If the Hamiltonian is given in the form

$$H_0 = \frac{\hbar^2}{2m} \sum_{x,y,z} (x^2 + y^2 + z^2) N_{x,y,z}, \quad (6.9)$$

one finds

$$\ddot{\rho}_{xy}(k) + C_{xy}^2 k_0^2 \rho_{xy}(k) = 0, \quad \ddot{\pi}_{xy}(k) + C_{xy}^2 k_0^2 \pi_{xy}(k) = 0, \quad C_{xy}^2 = \frac{\hbar^2}{m^2} (k_0^2 - x^2 - y^2). \quad (6.10)$$

When the interaction energy

$$V_{\text{int}} = \frac{1}{2} \sum_k G(k) \rho(k) \rho(-k) - \frac{1}{2} G(o) N,$$

is introduced, and further we assume

$$[\rho(k), \pi_{xy}(l)] = 0, \quad k \neq l, \quad (6.11)$$

which amounts to the assumption of small variation of the density (see (3.7)), the wave equations become

$$\rho_{xy}(k) + k^2 \sum_{x', y'} C_{xy, x' y'}^2(k) \rho_{x' y'}(k) = 0, \quad (6.12)$$

where

$$C_{xy, x' y'}(k) = \frac{\hbar^2}{m^2} (k_0^2 - x^2 - y^2) \delta_{x, x'} \delta_{y, y'} + \frac{\sqrt{N_{xy}} \sqrt{N_{x' y'}}}{m} G(k). \quad (6.13)$$

One readily finds that there are $\sqrt{x^2 + y^2} L$ degenerate oscillators belonging to the eigenfrequency $|k| C_{xy}$ which separate out by virtue of the interaction energy. Accordingly, one obtains $\sqrt{x^2 + y^2} L$ eigenvalues of the secular equation and corresponding proper variables $\tau_{s, i}(k)$ ($i=1, 2, \dots, S/L$), and their canonical conjugate variables $\sigma_{s, i}(k)$. Therefore the Hamiltonian of the oscillators:

$$H(k) = \frac{1}{2} \sum_{s, i} C_{xy}^2 \rho_{xy}(k) \rho_{xy}(-k) + \frac{1}{2} \sum_{s, i} k^2 \pi_{xy}(k) \pi_{xy}(-k) + \frac{1}{2} G(k) \rho(k) \rho(-k), \quad (6.14)$$

is written down in new variables

$$H(k) = \frac{1}{2} \sum_{s, i} \gamma_{s, i}^2(k) \tau_{s, i}(k) \tau_{s, i}(-k) + \frac{1}{2} k^2 \sum_{s, i} \sigma_{s, i}(k) \sigma_{s, i}(-k) + \frac{1}{2} G(k) \sum_s \tau_{s, 1}(k) \tau_{s, 1}(-k), \quad (6.15)$$

where $\gamma_{s, i}(k)$ is an eigenvalue and $\gamma_{s, 1}(k) = C_s$.

Next we ask for the normal coordinates which are obtained by a unitary transformation of variables $\tau_{s, i}(k)$, $\sigma_{s, i}(k)$ as follows:

$$\hat{\xi}_s(k) = (U^{-1}(k) \tau_1(k))_s, \quad \eta_s(k) = (\sigma_1(k) U(k))_s, \quad (6.16)$$

and also the eigenvalues designated by

$$\Gamma_s^2(k) = (U^{-1}(k) \gamma^2(k) U(k))_{ss}, \quad (6.17)$$

$$\gamma_{st}^2(k) = \frac{\hbar^2}{m^2} (k_0^2 - s^2) \delta_{s, t} + \frac{\sqrt{s N_s} \sqrt{t N_t}}{m} L G(k). \quad (6.18)$$

We calculate these eigenvalues by means of the perturbation method. If $\sqrt{s N_s} \sqrt{t N_t} G(k) L/m$ are considered as small quantities, Γ_s^2 is given by

$$\Gamma_s^2(k) = \sum_{p=1}^{\infty} (-1)^p \sum_{i \dots u} \underbrace{\frac{\mathcal{Q}_{si} \mathcal{Q}_{ui} \dots \mathcal{Q}_{u' i'} \mathcal{Q}_{u' s}}{(C_i^2 - C_s^2) \dots (C_u^2 - C_s^2)}}_p + C_s^2 + \mathcal{Q}_{ss}, \quad (6.19)$$

$$\mathcal{Q}_{si} = \frac{\sqrt{s N_s} \sqrt{i N_i} L G(k)}{m}. \quad (6.20)$$

Explicitly,

$$\Gamma_s^2(k) = C_s^2 + \mathcal{Q}_{ss} \{1 + D(s)\}^{-1}, \quad (6.21)$$

$$\begin{aligned} D(s) &= \frac{L^2}{2\pi} \int_0^{k_0} \frac{u N_u}{C_u^2 - C_s^2} du G(k) \\ &= \frac{L^3}{2\pi^2 \hbar^2} m G(k) \left(k_0 + \frac{\sqrt{k_0^2 - s^2}}{2} \log \frac{k_0 - \sqrt{k_0^2 - s^2}}{k_0 + \sqrt{k_0^2 - s^2}} \right). \end{aligned} \quad (6.22)$$

One finds creation and annihilation operators composed of normal coordinates as follows:

$$b_s^*(k) = \frac{1}{\sqrt{2\hbar}} \left\{ \sqrt{\frac{|k|}{\Gamma_s(k)}} \eta_s(k) + i \sqrt{\frac{\Gamma'_s(k)}{|k|}} \xi_s(-k) \right\}, \quad (6.23. a)$$

$$b(k) = \frac{1}{\sqrt{2\hbar}} \left\{ \sqrt{\frac{|k|}{\Gamma_s(k)}} \eta_s(-k) - i \sqrt{\frac{\Gamma'_s(k)}{|k|}} \xi_s(k) \right\}. \quad (6.23. b)$$

It is an interesting problem to compute the energy of the excited state and effects of external fields. These investigations are left to another occasion.

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Appendices

(1) Fermi statistics and Bose statistics

If our assembly is immersed in a large thermostat, it attains thermal equilibrium. Since we regard our objective as an ensemble of sound quanta, bosons, we must apply the Bose statistics. On the other hand it should be subjected to the Fermi statistics, because it is originally a fermion assembly. Mathematically, the distinction between them consists in the alternative whether we adopt the representation that the number of fermions is diagonal or the one that the number of sound quanta is diagonal. Which type of the two representations one may adopt, since one has only to do with traces of operators, one should obtain the unique description independent of mathematical treatments. If one adopts the Hamiltonian of the form

$$H = \frac{\hbar^2}{m} k_0 \sum_{|k| < k_0} |k| (N_k - 1) + \frac{\hbar^2}{m} k_0 \sum_{|k| > k_0} |k| N_k, \quad \rho_0 = N = \frac{V}{8\pi^3} \frac{4}{3} \pi k_0^3,$$

one gets the grand partition function

$$F(\beta, \xi) = \text{trace} \exp(-\beta H - \xi N), \quad N = \sum_k N_k,$$

which leads to the correct temperature dependence of the heat capacity. When we intend to apply the Bose statistics, we must lay stress on the assumption that the transitions occur at the neighborhood of the Fermi surface. This is the essential point discriminating the ensemble of sound quanta from those of harmonic oscillators in the conventional sense. The mean energy becomes

$$E = \sum_{|k|} \frac{\hbar \omega_k}{\exp(\beta \hbar \omega_k) - 1},$$

where the weight in the momentum space is not $\left(\frac{L}{2\pi}\right)^3 4\pi k^2$, but $\left(\frac{L}{2\pi}\right)^3 4\pi k_0^2$. And thus one gets the same result with the Fermi statistics. Indeed, the heat capacity becomes

$$C_v = \frac{\pi^2 2m}{3} \frac{k_0}{\hbar^2} \frac{k_0 T}{\beta^2}.$$

at low temperatures.

(2) The equation of motion with the interaction energy

Born and Green formulated the temporal behavior of the reduced density matrix in the configuration space. This formula is more directly given in our scheme. The equation corresponding to (2.7) becomes

$$\begin{aligned} \frac{\partial j(x)}{\partial t} &= \frac{i}{\hbar} [\bar{H}(x'), j(x)] = -\left(\frac{\hbar}{mi}\right)^2 \nabla \cdot [\nabla][\nabla] \rho(x) \\ &\quad - \int \frac{\partial G(x, x')}{\partial x} \{ \rho(x) \rho(x') - \rho(x) \delta(x-x') \} dx'. \end{aligned}$$

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Effect of Nucleon Recoil on Nuclear Forces

Gentaro ARAKI

*Department of Industrial Chemistry, Kyoto University
and Department of Physics, Kyushu University*

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The correction due to the nucleon recoil on nuclear forces is discussed according to the scalar and the pseudoscalar meson theory. The correction entirely changes the apparent form of two-nucleon interaction in case of the pseudoscalar theory. The singularity of the interaction becomes stronger in this case. However, the non-relativistic limit is the same as the well-known potential. The non-adiabatic correction includes the typical spin-orbit interaction in case of the scalar theory. The distinction of the present result from those which were obtained by other authors is discussed.

Introduction and Summary

Nuclear forces were usually derived in the meson theory by neglecting the effect of the nucleon recoil. Strictly speaking, the neglect of the recoil term can not be justified, in fact, even in the adiabatic limit because the two-nucleon interaction has matrix elements connecting positive and negative energy states of nucleons and the state of the two-nucleon system includes more or less the negative energy states. Therefore it is necessary to re-examine the derivation of nuclear forces by taking into account the effect of nucleon recoil.

On the other hand, the non-adiabatic effect of the nucleon recoil may play an important rôle in high energy processes between two nucleons. In such cases it must be necessary to take into account a coexisting meson field around the two-nucleon system and the force may not be represented by a potential, as was previously noted¹⁾. It may also be imagined that a small deviation from the adiabatic approximation has some effect even in lower energy processes if the phenomena are sensibly affected by a detailed behavior of two-nucleon wave functions. Such a consideration may be of interest, too, on a theoretical point of view in order to obtain a more cutting insight into the two-nucleon interaction.

The effect of the nucleon recoil on nuclear forces was briefly discussed in other place²⁾. The purpose of the present paper is to give a more detailed account of it. The certain considerations on the similar subject were previously given by Heitler³⁾, later by Toyoda⁴⁾, and recently by Dancoff⁵⁾. The present method and result are different from those given by them.

We begin with the transformation of the recoil term in the first section. We next discuss how the state of the two-nucleon system includes negative energy

states. The adiabatic two-nucleon potential is derived taking into account the recoil effect according to the scalar and the pseudoscalar meson theory in the third section. In case of the pseudoscalar theory the apparent form of the potential is entirely different from that which is derived neglecting the recoil effect, whereas there is no essential change in case of the scalar theory. It is found that the effect makes the singularity of the potential stronger in the former case.

The non-adiabatic effect of the nucleon recoil is discussed in the fourth section. In case of the scalar theory the correction includes the typical spin-orbit interaction. Dancoff⁵⁾ obtained the term of the same nature but the origin of his term was entirely different from that of the present one. Dancoff's spin-orbit interaction can be derived from the adiabatic part by the method of the Pauli approximation which is explained in the fifth section, though Dancoff's method was wholly different. Further it is shown there that Dancoff's term is completely cancelled by the other term which comes from the recoil effect. It is also found that the non-relativistic limit of the present result for the pseudoscalar theory is in agreement with the well-known nuclear forces.

In the last section the correction which can be obtained on the basis of an inadequate assumption of neglecting negative energy states of nucleons is discussed. It is found that the result is entirely different from those in the preceding sections, and that it involves Toyoda's result in a part of it. We see consequently that Toyoda's result is inadequate.

Our method can be applied to the case of the vector and the pseudovector theory, and further to the case of the electromagnetic interaction between electrons and protons. In the last case the effect of the electron or proton recoil is expected to influence the Lamb-Retherford shift of hydrogen levels. If the influence really exists its amount is expected to be of the order of 10 Mc/s. This will be quantitatively examined in the later article.

§ 1. Recoil Term

We shall consider the two-nucleon interaction which is transmitted by scalar or pseudoscalar mesons. Let this interaction energy be denoted by W . Its second order matrix elements were calculated by van Hove⁶⁾ in case of the pseudoscalar theory. Those in the scalar theory can be obtained in the same way. The results are given by*

$$W_{RA} = -f^2 \frac{4\pi}{V} \frac{(T\rho^{(1)}\rho^{(2)})_{BA}\delta(\mathbf{p}_A^{(1)} + \mathbf{p}_A^{(2)}, \mathbf{p}_R^{(1)} + \mathbf{p}_R^{(2)})}{\mu^2 + (\mathbf{p}_R^{(1)} - \mathbf{p}_A^{(1)})^2 - (\epsilon_R^{(1)} - \epsilon_A^{(1)})^2} \quad (1.1)$$

* We here omit van Hove's W_2 for the sake of simplicity.

where A and B are two arbitrary states of the two-free-nucleon system, \mathbf{p} and ϵ are the momentum and energy of a nucleon, μ is the mass of a meson, V is the volume of the normalization space, and all quantities are measured in $\hbar=c=1$ units. Quantities in the state A (or B) are indicated by the single suffix A (or B), and quantities of the first (or second) nucleon are indicated by the upper suffix (1) (or (2)). The matrix elements are indicated by the lower double suffices, for example, as $W_{BA}=(\psi_B, W \psi_A)$ where ψ_A and ψ_B are the appropriate (charge, spin, space, or total) parts of the eigenfunctions of A and B states respectively. Such notations will be used throughout the present paper. $\delta(\mathbf{p}, \mathbf{p}')$ is Kronecker's symbol which shows the momentum conservation. Further f , ρ and T are defined by

$$f = \begin{cases} g & \text{for the scalar theory} \\ f_3 = f_1 + (2M/\mu)f_2 & \text{for the pseudoscalar theory} \end{cases} \quad (1.2) \text{ (a)}$$

$$\rho = \begin{cases} \rho_3 & \text{for the scalar theory} \\ \rho_2 & \text{for the pseudoscalar theory} \end{cases} \quad (1.2) \text{ (b)}$$

$$T = \begin{cases} (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)})/2 & \text{for the charged theory} \\ (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)} + \tau_3^{(1)} \tau_3^{(2)})/2 & \text{for the symmetrical theory} \\ 1 & \text{for the neutral theory} \end{cases} \quad (1.2) \text{ (c)}$$

where M is the mass of a nucleon, g , f_1 and f_2 are the constants of the scalar, the pseudoscalar and the pseudovector coupling of mesons with nucleons, ρ_1 , ρ_2 and ρ_3 are the Dirac matrices of a nucleon, and τ_1 , τ_2 , and τ_3 are charge operators which have the same algebraic properties as ρ_1 , ρ_2 , and ρ_3 .

The above mentioned matrix elements are written in the form which is not symmetrical with respect to nucleons for the sake of practical convenience in the following calculation. They must be symmetrized with respect to nucleons in order to obtain the correct result. We shall first calculate using the asymmetrical form and then symmetrize the result. Van Hove⁶ obtained the above mentioned matrix elements in the centre-of-mass system assuming that $\epsilon_A^1 - \epsilon_A^2 = \epsilon_B^1 - \epsilon_B^2 = 0$. This assumption is satisfied if we neglect negative energy states, whereas the neglect of the negative energy states can not be justified for the reason which will be accounted for in the next section. We can show, however, that (1.1) is exactly valid in any system without this assumption.

The term, $(\epsilon_B^{(1)} - \epsilon_A^{(1)})^2$, of the energy difference of a nucleon in the denominator of (1.1) represents the effect of the nucleon recoil. Therefore we shall refer to this term as a recoil term. The purpose of the present paper is to study the effect of this term. One may imagine that a contribution from this recoil term to the binding energy of the deuteron may be negligible because the binding energy (2.17 MeV.) is very small compared with the rest energy of a meson, μ (146 MeV.), and that the adiabatic approximation may be considered to be

sufficient for a calculation of the binding energy. We shall find shortly that the recoil term will play an important rôle even in the adiabatic limit in case of the pseudoscalar theory.

In order to examine the effect we transform the denominator of (1.1). By rationalizing the denominator this can be carried out as follows:

$$\frac{1}{\mu^2 + \mathbf{k}^2 - (\epsilon - \epsilon_0)^2} = \frac{M^2 - (\mu^2/2) + \epsilon\epsilon_0 + \mathbf{p}\mathbf{p}_0}{2\{M^2(\mathbf{k}^2 + \lambda^2) + \mu^2\mathbf{p}\mathbf{p}_0 + \mathbf{p}^2\mathbf{p}_0^2 - (\mathbf{p}\mathbf{p}_0)^2\}} \quad (1.3)$$

where the following abbreviations are used: $\epsilon = \epsilon_H^{(1)}$, $\epsilon_0 = \epsilon_A^{(1)}$, $\mathbf{p} = \mathbf{p}_B^{(1)}$, $\mathbf{p}_0 = \mathbf{p}_A^{(1)}$, $\mathbf{k} = \mathbf{p} - \mathbf{p}_0$ and

$$\lambda = \mu\{1 - (\mu/2M)^2\}^{1/2} \quad (1.4)$$

The relation given by (1.3) is exactly valid. If p_0 is small compared with M , (1.3) can be written in the approximate form as follows:

$$\begin{aligned} \frac{1}{\mu^2 + \mathbf{k}^2 - (\epsilon - \epsilon_0)^2} &= \frac{1}{\mathbf{k}^2 + \lambda^2} \left\{ \frac{1}{2} - \left(\frac{\mu}{2M} \right)^2 + \frac{\epsilon\epsilon_0 + \mathbf{p}\mathbf{p}_0}{2M^2} \right\} \\ &- \frac{1}{M^2(\mathbf{k}^2 + \lambda^2)^2} \left\{ \frac{1}{2} - \left(\frac{\mu}{2M} \right)^2 + \frac{\epsilon\epsilon_0 + \mathbf{p}\mathbf{p}_0}{2M^2} \right\} \{ \mu^2\mathbf{p}\mathbf{p}_0 + \mathbf{p}^2\mathbf{p}_0^2 - (\mathbf{p}\mathbf{p}_0)^2 \} \end{aligned} \quad (1.5)$$

where the terms up to the second degree in p_0/M and μ/M are reserved. The approximation of the last relation is sufficient independently of how p is large so far as p_0 is very small compared with M . Since (1.3) is symmetrical with respect to \mathbf{p} and \mathbf{p}_0 , the argument is valid when \mathbf{p} and \mathbf{p}_0 are interchanged.

§ 2. On the Basic States of Two-Nucleon System

We have first to examine the states of two interacting nucleons in order to correctly obtain the two-nucleon interaction W . This has usually been derived taking into account only positive energy states of each nucleon. The reason may be that the matrix elements connecting positive and negative energy states correspond to the production or annihilation of the nucleon pair. One may think that there is no connection between negative energy states and the nuclear forces for this reason. This conclusion is not correct, however, because the nuclear forces are derived for the purpose of considering the bound states of the two-nucleon system.

When we consider a free nucleon we can distinctly separate its states of the positive energy from those of the negative energy. In this case the momentum and the energy commute, and we can consider the simultaneous eigenstate of the two quantities. If χ^+ and χ^- are the eigenfunctions of ρ_3 belonging to its eigenvalues $+1$ and -1 respectively, four basic states of a free nucleon belonging to the definite eigenvalue \mathbf{p} of its momentum are represented by the eigenfunctions of the following form:

$$\left. \begin{aligned} u_1^+(\mathbf{p}, q) &= (a_1^+ \chi^+ + a_1^- \chi^-) \varphi(\mathbf{p} \mathbf{x}) \\ u_2^+(\mathbf{p}, q) &= (a_2^+ \chi^+ + a_2^- \chi^-) \varphi(\mathbf{p} \mathbf{x}) \\ u_1^-(\mathbf{p}, q) &= (b_1^+ \chi^+ + b_1^- \chi^-) \varphi(\mathbf{p} \mathbf{x}) \\ u_2^-(\mathbf{p}, q) &= (b_2^+ \chi^+ + b_2^- \chi^-) \varphi(\mathbf{p} \mathbf{x}) \end{aligned} \right\} \quad (2.1)$$

where q stands for all coordinates, \mathbf{x} is the position vector, a 's and b 's are the spin parts, and φ is the space part defined by

$$\varphi(\mathbf{x}) = V^{-1/2} e^{i \mathbf{x} \cdot \mathbf{p}} \quad (2.2)$$

The spin functions a 's and b 's are the linear combinations of α and β , and their coefficients depend on \mathbf{p} and M only where α and β are the eigenfunctions of σ_3 belonging to its eigenvalues $+1$ and -1 respectively.

We assume that the Hamiltonian of the two-nucleon system is given by

$$H = H^{(1)} + H^{(2)} + W \quad (2.3)$$

where $H^{(1)}$ and $H^{(2)}$ are the Dirac Hamiltonian of the first and the second nucleon in the free state respectively: for example,

$$H^{(1)} = -i \rho_1^{(1)} \boldsymbol{\sigma}^{(1)} \nabla^{(1)} + M \rho_3^{(1)} \quad (2.4)$$

Let the eigenfunction of H be ψ . This function can be written in the form:

$$\psi = \psi^{++} \chi^+(1) \chi^+(2) + \psi^{+-} \chi^+(1) \chi^-(2) + \psi^{-+} \chi^-(1) \chi^+(2) + \psi^{--} \chi^-(1) \chi^-(2) \quad (2.5)$$

where, for example, $\chi^+(1)$ denotes the χ^+ function of the first nucleon. The functions ψ^{++} etc. depend on the spin and space coordinates. They can again be expanded into the Fourier series: for example,

$$\begin{aligned} \psi^{++} = & \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \{ c_1 a(1) a(2) + c_2 a(1) \beta(2) \\ & + c_3 \beta(1) a(2) + c_4 \beta(1) \beta(2) \} \varphi(\mathbf{p}_1 \mathbf{x}_1) \varphi(\mathbf{p}_2 \mathbf{x}_2) \end{aligned} \quad (2.6)$$

Four functions $\alpha \chi^+ \varphi$, $\beta \chi^+ \varphi$, $\alpha \chi^- \varphi$, and $\beta \chi^- \varphi$ are given by linear combinations of u_1^+ , u_2^+ , u_1^- , and u_2^- if we solve the simultaneous equations (2.1) for the former four. If we substitute (2.6) etc. into (2.5) and then again the above mentioned linear combinations into this result we have

$$\begin{aligned} \psi = & \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{k=1}^2 \sum_{j=1}^2 \{ c_{kj}^{++} u_k^+(\mathbf{p}_1, q_1) u_j^+(\mathbf{p}_2, q_2) + c_{kj}^{+-} u_k^+(\mathbf{p}_1, q_1) u_j^-(\mathbf{p}_2, q_2) \\ & + c_{kj}^{-+} u_k^-(\mathbf{p}_1, q_1) u_j^+(\mathbf{p}_2, q_2) + c_{kj}^{--} u_k^-(\mathbf{p}_1, q_1) u_j^-(\mathbf{p}_2, q_2) \} \end{aligned} \quad (2.7)$$

In the meson theory of nuclear forces the general form of W is given by

$$W = \sum_{k=0}^3 \sum_{j=0}^3 \rho_k^{(1)} \rho_j^{(2)} W_{kj} \quad (2.8)$$

where $\rho_0 = 1$. If we substitute this into (2.3) and then the result and (2.7) into the eigenvalue equation

$$H\psi = (E + 2M)\psi \quad (2.9)$$

we have a system of simultaneous equations for c_{kj}^{l+} etc. Owing to the fact that W actually involves ρ_1, ρ_2 or ρ_3 , c^{++} can not be determined independently of c^{+-} , c^{-+} , and c^{--} in general.

The eigenfunction ψ thus necessarily consists of negative as well as positive energy states. In other words, the vacuum nucleons are partially absorbed in the interacting two nucleons. The interaction operator W is completely represented by its matrix elements with respect to ψ 's. The orthonormal set which is equivalent to the set of all ψ 's is the direct product of two sets one of which consists of free states of the first nucleon including negative as well as positive energy states and the other also consists of negative and positive energy states of the second nucleon. Consequently we arrive at a conclusion that we have to take into account both positive and negative energy states of each nucleon when we consider the matrix elements of W . The following consideration will be based on this general stand point.

§ 3. Adiabatic Limit

We shall discuss the adiabatic limit of W in this section. The adiabatic limit here means the expression which is obtained from the exact interaction by substituting zeros for V 's operating on the two-nucleon wave function. The non-adiabatic correction will be considered in the next section.

The usual two-nucleon potential can be obtained from (1.1) if we neglect the recoil term altogether. In this case, making use of the relation

$$\frac{4\pi}{V} \frac{\delta(\mathbf{p}_A^{(1)} + \mathbf{p}_A^{(2)}, \mathbf{p}_B^{(1)} + \mathbf{p}_B^{(2)})}{(\mathbf{p}_B^{(1)} - \mathbf{p}_A^{(1)})^2 + \mu^2} = (\varphi_B, r^{-1} e^{-\mu r} \varphi_A) \quad (3.1)$$

we have

$$W = -f^2 T \rho^{(1)} \rho^{(2)} r^{-1} e^{-\mu r} \quad (3.2)$$

where r is the distance between two nucleons, and φ_A and φ_B are the space parts of eigenfunctions of the two-nucleon system in the states A and B respectively. It was shown by the present author that this potential is equivalent to the customary one for the large r in the non-relativistic limit.¹⁾

Such a potential is not the adiabatic one in our definition. When we consider the matrix elements of W connecting the two states in which momenta of nucleons are very small compared with M , the neglect of the recoil term is equivalent to the neglect of negative energy states of nucleons. However, we can not exclude the negative energy states from our consideration even in this limit because W contains ρ_2 in case of the pseudoscalar theory. In fact, the matrix element connecting the negative and the positive energy state is not smaller than that connecting the positive energy states in this case. If one of ϵ and ϵ_0 is positive and the other is negative the recoil term is by no means small compared with μ^2 . We have therefore to take into account both positive

and negative energy states of nucleons in order to obtain the correct result. How the neglect of the recoil term is inadequate can be seen in the following way.

The adiabatic limit of W can be obtained by replacing p_0/M in (1.1) by zero. In this case W_{BA} can be written in the form

$$W_{BA} = - (f^2/2) [T\rho^{(1)}\rho^{(2)}r^{-1}e^{-\lambda r} \{1 + (\epsilon\epsilon_0/M^2) - (\mu^2/2M^2)\}]_{BA} \quad (3.3)$$

where Eq. (3.1) is made use of. The constant λ is approximately equal to μ . Its correction factor is equal to 0.994 if we adopt $\mu/M=286/1840$. Comparing (3.2) with (3.3) we can find how the latter is changed into the former. If we replace $\epsilon\epsilon_0$ of (3.3) with M^2 we have the potential which is approximately the same with (3.2). The difference is a factor, $1 - (\mu/2M)^2$, and λ in the exponential. The substitution of M^2 for $\epsilon\epsilon_0$ can be justified if A and B are positive energy states and p is small compared with M , but $\epsilon\epsilon_0$ must be replaced by $-M^2$ when one of ϵ and ϵ_0 is negative and the other is positive. The latter case can occur when the meson field is pseudoscalar.

In order to correctly take into account this point we have to replace ϵ and ϵ_0 by the free-one-nucleon Hamiltonians, and the former must be placed on the left and the latter on the right of $\rho^{(1)}\rho^{(2)}$ exp $(-\lambda r)$ as follows:

$$(T\rho^{(1)}\rho^{(2)}r^{-1}e^{-\lambda r}\epsilon\epsilon_0)_{BA} = (H^{(1)}T\rho^{(1)}\rho^{(2)}r^{-1}e^{-\lambda r}H^{(1)})_{BA} \quad (3.4)$$

where $H^{(1)}$ is given by (2.4).

If $H^{(1)}$ is replaced by $M\rho_3^{(1)}$, the transform of $\epsilon\epsilon_0$ is equal to M^2 in case of the scalar theory whereas it is equal to $-M^2$ in case of the pseudoscalar theory because ρ is equal to ρ_3 in the former case and to ρ_2 in the latter case. The negative sign in the latter case is due to the anti-commutability of ρ_2 and ρ_3 . The sum $1 + (\epsilon\epsilon_0/M)$ in (3.3) is then equal to 2 in the scalar case whereas it vanishes in the pseudoscalar case. Therefore we have to adopt the complete Hamiltonian but not $M\rho_3^{(1)}$ as $H^{(1)}$ in the latter case even if we want to obtain the lowest degree term in $1/M$. The result is symmetrized with respect to nucleons for the reason mentioned in the first section. Thus we have the adiabatic two-nucleon potential as follows:

$$W_s = -g_s^2 T\rho_s^{(1)}\rho_s^{(2)}r^{-1}e^{-\lambda r} \quad (3.5) \quad (a)$$

$$W_{ps} = g_{ps}^2 T r^{-1} e^{-\lambda r} \{ (M\lambda/\mu^2) (1 + \lambda^{-1}r^{-1}) r^{-1} (\rho_2^{(2)}\sigma^{(1)}x - \rho_2^{(1)}\sigma^{(2)}x) + \rho_2^{(1)}\rho_2^{(2)} \} \quad (3.5) \quad (b)$$

where the subscripts s and ps are added in order to discriminate two cases for the scalar and the pseudoscalar theory respectively, x stands for $x^{(1)} - x$, σ is the Dirac matrix (vector), and

$$g_s = (\lambda/\mu)g = g\{1 - (\mu/2M)^2\}^{1/2}, \quad g_{ps} = (\mu/2M)f_3 = f_2 + (\mu/2M)f_1. \quad (3.6)$$

The former of these potentials is of the same form as (3.2) whereas the latter is entirely different from (3.2). The latter term in the brace of (3.5) (b) is of the same form as (3.2) but its sign is opposite to that of (3.2) and its

magnitude is smaller by a factor $(\mu/2M)^2$ than (3.2), as is seen from (3.6). The main term of W_{ps} is the former in its brace and the latter is a correction term as will be found in the fifth section. Therefore we see that the recoil term entirely changes the form of the adiabatic potential in case of the pseudoscalar theory. However, we shall find in the fifth section that the potential given by (3.5) (b) is effectively equivalent to the usual one for the large r in the non-relativistic limit.

The potential W_s of the scalar theory includes the term which is similar to the main term of W_{ps} , but the isolation of it from non-adiabatic terms injures the Hermitian property of the potential. Therefore we can not distinctly separate the adiabatic part from the non-adiabatic one. This has no essential effect on (3.5) (a) because the neglected terms are corrections which are smaller by a factor $(\mu/2M)^2$ than (3.5) (a) itself. It should be noted that W_{ps} given by (3.5) (b) has a singularity of r^{-2} in contradiction to the argument of van Hove⁽⁶⁾ while the potential given by (3.2) has a weaker singularity (the present W is equal to van Hove's W_1).

From the result just obtained, one may imagine that the recoil effect gives nuclear forces the qualitative change too in case of the vector and the pseudo-vector theory because the interactions involve ρ_1 and ρ_2 . The final conclusion can not immediately be obtained although matrix elements were calculated by van Hove⁽⁶⁾ and Enatsu⁽⁷⁾ according to these theories, because their results were obtained on the basis of an incorrect assumption that $\epsilon_A^{(1)} - \epsilon_A^{(2)} = \epsilon_B^{(1)} - \epsilon_B^{(2)} = 0$ where their E_{m1} and E_{m2} correspond to our $\epsilon_A^{(1)}$ and $\epsilon_A^{(2)}$ respectively. This assumption can not be justified even in the centre-of-mass system because, for example, $\epsilon_A^{(1)}$ is equal to $-\epsilon_A^{(2)}$ if one of them is negative and the other is positive. The definite conclusion can be obtained after the thorough examination of the matrix elements is carried out.

§ 4. Non-Adiabatic Correction

We shall next consider the non-adiabatic effect of the recoil term on nuclear forces. In this case we have to reserve p_0 in (1.3). Since we want to examine a small deviation from the above mentioned adiabatic limit, we consider the case in which p_0 is small compared with M , and we reserve the terms up to the second degree in p_0/M and μ/M , as is given by (1.5).

When we substitute the right-hand side of (1.5) for the denominator of (1.1), the terms due to the first line in the right-hand side of (1.5) can be written in the form of (3.3). The terms due to the second line can be transformed by making use of

$$\frac{4\pi}{V} \frac{\delta(\mathbf{p}_A^{(1)} + \mathbf{p}_A^{(2)}, \mathbf{p}_B^{(1)} + \mathbf{p}_B^{(2)})}{\{(\mathbf{p}_A^{(1)} - \mathbf{p}_A^{(2)})^2 + \lambda^2\}^2} = \frac{1}{2\lambda} (\varphi_B, e^{-\lambda r} \varphi_A) \quad (4.1)$$

Then we can replace ε and ε_0 by $\rho_1^{(1)}\sigma^{(1)}\mathbf{p} + M\rho_3^{(1)}$ and $\rho_1^{(1)}\sigma^{(1)}\mathbf{p}_0 + M\rho_3^{(1)}$ respectively if we place the former on the left and the latter on the right of $\rho^{(1)}$. If we further displace $\rho^{(1)}$ on the left end we have

$$\begin{aligned} (W_s)_{BA} = & -g_s^2 \{ T\rho_3^{(1)}\rho_3^{(2)}r^{-1}e^{-\lambda r}\}_{BA} \{ 1 - (\mu^2/2M^2\lambda^2)i\sigma_{BA}^{(1)}[\mathbf{k}\mathbf{p}_0] \} \\ & - g_s^2 (\mu^2/2M\lambda^2) \{ T\rho_3^{(2)}\rho_1^{(1)}r^{-1}e^{-\lambda r}\sigma^{(1)}(\mathbf{p} + \mathbf{p}_0) \}_{BA} \\ & + g_s^2 (2M^2\lambda)^{-1} \{ T\rho_3^{(1)}\rho_3^{(2)}e^{-\lambda r}\}_{BA} \{ \mu^2\mathbf{p}\mathbf{p}_0 + \mathbf{k}^2\mathbf{p}_0^2 - (\mathbf{k}\mathbf{p}_0)^2 \} \end{aligned} \quad (4.2) \quad (a)$$

$$\begin{aligned} (W_{ps})_{BA} = & -g_{ps}^2 (2M/\mu^2) \{ T r^{-1}e^{-\lambda r}\rho_3^{(2)}i\sigma^{(1)}\mathbf{k} \}_{BA} \\ & + g_{ps}^2 \{ T\rho_2^{(1)}\rho_2^{(2)}r^{-1}e^{-\lambda r}\}_{BA} \{ 1 + 2\mu^{-2}i\sigma_{BA}^{(1)}[\mathbf{k}\mathbf{p}_0] \} \\ & + g_{ps}^2 (M\lambda\mu^2)^{-1} \{ T e^{-\lambda r}\rho_2^{(2)}i\sigma^{(1)}\mathbf{k} \}_{BA} \{ \mu^2\mathbf{p}\mathbf{p}_0 + \mathbf{k}^2\mathbf{p}_0^2 - (\mathbf{k}\mathbf{p}_0)^2 \} \end{aligned} \quad (4.2) \quad (b)$$

where the thick square bracket denotes a vector product of two vectors embraced in it.

We replace $i\mathbf{p}$ and $i\mathbf{p}_0$ by $\nabla^{(1)}$ and place the former on the left and the latter on the right of $r^{-1}e^{-\lambda r}$ or $e^{-\lambda r}$. We symmetrize the result with respect to nucleons. Carrying out the operation by ∇ on $r^{-1}e^{-\lambda r}$ or $e^{-\lambda r}$ we have

$$\begin{aligned} W_s = & -g_s^2 T\rho_3^{(1)}\rho_3^{(2)} \frac{e^{-\lambda r}}{r} \left\{ 1 + \left(\frac{\mu}{2M} \right)^2 (D^{(1)} + D^{(2)}) + \left(\frac{\mu}{2M} \right)^2 \frac{1}{\lambda r} \left(1 + \frac{1}{\lambda r} \right) (\sigma^{(1)}\mathbf{L}^{(1)} + \sigma^{(2)}\mathbf{L}^{(2)}) \right\} \\ & - g_s^2 T\rho_3^{(2)}\rho_1^{(1)} \frac{e^{-\lambda r}}{r} \left(\frac{\mu}{2M} \right) \left\{ \frac{1}{2} \left(1 + \frac{1}{\lambda r} \right) \frac{i}{r} (\sigma^{(1)}\mathbf{x}) - \frac{i}{\lambda} (\sigma^{(1)}\nabla^{(1)}) \right\} \\ & + g_s^2 T\rho_3^{(1)}\rho_1^{(2)} \frac{e^{-\lambda r}}{r} \left(\frac{\mu}{2M} \right) \left\{ \frac{1}{2} \left(1 + \frac{1}{\lambda r} \right) \frac{i}{r} (\sigma^{(2)}\mathbf{x}) + \frac{i}{\lambda} (\sigma^{(2)}\nabla^{(2)}) \right\} \end{aligned} \quad (4.3) \quad (a)$$

$$\begin{aligned} W_{ps} = & g_{ps}^2 T\rho_2^{(2)} \frac{e^{-\lambda r}}{r} \left[\frac{M\lambda}{\mu^2} \left(1 + \frac{1}{\lambda r} \right) \frac{(\sigma^{(1)}\mathbf{x})}{r} + \left(\frac{\mu}{2M} \right) \frac{(\sigma^{(1)}\mathbf{x})}{r} d^{(1)} \right. \\ & \left. + \frac{\mu}{2M} \left\{ 1 - \frac{2}{\lambda r} \left(1 + \frac{1}{\lambda r} \right) \mathbf{x}\nabla^{(1)} \right\} \frac{(\sigma^{(1)}\nabla^{(1)})}{\lambda} \right] \\ & - g_{ps}^2 T\rho_2^{(1)} \frac{e^{-\lambda r}}{r} \left[\frac{M\lambda}{\mu^2} \left(1 + \frac{1}{\lambda r} \right) \frac{(\sigma^{(2)}\mathbf{x})}{r} + \left(\frac{\mu}{2M} \right) \frac{(\sigma^{(2)}\mathbf{x})}{r} d^{(2)} \right. \\ & \left. - \frac{\mu}{2M} \left\{ 1 + \frac{2}{\lambda r} \left(1 + \frac{1}{\lambda r} \right) \mathbf{x}\nabla^{(2)} \right\} \frac{(\sigma^{(2)}\nabla^{(2)})}{\lambda} \right] \\ & + g_{ps}^2 T\rho_2^{(1)}\rho_2^{(2)} \frac{e^{-\lambda r}}{r} \left\{ 1 - \frac{1}{\lambda r} \left(1 + \frac{1}{\lambda r} \right) (\sigma^{(1)}\mathbf{L}^{(1)} + \sigma^{(2)}\mathbf{L}^{(2)}) \right\} \end{aligned} \quad (4.3) \quad (b)$$

where

$$D^{(1)} = \lambda^{-2} d^{(1)} + (\lambda r)^{-1} \{ 1 + (\lambda r)^{-1} \} \mathbf{x}(\mathbf{x}\nabla^{(1)})\nabla^{(1)} - \mathbf{x}\nabla^{(1)} \quad (4.4) \quad (a)$$

$$d^{(1)} = \{ 1 + (\lambda r)^{-1} \} \{ D^{(1)} + (\lambda r)^{-2} \mathbf{x}(\mathbf{x}\nabla^{(1)})\nabla^{(1)} \} \quad (4.4) \quad (b)$$

$$\mathbf{L}^{(1)} = -i[\mathbf{x}\nabla^{(2)}], \quad \mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \quad (4.4) \quad (c)$$

and $D^{(2)}$, $d^{(2)}$, and $\mathbf{L}^{(2)}$ are obtained from $D^{(1)}$, $d^{(1)}$, and $\mathbf{L}^{(1)}$ by the interchange of the upper suffices (1) and (2).

Since we are interested in the internal energy of the two-nucleon system we write the result in the centre-of-mass system. For the sake of convenience for our later calculation we write the two-nucleon interaction in this system in the following form:

$$W_s = \rho_3^{(1)} \rho_3^{(2)} U_s + \rho_3^{(1)} \rho_1^{(2)} V_s^{(2)} + \rho_3^{(2)} \rho_1^{(1)} V_s^{(1)} \quad (4.5) \quad (a)$$

$$W_{ps} = \rho_2^{(1)} \rho_2^{(2)} U_{ps} + \rho_2^{(1)} V_{ps}^{(2)} + \rho_2^{(2)} V_{ps}^{(1)} \quad (4.5) \quad (b)$$

where

$$U_s = u \{ 1 + 2(\mu/2M)^2 D \} + (\mu/2M)^2 2v \mathbf{S} \mathbf{L} \quad (4.6) \quad (a)$$

$$V_s^{(1)} = -\frac{i}{4M} (\boldsymbol{\sigma}^{(1)} \nabla u + u \boldsymbol{\sigma}^{(1)} \nabla) \quad (4.6) \quad (b)$$

$$V_s^{(2)} = +\frac{i}{4M} (\boldsymbol{\sigma}^{(2)} \nabla u + u \boldsymbol{\sigma}^{(2)} \nabla) \quad (4.6) \quad (c)$$

$$U_{ps} = u - 2v \mathbf{S} \mathbf{L} \quad (4.7) \quad (a)$$

$$V_{ps}^{(1)} = - (M/\mu^2) [\boldsymbol{\sigma}^{(1)} \nabla, u] - (2M)^{-1} [\boldsymbol{\sigma}^{(1)} \nabla, u D] \quad (4.7) \quad (b)$$

$$V_{ps}^{(2)} = + (M/\mu^2) [\boldsymbol{\sigma}^{(2)} \nabla, u] + (2M)^{-1} [\boldsymbol{\sigma}^{(2)} \nabla, u D] \quad (4.7) \quad (c)$$

$$u = \begin{cases} -g_s^2 T r^{-1} e^{-\lambda r} & \text{for the scalar theory} \\ +g_{ps}^2 T r^{-1} e^{-\lambda r} & \text{for the pseudoscalar theory} \end{cases} \quad (4.8) \quad (a)$$

$$v = (\lambda r)^{-1} \{ 1 + (\lambda r)^{-1} \} u \quad (4.8) \quad (b)$$

$$D = \lambda^{-2} \Delta + (\lambda r)^{-1} \{ 1 + (\lambda r)^{-1} \} \boldsymbol{x} (\boldsymbol{x} \nabla) \nabla - \boldsymbol{x} \nabla \quad (4.8) \quad (c)$$

$$\mathbf{S} = (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})/2, \quad \mathbf{L} = -i[\boldsymbol{x} \nabla] \quad (4.8) \quad (d)$$

and the thin square bracket denotes the commutator while the thick bracket means the vector product.

\mathbf{S} represents the total spin angular momentum of the two-nucleon system and \mathbf{L} the orbital angular momentum. The last term of U_s represents the spin-orbit coupling which retains the same form in the Pauli approximation. Dancoff⁽⁵⁾ obtained the term of the same form but its origin was entirely different from the present one. The similar term is contained in U_{ps} but this can not retain its typical form in the Pauli approximation. These circumstances will be accounted for in the next section. If we omit the terms which contain ∇ 's operating on the two-nucleon wave function, W_{ps} given by (4.3) (b) and (4.5) (b) reduce to the adiabatic potential which is given by (3.5) (b), whereas W_s given by (4.3) (a) and (4.5) (a) does not reduce to (3.5) (a). The reason for the latter fact is that $V_s^{(1)}$ and $V_s^{(2)}$ can not be divided into the Hermitian adiabatic and the Hermitian non-adiabatic part as is seen from (4.6) (b) and (c). Thus

we were obliged to adopt (3.5) (a) as the adiabatic part of W_s . For the lower energy case, the main term is the first term, u , of U_s in case of the scalar theory and the first term of V_{ps} in case of the pseudoscalar theory, as will be seen in the next section.

§ 5. Pauli Approximation

We shall discuss in this section the Pauli approximation for the stationary state of the two-nucleon system in order to see the nature of every term included in the interactions which were obtained in the previous section. The method was already explained in the previous paper¹⁾.

The Hamiltonian of the two-nucleon system in the centre-of-mass system is given by (2.3) if $\nabla^{(1)}$ and $-\nabla^{(2)}$ are replaced by ∇ . The energy eigenvalue E is determined by (2.9), where E does not contain the rest energy of two nucleons. The eigenfunction is expanded into the linear combination which is given by (2.5), where the coefficient functions ϕ^{++} etc. depend on the relative position and the spin coordinates of two nucleons. The eigenvalue equation (2.9) then splits into four simultaneous equations for these coefficient functions.

We eliminate ϕ^{+-} , ϕ^{-+} , and ϕ^{--} from these simultaneous equations. We have then an equation for a single function ϕ^{++} in the following form:

$$H'\phi^{++} = E\phi^{++} \quad (5.1)$$

where H' is Hermitian and does not contain ρ_1 , ρ_2 , and ρ_3 . We can consider the operator H' as the effective Hamiltonian because the eigenvalue of H' is the same as that of $H - 2M$ where H includes the rest energy of two nucleons.

If we expand H' into power series in $1/M$ we have

$$H' = -\frac{1}{M}\Delta - \frac{1}{2M^3}\Delta^2 + \dots + W' \quad (5.2)$$

where W' depends on x , r or both. W' can be considered as the effective two-nucleon interaction for the same reason as that for H' . We refer to the limit of W' when $M \rightarrow \infty$ as the non-relativistic approximation of W and the terms of W' up to the second degree in $1/M$ as the Pauli approximation of W . The usual nuclear potential is the non-relativistic approximation. We consider in this section the Pauli approximation and omit higher degree terms in $1/M$ altogether. Further we retain only the terms which is proportional to f^2 because we have considered only such terms from the outset.

We begin with the scalar theory omitting the subscript s from W , U , and V of (4.5) (a) for the sake of simplicity. The elimination of ϕ^{+-} , ϕ^{-+} , and ϕ^{--} from the above mentioned simultaneous equations can easily be carried out. Then H' involves $(2M + E + U)^{-1}$, $(2M + E)^{-1}$, and $(4M + E)^{-1}$. These are expanded as follows:

$$(2M+E+U)^{-1} = (2M)^{-1} \{ 1 - (2M)^{-1}(E+U) + (E/2M)^2 + (EU/2M^2) + \dots \} \quad (5.3) \quad (a)$$

$$(2M+E)^{-1} = (2M)^{-1} \{ 1 - (E/2M) + (E/2M)^2 - \dots \} \quad (5.3) \quad (b)$$

The result involves the term such as OE where O is an operator. We can eliminate E from it by making use of $E\phi^{++} = H'\phi^{++}$. First we replace $OE\phi^{++}$ by $OH'\phi^{++}$ and then expand OH' in power series in $1/M$ retaining only the terms up to the second degree. In this way we have the effective two-nucleon interaction as follows:

$$\begin{aligned} W' = & U + (2M)^{-2} \{ 2Au + (\sigma^{(1)}V)u(\sigma^{(1)}V) + (\sigma^{(2)}V)u(\sigma^{(2)}V) \} \\ & + (2M)^{-1} i \{ (\sigma^{(2)}V)V^{(2)} + V^{(2)}(\sigma^{(2)}V) - (\sigma^{(1)}V)V^{(1)} - V^{(1)}(\sigma^{(1)}V) \} \end{aligned} \quad (5.4)$$

where the terms in the first line come from U of (4.5) (a) and those in the second line from V . Further U and V are given by (4.6).

It can easily be seen that (5.4) includes Dancoff's spin-orbit coupling in the brace of its first line which comes from the adiabatic part of W . The term which is exactly the same with Dancoff's except for its opposite sign is involved in the second line which comes from the non-adiabatic part of W . Therefore Dancoff's spin-orbit coupling is completely cancelled by the latter, and (5.4) reduces to

$$W' = U + (2M)^{-2} [Au] \quad (5.5)$$

There remains only the spin-orbit coupling in U . The origin of this spin-orbit coupling is entirely different from Dancoff's. It comes from the recoil term.

Carrying out the operation of A on u which is given by (4.8) (a) we have the Pauli approximation of the two-nucleon interaction in the scalar theory as follows:

$$\begin{aligned} W'_s = & -g^2 T \frac{e^{-\lambda r}}{r} \left\{ 1 + \left(\frac{\mu}{2M} \right)^2 \frac{1}{\lambda r} \left(1 + \frac{1}{\lambda r} \right) 2SL \right\} \\ & + g^2 T \frac{e^{-\lambda r}}{r} \left(\frac{\mu^2}{2M^2} \right) \left\{ \frac{1}{\lambda} \left(1 + \frac{1}{\lambda r} \right) \frac{\partial}{\partial r} - D \right\} \end{aligned} \quad (5.6)$$

Where g is the constant of scalar coupling, and T , D , S , and L are defined by (1.2) (c), (4.8) (c) and (d) respectively. The factor outside the brace of the first line is essentially the same as the usual potential. The remaining terms are the correction due to the recoil effect. Among them the correction term in the first line is the typical spin-orbit coupling which is just in agreement with Rosenfeld's expectation⁸⁾.

We next derive the Pauli approximation of W for the pseudoscalar theory. Eliminating ϕ^{+-} , ϕ^{-+} , and ϕ^{--} from the simultaneous equations for ϕ^{++} etc. we have

$$\begin{aligned}
 H' = & (2M+E)^{-1} \{ -2A + [(\sigma^{(1)}\nabla), V^{(2)}] - [(\sigma^{(2)}\nabla), V^{(1)}] \} \\
 & + (2M+E)^{-2} \{ (\sigma^{(1)}\nabla)U(\sigma^{(2)}\nabla) + (\sigma^{(2)}\nabla)U(\sigma^{(1)}\nabla) \} \\
 & - 2(2M+E)^{-1}(4M+E)^{-1} \{ (\sigma^{(1)}\nabla)(\sigma^{(2)}\nabla)U + U(\sigma^{(1)}\nabla)(\sigma^{(2)}\nabla) \} \quad (5.7)
 \end{aligned}$$

where the subscript ps is omitted from U and V which are given by (4.7). Further we eliminate E from the right-hand side making use of the expansion (5.3) and $E\phi^{++} = H'\phi^{++}$ as before. The result is given by (see (5.2))

$$W' = [(\sigma^{(1)}\nabla)[(\sigma^{(2)}\nabla)w]]\lambda^{-2} \quad (5.8)$$

where

$$w = u + (\mu^2/2M^2) \{ 3\lambda^{-2}uA + vx(x\nabla)\nabla - (u+2v)x\nabla + vSL \} \quad (5.9)$$

and u and v are given by (4.8) (a) and (b) respectively.

Carrying out the operation of ∇ on w we have the Pauli approximation of the two-nucleon interaction in the pseudoscalar theory as follows:

$$\begin{aligned}
 H'_{ps} = & g_{ps}^2 T \frac{e^{-\lambda r}}{r} \left\{ \frac{1}{3} \sigma^{(1)} \sigma^{(2)} + \phi(r) A \right\} \left\{ 1 + \frac{\mu^2}{2M^2} \left(\frac{4}{\lambda^2} - x\nabla \right) \right\} \\
 & + g_{ps}^2 T \frac{e^{-\lambda r}}{r} \left(\frac{\mu^2}{6M^2} \right) \left\{ \left(1 + \frac{6}{\lambda^2 r^2} \right) \phi(r) - \frac{1}{3} \right\} \mathcal{Q} \\
 & + g_{ps}^2 T \frac{e^{-\lambda r}}{r} \left(\frac{\mu^2}{2M^2} \right) \frac{3}{\lambda^2 r^2} \phi(r) \Gamma \quad (5.10)
 \end{aligned}$$

where T and g_{ps} are given by (1.2) (c) and (3.6) respectively, and $\phi(r)$, A , \mathcal{Q} and Γ are defined by

$$\phi(r) = 3^{-1} + (\lambda r)^{-1} + (\lambda r)^{-2} \quad (5.11)$$

$$A = 3r^{-2} (\sigma^{(1)}x) (\sigma^{(2)}x) - \sigma^{(1)}\sigma^{(2)} \quad (5.12) \text{ (a)}$$

$$\mathcal{Q} = r^{-2} (\sigma^{(1)}x) (\sigma^{(2)}x) \{ x(x\nabla)\nabla - x\nabla \} + 2^{-1} \{ (\sigma^{(1)}x) (\sigma^{(2)}\nabla) + (\sigma^{(2)}x) (\sigma^{(1)}\nabla) \} \quad (5.12) \text{ (b)}$$

$$\Gamma = A \{ x(x\nabla)\nabla - x\nabla \} - (\sigma^{(1)}x) (x\nabla) (\sigma^{(2)}\nabla) - (\sigma^{(2)}x) (x\nabla) (\sigma^{(1)}\nabla) \quad (5.12) \text{ (c)}$$

The first line of (5.10) consists of two parts. One is essentially the same as the usual potential. Another is the term which is proportional to μ^2/M^2 . This is the correction owing to the nucleon recoil. The second and the third line are the recoil corrections too. They are, so to speak, a sort of the spin-orbit coupling as well as the correction in the first line. Thus we see that the typical spin-orbit coupling in (4.3) (b) and (4.5) (b) disappears in (5.10). The spin dependent form of the recoil corrections is similar to that of the non-relativistic approximation. This is a contrast with the case of the scalar theory in which the correction includes the spin-orbit coupling and the non-relativistic approximation is independent of spins.

§ 6 Neglect of Negative Energy States

It was discussed from the general point of view in the second section that the neglect of the negative energy states is not adequate. In this section in order to compare the present result with Toyoda's⁴⁾ we shall examine the conclusion derived on this wrong assumption.

We consider the matrix element in which $(\varepsilon - \varepsilon_0)^2$ is smaller than $\mu^2 + k^2$. Such a case can occur when p and p_0 are smaller than M on account of the present assumption. In this case (1.1) can be expanded as follows:

$$W_{BA} = -f^2 \frac{4\pi}{V} \frac{(T\rho^{(1)}\rho^{(2)})_{BA}\delta}{\mu^2 + k^2} \sum_{n=0}^{\infty} \left\{ \frac{(\varepsilon - \varepsilon_0)^2}{\mu^2 + k^2} \right\}^n \quad (6.1)$$

where the argument of δ is omitted for the sake of brevity. This matrix element can be written in the following form:

$$W_{BA} = -f^2 (\psi_B, \sum_{n=0}^{\infty} \overbrace{[H^{(1)}[H^{(1)}[\dots[H^{(1)}, T\rho^{(1)}\rho^{(2)}U^{(n)}]\dots]]]}^{2n} \psi_A) \quad (6.2)$$

where the square bracket denotes a commutator, $H^{(1)}$ is the Dirac Hamiltonian given by (2.4), and $U^{(n)}$ is defined by

$$U^{(n)} = \frac{4\pi}{V} \sum_k \frac{\exp(i\mathbf{k}\mathbf{x})}{(\mu^2 + k^2)^{n+1}} \quad (6.3)$$

The two-nucleon interaction which is equivalent to (6.1) is thus given by

$$W = \sum_{n=0}^{\infty} W^{(n)} \quad (6.4)$$

where $W^{(0)}$ is the same as (3.2), and $W^{(1)}$ etc. are given by

$$W^{(n)} = -f^2 T \frac{(-1)^{n-1}}{n!} \frac{\partial^{n-1}}{(\partial \mu^2)^{n-1}} \frac{1}{2\mu} \overbrace{[H^{(1)}[H^{(1)}[\dots[H^{(1)}, \rho^{(1)}\rho^{(2)}e^{-\mu r}]\dots]]]}^{2n} \quad (6.5)$$

($n=1, 2, 3, \dots$)

These expressions must be symmetrized with respect to two nucleons.

According to van Hove⁽⁶⁾ any of $H^{(1)}$'s in (6.5) is replaced by $-H^{(2)}$ as follows. The right-hand side of (6.5) has a form $[H^{(1)}G]$ where G is a skew Hermitian operator which is the second degree in the coupling constants. If we denote $H^{(1)} + H^{(2)}$ by H_0 we can write as

$$[H^{(1)}G] = -[H^{(2)}G] + [H_0G] \quad (6.6)$$

The latter term, $[H_0G]$, is eliminated from the terms which are the second degree in the coupling constants by an unitary transformation though this can not be justified. Thus the first $H^{(1)}$ in (6.5) is replaced by $-H^{(2)}$. Since $H^{(1)}$ and $H^{(2)}$ are commutable, we can displace the first $H^{(2)}$ in any position of the commutators

of (6.5). If we want to write W in the symmetrical form with respect to two nucleons we can thus write as follows:

$$W^{(n)} = f^2 \frac{T}{n!} \frac{\partial^{n-1}}{(\partial \mu^2)^{n-1}} \frac{1}{2\mu} \left[\overbrace{H^{(1)}[H^{(1)}[\dots[H^{(1)}]]}^n \overbrace{H^{(2)}[H^{(2)}[\dots[H^{(2)}]]}^n, \rho^{(1)} \rho^{(2)} e^{-\mu r} \right] \dots \quad (6.7)$$

The difference between (6.5) and (6.7) is the ambiguity of this method.

Among these correction terms $W^{(1)}$ is the same as Toyoda's correction except for the sign⁴⁾. (Toyoda's sign is incorrect). If the influence of these correction terms, $W^{(n)}$ ($n=1, 2, 3, \dots$), is larger than that of $W^{(0)}$ we can not consider (6.4) to represent the two-nucleon interaction because the expansion given by (6.1) converges only provided that $(\epsilon - \epsilon_0)^2$ is smaller than $\mu^2 + k^2$. The main term is therefore $W^{(0)}$ in contrast with Toyoda's argument.

In order to see more concretely the inadequency of the present assumption we carry out the calculation of $W^{(1)}$ according to (6.5). The symmetrized result is given in the centre-of-mass system as follows:

$$W_s^{(1)} = -g^2 T \rho_3^{(1)} \rho_3^{(2)} r^{-1} e^{-\mu r} (1 - SL) + g^2 T \rho_3^{(1)} \rho_3^{(2)} e^{-\mu r} (2\mu^{-1} \Delta + 2^{-1} \mu - 2r^{-1} x \nabla) \\ + g^2 T e^{-\mu r} M i (\rho_3^{(1)} \rho_1^{(2)} \sigma^{(2)} - \rho_3^{(2)} \rho_1^{(1)} \sigma^{(1)}) \{ \mu^{-1} \nabla - (2r)^{-1} x \} \quad (6.8) \quad (a)$$

$$W_{ps}^{(1)} = f_3^2 T e^{-\mu r} \{ M (2r)^{-1} (\rho_2^{(1)} \sigma^{(2)} x - \rho_2^{(2)} \sigma^{(1)} x) - 2M^2 \mu^{-1} \rho_2^{(1)} \rho_2^{(2)} \} \\ - f_3^2 T \rho_2^{(1)} \rho_2^{(2)} r^{-1} e^{-\mu r} (1 - SL) + f_3^2 T \rho_2^{(1)} \rho_2^{(2)} e^{-\mu r} (2\mu^{-1} \Delta + 2^{-1} \mu - 2r^{-1} x \nabla) \quad (6.8) \quad (b)$$

These results are not in agreement with (4.3). For example, the correction terms in the (4.3) (a) are smaller by a factor $(\mu/2M)^2$ than (6.8) (a). That the correction terms obtained in this section are too large must be due to the inadequate assumption.

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Upper and Lower Bounds of Scattering Phases

Tosio KATO

Department of Physics, University of Tokyo

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A general method is developed for a rigorous estimation to upper and lower bounds of scattering phases in the variational methods applied to one-dimensional problems. It is also possible to estimate the mean error of the approximate wave function itself. An essential point in the method is the evaluation of mean square ϵ^2 of the left-hand side of the wave equation with respect to a weight function chosen appropriately. Also we have need of certain auxiliary constants α , β defined in connection with an eigenvalue problem associated with the wave equation. However, only a rough estimate is required of these quantities, and some general methods are given for their estimation. As an example, the scattering of slow electrons by hydrogen atoms is treated in the one-body approximation. It turns out that phases are determined rigorously with the possible errors of 10^{-3} or 10^{-4} by assuming a very simple trial function containing only two parameters, and that the approximate wave function is also exact within about 10^{-2} .

Introduction

In these years variational methods have come to be applied to scattering problems by many authors,¹⁾ and it seems that the methods have reached a stage of importance comparable to the famous Rayleigh-Ritz method in eigenvalue problems.

As an essential difference from the latter, it was pointed out²⁾ that the scattering phase as given by these variational methods are in general neither an upper nor a lower bound to the correct value, whereas in the case of eigenvalue problems the Rayleigh-Ritz method leads always to an upper bound of the correct value at least in the case of the lowest eigenvalue. This fact causes some inconvenience in the application, for one is not certain whether the approximation improves steadily when the number of the variation parameters is increased.

In a previous note of the writer,³⁾ however, it was shown that there is an exception to this general rule; one of the variational methods introduced by Schwinger⁴⁾ proved to be capable of giving either an upper or a lower bound of the scattering phase under very general conditions. Furthermore, it was shown that the same method, when pushed on to the second approximation, can give the *opposite* bound of the phase, thus enabling us to obtain *both* upper and lower bounds of the phase.

In the present paper, we shall consider the problem from a wider point of view, and derive formulae giving both upper and lower bounds of the scattering

phases in a symmetric form. These formulae contain as a special case the Schwinger formula just mentioned, but in general they give more accurate upper and lower bounds than the latter. It will be remarked that the formulae bear considerable resemblance with the corresponding formulae in eigenvalue problems derived by Temple, Weinstein and others.⁵⁾

As an application of these formulae, we shall consider the scattering of slow electrons by hydrogen atoms in the one-body approximation. It turns out that a very simple trial function containing only two parameters leads to almost exact solutions of the wave equation, and the numerical results are in good agreement with those of McDougall⁶⁾ obtained by numerical integration. Since our solutions are very accurate and nevertheless have simple analytic expression, they may be useful as the basis for further approximations where polarization and exchange effects are taken into account.

§ 1. General formulae

For the moment we consider the radial wave equation for S -scattering by a center of force:

$$L[u] \equiv d^2u/dr^2 + [k^2 + W(r)]u = 0, \quad (1)$$

where k is the wave number and $W(r)$, simply referred to as the "potential" in the following, is actually the potential energy multiplied by $-2m\hbar^{-2}$ (m = reduced mass). We assume that the integral $\int |W(r)| dr$ is convergent at $r \rightarrow \infty$, while W may be $O(r^{-1})$ at $r \rightarrow 0$.

We denote by $\bar{u}(r)$ the correct solution of (1) with the boundary condition $\bar{u}(0) = 0$ and normalized to unit amplitude at $r \rightarrow \infty$. The scattering phase $\bar{\eta}$ is defined from the asymptotic form of $\bar{u}(r)$ by the relation $\bar{u}(r) \rightarrow \sin(kr + \bar{\eta})$. It will be noted that $\bar{\eta}$ is thereby determined only up to an integral multiple of π , but the latter is completely fixed if we pursue the behavior of $\bar{u}(r)$ from $r=0$ to $r \rightarrow \infty$ continuously. It is well known⁷⁾ that the phase $\bar{\eta}$ thus defined is positive (negative) if $W(r)$ is everywhere ≥ 0 (≤ 0). More generally, it is easily shown that $\bar{\eta}$ is a monotonically increasing function of the "potential" $W(r)$.

In order to apply the variational method to our problem, we introduce a trial function $u(r)$ such that

$$u(0) = 0; \quad u(r) \rightarrow \sin(kr + \eta), \quad r \rightarrow \infty, \quad (2)$$

i. e., $u(r)$ is assumed to be normalized to unit amplitude at infinity. In what follows we have also to consider different normalizations on the same trial function, so that it is convenient to introduce the following notations: we denote by $u_\theta(r)$ the same function $u(r)$ normalized according to

$$u_\theta(r) \rightarrow \cos(kr + \theta) + \lambda_\theta \sin(kr + \theta), \quad r \rightarrow \infty, \quad (3)$$

where θ is a fixed constant and λ_θ is one of the variation parameters. The following relations are obvious:

$$\lambda_\theta = \cot(\eta - \theta), \quad (4)$$

$$u_\theta(r) = u(r) / \sin(\eta - \theta) = \pm (1 + \lambda_\theta^2)^{1/2} u(r). \quad (5)$$

We use the corresponding notations $\bar{u}(r)$, $\bar{u}_\theta(r)$, $\bar{\lambda}_\theta$ etc. for the correct wave function. We shall refer to (2) and (3) as the *absolute* and θ -*normalizations* respectively.

Applying Green's formula

$$\int_0^\infty (fL[g] - L[f]g) dr = [f dg/dr - (df/dr)g]_0^\infty \quad (6)$$

to $f = \bar{u}_\theta$ and $g = u_\theta$, it follows easily that

$$\int_0^\infty \bar{u}_\theta L[u_\theta] dr = k(\lambda_\theta - \bar{\lambda}_\theta). \quad (7)$$

Setting

$$w_\theta = u_\theta - \bar{u}_\theta \quad (8)$$

and noting

$$L[w_\theta] = L[u_\theta], \quad (9)$$

we obtain from (7)

$$k\bar{\lambda}_\theta = k\lambda_\theta - \int_0^\infty u_\theta L[u_\theta] dr + \int_0^\infty w_\theta L[w_\theta] dr. \quad (10)$$

The first two terms on the right of (10) can be calculated by specifying the trial function u_θ and give an approximate value of the quantity $k\bar{\lambda}_\theta = k \cot(\bar{\eta} - \theta)$, while the last term represents the error involved. This error is of the order w_θ^2 and would be small if u_θ is a good approximation. In fact, (10) is nothing but the formula⁸⁾ of the usual variational method expressed in finite (non-infinitesimal) terms, and coincides essentially with the expressions of Kohn, Huang, Hulthén and Schwinger according to different choices of θ .⁸⁾

Since (10) is an exact formula, we shall be able to obtain rigorous upper and lower bounds of λ_θ , and hence of $\bar{\eta}$, if we can estimate the residual term $\int w_\theta L[w_\theta] dr$. For this purpose it will be noted that, although w_θ itself is an unknown function, we have at least the following informations about w_θ : 1) $L[w_\theta]$ is known by (9); 2) by virtue of (8) and (3) w_θ satisfies the boundary conditions

$$w_\theta(0) = 0; \quad w_\theta(r) \rightarrow \text{const.} \sin(kr + \theta), \quad r \rightarrow \infty.$$

We shall now show that the desired estimation of the error can be obtained by

considering a certain eigenvalue problem associated with the wave equation (1).

§ 2. The associated eigenvalue problem

Let us consider the following eigenvalue problem with eigenvalues μ and eigenfunctions $\varphi(r)$:

$$L[\varphi] + \mu \rho \varphi = 0, \quad 0 \leq r < \infty, \quad (11)$$

where $\rho = \rho(r) \geq 0$ is some fixed weight function such that the integral $\int \rho dr$ is convergent at $r \rightarrow \infty$, and is to be chosen appropriately in each case. As will be shown below, $\rho = |W|$ is a convenient choice at least if $W(r)$ is of constant sign. The eigenfunctions are required to satisfy the homogeneous boundary conditions:

$$\varphi(0) = 0; \quad \varphi(r) \rightarrow \text{const.} \sin(kr + \theta), \quad r \rightarrow \infty, \quad (12)$$

with a fixed θ , i. e., $\varphi(r)$ should have the asymptotic phase $\theta + n\pi$, $n = 0, \pm 1, \pm 2, \dots$. If f, g are two functions satisfying the boundary conditions (12), it follows from Green's formula (6) that

$$\int_0^\infty f L[g] dr = \int_0^\infty L[f] g dr. \quad (13)$$

Let us denote by H_0 the operator $-L$ restricted by the boundary conditions (12). Then (13) implies that H_0 is Hermitian, and hence all its eigenvalues are real. Indeed, owing to the assumed integrability of the weight function ρ , H_0 has only discrete, non-degenerate eigenvalues

$$\dots < \mu_{-1} < \mu_0 < \mu_1 < \mu_2 < \dots,$$

in general ranging from $-\infty$ to $+\infty$. This will easily be seen if one notes that μ_n is the value of μ such that the asymptotic phase $\delta(\mu)$ associated with the wave equation

$$d^2u/dr^2 + [k^2 + W(r) + \mu \rho(r)]u = 0 \quad (14)$$

is just equal to $\theta + n\pi$ ($\delta(\mu)$ is a monotonically increasing function of μ). The associated eigenfunctions φ_n satisfy the orthogonality relations with the weight function ρ :

$$\int_0^\infty \varphi_m \varphi_n \rho dr = \delta_{mn}, \quad m, n = 0, \pm 1, \pm 2, \dots \quad (15)$$

It can also be shown that these eigenfunctions form a complete system, and the Parseval identities hold for any functions f, g :

$$\int f^2 \rho dr = \sum a_n^2, \quad \int f g \rho dr = \sum a_n b_n, \quad \int g^2 \rho dr = \sum b_n^2, \quad (16)$$

where a_n, b_n are Fourier coefficients of f, g :

$$a_n = \int \varphi_n f \rho dr, \quad b_n = \int \varphi_n g \rho dr. \quad (17)$$

Now let f satisfy the boundary conditions (12) and set $g = \rho^{-1} L[f]$. Then

$$b_n = \int \varphi_n L[f] dr = \int L[\varphi_n] f dr = -\mu_n \int \varphi_n f \rho dr = -\mu_n a_n,$$

and (16) becomes

$$\int f^2 \rho dr = \sum \mu_n^{-2} b_n^2, \quad (18a)$$

$$\int f L[f] dr = -\sum \mu_n^{-1} b_n^2, \quad (18b)$$

$$\int (L[f])^2 \rho^{-1} dr = \sum b_n^2. \quad (18c)$$

Let α_0 be the smallest *positive* eigenvalue, and let $-\beta_0$ be the smallest (in absolute value) *negative* eigenvalue*, $\alpha_0 > 0, \beta_0 > 0$. Then we have $-\mu_0^{-1} \leq \mu_n^{-1} \leq \alpha_0^{-1}$ for all n , and it follows from (18b, c) that

$$-\alpha_0^{-1} \int (L[f])^2 \rho^{-1} dr \leq \int f L[f] dr \leq \beta_0^{-1} \int (L[f])^2 \rho^{-1} dr. \quad (19)$$

Similarly, we have from (18a, c)

$$\int f^2 \rho dr \leq \text{Max}(\alpha_0^{-2}, \beta_0^{-2}) \int (L[f])^2 \rho^{-1} dr. \quad (20)$$

Of course these inequalities are false unless f satisfies the boundary conditions (12).

§ 3. Estimation of the phase and the wave function

After these preparations we can proceed to the estimation of the error term in (10). Since the function w_0 satisfies the boundary conditions (12) as we noted at the end of § 1, we can use the inequalities (19) and (20) for $f = w_0$. Thus (19) gives

$$-\alpha_0^{-1} \epsilon_0^2 \leq \int w_0 L[w_0] dr \leq \beta_0^{-1} \epsilon_0^2, \quad (21)$$

where

$$\epsilon_0^2 = \int_0^\infty (L[u_0])^2 \rho^{-1} dr \quad (22)$$

* We assume that all $\mu_n \neq 0$.

(note (9)). From (10) and (21) we obtain the rigorous estimation

$$-a_0^{-1} \epsilon_0^2 \leq k\bar{\lambda}_0 - k\lambda_0 + \int_0^\infty u_0 L[u_0] dr \leq \beta_0^{-1} \epsilon_0^2. \quad (23)$$

This is the desired formula giving upper and lower bounds of $k\bar{\lambda}_0$, and hence of $\bar{\eta}$. Also the mean error of u_0 itself can be estimated with respect to the weight function ρ from (20), (22) and (8):

$$\int_0^\infty (u_0 - \bar{u}_0)^2 \rho dr \leq \text{Max}(a_0^{-2}, \beta_0^{-2}) \cdot \epsilon_0^2. \quad (24)$$

In these formulae the quantity ϵ_0^2 can be calculated exactly by specifying the trial function, though it might be sometimes laborious. Also they contain the auxiliary quantities a_0^{-1}, β_0^{-1} , which are in general difficult to evaluate exactly. But it is important to note that we have only to know their rough upper bounds since they appear only multiplied by the small quantity ϵ_0^2 , and there is no real difficulty (see § 4).

Hitherto we have left the constant θ unspecified. Indeed, even if we use the same trial function u , (23) or (24) represents essentially different formulae according to different choices of θ , and there is no a priori reason to prefer one to other values of θ .

Therefore it is sometimes convenient to rewrite these formulae in terms of u (absolute normalization) instead of u_0 . Substitution of (4) and (5) into (23) and (24) yields

$$-a_0^{-1} \frac{\epsilon^2}{\sin^2(\eta - \theta)} \leq k \cot(\bar{\eta} - \theta) - k \cot(\eta - \theta) + \frac{\int u L[u] dr}{\sin^2(\eta - \theta)} \leq \beta_0^{-1} \frac{\epsilon^2}{\sin^2(\eta - \theta)}, \quad (25)$$

$$\int \left\{ u - \frac{\sin(\eta - \theta)}{\sin(\bar{\eta} - \theta)} \bar{u} \right\}^2 \rho dr \leq \text{Max}(a_0^{-2}, \beta_0^{-2}) \cdot \epsilon^2, \quad (26)$$

with

$$\epsilon^2 = \int_0^\infty (L[u])^2 \rho^{-1} dr. \quad (27)$$

Thus the most logical procedure would be to calculate $\int u L[u] dr$ and ϵ^2 by assuming the trial function u with some undetermined parameters c_1, c_2, \dots (η being one of them) in conformity with the absolute normalization (2), and then to look for the values of c_j together with θ for which (25) provides the most favorable estimate of $\bar{\eta}$, independently for the upper and lower bounds. But this procedure is not practical, for on the one hand the absolute normalization (2) is not a convenient one for performing the variation because it contains the non-linear parameter η , while on the other hand the expression (25) is a rather complicated function of θ .

Thus it seems that the variation of θ is practicable only in exceptional cases

and we are obliged to assume some special value (zero or $\pi/2$, for instance) of θ from the beginning. Then the formula (23) with θ -normalization is more convenient than (25) with the absolute normalization. However, it will be explicitly repeated that (23) represents essentially independent formulae for different values of θ , and that in consequence we can often obtain much more accurate estimation of $\bar{\gamma}$ by considering two or more different values of θ than by taking only one value into account.

For these reasons we shall adopt two special values $\theta=0$ and $\theta=\pi/2$ in the examples treated below, and take as the standard form the 0-normalization u_0 which corresponds to the method of Hulthén.⁹⁾ Setting $\theta=0$ in (3) and (23) and introducing the parameter $\gamma=k\lambda_0$ in place of λ_0 , we obtain

$$u_0(r) \rightarrow \cos kr + \gamma k^{-1} \sin kr, \quad r \rightarrow \infty, \quad (28)$$

$$-a_0^{-1}\epsilon_0^2 \leq k \cot \bar{\gamma} - \gamma + \int_0^\infty u_0 L[u_0] dr \leq \beta_0^{-1}\epsilon_0^2. \quad (29)$$

For $\theta=\pi/2$ we note that $\lambda_{\pi/2}=-\lambda_0^{-1}$, $u_{\pi/2}=-\lambda_0^{-1}u_0$ and $\epsilon_{\pi/2}^2=\lambda_0^{-2}\epsilon_0^2$ by (3), (4) and (22). Hence (23) becomes for $\theta=\pi/2$, after multiplication by k^{-2} ,

$$-a_{\pi/2}^{-1} \frac{\epsilon_0^2}{\gamma^2} \leq -\frac{1}{k \cot \bar{\gamma}} + \frac{1}{\gamma} + \frac{1}{\gamma^2} \int_0^\infty u_0 L[u_0] dr \leq \beta_{\pi/2}^{-1} \frac{\epsilon_0^2}{\gamma^2}. \quad (30)$$

In the same way we obtain from (24) specialized to $\theta=0$ and $\theta=\pi/2$

$$\int_0^\infty (u_0 - \bar{u}_0)^2 \rho dr \leq \text{Max}(a_0^{-2}, \beta_0^{-2}) \cdot \epsilon_0^2 \quad (31)$$

and

$$\int_0^\infty \left(u_0 - \frac{\gamma}{k \cot \bar{\gamma}} \bar{u}_0 \right)^2 \rho dr \leq \text{Max}(a_{\pi/2}^{-2}, \beta_{\pi/2}^{-2}) \cdot \epsilon_0^2. \quad (32)$$

After thus having restricted the value of θ , the best estimate of $\bar{\gamma}$ is obtained by adjusting the parameters c_j (γ being one of them) contained in u_0 in such a way that the upper (lower) bound of $\bar{\gamma}$ as given by (29) or (30) is minimum (maximum). More convenient though somewhat less accurate procedure is to determine the parameters by requiring ϵ_0^2 to be minimum, and then to calculate $\int u_0 L[u_0] dr$ with u_0 thus determined. Of course one can devise other convenient methods according to necessity.

Remark 1. Since our formulae are rigorous, they have both their merits and defects inherent in rigorous formulae. In general they give too conservative estimates, and sometimes it occurs that they are too crude to be useful. Usually it is plausible that the true errors are much smaller than their limits given by (29) and (30). On the other hand, if a small limit of error is attained by a simple trial function, we can safely stop there and need not go on with further approximations to ensure the convergence, which is necessary in the conventional procedures where no exact estimation is available.

2. Obviously the accuracy of the formulae depends on the smallness of ϵ_0^2 or ϵ^2 . However, even if ϵ_0^2 is not very small, fairly good approximation is often obtained by performing independent variations for the upper and lower bounds independently and with two or more values of θ , as stated above.

3. In general, the calculation of ϵ_0^2 is rather tedious since the integrand contains the expression $(L[u_0])^2$. But it should be noted that we need only a rough value of ϵ_0^2 . Thus, if the order of magnitude of $L[u_0]$ is known for all r , ϵ_0^2 can be estimated without difficulty.

4. The parameter γ in (28) is equal to $k \cot \bar{\eta}$ for the *correct* wave function. For the trial function, however, γ differs from $k \cot \bar{\eta}$ by a first-order quantity; it should be regarded as a mere parameter and must be strictly distinguished from the accurate value of $k \cot \bar{\eta}$ as given by (29) or (30).¹⁰⁾

§ 4. Auxiliary constants α_0 and β_0

According to the definition (see the end of § 2), α_0 is the smallest positive value of μ such that the asymptotic phase $\delta(\mu)$ associated with the wave equation (14) is equal to $\theta + n\pi$ with *some* integer n , while $-\beta_0$ is the smallest (in absolute value) negative value of μ with the same property. Combining this definition with the principle that the phase associated with a wave equation is a monotonically increasing function of the "potential", it is not difficult to obtain rough upper bounds of α_0^{-1} and β_0^{-1} in each case, as will be illustrated by the example given below.

However, there are some important special cases where at least one of α_0 and β_0 can easily be estimated exactly or at least with considerable precision.

i) The case $W(r) \geq 0$. Then $\bar{\eta} > 0^*$. It is convenient to take $\rho = W$. Then the phase $\delta(\mu)$ associated with (14) is positive as long as $\mu > -1$ and equal to zero for $\mu = -1$. Hence $\mu = -1$ is the smallest negative value of μ for which $\delta(\mu) \equiv 0 \pmod{\pi}$, provided that the correct phase $\bar{\eta} = \delta(0)$ of (1) is less than π . Thus $\beta_0 = 1$ in this case, and it follows from (29) that

$$\gamma - \int_0^\infty u_0 L[u_0] dr - \alpha_0^{-1} \epsilon_0^2 \leq k \cot \bar{\eta} \leq \gamma - \int_0^\infty u_0 L[u_0] dr + \epsilon_0^2. \quad (33)$$

As we have shown in a previous note,⁸⁾ the right side is actually independent of the parameter γ and equivalent to the formula of $k \cot \bar{\eta}$ in Schwinger's variational method.⁴⁾ Thus we have shown again¹¹⁾ that his formula gives an upper bound of $k \cot \bar{\eta}$ provided that $W \geq 0$ and $\bar{\eta} < \pi$. Also it is easily seen that the left side of (33) coincides with the lower bound derived by a different method in the previous note.¹¹⁾

It should be remarked, however, that $\theta = 0$ is not necessarily the most favora-

* Except in the trivial case $W(r) \equiv 0$.

ble choice under the present assumptions. If it is certain that $\bar{\eta} < \theta$ for some $\theta < \pi$, it is usually better to adopt the θ -normalization with this θ . This is due to the fact that $\beta_0 > 1$ because $\delta(\mu) \equiv \theta \pmod{\pi}$ for $-1 \leq \mu \leq 0$. To obtain more accurate lower bound of β_0 , it is often convenient to compare $W(r)$ with r^{-2} . Let there be a constant b such that

$$W(r) \leq br^{-2}, \quad 0 \leq r < \infty. \quad (34)$$

Then

$$0 \geq (1+\mu)W \geq \frac{-(1+\mu)b}{\nu(\nu+1)} \cdot \frac{-\nu(\nu+1)}{r^2}, \quad \nu = 2 - \frac{2\theta}{\pi} > 0,$$

for $1+\mu < 0$. But as the phase associated with the "potential" $-\nu(\nu+1)r^{-2}$ is well known to be $-\nu\pi/2 = -\pi + \theta$ for all k , the phase associated with the "potential" $(1+\mu)W$ is larger than $-\pi + \theta$, provided that $0 \leq -(1+\mu)b/\nu(\nu+1) < 1$ or $-1 \geq \mu > -1 - b^{-1}\nu(\nu+1)$. Combined with the result stated above, it follows that $\delta(\mu) \equiv \theta \pmod{\pi}$ for $0 \geq \mu > -1 - b^{-1}\nu(\nu+1)$. In other words

$$\beta_0 \geq 1 + b^{-1}\nu(\nu+1) = 1 + b^{-1}(2 - 2\pi^{-1}\theta)(3 - 2\pi^{-1}\theta), \quad (35)$$

provided that $W \geq 0$ and $\eta < \theta < \pi$. In particular if it is certain that $\bar{\eta} < \pi/2$, we have ($\theta = \pi/2$)

$$\beta_{\pi/2} \geq 1 + 2b^{-1}. \quad (36)$$

Another remark applies also to the estimation of β_0 . Suppose that $W(r) = 0$ for $r \geq a$, say. Then it is easily seen that $\delta(\mu) > -ka$ for all μ . Therefore, if $\bar{\eta} < \theta < \pi$ and $ka < \pi - \theta$, there is no $\mu < 0$ for which $\delta(\mu) \equiv \theta \pmod{\pi}$. This means that we may take $\beta_0 = \infty$, and it follows from (23) that

$$k \cot(\bar{\eta} - \theta) \leq k\lambda_0 - \int_0^\infty u_0 L[u_0] dr. \quad (36a)$$

Thus we have an upper bound of $k \cot(\bar{\eta} - \theta)$ without calculating ϵ_0^2 . Of course the trial function u_0 must satisfy $L[u_0] = 0$ exactly for $r \geq a$ in this case. Even if $W(r)$ does not vanish exactly outside the range of force, it is natural to expect that β_0 would be very large under similar conditions (weak potential and small k), and it is very plausible that (36a) is still valid.

As regards the estimation of u_0 , there seems to be no such simple formula and particular method must be devised for each case.

ii) The case $W(r) \leq 0$. Then $\bar{\eta} < 0$. It is convenient to take $\rho = -W$, and the estimation of u_0 and β_0 goes parallel to the case $W \geq 0$, except that the rôles of u_0 and β_0 are interchanged. For instance, it follows that the Schwinger method¹⁴⁾ gives a lower bound of $k \cot \bar{\eta}$ provided that $\bar{\eta} > -\pi$.

§ 5. Generalizations

Our method is applicable to more general wave equation .

$$L[u] \equiv d^2u/dr^2 + [k^2 - l(l+1)r^{-2} + U(r) + W(r)]u = 0, \quad (1')$$

where $-l(l+1)r^{-2}$ is the centrifugal force and $U(r)$ represents some long-range potential which is conveniently distinguished from the short-range one $W(r)$. In this case we denote by $\bar{\eta}$ the *relative* phase shift. In other words, the correct solution of (1') is assumed to have the asymptotic form $\sin(kr - l\pi/2 + \bar{\delta} + \bar{\eta})$, where $\bar{\delta}$ is the phase shift due to $U(r)$ alone. Then, except for the obvious substitution of kr by $kr - l\pi/2 + \bar{\delta}$ in (2), (3), (12) and of $W(r)$ by $-l(l+1)r^{-2} + U(r) + W(r)$ in (14), no formal change is necessary in all our arguments and formulae; In particular, it follows that the Schwinger method mentioned in § 4 also gives an upper (lower) bound of $k \cot \bar{\eta}$ provided that 1) *only the short range potential* $W(r)$ is everywhere ≥ 0 (≤ 0) and 2) the correct relative phase $\bar{\eta}$ is less than π in absolute value. Of course the calculation of such quantities as $\int u_0 L[u_0] dr$ and ϵ_0^2 as well as the estimation of u_0 , β_0 would be in general more complicated than before.

Our method may also be generalized to more complicated problems containing compound systems. At present, however, it seems to be of little practical use in view of the tedious calculation necessary for evaluating ϵ_0^2 and the difficulty in estimating such quantities as u_0 , β_0 , and here we will not attempt to derive explicit formulae.

§ 6. Application to the scattering of slow electrons by hydrogen atoms

As an application of our formulae, let us consider the elastic scattering of slow electrons by hydrogen atoms in the one-body approximation. The atom is represented by a center of force and, if we take the radius a_0 of the first Bohr orbit as the unit of length, the radial wave equation for S -state takes on the form (1) with¹²⁾

$$W(r) = 2(r^{-1} + 1)e^{-2r}. \quad (37)$$

We note the inequality

$$W(r) < W_H(r) \equiv 4e^{-2r}(1 - e^{-2r})^{-1}, \quad 0 < r < \infty. \quad (38)$$

The right side is the potential introduced by Hulthén.¹³⁾ It is known that the wave equation

$$d^2u/dr^2 + [k^2 + aW_H(r)]u = 0 \quad (39)$$

can be solved explicitly by making use of hypergeometric functions. The corresponding phase shift is shown to be

$$\begin{aligned} \zeta(a) = \frac{\pi}{2} + \arg \Gamma(ik) - \arg \Gamma\left(1 + \left(a - \frac{k^2}{4}\right)^{\frac{1}{2}} + \frac{ik}{2}\right) \\ - \arg \Gamma\left(1 - \left(a - \frac{k^2}{4}\right)^{\frac{1}{2}} + \frac{ik}{2}\right), \end{aligned} \quad (40)$$

where Γ is the gamma function.

It follows from (38) that the correct phase $\bar{\eta}$ of (1) satisfies the inequality

$$\bar{\eta} < \zeta(1). \quad (41)$$

This gives a rough upper bound of $\bar{\eta}$. It is seen that $\zeta(1) \leq \pi/2$ for every k so that we have

$$\bar{\eta} < \pi/2. \quad (42)$$

Since $W(r) > 0$, we choose $\rho = W$ according to § 4, i). We shall now estimate the auxiliary constants α_0 and β_0 . α_0 is the smallest positive value of μ such that $\delta(\mu) \equiv \theta \pmod{\pi}$. Since $\delta(0) = \bar{\eta} < \pi/2$, we have $\delta(\alpha_0) = \pi$. But $W < W_H$ shows that $\zeta(1 + \mu) > \delta(\mu)$ and hence $\zeta(1 + \alpha_0) > \pi$. If we define α'_0 by $\zeta(1 + \alpha'_0) = \pi$, it follows that $\alpha_0 > \alpha'_0$, for $\zeta(a)$ is a monotonically increasing function of a . Thus we have obtained a lower bound of α_0 . α'_0 is easily estimated from the table of the Γ -function of complex arguments.¹⁴⁾ In this way we have the result

$$\begin{array}{cccccc} k = & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\ \alpha_0 > & 1.25 & 1.45 & 1.7 & 1.9 & 2.1 & 2.3 \end{array} \quad (43)$$

On the other hand, this method gives lower bounds of $\alpha_{\pi/2}$ which are very small and useless.

The estimation of β_0 is much simpler. It is easily shown that

$$W(r) \leq br^{-2} \quad \text{with } b = (1 + \sqrt{2}) \exp(-\sqrt{2}) = 0.5871.$$

Since we know that $\bar{\eta} < \pi/2$, we can use (36) and obtain

$$\beta_{\pi/2} \geq 1 + 2/0.5871 = 4.407 \quad (44)$$

for every k . Thus $\beta_{\pi/2}$ is larger than β_0 which is just equal to unity as was shown in § 4, i).

In this way we see that the left inequality of (29) is adequate to give a lower bound of $k \cot \bar{\eta}$, while the right inequality of (30) is more convenient to obtain an upper bound of $k \cot \bar{\eta}$.

We next introduce the trial function in the 0-normalization (28):

$$\begin{aligned} u_0(r) &= \cos kr - y(r) + \gamma k^{-1} \sin kr, \\ y(0) &= 1, \quad y(r) \rightarrow 0, \quad r \rightarrow \infty. \end{aligned} \quad (45)$$

Then we have

$$\int u_0 L[u_0] dr = P - Q + (2S - 1)\gamma + T\gamma^2, \quad (46)$$

$$\epsilon_0^2 = P - 2Q + R + 2(S - 1)\gamma + T\gamma^2, \quad (47)$$

with

$$\begin{aligned} P &= \int (\cos kr - y)^2 W dr, & Q &= \int [(dy/dr)^2 - k^2 y^2] dr, \\ R &= \int (d^2 y/dr^2 + k^2 y)^2 W^{-1} dr, \end{aligned} \quad (48)$$

$$S = k^{-1} \int \sin kr (\cos kr - y) W dr, \quad T = k^{-2} \int \sin^2 kr W dr.$$

It will be noted that

$$\epsilon_0^2 - \int u_0 L[u_0] dr = -Q + R - \gamma. \quad (49)$$

We assume the following simple form for y :

$$y(r) = e^{-2r}(1 + c_1 r). \quad (50)$$

Direct calculation yields

$$\begin{aligned} P &= P_{00} + 2P_{01}c_1 + P_{11}c_1^2, \\ Q &= Q_{00} + 2Q_{01}c_1 + Q_{11}c_1^2, \\ R &= R_{00} + 2R_{01}c_1 + R_{11}c_1^2, \\ S &= S_0 + S_1c_1, \end{aligned} \quad (51)$$

with¹⁵⁾

$$P_{00} = -\frac{5}{6} + 2 \log \frac{4}{3} + \frac{1}{2(1+k^2)} - \frac{16}{16+k^2} + 2 \log \left(1 + \frac{k^2}{16}\right) - \frac{1}{2} \log(1+k^2),$$

$$P_{01} = \frac{7}{18} - \frac{6}{16+k^2} - \frac{64}{(16+k^2)^2}, \quad P_{11} = \frac{2}{27},$$

$$Q_{00} = \frac{1}{4}(4-k^2), \quad Q_{01} = -\frac{1}{16}(4+k^2), \quad Q_{11} = \frac{1}{32}(4-k^2),$$

$$R_{00} = \xi(4+k^2)^2, \quad R_{01} = \frac{1}{8}(4+k^2)^2 - \xi(4+k^2)(8+k^2),$$

$$R_{11} = -(4+k^2) + \xi(8+k^2)^2,$$

$$\left(\xi = \frac{1}{4} [1 + 2e^2 \operatorname{Ei}(-2)] = 0.069335691 \right) \quad (52)$$

$$S_0 = \frac{1}{2(1+k^2)} - \frac{2}{16+k^2} - \frac{1}{k} \operatorname{arctan} k - \frac{2}{k} \operatorname{arctan} \frac{k}{4},$$

$$S_1 = -\frac{2}{16+k^2} - \frac{16}{(16+k^2)^2},$$

$$T = \frac{1}{2(1+k^2)} + \frac{1}{2k^2} \log(1+k^2).$$

By (47), (51) and (52), ϵ_0^2 is an inhomogeneous quadratic form in the two parameters γ and c_1 with coefficients which are elementary functions of k . These parameters are determined in such a way that ϵ_0^2 becomes minimum for several values of k chosen in accordance with McDougall.⁶⁾ The results are shown in Table 1.

energy in volts	k	γ	c_1	ϵ_0^2
0	0	0.10702	-0.09589	0.000361
0.0625	0.068	0.11053	-0.09926	0.000332
0.25	0.136	0.12103	-0.10929	0.000259
1	0.272	0.16245	-0.14883	0.000067
2	0.384	0.21599	-0.19945	0.000029
5	0.608	0.37101	-0.33837	0.000670
13.54	1.000	0.78074	-0.61523	0.005450

Table 1. The minimum value of ϵ_0^2 and the corresponding values of the parameters γ and c_1 .

It will be seen that ϵ_0^2 is very small, particularly in the range $k \lesssim 0.5$.

As stated before, the upper bound of $k \cot \bar{\eta}$ is calculated by the right inequality of (30) and the lower bound by the left inequality of (29), α_0 and $\beta_{\pi/2}$ being taken from (43) and (44). The results are given in Table 2, and compared with the results of McDougall⁽⁶⁾ obtained by numerical integration. It will be seen that our results are more accurate than the latter in the range $k \approx 0.3$ —0.6. The mean error of the function u_0 itself can also be estimated by (31), where $\text{Max}(\alpha_0^{-2}, \beta_0^{-2}) = 1$ since α_0 is given by (43) and $\beta_0 = 1$ (see § 4, i)).

k	$k \cot \bar{\eta}$		$\bar{\eta}$		
	lower bound	upper bound	lower bound	upper bound	McDougall
0	0.10560	0.10598	0	0	
0.068	0.10917	0.10952	0.55565	0.55706	0.557
0.136	0.11982	0.12008	0.84749	0.84859	0.847
0.272	0.16186	0.16192	1.03383	1.03400	1.034
0.384	0.21629	0.21632	1.05776	1.05781	1.060
0.608	0.37414	0.37468	1.01851	1.01916	1.007
1.000	0.78278	0.78643	0.90438	0.90665	0.905

Table 2. Upper and lower bounds of $k \cot \bar{\eta}$ and $\bar{\eta}$ as given by (30) and (29), together with the results by McDougall.

We have also derived the formula of $k \cot \bar{\eta}$ as a power series in k^2 . For this purpose we simply made use of the approximate formula

$$k \cot \bar{\eta} = \gamma - \int u_0 L[u_0] dr,$$

and evaluated the stationary value of the right side (this is equivalent to Hultén's method⁽⁹⁾, see reference 8)). The result is

$$k \cot \bar{\eta} = 0.1059 + 0.7665 k^2 - 0.1270 k^4. \quad (53)$$

It is verified that this formula is correct to about 10^{-3} for $k \lesssim 0.6$. For $k \approx 1$

it contains an error of about 5 %; of course this error is due to the bad convergence of the power expansion and has nothing to do with the variational method itself.

In conclusion the writer wishes to express his sincere thanks to Professor T. Yamanouchi for his interest in this work, and to Mr. H. Horie for valuable discussions. This work was supported by the Grant in Aid for Fundamental Scientific Research.

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Nuclear Shell Model and the β -Decay Schemes, I*

Minoru UMEZAWA, Seitaro NAKAMURA

Department of Physics, University of Tokyo

Yoshio YAMAGUCHI

Osaka City University

and

Mituo TAKETANI

Tokyo

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In § 1: the spin-orbit coupling shell model is introduced for nuclei with even mass number besides that for odd mass nuclei.

In § 2; to summarize, the overall agreement between Mayer's shell model and the Fermi theory of β -decay is very satisfactory. It is also shown that the type of nuclei predicted by the spin-orbit shell theory is useful for the study of analysis of β -decay schemes and the nuclear spectroscopy.

Introduction

To obtain a definite answer for the check of the Fermi theory of beta decay, exact informations of nuclear states concerned have been desired for a long time. Mayer,¹⁾ Feenberg and Hammack,²⁾ and Nordheim³⁾ developed specification of various nuclei by means of their original ideas concerning the nuclear shell model. However, the matter under consideration was essentially the ground state character of nuclei and the magnetic moments, although a discussion was also made on beta decay schemes.

In this paper, in order to provide a basis for analysis of beta decay, consideration will be given to the importance of beta emitters and their daughters with respect to ground states.

§ 1. Shell models

As far as odd mass nuclei are concerned, the general principle of level schemes was already established by Mayer.¹⁾ However, as far as even mass nuclei are concerned, it has not been definitely determined. New reasonable assumptions will, therefore, be added in order to complete the level assignments.

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Odd mass nuclei

The following is a summary of Mayer's shell model in view of its bearing upon our researches.

It was first pointed out by Schmidt⁴⁾ that the odd mass nucleus have the following magnetic moment:

$$\mu_i = g_l l + g_s \quad \text{for } J = l + 1/2,$$

$$\mu_i = g_l \frac{(l+1)(2l-1)}{2l+1} - g_s \frac{2l-1}{2l+1} \quad \text{for } J = l - 1/2,$$

where $g_l = 1$ for the proton, $g_l = 0$ for the neutron, and g_s is the intrinsic magnetic moment of the proton or the neutron. Deviations of experimental values from those predicted by Schmidt are found to be considerable. These deviations suggest that the angular momentum l may not be a good quantum number. In spite of this fact, the general tendencies indicated speak strongly in favor of the proposal that the nuclear configuration can be expressed in terms of the angular momentum

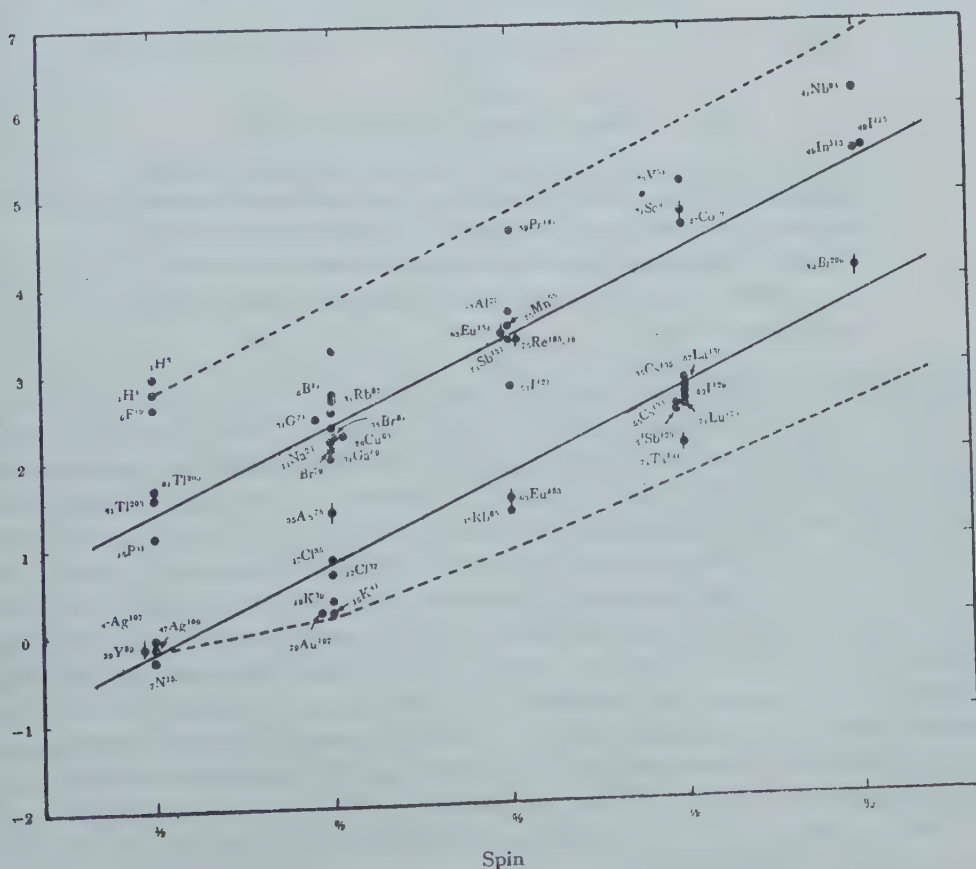


Fig. 1, a. Magnetic moment of odd proton nuclei.

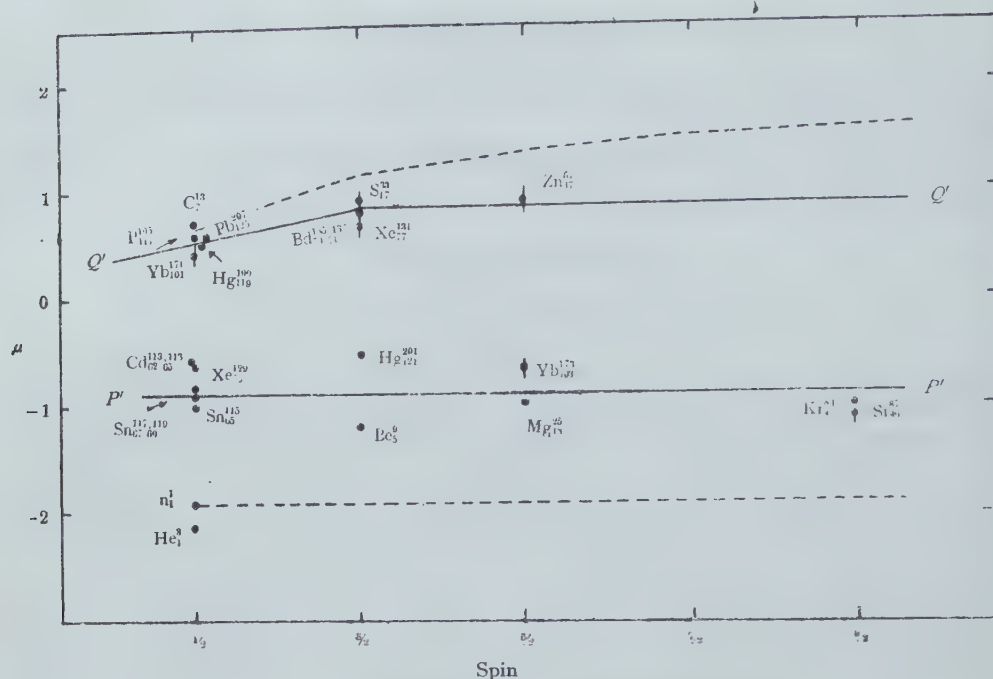


Fig. 1, b. Magnetic moment of odd neutron nuclei.

Fig. 1 μ -values for odd proton nuclei lie along the two lines \overline{PP} and \overline{QQ} , whose angle of inclination is 45° , and μ -values for odd neutron nuclei lie along the two horizontal lines $\overline{P'P'}$ and $\overline{Q'Q'}$. For the nucleus lying along \overline{PP} and $\overline{P'P'}$, the configuration of which has the angular momentum $l = J - \frac{1}{2}$ for the nucleus along \overline{QQ} and $\overline{Q'Q'}$, the configuration of which has the angular momentum $l = J + \frac{1}{2}$.

quantum number, though approximately. (see Fig. 1)

Moreover, it was mentioned that isomers or nuclei which have the same number of neutrons have the same spin value each other with few exceptions of (Rb^{85} , Rb^{87}), (Sb^{121} , Sb^{123}), and (I^{127} , I^{129}). Moreover such nuclei have same configuration with the exceptions of nuclei which have N or $Z = 51 \sim 75$ and also Rb^{86} . In Fig. 2, 3, following facts should be looked at with care. A) Up to the closed shell 20, configurations are completely the same for nuclei with odd protons and for those with odd neutrons. B) The order of levels is $1s$, $2p$, $2s$, $3d$, $4f$, $3p$, $5g$, & $4d$, $3s$, $6h$, etc. This is the order of levels between the oscillator and the square well potentials. C) The levels in which $J = l + 1/2$ are generally lower than those $J = l - 1/2$. A strong spin-orbit coupling may provide their explanation. In spite of this, it is looked at with care that there are, on one hand, 6×2 of $3p$ state, 36 of $5g + 4d$ state, and on the other hand only 4×2 of $4f$ state and none of $4f_{5/2}, 6h_{5/2}$ state. This was a reason why Mayer⁵⁾ had to introduce pairing energy.

	No. of P, N	1	3	5	7	9	11	13	15	17	19	21	23	25
odd P	orbit spin	$s_{1/2}$	$p_{3/2}$	$p_{3/2}$	$p_{3/2}$	$s_{1/2}$	$f_{7/2}$	$d_{5/2}$	$s_{1/2}$	$d_{3/2}$	$d_{3/2}$	$f_{7/2}$	$f_{7/2}$	$d_{5/2}$
	element	H ¹ H ³	Li ⁷	B ¹¹	N ¹⁵	F ¹⁹	Na ²³	Al ²⁷	P ³¹	Cl ³⁵ Cl ³⁷	K ³⁹ K ⁴¹	Sc ⁴⁵	V ⁵¹	Mn ⁵⁵
odd N	orbit spin	$s_{1/2}$		$f_{7/2}$	$p_{3/2}$	$1/2$	$f_{7/2}$	$5/2$	$1/2$	$d_{3/2}$	$3/2$			
	element	n ¹ He ³		Be ⁹	C ¹³	O ¹⁷	Ne ²¹	Mg ²⁵	Si ²⁹	S ³³	S ³⁵			

	No. of P, N	27	29	31	33	35	37	39	41	43	45	47	49	51
odd P	orbit spin	$f_{7/2}$	$f_{7/2}$	$f_{7/2}$	$f_{7/2}$	$f_{7/2}$	$f_{5/2}$ $f_{7/2}$	$f_{7/2}$	$g_{9/2}$			$p_{1/2}$	$g_{9/2}$	$d_{5/2}$ $g_{7/2}$
	element	Co ⁵⁹	Cu ⁶³	Ga ⁶⁹ Ga ⁷¹	As ⁷⁵	Br ⁷⁹ Br ⁸¹	Sb ⁸⁵ Rb ⁸⁷ Rb	Y ⁸⁹	Nb ⁹³			Ag ¹⁰⁷ Ag ¹⁰⁹	In ¹¹² In ¹¹⁵	Sb ¹²¹ Sb ¹²³
odd N	orbit spin						$f_{5/2}$					$g_{9/2}$	$g_{9/2}$	$5/2$
	element						Zn ⁶⁷					Kr ⁸³	Sr ⁸⁷	Zr ⁹¹

	No. of P, N	53	55	57	59	61	63	65	67	69	71	73	75	77
odd P	orbit spin	$d_{5/2}$ $g_{7/2}$	$g_{7/2}$	$g_{7/2}$	$d_{5/2}$		$d_{5/2}$ $f_{7/2}$	$3/2$	$7/2$	$1/2$	$g_{7/2}$	$g_{7/2}$	$d_{5/2}$	
	element	P ¹²⁷ I ¹²⁹	Cs ¹³³ Cs ¹³⁵ Cs ¹³⁷	La ¹³⁹	Pr ¹⁴¹		Eu ¹⁵¹ Eu ¹⁵³	Tb ¹⁵⁹	Ho ¹⁶⁵	Tm ¹⁶⁹	Lu ¹⁷⁵	Ta ¹⁸¹	Re ¹⁸⁵ Re ¹⁸⁷	
odd N	orbit spin						$s_{1/2}$	$s_{1/2}$	$s_{1/2}$	$s_{1/2}$	$1/2$	$1/2$	$s_{1/2}$	$d_{3/2}$
	element						Cd ¹¹¹	Cb ¹¹³ Sn ¹¹⁵	S ¹¹⁷	Sn ¹¹⁹	Te ¹²³	Te ¹²⁵	Xe ¹²⁹	Xe ¹³¹

	No. of P, N	79	81	83	85	87	89	91	93	95	97	99	101	
odd P	orbit spin	$d_{3/2}$	$s_{1/2}$	$h_{9/2}$				$3/2$	$5/2$					
	element	Au ¹⁹⁷	Tl ²⁰³ 205	Bi ²⁰⁹				Pa ²³¹	Np ²³⁷					
odd N	orbit spin	$d_{3/2}$	$d_{3/2}$										$p_{1/2}$	
	element	Ba ¹³⁵	Ba ¹³⁷										Yb ¹⁷¹	

	No. of P, N	103	105	109	111	113	115	117	119	121	123	125	127
odd N	orbit spin	$d_{5/2}^6$		$\frac{1}{2}$		$\frac{1}{2}$		$p_{3/2}^2$	$p_{3/2}^2$	$p_{3/2}^2$		$p_{3/2}^2$	
	element	Yb ¹⁷³		W ¹⁸³		Os ¹⁸⁹		Pt ¹⁹⁵	Hg ¹⁹⁹	Hg ²⁰¹		Pb ²⁰⁷	

Fig. 2 Orbits of even-odd nuclei. When two or more isotopes occur, individual spin values and orbits are given only when they are different.

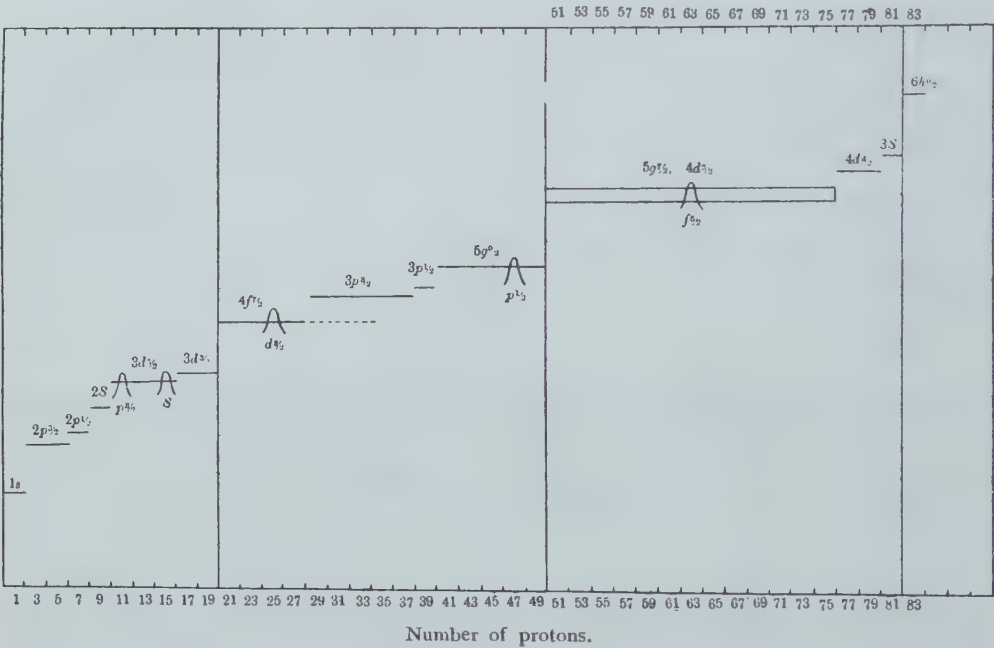


Fig. 3 Orbits of odd P-even N nuclei \wedge represent discrepancies.

Mayer's assumptions are as follows. i) An even number of identical nucleons in any orbit with total angular momentum quantum number J will always couple to give a spin zero and no contribution to the magnetic moment. ii) For a given nucleus, the "pairing energy" of the nucleons in the same orbit will be greater for the orbits with larger J . These assumptions should provide a basis for explaining the order of levels, mentioned above. Thus in the region where the number of the protons or the neutrons amounts from 21 to 38, the order of levels is $p_{3/2}$, $f_{5/2}$. However, with respect to the nuclei which has even number of protons or neutrons in the shell, the level $f_{5/2}$ is lower than $p_{3/2}$. On account of pairing energy, $f_{5/2}$ orbits will, therefore, be filled preferentially in pairs. Moreover, $4(3p_{3/2}) + 6(4f_{5/2}) = 10*$, which is just equal to $38 - 28$. On the other

hand, in the region where 51 to 76 protons are included, the order of levels is $4g_{7/2}$ and $4d_{5/2}$, $6h_{11/2}$. On account of the large pairing energy, the $6h_{11/2}$ orbits fill preferentially in pairs. Moreover, $12(6h_{11/2}) + 8(6g_{7/2}) + 6(4d_{5/2}) = 26$ which is equal to 76-50.

On the case of neutron shell which contains more than 51 neutrons, it is necessary to investigate in detail some more.

Even mass (odd-odd) nuclei

On the case of those nuclei, nuclei, of which magnetic moment is measured, are very few. However, there are many data of beta decay which suggest us the spin parity of those nuclei.⁽⁶⁾ Following assumptions are adequate to analyse the data of β -decay in all.

1) The odd-odd nucleus, the extra proton of which is in j -orbit and the extra neutron in j' -orbit, will have spin J , where

$$|j - j'| \leq J \leq |j + j'|.$$

2) Parity of odd-odd nucleus will be equal to the product of the parity of the orbits which are occupied by the extra proton and that by neutrons.

§ 2. Shell structure and β -decay scheme

As for the study of β -decay itself, only forbidden decays are interested. The analysis of allowed type β -decays are important only to examine the nuclear shell model.

I. Nuclear energy levels of N, P in nuclei and types of transitions in β -decay.

1). The nuclei in which the number of protons up to 20 and level states of neutrons are below $f_{7/2}$. (Namely the mass number is up to about 42.)

i. If two odd nuclei have the same mass and their protons and neutrons exist up to the same shell (of course, numbers of protons and neutrons in the last shell are different each other), the mass defect of such nucleus is larger as the nucleus is rich in neutrons.

Here, ($_{12}\text{Mg}^{27}$, $_{13}\text{Al}^{27}$), ($_{14}\text{Si}^{29}$, A^{29}), ($_{16}\text{P}^{31}$, $_{14}\text{Si}^{31}$) are exceptions. $_{14}\text{Si}^{29}$ and $_{15}\text{P}^{31}$ whose spins are, both 1/2, are a discrepancy for the Mayer's shell model. $_{12}\text{Mg}^{27}$ is expected to be a spin discrepancy for the Mayer's shell model, since $_{12}\text{Mg}^{27}$ has 15 neutrons.

i'. Especially, it is important that, in pairs of mirror nuclei, the mass defect is larger for neutron-rich nucleus.

ii. If two even mass nuclei are of the same mass number and their protons and neutrons are even and exists up to the same shell (of course, number of protons and neutrons in the last shell is different each other), the mass defect is larger as the nucleus is more rich in neutrons.

Here, (Be^{10} , C^{10}) are exceptions. Since the mass number of these nuclei is so small, the rule shall not hold with these nuclei.

We can determine the order of energy levels of each shell under the light of features mentioned above (see Fig. 4).*

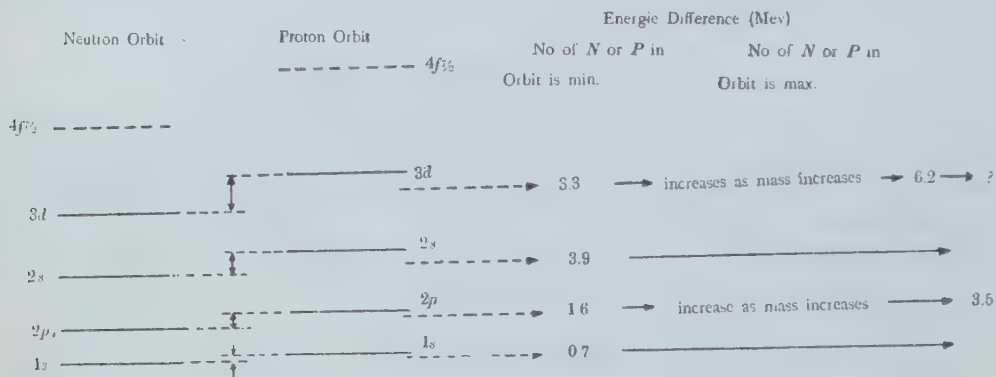


Fig. 4

From Fig. 4, following facts may be suggested. If the two nucleus have the same mass number and their neutrons are up to $f_{7/2}$ state and their protons are up to d -state, the mass defect is smaller as nucleus is more rich in neutrons, since the energy level of $f_{7/2}$ is higher than that of d . Indeed, this is indicated in the case of ($_{17}\text{Cl}^{37}$, $_{16}\text{S}_{21}^{37}$). Also following things must be noticed in light of Fig. 4. For instance, there is no nucleus in which last neutron exists in $3d$ state and the last proton exists in $2p$ shell. Generally there appear no nuclei, in which other level exists between two levels in which last neutron and last proton exists.

Since such a nucleus has smaller mass defect than other nucleus which has same mass number and smaller number of nucleons which exist in higher level, such a nucleus is able to be N or P emitter.**)

On account of these facts nuclei which really exist are limited to one of the following types.***) ($4f$, $3d$), ($3d$, $3d$), ($3d$, $2s$), ($2s$, $2s$), ($2s$, $2p$), ($2p$, $2p$), ($2p$, $1s$), ($1s$, $1s$).

* In Fig. 4, the spacings between energy levels of the same orbits of N and P are exactly determined from mirror nuclei, but the spacing of the energy levels of different orbits of N and P may be difficult to be estimated.

** Although we must expect same circumstances for the nuclei, the last N of which exists in d (or p) state and the last P exists in $2s$ (or $1s$) state, but, since only two nucleons are able to enter into s state, there are not two nuclei which satisfy such a condition.

*** The nuclei, where neutrons exist up to g state and protons exist up to r state, are shown by (g , r).

β -decay schemes then, are limited to one of the following types.*) ($4f_{7/2} \longleftrightarrow 3d$), ($3d \longleftrightarrow 3d$), ($3d \longleftrightarrow 2s$), ($2s \longleftrightarrow 2s$), ($2s \longleftrightarrow 2p$), ($2p \longleftrightarrow 2p$), ($2p \longleftrightarrow 1s$), ($1s \longleftrightarrow 1s$):

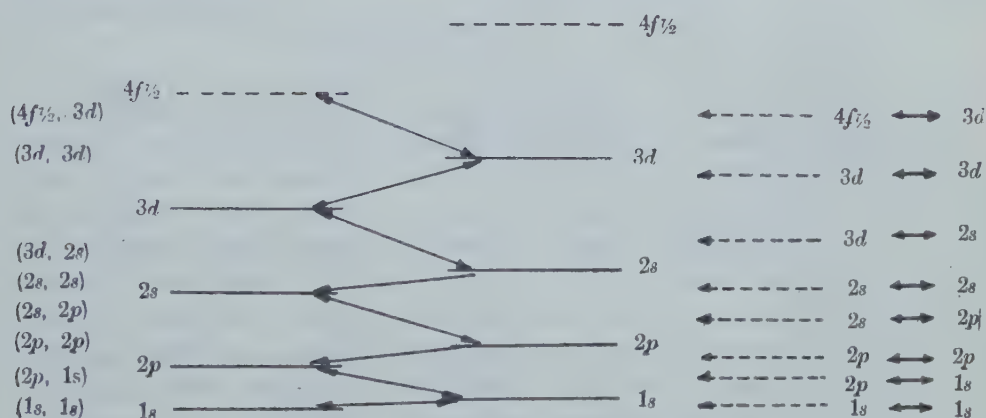
Thus the analysis of β -decay schemes will do come to examine the transition of the above mentioned types.

Nuclei, with $P > 21$

There is no mirror nucleus in this region. It is, therefore, difficult to estimate exact energy levels. In this region, nuclei which really exist are limited to one of the following types (see Fig. 6):

$$\begin{aligned} & (4f_{7/2}, 4f_{7/2}), (3p \text{ \& } 4f_{5/2}, 4f_{7/2}), (3p \text{ \& } 4f_{5/2}, 3p \text{ \& } 4f_{5/2}), \\ & (5g \text{ \& } 4d \text{ \& } 3s \text{ \& } 6h_{11/2}, 3p \text{ \& } 4f_{5/2}), \\ & (5g \text{ \& } 4d \text{ \& } 3s \text{ \& } 6h_{11/2}, 5g \text{ \& } 4d_{5/2} \text{ \& } 6h_{11/2}), \\ & (6h_{9/2} \text{ \& } 5f \text{ \& } 4p \text{ \& } 7i_{13/2}, 5g \text{ \& } 4d \text{ \& } 3s) \\ & (7i_{11/2} \text{ \& } \dots, 4d_{5/2} \text{ \& } 3s) \quad (7i_{11/2} \text{ \& } \dots, 6h_{9/2} \text{ \& } 5f \text{ \& } 4p \text{ \& } 7i_{13/2}) \end{aligned}$$

Type of nucleus

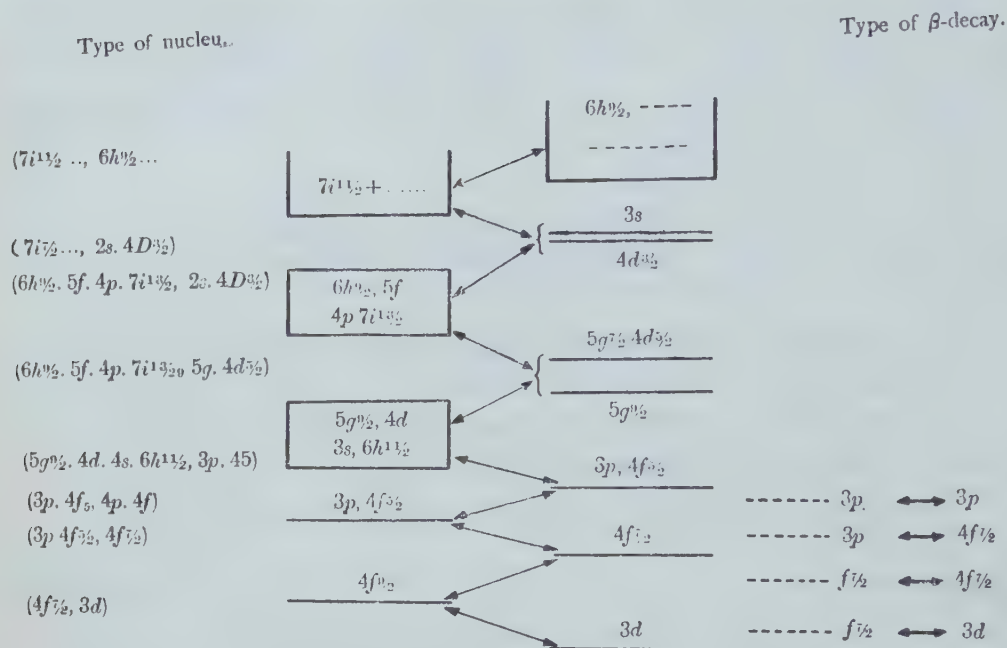
 Type of β -decay

 Fig. 5 Types of nuclei and of β -decay.

From these circumstances, we can roughly estimate the order of levels, which may be shown as Fig. 6.

Thus, there will exist β -decays, types of which may correspond to the above mentioned type of nuclei.

In the present discussion, we confine our attention to nuclei with the neutron number $N < 41$. As far as nuclei with $N > 41$ are concerned where the $g_{9/2}$ state begins to appear the theory remains much to be studied. In this paper, only

*) β -decay of nucleus (q, r) is shown by $q \longleftrightarrow r$. For the odd mass nuclei, this corresponds to the transition from q to r state (or contrary). For even mass nuclei this is complicated.

Fig. 6 Type of nuclei and β -decay.

the transition of type $f_{7/2} \longleftrightarrow f_{7/2}$, $3p \longleftrightarrow f_{7/2}$, $3p \longleftrightarrow 3p$, will be dealt with.

II. β -decay scheme — up to type $(3p, 3p)$. From the Fermi theory of β -decay, one can readily ascertain the validity of classification of nuclear states that will be obtained from the theory of shell structure. The Fermi interaction, i. e. $\{ \sigma$ as derived from tensor or axial vector coupling. However, for explaining a certain forbidden spectrum such as RaE, more refined types of coupling may be necessary. Hence, in addition to $\{ \sigma$ all the matrix elements that are actually involved in scalar, polar vector, tensor and pseudoscalar couplings, will tentatively be used for the Fermi interaction^{*}) throughout this paper. The results will be indicated for odd mass nuclei and even mass nuclei separately as in Fig. 7 and 8. We shall discuss them in the order of specific transitions. Firstly, β -decay of the even mass nuclei (Fig. 7) then for the even mass nuclei will be discussed.

Odd mass nuclei

(i) As regards to the decay of type $1s \longleftrightarrow 1s$, $2p \longleftrightarrow 2p$, $2s \longleftrightarrow 2s$, $3d \longleftrightarrow 3d$, $4f_{7/2} \longleftrightarrow 4f_{7/2}$, $3p \longleftrightarrow 3p$, direct transition to ground state (namely the highest group of no γ rays) must be the allowed transition.

^{*}) It is interesting to note the availability of pseudoscalar coupling. As will be indicated later, the contribution of $\{ \tau_5$ as derived from pseudoscalar coupling is limited only in the case (ii). For in (ii), (ii') the allowed transition due to $\{ \tau_5$ are rejected on account of the selection rule, because $\{ \tau_5$ require zero spin change. In (iii), (iii'), (iv'), the 1st forbidden transition due to PS interaction (matrix element $\{ \tau_5 \tau$) are rejected on account of the selection rule, because $\{ \tau_5 \tau$ require spin change 0 or ± 1 .

(ii) As regards the decay of type $4f_{7/2} \leftarrow 3d_{3/2}$, direct transition to ground state (namely, highest group followed by no γ ray) must be 1st forbidden.

(ii) As regards the decay of type $2s \leftarrow 2p_{1/2}$, direct transitions to ground

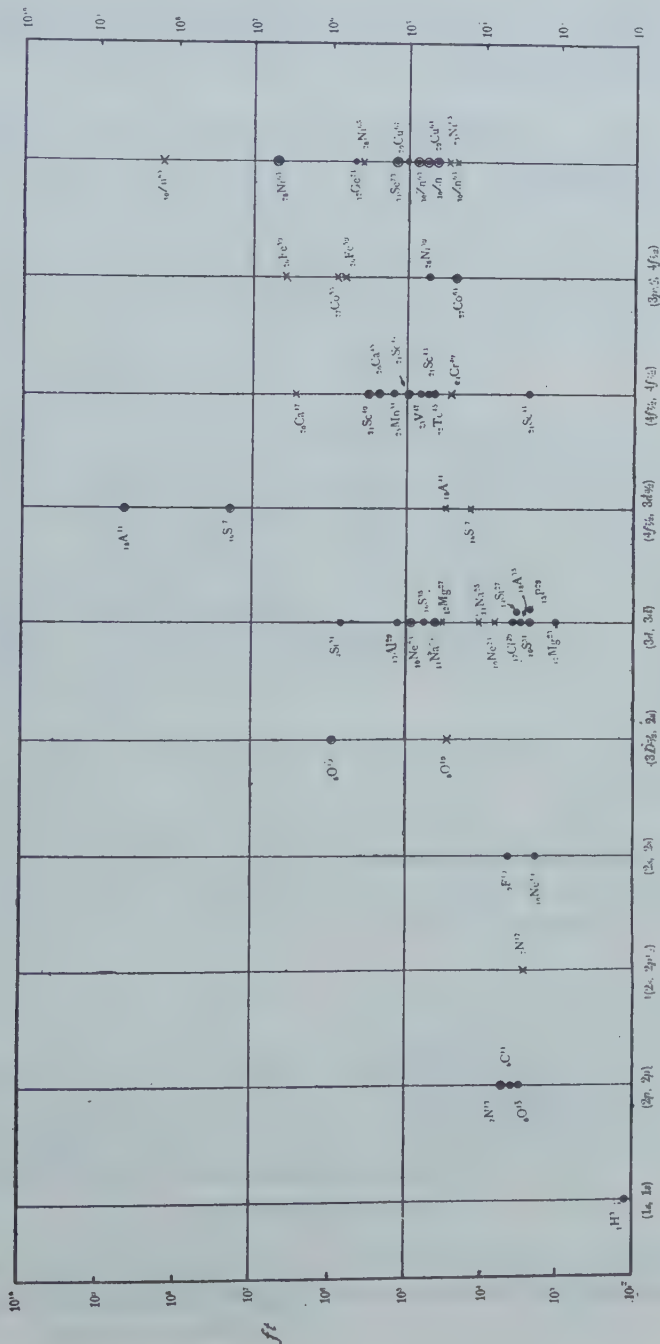


Fig. 7 \odot ... Direct transition to ground state. \bullet ... Transition that is probably direct transition to ground state but is not yet determined. \times ... Transition to excited state. The nuclei, which neutrons exist up to q orbit and protons up to r orbit, are shown by ($q; r$).

states should be 1st forbidden, provided PS interaction dose not exist. If PS interaction exists, allowed transition due to PS interaction (matrix element $\{r_5\}$) should be possible.

(iii) As regards the decay of type $3d_{5/2} \longleftrightarrow 2s$, $3p_{3/2} \longleftrightarrow 4f_{7/2}$, direct transition to the ground state must be 2nd forbidden. Then, beside the particular cases,* this transition can not be observed.

Even mass nuclei

(i)' As regards the decay of type $(2s, 2s)$ direct transition to the ground state must be the allowed transition.

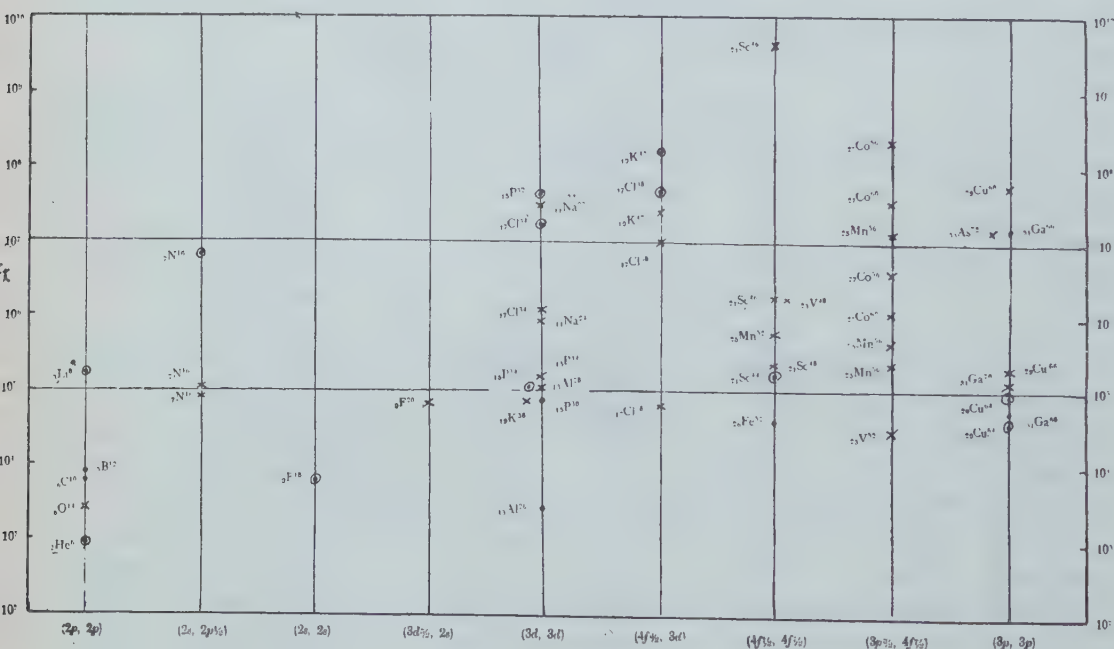


Fig. 8 Type of nuclei

(ii)' As regards the decay of nuclei of type $(2s, 2p_{1/2})$, direct transition to the ground state must be 1st forbidden.

(iii)' As regards the decay of nuclei of type $(3d_{5/2}, 2s)$, direct transition to the ground states must be the 2nd forbidden.

(iv)' As regards the decay of nuclei of type $(4f_{7/2}, 3d_{3/2})$, $(3p_{3/2}, 4f_{7/2})$ direct transitions to ground states must be the 1st, 2nd or more highly forbidden, respectively.

* Especially, if the parent nucleus can not afford sufficient energy for transition to the excited state of the daughter nucleus, direct transition to the ground state will be liable to be observed.

In light of Fig. 7 and 8 we can easily ascertain*) the above things with few discrepancies.**)

(1) Odd mass nuclei

Decay of nuclei of the types $(1s, 1s)$, $(2p, 2p)$, $(2s, 2s)$:

As above expected, direct transition to the ground state are all allowed transitions.

Decay of nuclei of the type $(2s, 2p)$:

N^{17} decays to the excited state of O^{17} , and direct transition to the ground state is not observed (experiment is not yet definite). If PS interaction does not exist, this is in agreement with the expectation. On the other hand, when PS interaction exists, we must understand this as follows, namely allowed transition due to PS interaction (matrix element $\{ \gamma_5 \}$) loses in a competition with that to the excited state of O^{17} and will not be observed since the matrix element $\{ \gamma_5 \}$ is considerably smaller than $\{ 1 \}$ and $\{ \gamma_5 \}$.

Decay of nuclei of type $(3d_{5/2}, 2s)$:

${}_8O^{19}$ is in discrepancy. ft value of direct transition to the ground state is too small to be that of second forbidden. No. of neutrons is 11, and this is discrepancy for Mayer's shell model. Thus it may not be surprising to expect this discrepancy. The ft value is between that of the allowed and the 1st forbidden. If this is allowed, there should exist an s state above $2s$ state, since transition to the excited state is also allowed.

Decay of nuclei of type $(3d, 3d)$:

As expected, direct transition to the ground state is all allowed transition but Mg^{27} , ${}_{14}Si^{31}$ ***)

As regards ${}_{12}Mg^{27}$, since No. 15 is discrepancy for Mayer's shell, ${}_{12}Mg^{27}$ shall be discrepancy for Mayer's shell. If this is true, it is explained that direct transition to the ground state is not observed.

As regards ${}_{14}Si^{31}$, ft value of which is between allowed and 1st forbidden, this decay shall be attended by spin change 1 and no parity change and then allowed, since ground nucleus ${}_{15}P^{31}$ is in discrepancy for Mayer's shell model and has s state.

*) In spite of this success, some difficulties appear as soon as the number of N exceeds 41. The regions 41-50 are theoretically expected to be $g_{9/2}$ state. The difficulties are due to the consequences of the following three facts, namely-(1), in the case of N -shell, states in the region of over 41, have not yet been determined experimentally, (2) in this region, many states with low spin value came in light, (3) in the region of over 41, decay scheme is very complex on account of small energy space between relevant levels. These shall be treated in the later paper.

**) In the case of β -decay discrepancy, either parent, or daughter nucleus should be discrepancy for Mayer's shell model.

***). Beside these ground nucleus of Al^{29} decay is Si^{29} which is discrepancy for Mayer's shell Model. Then ${}_{13}Al^{29}$ decay will probably be in discrepancy and be followed by γ -ray, although the evidence is not definite.

Decay of nuclei of type ($4f_{7/2}$, $3d_{3/2}$) :

As above expected, direct transitions to ground states are all 1st forbidden. Moreover, these transitions require spin change 2 and parity change, then only matrix element B_{ij} is able to rise these transitions. Then the shape of spectrum of β -decay must be 'a'-shape. Indeed A^{39} has 'a'-shape spectrum. This is a very fair success.

Decay of nuclei of type ($4f_{7/2}$, $4f_{7/2}$) :

As above expected, direct transitions to ground states are all allowed transitions but $^{20}\text{Ca}^{47}$, $^{24}\text{Cr}^{49}$.

$^{24}\text{Cr}^{49}$ may probably be in discrepancy for Mayer's shell model.

It is not yet undetermined that either of Ca^{47} and Sc^{47} is in discrepancy.

Decay of nuclei of type ($3p_{3/2}$, $4f_{7/2}$) :

As above expected, Fe^{59} and Co^{55} decay can not go directly to the ground state.

$^{27}\text{Co}^{61}$ is in discrepancy. Either of Ni^{61} and Co^{61} must be in discrepancy for Mayer's shell.

$^{28}\text{Ni}^{67}$ should be in discrepancy if this decay is not accompanied by γ -ray.

Decay of nuclei of type ($3p$, $3p$) :

As above expected, direct transitions to ground states are allowed transitions with few discrepancies.

$^{30}\text{Zn}^{65}$, $^{28}\text{Ni}^{65}$ are in discrepancy. According to Mayer's shell model, $^{29}\text{Cu}^{65}$ is experimentally known to have $p_{3/2}$ state. Then $^{30}\text{Zn}^{65}$ and $^{28}\text{Ni}^{65}$ shall probably be in discrepancy for Mayer's shell model.

(2) Even mass nuclei*

Decay of nuclei of type ($2s$, $2s$) :

As above expected, direct transitions to ground states are all allowed.

Decay of nuclei of type ($2s$, $2p$) :

As above expected, direct transitions to ground states are all 1st forbidden.

Decay of Nuclei of type ($3d_{5/2}$, $2s$) :

As above expected, direct transitions to ground states are not observed.

Decay of nuclei of type ($4f_{7/2}$, $3d_{3/2}$) :

As above expected, direct transitions to ground states are 1st forbidden (K^{42} , Cl^{35}) or 3rd forbidden, (K^{40}).⁸⁾

Moreover, decay ($4f_{7/2}$, $3d_{3/2}$) are attended with more than two spin change. Then direct transitions to the ground state are able to rise only by the matrix element B_{ij} which belongs to axial vector or tensor interaction. Then the shape of spectrum must take 'a'-shape.

*) As regards the scheme of β -decay, especially the excited states of daughter nuclei, the nuclei which belong to the same type are very similar to each other (of course, these are completely different from those which lie in the different type). These things will be treated with the analysis of excited states in a later paper.

Indeed, K^{42} and Cl^{32} have the spectrum of shape 'a'. This is a fair success.
Decay of nuclei of type $(3p_{3/2}, 4f_{7/2})$:

As above expected, direct transitions to the ground states are not observed.

(3) Estimation of nuclear spin and decay matrix element

Odd mass nuclei.

As regards the odd mass nuclei, it is needless to say, since matrix elements have been determined by spin and parity corresponding to the type of nuclei.

But it is worth while to notice that decay of nuclei of type $(f_{7/2}, 3d_{3/2})$ is accompanied, by 2 spin change and parity change. Indeed, A^{39} is known to have 'a'-shape spectrum. A^{41} and S^{37} , which belong to $(f_{7/2}, 3d_{3/2})$ and whose spectrum are not determined experimentally, must have spectrum of shape 'a' and rise due to matrix element B_{ij} .

Even mass nuclei.

As previously mentioned, we assumed that the nuclei of type (q, r) have various spin value J which $|J_q - J_r| \leq J \leq J_q + J_r^*)$ and are not definite. Then, it is necessary to estimate spin and decay matrix element for each nucleus. When nucleus has considerably small J value, we can estimate spin value J for each nucleus as follows. Moreover, we can estimate matrix elements pretty well too, in view of the fact that ground nuclei (even-even nuclei) always spin zero.

Decay of nuclei of type $(2s, 2s)$:

F^{18} is obviously estimated to have spin 1 and decay due to matrix element $\int \sigma$.

Decay of nuclei of type $(2s, 2p_{1/2})$:

This is the decay which is attended by 1 spin change and parity change and is 1st forbidden. Then N^{16} is estimated to have spin 1. (There are several matrix elements corresponding to this decay). (see Konopinski.)

Decay of nuclei of type $(3d_{5/2}, 2s)$:

Nuclei of this type have spin 2-3. F^{20} is 2nd or more highly forbidden then F^{20} is estimated to have spin $J=2$ or 3 (there are several of matrix element corresponding to this decay.)

Decay of nuclei of type $(4f_{7/2}, 3d_{3/2})$

According to assumptions previously mentioned as regards the decay of this type, direct transitions to ground states are attended by 2 or more spin change and parity change. Then, direct transitions to ground states which are 1st forbidden, be able to rise only due to matrix element B_{ij} , and must have spectrum of shape 'a'.

*) Then there are many transitions which are forbidden, but to the higher order, in point of fact, almost all the parent nuclei have sufficient energy for transition to excited state of ground nuclei. Then the direct transition to ground state lose in a competition with transitions to excited state and is not observed in experiment. Merely, direct transitions to ground states are observed. As regards Be^{13} , Cl^{36} , Cl^{44} , K^{40} , since these have not sufficient energy for transitions to excited states of ground nuclei, direct transition to ground state are observed.

This is a very important touch stone! Cl^{38} and K^{42} are 1st forbidden and have spectrum of shape ' α '. Well done!⁷⁾ (see also Decay of odd nuclei of this type).
Decay of nuclei of type $(3p_{3/2}, 4f_{7/2})$:

As regards Co^{60} , Mn^{56} , Co^{62} , V^{52} , direct transitions to ground states are not yet observed. Then, as above expected these are 2nd or more highly forbidden.

Then these nuclei are estimated to have spin J , $5 \geq J \geq 3$.

Decay of nuclei of type $(2p, 2p)$:

According to assumption, The nuclei of this type have spin J , $1 \leq J \leq 3$. As regards Li^8 , Li^6 (and probably B^{12} too), direct transitions to ground state are allowed transitions, then are estimated to have spin 1 and decay due to matrix element $\{ \sigma \}$.

Decay of nuclei of type $(3d, 3d)$:

According to the assumption, the nuclei of this type have spin J , $1 \leq J \leq 3 \sim 5$.

As regards P^{34} , (P^{30} , Al^{26} are not accurate in experiments) direct transitions to ground state are allowed, then P^{34} (P^{30} , Al^{26}) are expected to have spin 1 and decay due to matrix element $\{ \sigma \}$. P^{32} and Cl^{34} may be discrepancies. As regards P^{32} and Cl^{34} , ft value of direct transitions are too large to be regarded as allowed transitions. Moreover, P^{32} is known to have allowed shape spectrum. Then P^{32} (probably Cl^{34} too) are expected to have spin 1 and decay due to matrix element $\{ \alpha \}$. But $\{ \alpha \}$ requires parity change. This is in contradiction to the assumption that decay of $(3d, 3d)$ type requires no spin change. From these circumstances, although ft value are large, P^{32} and Cl^{34} may be allowed transitions.

As regards Na^{24} , Al^{28} , K^{38} , the direct transitions to ground state are not observed in experiment then are probably 2-nd or more highly forbidden. Then Na^{24} , Al^{28} are estimated to have spin $5 \geq J \geq 2$, and K^{38} , $3 \geq J \geq 2$.

Decay of nuclei of type $(4f_{7/2}, 4f_{7/2})$:

According to assumptions, nuclei of this type have spin J , $1 \leq J \leq 7$.

As regards Sc^{44} (Fe^{55} decay schemes are not accurate) direct transitions to ground states are allowed transitions. Then Sc^{44} (Fe^{55}) are estimated to have spin 1 and decay due to matrix element $\{ \sigma \}$.

As regards Sc^{46} , V^{48} , Mn^{52} , Sc^{48} , direct transitions to ground state are not observed, then, these are estimated to have spin J , $7 \geq J \geq 2$.

Decay of nuclei of type $(3p, 3p)$:

According to assumptions, nuclei of this type have spin J , $1 \leq J \leq 3$.

As regards Cu^{64} (Ga^{68} decay scheme is not accurate) direct transitions to ground states is allowed, then Cu^{64} is estimated to have spin 1, and decay due to matrix element $\{ \sigma \}$.

As regards Cu^{60} , As^{72} , Cu^{66} , Ga^{70} , direct transitions to ground states are not observed, then shall be 2nd or more highly forbidden. Then Cu^{60} , As^{72} , Cu^{66} Ga^{70} have spin 2 or 3.

III. Selection rules and spectra in the forbidden case

1) 2nd or more highly forbidden case*

In this case, decay due to PS interaction is not observed, since, n -th forbidden decay due to PS interaction loses in competition with $n-1$ -th forbidden decay due to other interactions.

K^{40} : K^{40} belongs to $(f_{7/2}, d_{3/2})$ type, and has spin 4, then K^{40} decay must be 3rd forbidden and rise only due to matrix element S_{ijkl} . Indeed this is the fact.¹⁰⁾
 Cl^{36} : Cl^{36} belongs to type $(2d, 2d)$, and has spin 2. Then Cl^{36} -decay must be 2nd forbidden. Since matrix element S_{ijk} are forbidden for the case $2 \rightarrow 0$, Cl^{36} can not decay due to S_{ijk} , although its shape of spectrum is similar to c -shape,⁹⁾ and only A_{ij} , T_{ij} , R_{ij} are able to contribute to this decay.

Be^{10} : Be^{10} belongs to type $(2p_{3/2}, 2p_{3/2})$ and has spin 3. Then Be^{10} decay must be 2nd forbidden and rise only due to matrix element S_{ijk} .¹¹⁾ Indeed this is the fact.

$_{41}Cb^{94}$, $_{6}C^{14}$: ghost haunts! Although they have very large ft values, the shape of energy spectrum are similar to that of allowed type, yet unsolved problem these are.

2) Case of 1st forbidden and shape 'a'. Up to the present, 8 elements (see Fig. 9) are found to belong to this case.

Among these, Y^{90} , Sr^{89} , Sr^{90} , Y^{91} and Cs^{137} are solved by Feenberg and Nordheim²⁾. The situations are also adequate in Mayer's shell model. Cl^{38} , K^{42} and A^{39} are solved on this paper.

3) Case where shape are allowed but ft are larger.

In this case we have not arrived to conclusion. Following possibilities are

Fig. 9 1st forbidden transitions of a shape spectra

element		$W_0(mc^2)$	$t(sec)$	Type of spectra	ft	$\bar{f}t$	Matrix element	Selection Rules
$_{39}Y^{90}$	t, g	5.29	2.2×10^5	a	1.8×10^8	3×10^7	B_{4f}	$\Delta J = \pm 2$ yes
$_{38}Sr^{90}$		2.05	8×10^8	a		7×10^7		
$_{38}Sr^{89}$		3.88	4.7×10^8	a	3.8×10^8	6.2×10^7		
$_{39}Y^{91}$		4.01	5.3×10^6	a		7×10^7		
$_{17}Cl^{38}$		10.43	2.3×10^3	a		5×10^7		
$_{19}K^{42}$		8.0	4.5×10^4	a		1.3×10^8		
$_{18}A^{39}$	t, e	2.11	4.5×10^8	a		$< 4.5 \times 10^7$		
$_{55}Cs^{137}$		2.01	1.2×10^9	a	4.7×10^8	8.7×10^7		

ft shows corrected value on account of its shape a .

* RaE^{210} is treated in the later paper.

The transitions of about 1st forbidden ft value and of allowed shape spectra.

element		$W_0(mc^2)$	$t(\text{sec})$	ft
Pm ¹⁴⁷		1.44	1.2×10^8	1.8×10^6
²³ V ⁴⁸	<i>t. e</i>	2.42	1.4×10^6	1.8×10^6
⁵⁹ Pr ¹⁵³	<i>t. g.</i>	2.83	1.2×10^6	3.9×10^6
⁷⁵ Re ¹⁸⁶	<i>t. e</i>	3.1	3.2×10^6	1.86×10^7
¹¹ Na ²²	<i>t. e</i>	2.12	8.2×10^7	3.9×10^7
¹⁵ P ³²	<i>t. g.</i>	4.37	1.2×10^6	4.96×10^7
⁶⁹ Tm ¹⁷⁰	<i>t. g.</i>	2.94	1.2×10^7	5.2×10^7

t. e shows the transition to excited state,
t. g. " " " " ground " .

considered: Some of the cases might be allowed transitions, although ft value is large, while some of them are 1st forbidden which are caused by the matrix element $f\alpha$.¹²⁾

V. Concluding remarks^{13) 14)}

Through the whole of this paper, mirror nuclei fall in the neighborhood of $ft=5 \times 10^3$. The region of $ft=10^{4-6}$ are filled by the other allowed transitions. The difference of ft values may be due to the incomplete overlapping of the radial wave functions. The 1st forbidden decays fall in the region of $ft=10^{7-9}$.

As shown in Fig. 10, density- ft curve has four peaks; namely,

- A $ft \sim 5 \times 10^3$ —almost of these are mirror nuclei,
- B $ft \sim 2 \times 10^5$ —the other allowed transition,
- C $ft \sim 5 \times 10^6$,
- D $ft \sim 5 \times 10^7$.

From this situation, almost all elements in C, D have been considered to be the 1st forbidden; the elements which have 'a'-shape spectrum are all in D, their ft values being almost the same order if one uses the corrected 1st forbidden f -functions.¹²⁾

In conclusion, we would like to express our sincere thanks to Prof. Muneo Sasaki for his valuable discussions.

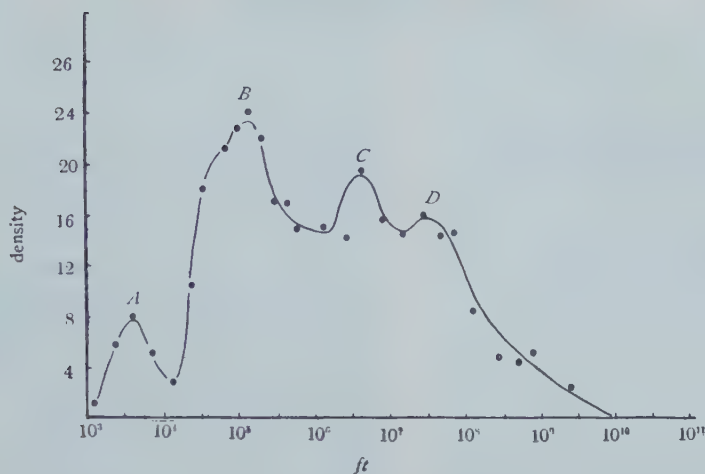


Fig. 10 ft -density curve of β -decay throughout the whole.

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Letters to the Editor

On the Meson Cloud around the Nucleons

H. Umezawa, T. Takahashi
and S. Kaméfuchi

*Institute of Theoretical Physics,
Nagoya University*

April 5, 1951

Hitherto, the research of the "model" of the elementary particles, that is, the study of the kinds of the elementary particles existing in the nature and their internal correlation, have been an important problem in the meson theory. Recently, various theories have been presented, which aim to investigate the "structure" of the elementary particles within the framework of the known models of the elementary particles or in connection with the divergence difficulties in the quantum field theory.¹⁾ Since the most problems in the present quantum field theory are related to the system of the field around its source, one of the most important researches of the elementary particles in the correspondence theoretical view point is the investigation of the properties of the meson cloud of the nucleon system. In such problem we must be concerned with the field at a local space-time which is not always on the surface at the time $+\infty$ or $-\infty$ and so the Heisenberg representation is suitable for our purpose. In order to get the intuitive picture of the meson cloud we shall describe the meson cloud by $U(x, \sigma)$, which satisfies the homogeneous equation of the free meson field and coincides, on the world point x/σ on the space-like surface σ , with the field operator $U(x)$ in the Heisenberg representation. Such operator $U(x, \sigma)$ was already introduced by Yang and Feldman.²⁾ When we denote $U(x, -\infty)$

as $U_{in}(x)$, then there is an unitary transformation $S(\sigma)$:²⁾

$$U(x, \sigma) = S^{-1}(\sigma) U_{in}(x) S(\sigma). \quad (1)$$

Since $U(x, \sigma)$ is the free field operator, we can decompose it into the positive and negative frequency parts $+U(x, \sigma)$ and $-U(x, \sigma)$. In the case of the spinless meson, of which interaction Lagrangian is $-g\bar{\psi}\phi\psi U$ (scalar coupling) or $-g\bar{\psi}\phi_\mu\psi \frac{\partial U}{\partial x_\mu}$ (vector coupling), the relation between the meson cloud and its source (nucleon field) is shown to be as follows:

$$\pm U(x, \sigma) = \pm U_{in}(x)$$

$$+ \begin{cases} g \int^\sigma \pm \Delta(x-x') dx' O(x'), & (\text{scalar coup.}) \\ g \int^\sigma \frac{\partial \pm \Delta(x-x')}{\partial x_\mu} dx' O_\mu(x') & (\text{vector coup.}) \end{cases} \quad (2)$$

$$\equiv \pm U_{in}(x) + \int_{-\infty}^\sigma \pm u(x, \sigma; x') dx', \quad (3)$$

$$O(x) = [S^{-1}(\sigma) \bar{\psi}_{in}(x) \phi \psi_{in}(x) S(\sigma)]_{x/\sigma}. \quad (4)$$

By means of $+U(x, \sigma)$ we can define the number operator $N(\sigma)$ of the meson cloud on a space-like surface σ :

$$N(\sigma) \equiv \frac{2}{i} \left[\int_\sigma (+U(x, \sigma))^* \frac{\partial +U(x, \sigma)}{\partial x_\mu} d\sigma_\mu \right]_{x/\sigma}, \quad (5)$$

$$N(\sigma) = \int N(\mathbf{k}) d(\mathbf{k}), \quad (6)$$

where $*$ means the Hermite conjugate operator and $N(\mathbf{k})$ is number operator in the momentum space.

As it is seen in (5) that $+U(x, \sigma)$ gives intuitive picture of the free meson for the cloud, we call $+U(x, \sigma)$ the "cloud spectrum."³⁾ On the other hand, since $O(x)$ represents the spin matrix O of the nucleon

accompanying the meson cloud as is shown in (4), we call $O(x)$ the "effective spin (e-spin)" of the nucleon with the meson cloud. With the aid of c-spectrum and e-spin, we can treat the meson cloud and the nucleon in the meson cloud. These quantities are related each other by the integral equation (2), and so we can obtain $O(x)$ from a given c-spectrum.

In this formalism, the investigations of the phenomena of the nucleon with the meson cloud are achieved by rewriting them in terms of the operator $U(x, \sigma)$ and the similar nucleon operator $\psi(x, \sigma)$. The phenomenological investigation will be performed by a priori giving the phenomenological meson cloud $U(x, \sigma)$.

The relation between this treatment and the usual one in the Heisenberg representation is as follows:

$$[U(x, \sigma)]_{x/\sigma} = U(x),$$

$$\left[\frac{\partial U(x, \sigma)}{\partial x_4} \right]_{x/\sigma} = -i\Pi(x), \quad (7)$$

where $\Pi(x)$ is the canonical conjugate operator of $U(x)$. By use of (7) or (2) we can obtain $U(x, \sigma')$ from a given $U(x/\sigma')$ by replacing $\exp(\pm ik_0 t')$ by $\exp(\pm iKt)$ $\exp(\pm i(k_0 - K)t')$ ($K \equiv \sqrt{k^2 + \kappa^2}$). By means of this rule, we can obtain $U(x, \sigma)$ from the $U(x)$, which may be calculated in the convenient approximation (for example, weak, intermediate or strong coupling approximation).

When we represent the energy-tensor T_{44} of the system of the meson and the nucleon by means of $U(x, \sigma)$ and $\psi(x, \sigma)$, we find that the total energy consists of the kinetic energies of free meson and nucleon fields ($U(x, \sigma)$, $\psi(x, \sigma)$) and the interaction energies between them. In the non-relativistic approximation we can show that this total energy coincides with the self-energy obtained by the Block-Nordsieck transformation.⁵⁾

In the treatment of the meson cloud it is important to separate the meson cloud in

the absence of an external field from the one in the presence of the external field $\bar{\phi}V\phi$. We denote $S(\sigma)$, ${}^+U(x, \sigma)$, ${}^+u(x, \sigma; x')$ and $O(x)$ in the absence of the external field by $S_0(\sigma)$, ${}^+_0U(x, \sigma)$, ${}^+_0u(x, \sigma; x')$ and $\bar{O}(x)$ respectively. The state Ψ_0 in which a nucleon and no mesons exist is used for the meson cloud of the single nucleon.

When the external effect is present, the e-spin $O(x)$ deviates from the $O(x)$ owing to the various effects in the past processes. When this $O(x)$ is described by $\bar{O}(x)F[\sigma(x)]$, we denote ${}^+U(x, \sigma)$ by $[{}^+_0UF](x, \sigma)$. $\bar{O}(x)F[\sigma(x)]$ represents the forced vibration of e-spin caused by the past various effects. $[{}^+_0UF](x, \sigma)$ satisfies also the homogeneous free field equation.

The meson production by γ -nucleon or nucleon-nucleon collisions and the magnetic moment of the nuclear system are the important problems for the research of the properties of the meson cloud. In the following letter we shall consider the multiple production of mesons due to the nucleon-nucleon collision.

The authors wish to express their cordial thanks to Prof. S. Tomonaga and Prof. S. Sakata for their constructive criticisms throughout this work.

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- 2) C. N. Yang and D. Feldman, Phys. Rev. **79** (1950), 972.
- 3) We may expect that, if the interaction between the meson and the nucleon fields is switched off by means of a certain external disturbance, then mesons would be emitted according to the c-spectrum amplitude. This expectation was pointed out by Prof. S. Tomonaga.
- 4) W. Heisenberg and W. Pauli, ZS. f. Phys. **56** (1929), 1; W. Heisenberg, Ann. d. Phys. **5** (1931), 339.
- 5) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54.

On the Multiple Production of Mesons¹⁾

H. Umézu, Y. Takahashi
and S. Kamefuchi

*Institute of Theoretical Physics,
Nagoya University*

April 5, 1951

Heisenberg pointed out in his famous paper²⁾ that the problem of the multiple production is closely related to the properties of the meson cloud. His discussion was based on two assumptions which were introduced in taking into account the Bloch-Nordsieck's result³⁾: (i) the spectrum of the meson cloud may be classically calculated, (ii) the produced mesons obey the Poisson distribution. Lewis, Oppenheimer and Wouthuysen⁴⁾ treated this problem in detail by means of the Bloch-Nordsieck's transformation from the view point that the multiple production is caused by the spin inertia due to the meson cloud. We shall investigate this problem in connection with above various theories of the multiple production of mesons and research the formula of this problem which is expressed in terms of the spectrum of the meson cloud introduced in the preceding letter. Further, we shall discuss the relation between the energy spectrum of the phenomenologically given ϵ -spectrum of the meson cloud and the multiplicity of mesons.

We denote the transition matrix element for the production of n_1, n_2, \dots, n_m ($\sum n_i = n$) mesons with momentum $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_m$ respectively by $S(n_1, \dots, n_m; \mathbf{k}_1, \dots, \mathbf{k}_m)$ and introduce the following $S^n(x/\infty)$:

$$S^n(x/\infty) = \int S(n_1, \dots, n_m; \mathbf{k}_1, \dots, \mathbf{k}_m) \times \exp \{i \sum n_i (\mathbf{k}_i \mathbf{X} - K_i t)\} \times \prod_{i=1}^m (\sqrt{2K_i} d\mathbf{k}_i)^{n_i}. \quad (1)$$

Through some calculations, we rewrite the above $S^n(x/\infty)$ in terms of the "spectrum" as follows:

$$S^n(x/\infty) = \left\langle \sum_{m=0}^n (-i)^{n+1} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_m \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx_{m+1} \cdots \int_{-\infty}^{\infty} dx_n {}^+_0 u(x, \infty; x_1), \dots, {}^+_0 u(x, \sigma_{m-1}; x_m) V(\sigma_m; y) {}^+_0 u(x, \sigma_y; x_{m+1}), \dots, {}^+_0 u(x, \sigma_{n-1}; x_n) \right\rangle_{m=0}, \quad (2)$$

where

$$V(\sigma, y) = \{1 + \epsilon(\sigma, y)/2\} S^{-1}(\sigma_y) \times \bar{\psi}_{in}(y) V(y) \psi_{in}(y) S(\sigma_y). \quad (3)$$

represents the external interaction with the nucleon accompanying the meson cloud. In (2), for simplicity, we restrict ourselves to the approximation of the first order with respect to the external potential $V(y)$. $\langle \rangle_{m=0}$ means the expectation value for the state in which there are no mesons.

We can rewrite (2) as follows:

$$S^n(x/\infty) = \left\langle \sum_{m=0}^n (-i)^{n+1} [({}^+_0 U)^m \times V({}^+_0 U)^{n-m}] (x, \infty) \right\rangle_{m=0}, \quad (4)$$

where we have used the notation

$$[A^1, A_2, \dots, A_n] \equiv [A_1 [A_2 [\cdots [A_{n-1}, A_n]] \cdots]. \quad (5)$$

(4) shows that the multiple production of mesons is caused by the iteration of the ϵ -spectrum, in which the vibration of the ϵ -spin of the nucleon is forced by the past effects. Thus, it is to be noticed that (5) is the quantum theoretical relation which corresponds to the classical relation discussed by Heisenberg.²⁾ ${}^+_0 U$ and ${}^+_0 u$ are found to be the non-linear operators with respect to the meson field operator U_{in} .

Further, we can write (4) in an alternative form:

$$S(n_1, \dots, n_m; \mathbf{k}_1, \dots, \mathbf{k}_m) = \sum_{a=0}^n \left\{ \prod_{i=1}^m \frac{1}{n_i! (2K_i)^{n_i}} \right\} \sum_{\text{Perin}}^{(1, \dots, n)}$$

$$\langle \mathbf{O}(k_1), \dots, \mathbf{O}(k_n) v(l) \mathbf{O}(k_{n+1}) \dots \mathbf{O}(k_n) \rangle_{m=0} \delta(\sum b_i + l_0) dl_0 / b_1(b_1 + b_2) \dots (b_1 + \dots + b_n + l_0) \dots (b_1 + \dots + b_{n-1} + l_0), \quad (6)$$

where $\sum_{\text{Perm}}^{(1, \dots, n)}$ means that the summation over all permutations of all possible group (k_1, \dots, k_n) which are arbitrarily selected from the group (k_1, \dots, k_n) and $b_i \equiv k_{ni} - K_i$. $v(l)$ is the Fourier component of the external interaction $\bar{\psi}(x)V(x)\psi(x)$ and so l_0 is the energy transfer of the nucleon accompanying the meson cloud during the scattering process by the external potential. $\mathbf{O}(k)$ is defined by $\bar{O}(k)$ (scalar coupling) or $iK_\mu \bar{O}_\mu(k)$ (vector coupling) respectively. $\mathbf{O}(k)$ and $\bar{O}_\mu(k)$ are the Fourier component of $\bar{O}(x)$ and $\bar{O}_\mu(x)$. (6) gives the intuitive picture of the multiple production of mesons: Meson-production is caused by the oscillation of the e-spin of the nucleon accompanying the meson cloud in the external potential.

When the following two conditions are approximately satisfied: (i) $\mathbf{O}(k_i)$ ($i=1, \dots, n$) and $v(l)$ are commutable each other, (ii) there are no interferences between operators $\mathbf{O}(k_i)$ ($i=1, \dots, n$) and $v(l)$, we obtain the Poisson distribution for the mesons produced:

$$(\text{probability}) \quad w = \left\langle \frac{1}{n!} \left\{ \int N(\mathbf{k}) d(\mathbf{k}) \right\}^n \times |V(l)|^2 \right\rangle, \quad (7)$$

where the $3n$ -fold integration in momentum space is to be performed in such a way as to satisfy the energy-momentum conservation law. In the case of the Bloch-Nordsieck and Lewis, Oppenheimer and Wouthuysens' approximation,^(3,4) the conditions (i) and (ii) are satisfied.

When we assume the energy-dependency in the high energy region of the phenomenologically given c-spectra as follows:

$$+U(k) \propto (f(a))^{1/2} K^q / \sqrt{K}, \quad f(a) \propto (\epsilon_{max})^p, \quad (10)$$

where $f(a)$ depends only on the property of the nucleon and not on the energy-momentum of mesons. ϵ_{max} is the maximum energy which is transferred to the whole mesons. Then we obtain the most probable number n of the produced mesons:

$$\bar{n} \propto \begin{cases} (\epsilon_{max})^{2q+3+p/2q+4}, & \text{for } q > -\frac{3}{2}, \\ (\epsilon_{max})^p \log \epsilon_{max}, & \text{for } q = -\frac{3}{2}. \end{cases}$$

$(p=0, q=-\frac{3}{2})$ gives $\bar{n} \propto \log(\epsilon_{max})$ which is obtained, for example, in the case of scalar meson with the scalar coupling in the approximation of Lewis et al.⁽⁴⁾ $(p=0, q=-\frac{1}{2})$ gives $\bar{n} \propto (\epsilon_{max})^{2/3}$ which is obtained, for example, in the case of the pseudo-scalar meson with the pseudovector coupling in their approximation. The spectrum, which is obtained from the leading term of the Pauli-Dancoff's strong coupling theory⁽⁵⁾ for the pseudo-scalar meson with the pseudo-vector coupling also gives $\bar{n} \propto (\epsilon_{max})^{2/3}$. ($p=-1, q=0$)⁽⁶⁾ gives $\bar{n} \propto (\epsilon_{max})^{1/2}$. When $N(\mathbf{k})d(\mathbf{k}) \propto d\mathbf{k}/K$ (the Lorentz invariant volume), we find $\bar{n} \propto (\epsilon_{max})^{2/3}$. Detailed account will appear elsewhere.

The author wish to express their cordial thanks to Prof. S. Tomonaga and Prof. S. Sakata for their constructive criticisms through this work.

- 1) In this letter we use the same notations as in the preceding letter.
- 2) W. Heisenberg, ZS. f. Phys. **113** (1939), 61.
- 3) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54.
- 4) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthuysen, Phys. Rev. **73** (1948), 127.
- 5) W. Pauli and S. M. Dancoff, Phys. Rev. **62** (1942), 85.
- 6) E. Fermi, Prog. Theor. Phys. **5** (1950), 570.

On the Intervals of Na 2D and Cu 2F

G. Araki and S. Tutihasi

Faculty of Engineering, Kyoto University

April 23, 1951

It was tried to account for the inversion of the $4f$ 2F levels of the copper atom by the over-screening due to the spin-orbit interaction between a valence and core electrons.¹⁾ Ta-You Wu²⁾ pointed out that the values of the parameters used in the above mentioned calculation are incorrect. In compliance of his criticism one of the present writers recalculated the values of the parameters assuming the hydrogen-like radial functions with certain screening constants.³⁾ The result seemed to require no change in the previous conclusion, but the definite conclusion could not be obtained because the radial functions used in the computation were too rough.

In order to get a more definite conclusion one (S.T.) of the present writers was engaged in the numerical computation of the parameters employing the analytical expressions of the self-consistent radial functions which were given by Fock and Petrashen⁴⁾ for Na and by Slater⁵⁾ for Cu. The result for Na $3d$ 2D and Cu $4f$ 2F will be reported in the following,⁶⁾ in which the notations will be the same with those in the previous papers.¹⁾³⁾ A preliminary short account⁷⁾ was already given in the form of a note added to the second criticism of Ta-You Wu.⁸⁾

We first report the result for Na $1s^2 2s^2 2p^6 3d$ 2D . Fock and Petrashen⁴⁾ gave only the core functions. We have accordingly to assume an effective charge, $Z-s$, for the $3d$ radial function. The calculated values of various parameters on the assumption $Z-s=1$ are given in Table 1. The value of σ can not exceed the atomic number, 11, of Na and we have the normal doublet which is not in agreement with the experimental fact.⁷⁾

Next we consider Cu $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4f$ 2F . The result for three assumed

values of the effective charge for the $4f$ function is shown in Table 2. The inversion still can not occur. Since σ becomes large when the effective charge becomes smaller we may expect, by the extrapolation, the inversion of the doublet probably for a value smaller than 3 of the effective charge. However, in this case, the absolute value of the doublet interval is expected to be too small in comparison with the observed value⁸⁾ as was pointed out by Ta-You Wu,²⁾ and the theory may still not agree with experiment.

The self-consistent functions which are employed in the above mentioned computation were calculated for the singly ionized atoms of Na and Cu. Strictly speaking, we should employ the functions of the neutral atoms. The correction for this point may give certain change to the values of the parameters.

Table 1 Na ($Z=11$, $Z-s=1$)

$\zeta_{10, 32}=0.99999$	$\tilde{\zeta}_{10, 32}=0.00002$
$\zeta_{20, 32}=0.99756$	$\tilde{\zeta}_{20, 32}=0.00320$
$\zeta_{21, 32}=0.99699$	$\tilde{\zeta}_{21, 32}=0.00354$
$\sigma=10.06008$	$\tilde{r}^{-2}=0.002462$
atomic units	
$^2D_{3/2}-^2D_{5/2}=0.0337 \text{ cm}^{-1} \text{ (calc),}$	
$-0.0494 \text{ cm}^{-1} \text{ (obs)}^7)$	

- 1) G. Araki, Proc. Phys. Math. Soc. Japan **21** (1939), 508.
- 2) Ta You Wu, J. Phys. Soc. Japan **4** (1949), 343.
- 3) G. Araki, J. Phys. Soc. Japan **4** (1949), 345.
- 4) V. Fock and M. J. Petrashen, Sowj. Phys. **6** (1934), 368.
- 5) J. C. Slater, Phys. Rev. **42** (1932), 33.
- 6) Ta-You Wu, J. Phys. Soc. Japan **5** (1950), 421.
- 7) E. R. Thackeray, Phys. Rev. **75** (1949), 1840.
- 8) R. F. Bacher and S. Goudsmit, *Atomic Energy States* (1932).

* It would have been better if we had published this report in the Journal of the Physical Society of Japan, but too many manuscripts are now submitted to the journal. This is the reason why we report the result here.

Table 2 Cu ($Z=29$)

	$Z-s=7$		$Z-s=6$		$Z-s=5$	
nl	$\zeta_{nl,43}$	$\tilde{\zeta}_{nl,43}$	$\zeta_{nl,43}$	$\tilde{\zeta}_{nl,43}$	$\zeta_{nl,43}$	$\tilde{\zeta}_{nl,43}$
10	1.00000	0.00000	1.0000	0.0000	1.00000	0.00000
20	0.99909	0.00111	0.9996	0.00059	0.99982	0.00024
21	0.99933	0.00081	0.9996	0.00042	0.99987	0.00017
30	0.92212	0.04370	0.9505	0.03155	0.97528	0.01925
31	0.90750	0.04629	0.9404	0.03423	0.96835	0.02216
32	0.76393	0.05185	0.8216	0.04762	0.87886	0.04344
σ	27.59486		28.0022		28.3702	
r^{-3}	0.12758 atomic units		0.080357 atomic units		0.04650 atomic units	
${}^2F_{5/2}-{}^2F_{7/2}$	3.65 cm^{-1}		1.63 cm^{-1}		0.597 cm^{-1}	

$$({}^2F_{5/2}-{}^2F_{7/2})_{obs} = -3.6 \text{ cm}^{-1}$$

Elektronoj ekvilibras kun mezonoj en malalta atmosfero

Y. Takahashi kaj T. Miyazima

*La Instituto de Fiziko,
Tokyo Bunrika Daigaku*

May 1, 1951

Kiam μ mezono rompiĝas al unu elektrono kaj unu neŭtrino, pri elektronoj ekvilibras kun mezonoj en malalta atmosfero estis kalkulanta de multaj homoj. Sed nuntempe μ mezono estas certiganta rompiĝi al unu elektrono kaj du neŭtrinoj de multaj eksperimentoj. Konsekence la spektro de elektrono estas daŭra.

Ni kalkulis nombron de la elektronoj ekvilibras kun mezonoj en malalta atmosfero en la okazo de la daŭra spektro de elektro:

La daŭra spektro en senmova sistemo de mezono

$$I_0(E)dE \sim E^2 [3\mu^2 - 6\mu E + 2E^2] dE,$$

en laboratoria sistemo

$$I(E)dE \sim \frac{\sqrt{1-\beta^2}}{1-\beta \cos \theta} I_0 \left(\frac{1-\beta \cos \theta}{\sqrt{1-\beta^2}} E \right) dE.$$

Kalkulada maniero utilizis koncepton de la elektronona-trairitra longeco (electron-track length) kiel okazo de Rossi-Greisen, Rossi-Klapman kaj Bernardini.

Nombro de elektronoj ekvilibras estas

$$n_R = \int_E^{E_{\max}} \chi(E_K) Z(E_K, E) dE_K,$$

Tiam $\chi(E_K)$ estas probableco (probability) ke forĵentinta elektrono havas energion E_K sur ciu unua longeco. $Z(E_K, E)$ estas trairita longeco de elektrono kiu havas energion pli alta ol E kaj komencan energion E_K .

Ni uzis spektro de Ferretti¹⁾ kiel $Z(E_K, E)$. Tio estas

$$Z(E_K, E) = \frac{E_K}{\beta} \left\{ 1 - \frac{E}{E_K} \exp \left[-\frac{2(E_K - E)}{E_c} \right] - \frac{2E}{E_c} \exp \left[\frac{2E}{E_c} \right] \left[E_1 \left(-\frac{2E_K}{E_c} \right) - E_1 \left(-\frac{2E}{E_c} \right) \right] \right\}.$$

Tiu ci estas tre simila al spektro de Tamm-Belenky. Ni prenis 10 MeV kiel E . Tio respondas al okazo de utilizo de numerilo (counter) de 5 g/cm²-laŭtro.

Kiam ni komparas kun rezultato de Ferretti kaj Rossi-Klapman, ni akiras du rezultatoj tre simila kiel ĉisubaj skriboj

$$\begin{aligned} n_R &= \frac{\mu}{\tau_{0\theta}} \times 0.34302 \times \frac{30.9}{10^7}, \quad (\text{de Ferretti}) \\ &= \frac{\mu}{\tau_{0\theta}} \times 0.34202 \times \frac{30.7}{10^7}. \end{aligned}$$

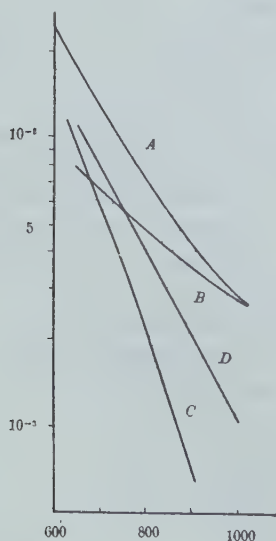
(de Rossi-Klapman)

Tiam E estas 10 MeV, μ kaj E_c estas 100 MeV kiel Rossi-Klapman. Sed en la detela kalkulado, ni prenis 10 MeV, 110 MeV kaj 86 MeV kiel E , μ , kaj E_c . Kiel vivtempo de μ mezono prenis 2.10^{-6} .

Rezultato estas montranta en Fig. 1. Tiam ni prenis 7% de mezono kiel bata elektrono (knock-on electron). Je mara-alta nivero, kalkula rezultato estas 15% pli granda al eksperimenta varolo. Ĉi tiu kauzo unue estas ekscesa takso energio transiĝa al elektrono el μ mezono. Due la spektro de Ferretti persone havas eronon. Sed ni povas dire nenion pri tiu ĉi punkto.

Ĉar ni ne scias kiel la spektro de Ferretti estis ludanta, ni normigis kalkulan rezultaton al eksperimenta varolo de mara-alta nivero.

Kurbo A estas eksperimenta varolo de la elektrono kaj kurbo B nia varolo kalkula. Kurbo C montras varon $A-B$. Kurbo D estas varolo de elektrono de neŭtrecaj mezonoj kiu estis kalkulanta de Hayakawa²⁾. Sed diferenco de kurbo C kaj kurbo D ne havas absolutan sencon, ĉar nia kalkulado estas normiganta al eksperimenta varolo de mara-alta nivero.



- 1) G. Bernardini, B. N. Cacciapuoti and R. Querzoli, Phys. Rev. **73** (1948), 335.
- 2) S. Hayakawa, Prog. Theor. Phys. **5** (1950), 158.

Note on Belinfante's New Theory

K. Husimi and R. Utiyama

Department of Physics, Osaka University

May 9, 1951

Recently Belinfante¹⁾ proposed a new idea in this journal in order to construct a divergence-free field theory. It is based on the following two principles: i) to obtain the S -matrix, we should only calculate matrix elements corresponding to the so-called irreducible diagrams; ii) radiative corrections are to be taken into account by modifying some factors in the integrands of the S -matrix. But his second rule is stated in quite a vague way. It may perhaps mean that one should replace S_F , D_F and γ_μ by S_F' , D_F' and Γ_μ respectively, as Dyson had proposed in his paper.²⁾ Then the contents of Belinfante's theory will be identical with those of Tomonaga-Schwinger theory so far as the $S(\infty)$ -matrix is concerned.

Now Belinfante insists on the validity of the inequality (4), i.e., $U(t_2 t_0) \neq U(t_2 t_1) \times U(t_1 t_0)$, rather than the equality, when one considers a transformation matrix $U(t_2 t_0)$ for a finite time-interval $(t_2 - t_0)$.

If we make a matrix product of b and c (cf. Fig. 1), then it contains contributions

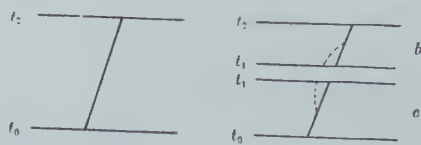


Fig. 1.

from those unallowable processes which should be abandoned in the case a . However, by virtue of the very character of the self-con-

sistency of the integral equation which defines S_F' , we may assert the equation $U(t_0 t_0) = U(t_0 t_1)U(t_1 t_0)$, if we assume the second rule ii) to be the same with that given by Dyson.³⁾ On the other hand, if our assumption concerning to this point is not meant by Belinfante, then the inequality (4) might in general hold, and prohibit us to give a uniquely determined meaning to the operator $U(t_0 t_1)$. But in this case it seems impossible to construct a differential equation of the type (6) as will be stated later.

In the ideal experiment stated in § 2, Belinfante seems to miss his object. The content of his experiment can be expressed symbolically as follows;

$$U(t_0 t_0) \neq U(t_0 t_1) \circ U(t_1 t_0),$$

where \circ is a symbol (not an operator) representing an accidental change of the state vector caused by an observation;

$$\Psi_{i_0}(t_1) = U(t_1 t_0) \Psi_{i_0}(t_0) \rightarrow \Psi_{i_1}(t_1) \quad (\text{observation}).$$

This is an essential character of the ordinary quantum mechanics, and is independent on the rule i).

Finally let us consider the eq. (6).

He asserts the existence of a mystic operator $B(tt_0)$ which possesses memories about the state vector in the past, and he attributes this character of B to the inequality (4). The deduction of his eq. (6) is

$$\begin{aligned} i \frac{\partial \Psi_{i_0}(t)}{\partial t} &= i \frac{\partial U(tt_0)}{\partial t} \cdot U^{-1}(tt_0) U(tt_0) \Psi_{i_0}(t_0) \\ &= B(tt_0) \Psi_{i_0}(t), \end{aligned}$$

where

$$\begin{aligned} \Psi_{i_0}(t) &= U(tt_0) \Psi_{i_0}(t_0), \\ B(tt_0) &= i \frac{\partial U(tt_0)}{\partial t} U^{-1}(tt_0). \end{aligned} \quad (6)$$

Here let us interpret all the product expressions in the above equations as ordinary matrix products. Then from the fact that U is unitary and $U^{-1}(tt_0)$ is the left inverse

of $U(tt_0)$, $U^{-1}U$ and UU^{-1} are both unit operators and do not depend on the time t_0 . Then the operator B can be written as follows,

$$B(tt_0) = \lim i \{ U(t+dt, t_0) U^{-1}(tt_0) - 1 \} / dt.$$

From the rules, the operator $U(t+dt, t_0) \times U^{-1}(t, t_0)$ will in general depend on t_0 , and is not equal to $U(t+dt, t)$, if the inequality (4) holds. The eq. (6) gives us a rule of the change of the state vector from t to $t+dt$, i.e., (cf. Fig. 2)

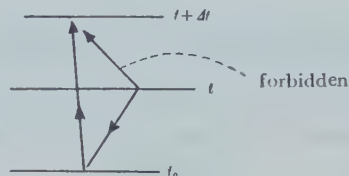


Fig. 2.

$$\begin{aligned} \Psi_{i_0}(t+dt) &= U(t+dt, t_0) U^{-1}(tt_0) \Psi_{i_0}(t) \\ &(\neq U(t+dt, t) \Psi_{i_0}(t)). \end{aligned}$$

From this point of view the operator B can only be determined by the knowledge of the operator $U(tt_0)$ and the latter is obtained from the usual Feynman-Dyson theory by making use of the two rules. Then his theory has no point of excellence compared with the F - D theory in case of the practical application. Whereas from the logical stand point, the operator B should be given a priori. But it seems very difficult to give the operator B in such a way as to make the operator U obtained by the integration of the eq. (6) satisfy the two rules.

On the other hand if we interpret the product expressions of the eq. (6) in the meaning stated about the ideal experiment, then it is very difficult and perhaps impossible to define $U^{-1}(tt_0)$. And the operator B depends on the accident at the time t_0 , though it may be possible to define U^{-1} . This situation makes it possible to attribute any mathematical expression to the state vector.

Though his ambitious trial exploring into

the foundation of the field theory is very valuable, it seems necessary further to investigate the character of $U(t t_0)$ in the ordinary Feynman-Dyson theory before any modification of F - D theory will be tried.

- 1) F. J. Belinfante, *Prog. Theor. Phys.* **6** (1951), 202.
- 2) F. J. Dyson, *Phys. Rev.* **75** (1949), 1676.
- 3) $S_{F'}$, $D_{F'}$, Γ_μ here introduced are in general dependent on the time t_2, t_1, t_0 .

On the Self-Energies of Nucleons

H. Enatsu

Department of Physics, Kyoto University

May 21, 1951

In the previous letter,¹⁾ we have investigated the self-energies of nucleons due to electromagnetic and cohesive mesonic fields, and shown that the expression obtained leads to the mass difference of proton and neutron which has the right sign and order of magnitude for the definite C meson mass.

It is the aim of this note to envisage whether the method of eliminating infinities in the self-energies by means of compensation²⁾ is effective to ones due to π mesons.

For the π mesons it would be not unreasonable to choose the symmetrical pseudoscalar type with pseudovector coupling which seems to be most promising from some evidences.³⁾ Our computations are carried out for all types of mesons, and in order to avoid the complicity, we confine ourselves to neutral fields, because in a symmetrical theory the first order approximation to self-energy is twice that of neutral theory. It is well known that there exist ambiguities of evaluations in the Feynman-Dyson formalism.⁴⁾ We adopted here the following way in connection with the cases of electromagnetic and cohesive mesonic ones. Namely, according to Karplus and Kroll,⁵⁾ the surface integral,

$$\int f(k_\nu) d^4k = \int \left(1 - u p_\nu \frac{\partial}{\partial k_\nu} + \frac{1}{2!} u^2 p_\nu p_\nu \frac{\partial^2}{\partial k_\nu \partial k_\nu} - \dots \right) f(k_\nu + u p_\nu) d^4k' \quad (1)$$

has been taken into account. In the course of this treatment Koba's P^* symbol⁶⁾ has been used by means of which the self-energy can be easily computed for any interaction with the derivative of the meson field wave function.

The expression for the self-energy of nucleon due to π meson is found to be:

$$\begin{aligned} \Delta W_1 = & m c^2 \left(\frac{F^2}{4\pi\hbar c} \right) \left(\frac{1}{\lambda^2} \right) \left[-\frac{3}{8\pi} \left(\frac{K K_0}{M^2} \right) \right. \\ & \left. - \log \frac{K + K_0}{M} \right] + \frac{1}{120\pi} (75\lambda^2 + 20) \\ & \times \left(\log \frac{K + K_0}{M} \right) + \text{finite term} \quad (2) \end{aligned}$$

where m : Proton Mass,
 κ : Meson Mass,
 F : Coupling constant of meson,

$$\lambda = \frac{\kappa}{m}, \quad M = \frac{mc}{\hbar}, \quad K_0 = \sqrt{M^2 + K^2}, \quad K \rightarrow \infty.$$

As is expected, the pseudovector coupling gives rise to quadratic and logarithmic divergences. After some calculations, it was found that the only meson field with single coupling which fulfils the conditions for the compensation is of the scalar type with vector coupling. It is given as follows:

$$\begin{aligned} \Delta W_2 = & m c^2 \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{1}{\lambda'^2} \right) \left[\frac{1}{8\pi} \left(\frac{K K_0}{M^2} \right) \right. \\ & \left. - \log \frac{K + K_0}{M} \right] - \frac{1}{120\pi} (5\lambda'^2 + 4) \left(\log \frac{K + K_0}{M} \right) \\ & + \text{finite term} \quad (3) \end{aligned}$$

where μ and f are the meson mass and coupling constant respectively, and $\lambda' = \frac{\mu}{m}$.

Thus the nucleons have, in first appro-

ximation, a finite self-energy provided that

$$\left. \begin{aligned} f^2 &= 3F^2, \\ \text{and } 15\mu^2 &= 8m^2 + 75\kappa^2, \end{aligned} \right\} \quad (4)$$

which denote two relations between the coupling constants and between masses of nucleon and mesons. It must be remarked that, contrary to the electromagnetic case, as we assume the symmetrical theory for π mesons, the cohesive fields for them should be considered as the symmetrical one.

Noting that

$$m = 1837 m_e, \quad \kappa = 276 m_e, \quad (5)$$

we obtain for the mass of cohesive meson,

$$\mu = 1474 m_e. \quad (6)$$

From the realistic point of view, one may inquire into the possibility of explanation of the existence of heavy mesons (τ mesons),⁷⁾ at the present stage, however, one will see that, owing to the lack of knowledge about heavy mesons, one may need further investigations. However it may be, we should like to point out that we obtain finite results for the self-energy of nucleons due to π mesons by introducing "heavy mesons."

Full accounts will soon appear in this journal.

The author is grateful to Dr. A. Pais for his valuable communication with respect to the self-energy problem.

- 1) H. Enatsu and P. Y. Pac, *Prog. Theor. Phys.* **6** (1951), 261.
- 2) A. Pais, *Verh. Kon. Ak. Wetenschappen, Amsterdam* **19** (1947), 1.
- 3) K. Aidzu, Y. Fujimoto, H. Fukuda, S. Hayakawa, K. Takayanagi, G. Takeda and Y. Yamaguchi, *Prog. Theor. Phys.* **6** (1950), 931.
- 4) D. C. Peaslee, *Phys. Rev.* **81** (1951), 94.
- 5) R. Karplus and N. M. Kroll, *Phys. Rev.* **77** (1950), 536.
- 6) Z. Koba, *Prog. Theor. Phys.* **5** (1950), 696.
- 7) R. Brown, U. Camerini, P. H. Fowler, H. Muirhead, C. F. Powell and D. N. Riston, *Nature* **163** (1949), 48, 82.

L. Leprince-Ringuet, Hoang Tchang-Fong, L. Janean and D. Morellet, *C. R.* **226** (1948), 1897.

On the Mass of Cohesive Meson and the Mass Difference of Nucleons, II

H. Enatsu and P. Y. Pac

Department of Physics, Kyoto University

May 26, 1951

In the previous note,¹⁾ we have shown that the right sign and order of magnitude of the mass difference of proton and neutron would be given provided that we took the values

$$\left. \begin{aligned} \mu &= 110 m_e, \\ \text{and } f^2 &= 2e^2, \end{aligned} \right\} \quad (1)$$

for the mass of cohesive meson and its coupling constant respectively. However, as was pointed out by Wightman,²⁾ it is the serious difficulty that the range of cohesive force which corresponds to the mass value of $110 m_e$ is too large to be in accordance with experimental informations of the so-called mirror nuclei.

In this note we try to investigate another possibility for the derivation of the mass difference of nucleons. As Pais has described,³⁾ one can get compensation by assuming the cohesive field to be of the pseudovector type with pseudovector coupling.

The method of calculation is the same as that of I. The expressions for the mass corrections due to electromagnetic and cohesive mesonic self-energies are as follows:

$$\delta m = \delta m_1 + \delta m_2 + \delta m_3, \quad (2)$$

$$\delta m_1 = m \left[\frac{3}{2\pi} \left(\frac{e^2}{4\pi\hbar c} \right) \left(\log \frac{K+K_0}{M} \right) - \frac{5}{2\pi} \left(\frac{g^2}{4\pi\hbar c} \right) \left(\log \frac{K+K_0}{M} \right) \right], \quad (3)$$

$$\delta m_2 = -m \left[\left(\frac{1}{4\pi} \right) \left(\frac{e^2}{4\pi\hbar c} \right) + \left(\frac{1}{4\pi} \right) \left(\frac{g^2}{4\pi\hbar c} \right) \right]$$

$$\times \left\{ 1 - \lambda^2 + (\lambda^4 - 8\lambda^2) \log \lambda + (-\lambda^5 + 10\lambda^3 - 24\lambda) \frac{1}{\sqrt{4 - \lambda^2}} \tan^{-1} \left(\frac{\sqrt{4 - \lambda^2}}{\lambda} \right) \right\} \quad (4)$$

$$\delta m_3 = -m \left(\frac{1}{8\pi} \right) \left[\left(\frac{e^2}{4\pi\hbar c} \right) + \left(\frac{g^2}{4\pi\hbar c} \right) \right] \quad (5)$$

where $\lambda = \frac{\alpha}{m}, \quad M = \frac{mc}{\hbar},$

m : proton mass,
 α : C -meson mass.

The conditions for the elimination of divergences and the derivation of the right mass difference of nucleons are

$$\left. \begin{aligned} g^2 &= \frac{3}{5} e^2, \\ \alpha &= 13 m_e. \end{aligned} \right\} \quad (6)$$

and

Therefore, the range of it is larger than that of the meson of scalar type with scalar coupling. Even if we neglect the effect of the surface integral, i.e. δm_3 , it may be shown that the most near-by value for mass difference, is only

$$\delta m = -1.70 m_e \quad (7)$$

for the zero mass of cohesive meson, which is unsatisfactory in itself.

Then, one may state that the present form of cohesive meson theory is not suitable for the description of electromagnetic phenomena for heavy nuclei.

It is pleasure to acknowledge the encouragement of Dr. A. Pais who suggested this problem.

1) H. Enatsu and P. Y. Pac, *Prog. Theor. Phys.* **6** (1951), 261.

This letter will be referred to as I.

2) A. S. Wightman, *Phys. Rev.* **71** (1947), 447.

3) A. Pais, *Verh. Kon. Ac. Wetenschappen, Amsterdam* **19** (1947), 1.

Note on the "Champ Soustractif" of Louis de Broglie

S. Endo and H. Kanazawa

Faculty of General Culture, Tokyo University

May 30, 1951

Recently, de Broglie¹⁾ has published a tentative theory of mixing fields. He claims that there occur no divergence difficulties in quantum electrodynamics according to his theory. But his theory, as it is, has some ambiguities and inconsistencies; for example, we cannot understand how to derive the Hamiltonian employed by him from his original idea. So we shall reformulate his theory according to his idea.

We write the equations of two vector fields:

$$\square \varphi_\mu^{(1)}(X) = -\frac{1}{c} j_\mu(X), \quad \frac{\partial \varphi_\mu^{(1)}(X)}{\partial X_\mu} = 0, \quad (1)$$

$$\begin{aligned} \square \varphi_\mu^{(2)}(X) - \lambda^2 \varphi_\mu^{(2)}(X) &= \frac{1}{c} j_\mu(X), \\ \frac{\partial \varphi_\mu^{(2)}(X)}{\partial X_\mu} &= 0, \end{aligned} \quad (2)$$

where j_μ is the charge-current density of the electron. Summing (1) and (2), we have an equation of the form

$$\begin{aligned} \square A_\mu(X) &= \square(\varphi_\mu^{(1)}(X) + \varphi_\mu^{(2)}(X)) \\ &= \lambda^2 \varphi_\mu^{(2)}(X). \end{aligned} \quad (3)$$

On the other hand, we get from (2)

$$\varphi_\mu^{(2)}(X) = -\frac{1}{c} \int \bar{\Delta}_\lambda(X - X') j_\mu(X') dX' \quad (4)$$

using the property of the function $\bar{\Delta}_\lambda(X)$;

$$(\square - \lambda^2) \bar{\Delta}_\lambda(X) = -\delta(X). \quad (5)$$

Substituting (4) into (3), we obtain an equation of the electromagnetic field:

$$\square A_\mu(X) = -\frac{1}{c} j_\mu^{(a)}(X), \quad \frac{\partial A_\mu(X)}{\partial X_\mu} = 0. \quad (6)$$

$$j_{\mu}^{(d)}(X) = \lambda^2 \int \Delta_{\lambda}(X-X') j_{\mu}(X') dX' \quad (7)$$

where $j_{\mu}^{(d)}$ satisfies the conservation law $\frac{\partial j_{\nu}^{(d)}}{\partial X_{\nu}} = 0$. The equation (6) is equivalent to the following equation;

$$(\square - \lambda^2) \square A_{\mu}(X) = \frac{\lambda^2}{e} j_{\mu}(X). \quad (8)$$

But if one quantizes the fields starting from (8) as the field equation, one meets with the well known difficulty of negative energy. We take (6) as the fundamental field equation.

Using the action principle

$$\delta \int L dX = \delta \left[\int (L_{\text{electron}} + L_{\text{photon}}) dX \right. \\ \left. + \frac{\lambda^2}{e} \int \int A_{\mu}(X) \Delta_{\lambda}(X-X') j_{\mu}(X') dX dX' \right] \\ = 0, \quad (9)$$

we get (6) and the following equation;

$$\left(\gamma_{\mu} \frac{\partial}{\partial X_{\mu}} + \kappa_0 \right) \psi(X) = \frac{ie}{\hbar c} A_{\mu}^{(d)}(X) \gamma_{\mu} \psi(X), \quad (10)$$

$$A_{\mu}^{(d)}(X) = \lambda^2 \int \Delta_{\lambda}(X-X') A_{\mu}(X') dX'. \quad (11)$$

By means of (10) we can calculate the self energy of the electron according to Feynman's method. The static potential of electron is given by $\frac{e}{r}(1-e^{-\lambda r})$, so the amplitude of virtual photon gets the form of

$$\frac{-1}{k_{\mu} k_{\mu}} - \frac{-1}{k_{\mu} k_{\mu} + \lambda^2} = \frac{-1}{k_{\mu} k_{\mu}} \frac{\lambda^2}{k_{\mu} k_{\mu} + \lambda^2}.$$

The convergence factor $\frac{\lambda^2}{k_{\mu} k_{\mu} + \lambda^2}$ is just the same as Feynman's one. The introduction of this factor can be understood also from the following formula;

$$\lambda^2 \int e^{i\kappa X'} \Delta_{\lambda}(X-X') dX' = \frac{\lambda^2 e^{i\kappa X}}{k_{\mu} k_{\mu} + \lambda^2}. \quad (12)$$

Thus the self energy is given by

$$\Delta m = m \frac{e^2}{\hbar c} \frac{1}{2\pi} \left(3 \ln \frac{\lambda}{m} + \frac{4}{3} \right). \quad (13)$$

But in the problem of vacuum polarization the divergence cannot be perfectly removed out.

- 1) L. de Broglie, Jour. de Phys. et Rad. 11 (1950), 431.

On the Photo-Meson Production from Heavy Nuclei

S. Machida and T. Tamura

Department of Physics, University of Tokyo

May 31, 1951

According to a recent experiment,¹⁾ it seems that the positive photo-meson production cross sections from various heavy nuclei obey the $\sigma \propto Z^{2/3}$ law. Former treatments of Fujimoto et al²⁾ and Lax and Feshbach,³⁾ which took into account the effect of nuclear binding to meson-producing nucleons, could show the decrease of efficiency of photo-meson production from heavy nuclei compared to that from free protons, yet the decrease did not depend on Z , so it seems necessary to consider the effect of interaction between the mesons and the nuclei producing them besides the one mentioned above. Since none of the known photonuclear reactions have cross sections proportional to nuclear area at these energies, the nuclear opacity to photons would not be responsible for the point under consideration.

A little ago, Nambu⁴⁾ has shown that, if we use the Fermi gas model for nuclei, mesons which entered into nuclei be observed as though their self-energies have increased from the one when they were free. That is, mesons may be considered to suffer an effect of, it might be called, dimesonic constant when they are in nuclear matter, which is the

analogue to dielectric constant to photons. Phenomenologically this effect may be replaced by a rather strong repulsive potential to mesons in nuclei.

In somewhat excited nuclei, nucleons will show a spatial distribution shown in Fig. 1 by dotted line approximately, so it may be justified to assume a repulsive potential as shown in the same figure by full line. In the following, for reason of easiness of calculation, we divide the latter into two parts as in Fig. 2.

As the matrix element of photo-meson production is proportional to the amplitude of the wave function of the meson at the producing point, so the decrease of the cross section by this effect may be considered to

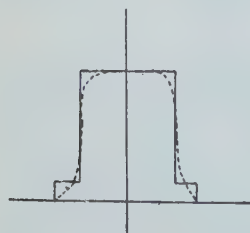


Fig. 1

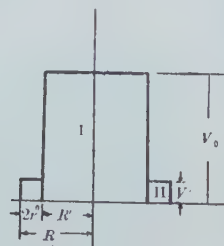


Fig. 2

be proportional to $(A'/A)^2$, where A and A' are the amplitudes of meson waves when they are free or suffered from repulsive potential respectively. If we consider the effects of Part I of Fig. 2, the condition that the meson wave function is smooth at R' gives

$$\left(\frac{A'}{A}\right)^2 = \frac{k^2}{k^2 \sinh^2(k'R') + k'^2 \cosh^2(k'R')} \quad (1)$$

where

$$-k'^2 = k^2 - (2\mu V_0/\hbar^2) < 0, \quad (2)$$

μ being the meson mass, and $R' \approx R - 2r_0$ (r_0 = radius of a nucleon; R = nuclear radius). If $k'R' \gg 1$, then

$$(A'/A)^2 \approx (4k^2/k^2 + k'^2) \exp(-2k'R') \quad (3)^*$$

$$= (4\hbar^2 k^2 / 2\mu V_0) \exp(-2k'R'). \quad (4)$$

As is seen from (4) $(A'/A)^2 \propto (1/V_0) \exp(-2k'R')$ and, from (2), k' increases with increasing V_0 , so $(A'/A)^2$ decreases very rapidly with increasing V_0 . A measure of the order of magnitude of V_0 is given as follows:

$$V_0 = \hbar^2(k^2 + k'^2)/2\mu \geq \hbar^2(k^2 + 1/R'^2)/2\mu$$

$$\begin{cases} \approx 76 + 20 \approx 100 \text{ MeV} \\ \quad \text{(for meson energy 76 MeV),} \\ \approx 42 + 20 \approx 65 \text{ MeV} \\ \quad \text{(for meson energy 42 MeV).} \end{cases} \quad (5)$$

As is seen from (5), the lower V_0 value is sufficient to decrease the ratio $(A'/A)^2$, the heavier the nuclei. This value might seem to be rather high, but according to the calculation of Nambu,⁴⁾ if we take the pseudoscalar meson theory, the increase of self-energy mentioned above comes to about 100 MeV. In addition, recent experiments of large angle scattering of mesons by nuclei⁵⁾ seem to support this large value.**

The contribution to the photo-meson cross section from Part II of Fig. 2 can easily be estimated as that from Part I, and is known to be approximately

$$\left(\frac{A''}{A}\right)^2 = \frac{k^2}{k^2 \sin^2(2k'r_0) + k'^2 \cos^2(2k'r_0)}, \quad (6)$$

where

$$k'^2 = k^2 - (2\mu V'/\hbar^2) > 0. \quad (7)$$

It is legitimate to consider $V' \ll k$, so $k \approx k'$, and

$$(A''/A)^2 \approx 1. \quad (8)$$

From (4) and (7) it is seen that by assuming repulsive potential, which is legitimate by several considerations stated above, the main contribution to photo-meson production comes from the surfaces of nuclei, and it is clear that it gives the desired $\sigma \propto Z^{2/3}$ law.

Recent experiments on γ -star performed by Kikuchi⁶⁾ and Miller⁷⁾ also seem to support our model. According to them, excitation function of stars produced by high energy γ -rays are very similar to that produced by mesons, and it could be interpreted by assuming that the mesons produced at inner parts of nuclei were absorbed by the latter. Meson tracks, which associated with stars, were about 5 percent of stars,⁶⁾ and about 1/4 of the low energy π^- -mesons found which were produced in the emulsion had as many as 3 other prongs associated with their production.⁷⁾ Considering the fact that about half of mesons produced at the surface can go out of nuclei, and the effect of nuclear binding make the efficiency decrease to about 1/3 in these energy region, above results are consistent with our model.

* Even when $k/k'=1$ the difference between (1) and (3) is only 10 percent of (1) and it decreases very rapidly with increasing k' .

** Nambu et al have shown that repulsive potential of about 15 MeV height is sufficient to account for these experiments (Y. Nambu, T. Nakano and Y. Yamaguchi, lecture at the semi-annual meeting of the Phys. Soc. Jap., May, 1951). But since the effect of dimesonic constant can be very different to real and virtual mesons respectively, it is not permissible to compare their results with ours directly.

- 1) R. F. Mozley, Phys. Rev. **80** (1956), 493.
- 2) Y. Fujimoto, K. Nishijima, T. Okabayashi, K. Takayanagi and Y. Yamaguchi, Prog. Theor. Phys. **5** (1950), 870.
- 3) M. Lax and H. Feshbach, Phys. Rev. **81** (1951), 189.
- 4) Y. Nambu, Researches on the Theory of Elementary Particles **2** (1950), No. 2, 170. (in Japanese)
- 5) H. Bradner and B. Rankin, Phys. Rev. **80** (1950), 916; G. Bernardini, E. T. Booth, L. Lederman and J. Tinlot, Phys. Rev. **80** (1950), 924; **82** (1951), 105.
- 6) S. Kikuchi, Phys. Rev. **80** (1950), 492; Bull. Am. Phys. Soc. **26** No. 3 (1951), A3.
- 7) R. D. Miller, Phys. Rev. **82** (1951), 260.

Remarks on the Neutron-Proton Scattering with Tensor Force

Y. Yamaguchi

Department of Physics, Osaka City University

June 3, 1951

While many authors have discussed the nucleon-nucleon scattering with central force under the so-called shape independent approximation,¹⁾ tensor force scattering has scarcely been discussed.²⁾ In this note we want to add some remarks on shape independent approximation for tensor force. The notations used here are the same as in Christian's paper.³⁾

Christian gave the following formula for " S' "-phase shift δ_1 (i.e., the phase shift for the S predominating state)

$$x \cot \delta_1 = \frac{1}{a_1} + x^2 \int_0^\infty \left[\left(\frac{x}{a_1} + 1 \right)^2 - V_1 V_1 \right] dx + O(x^4). \quad (1)$$

Similarly we get the analogous formula for " D "-phase shift δ_3

$$x^5 \cot \delta_3 = \frac{1}{a_3} + x^2 \int_0^\infty \left[\left(\frac{R_0 x}{a_3} \right)^2 + \left(\frac{x^3}{15a_3} - \frac{3}{x^5} \right)^2 - \frac{9}{x^4} - V_3 V_3 \right] dx + O(x^4). \quad (2)$$

If we use the deuteron wave function

$$V_D = \left[\frac{u_D}{w_D} \right], \quad (3)$$

$$\left. \begin{aligned} u_D &\rightarrow e^{-\alpha x} \\ w_D &\rightarrow a^2 R_D e^{-\alpha x} \left(1 + \frac{3}{ax} + \frac{3}{a^2 x^2} \right) \end{aligned} \right\} (x \rightarrow \infty),$$

instead of V_1 , we get another expression for δ_1 :

$$\begin{aligned} x \cot \delta_1 &= -a + (x^2 + a^2) \int_0^\infty (e^{-2\alpha x} - u_D^2 - w_D^2) dx + \dots \\ &\doteq -a + (x^2 + a^2) \left(\frac{1}{2} r_i^{(c)} - \frac{P_D}{N_D^2} \right), \end{aligned} \quad (1')$$

where

$$r_t^{(c)} = 2 \int_0^\infty (e^{-2ax} - u_j^2) dx,$$

$$N_g^2 = \frac{2a}{1 - ar_t^{(c)}}$$

is the triplet effective range and normalization constant if only the central force is assumed. P_D means the D -state probability:

$$P_D = \frac{\int_0^\infty w_j^2 dx}{\int_0^\infty (u_j^2 + w_j^2) dx}.$$

The result (1') was also derived by Bethe and Longmire.³⁾

Finally we can derive the shape independent formula for the relative amplitude R :

$$\begin{aligned} \frac{R_0}{a_3} - \frac{R}{a_3} = x^2 \int_0^\infty \left[\left(\frac{x}{a_1} + 1 \right) \left(-\frac{R_0 x}{a_3} \right) \right. \\ \left. + \frac{R_0 x}{5a_3} - V_1 V_3 \right] dx + \dots, \quad (4) \end{aligned}$$

$$R_j x^5 \cot \delta_3 - R x^5 \cot \delta_3$$

$$= (x^2 + a^2) \int_0^\infty \left[-\frac{R_0 x}{a_3} e^{-xx} - \frac{x^3}{15a_3} a^2 R_0 e^{-ax} \right.$$

$$\left. \times \left(1 + \frac{3}{ax} + \frac{3}{a^2 x^2} \right) - V_1 V_3 \right] dx + \dots \quad (4')$$

Next let us consider the low energy triplet neutron-proton scattering. The differential cross section is easily calculated by the method described by Rarita and Schwinger.⁴⁾ At very low energy region we can put $\delta_3 = 0$ and must observe only the " S " scattering. However, the small D -mixture accompanying main S state will cause small asymmetric scattering. In fact one finds for average differential cross section in the center of mass system

$$d\sigma \doteq \frac{\sin^2 \theta}{x^2} \left[1 + \frac{R x^2}{2} (3 \cos^2 \theta - 1) \right] d\Omega, \quad (5)$$

where θ is the scattering angle in the center of mass system. From (4) and (4') we get

$R \approx R_0 \approx R_g$. And, for instance, Rarita-Schwinger potential⁴⁾ gives

$$a^2 R \approx a^2 R_g = 0.02.$$

According to Christian and Hart⁵⁾ neutron-proton forces in odd state are very small. Thus the formula (5) or more exact expression including δ_3 may be useful for the analyses of very accurate experiments at some energy region, say, ~ 10 MeV.

The effect of tensor force on reaction $d(\gamma, n)p$ will be discussed in another paper.

- 1) G. F. Chew and M. L. Goldberger, Phys. Rev. **75** (1949), 1937.
J. M. Blatt and J. D. Jackson, Phys. Rev. **76** (1941), 18.
H. A. Bethe, Phys. Rev. **76** (1949), 33.
- 2) R. S. Christian, Phys. Rev. **75** (1949), 1675.
- 3) H. A. Bethe and C. Longmire, Phys. Rev. **77** (1950), 647.
- 4) W. Rarita and J. Schwinger, Phys. Rev. **59** (1941), 436, 556.
- 5) R. S. Christian and E. W. Hart, Phys. Rev. **77** (1950), 441.

On Internal Pair Creation following Some Beta-Decay

T. Nakano

Department of Physics, Osaka City University

June 9, 1951

When the resultant nucleus of complex β -decay has ground and excited state both having zero angular momenta, gamma radiation is strictly forbidden. But if the parities of the two states are the same and the energy difference is more than $2mc^2$, there would occur internal pair creation or emission of an orbital electron.

Yukawa and Sakata¹⁾ calculated the process for RaC' and obtained the result that the ratio of the probability of pair-creation to that of K -electron emission ρ is

$$\rho \cong 3.2 \times 10^{-3}.$$

But the value obtained from the experiments of Ellis, and Aston²⁾ and Alichanow, Alichanian, and Kosodaew³⁾ is

$$\rho \approx 1.$$

Because of the great discrepancy of the results of the theory and the experiments, Yukawa and Sakata gave up to explain the phenomenon by the above-mentioned process only.

As the experiment on RaC is so complicated that identification of the data might be difficult, more simple source of β -emission is favorable to test the theory.

Several authors¹⁾ have reported the existence of positively charged particles in the cloud chamber tracks from P^{32} . On the other hand, Agnew⁵⁾ and Warshaw, Chen, and Appleton⁶⁾ have observed the rise at low energy part of β -spectrum of P^{32} , and Warshaw et al. reported that the end point energy of the low energy component must be 330-KeV. If this rise is due to the excited level in S^{32} , as no γ -radiation have been observed, the level would probably have zero angular momentum. Supposing the parity of two states is the same, internal pair creation or emission of an orbital electron may be expected since the energy difference of two states becomes 1.38-MeV.

Calculation by the same assumption with Yukawa and Sakata shows that ratio of electron pairs to K -electrons is

$$\rho \approx 1.6.$$

The type of β -decay of P^{32} has not been known, but the spectrum shows nearly allowed type; so we may have approximate value of the ratio of main to additional part of β -spectrum by the formula for allowed type transition. Thus we have as ratio of positive to negative particles

$$\frac{N_+}{N_-} \approx \frac{P_\beta'}{P_\beta} \cdot \frac{P_{\text{pair}}}{P_{\text{pair}} + P_0} \approx 1.3 \times 10^{-3},$$

where P_β , P_β' , P_{pair} and P_0 are probabilities

of high and low energy part of β -decay, internal pair creation and orbital electron emission respectively. This result is in good agreement with experiments. Though so far no K -electron have been observed for the decay of P^{32} , if our assumption is accepted, there should be K -electrons of energy 1.38-MeV with probability of 0.8×10^{-3} /disintegration.

There remain many positive particles associated with β -decay which have as yet no satisfactory account. On this point some analysis will be made in near future.

In conclusion, I should like to express my deep gratitude to Prof. Y. Nambu, Messrs. Y. Yamaguchi and Yoshizawa for their kind interests in this work. Further the author is indebted to Mr. Yanabu and his collaborators who showed him their experimental material.

- 1) H. Yukawa and S. Sakata, Proc. Phys.-Math. Soc. Japan **17** (1935), 397.
- 2) Ellis and Aston, Proc. Roy. Soc. **A129** (1930), 180.
- 3) Alichanow, Alichanian and Kosodaew, J. Phys. U.S.S.R. **7** (1934), 163.
- 4) E. g. Smith and Groetzinger, Phys. Rev. **70** (1946) 96; Pi and Chao, Phys. Rev. **72** (1947), 639; Barlow and Rogers, Phys. Rev. **74** (1948), 700; Yanabu et al., private communication.
- 5) H. M. Agnew, Phys. Rev. **77** (1950), 656.
- 6) S. d. Warshaw, J. J. L. Chen, and G. L. Appleton, Phys. Rev. **80** (1950), 238.

Note on Some Calculations in Quantum Field Theory

T. Epstein

Department of Physics, Columbia University
New York, U.S.A.

June 13, 1951

In a recent paper Yukawa and Umezawa have calculated $\langle d^3x \phi^* \beta \phi \rangle$ for several coupling schemes. In this note we wish to show that, contrary to the impression given

in their paper, their results can be derived using the methods of Pais and Epstein, i.e. using the general theorem²⁾

$$\langle \frac{\partial H}{\partial \mu} \rangle = \frac{\partial \langle H \rangle}{\partial \mu},$$

where H is the hamiltonian and μ is a parameter which we take, for our purposes, to be the mass of the ϕ field.

For the problem at hand H is, in obvious notation,³⁾

$$H = H(\phi) + H(\psi) + g \int d^3x \\ \times (\psi^* O \psi - \langle \psi^* O \psi \rangle_{\text{vac}}) \phi$$

(we have omitted any possible tensor indices) from which it follows that

$$\langle d^3x \psi^* \beta \psi \rangle = \frac{\partial \langle H \rangle}{\partial \mu} \\ + g \int d^3x \langle \phi \rangle \frac{\partial}{\partial \mu} \langle \psi^* O \psi \rangle_{\text{vac}}. \quad (1)$$

For both the pseudo-scalar and vector theories the second term on the right hand side is identically zero. In the former case it is trivially zero since the vacuum current vanishes, while in the latter case one can see that it is zero either by direct calculation or by using the fact that the subtraction of the vacuum current can also be carried out by charge symmetrization⁴⁾ in which case the interaction term is explicitly independent of μ , whence only the first term will appear in (1).

For the scalar theory, however, neither of these circumstances occurs; the second term does contribute and indeed, as a simple calculation shows, to order g^2 it contributes just the "V.P." term of Sawada⁵⁾ and of Yukawa and Umezawa.¹⁾ The result (1) is thus in complete agreement with the calculations of these authors.

1) J. Yukawa and H. Umezawa, *Prog. Theor. Phys.* **6** (1951), 112.

2) R. P. Feynman, *Phys. Rev.* **56** (1939), 340.

A. Pais and S. T. Epstein, *Rev. Mod. Phys.* **21**, (1949), 445.

In applying this result we will be interested in $\langle \frac{\partial H}{\partial \mu} \rangle_{\text{vac}+1} - \langle \frac{\partial H}{\partial \mu} \rangle_{\text{vac}}$, whence additive C numbers in H will play no role.

- 3) This subtraction of the vacuum current is implicitly implied in all calculations made with these theories since, in the language of Feynman diagrams, one always omits diagrams in which a point connects directly with itself (see F. J. Dyson, *Phys. Rev.* **75** (1949), 486.) If one would include such diagrams then, of course, most quantitative results would be changed.
- 4) J. Schwinger, *Phys. Rev.* **75** (1949), 651.
- 5) K. Sawada, *Prog. Theor. Phys.* **5** (1950), 236.

Zero-zero Transition and Electron-Neutron Interaction

Y. Yamaguchi

Department of Physics, Osaka City University

June 14, 1951

As is well known, the zero-zero transitions can be explained in two different manners¹⁾; (i) the ordinary electro-magnetic interaction between protons and electrons and (ii) the hypothesis of direct coupling between nucleons and electrons. As for ${}^{\pi}\text{O}^{16}$, the energy spectrum,²⁾ the angular correlation of the positron with respect to the electron³⁾ and the life time⁴⁾ are all in good agreement with calculation based on (1). It seems, however, to be of some interest to discuss the assumption (ii) in connection with the electron neutron interaction established by the excellent experiment of Rabi et al.⁵⁾

Let us assume the direct coupling of the type¹⁾⁶⁾

$$g(\Psi^* O \Psi)(\phi^* O \phi), \quad (1)$$

where Ψ and ϕ are the nucleon and electron wave functions, respectively, analogous to the interaction responsible to β -decay. This type of the interactions was extensively examined by Nomoto.⁶⁾ According to him,

the first forbidden transition of axial vector coupling ($O=(\sigma, r_5)$) with only the nuclear matrix element $\int r_5$ and the allowed transition of pseudo-scalar coupling ($O=r_5$, nuclear matrix element $\int \beta r_5 \approx \int r_5$) give the desired energy spectrum²⁾ and all other cases are rejected. (In the allowed transitions of scalar and vector cases the nuclear matrix elements $\int \beta$ and $\int 1$ vanish in contrast to the corresponding cases of β -decay.) Nuclear matrix element $\int r_5$ changes the parity of the nucleus, while the nuclear matrix element $\int \Psi_{\sigma}^* \sum_{\text{all protons}} r^2 \Psi_e$ (which we shall write briefly as $\int r^2$ hereafter) in the ordinary electro-magnetic case does not change the parity.

The parity change of the nucleus is the most direct check of two explanations (i) and (ii).¹⁾ But there is no definite experimental evidence concerning parity change of zero-zero transition. The current nuclear model favours to parity change "no" though it is not decisive.

The angular correlation is as follows⁷⁾:

$$1 + 0.95 \cos \theta \text{ for electro-magnetic case,} \\ 1 + 0.91 \cos \theta \text{ for axial vector case, (2)}$$

and

$$1 - 0.91 \cos \theta \text{ for pseudo-scalar case;}$$

where θ means the angle between electron and positron. In these calculations we use the plane wave for electron and positron wave functions. Comparing the above results with experimental one $1 + 0.85 \cos \theta$,³⁾ we can reject the pseudo-scalar case. If we assume $|\int r_5|^2 \sim 10^{-2}$, as in the case of β -decay, the axial vector coupling constant must be the following order of magnitude

$$g \sim 5 mc^2 (e^2/mc^2)^3, \quad m = \text{electron mass, (3)}$$

in order to fit the observed life time of πO^{16} $((7 \pm 0.7) 10^{-11})$.⁴⁾

On the other hand we can explain the electron-neutron scattering on the basis of phenomenological direct interaction of the type (1). If we adopt the axial vector type, the coupling constant is determined by the

electron-neutron scattering⁵⁾ as follows:

$$g \sim 0.02 mc^2 (e^2/mc^2)^3. \quad (4)$$

Since (4) is quite small compared with (3), we must give up to explain the zero-zero transitions on the assumption of direct electron-nucleon coupling, unless the direct (non-electro-magnetic) interaction between electrons and protons is unreasonably greater than the one between electrons and neutrons. Therefore we may safely conclude that *the assumption (ii) must be abandoned and only the more reasonable explanation (i) remains.* Thus *the parity change in zero-zero transition is "no"* and the parity of, e.g., πO^{16} must be even.

As is discussed above the zero-zero transition is certainly caused by the ordinary electro-magnetic interaction. Then we can determine the values of nuclear matrix element $\int r^2$ from the observed life times and the *exact* expression for zero-zero transition probability:

$$|\int r^2|^2 = \begin{cases} 2.3 \left(\frac{e^2}{2mc^2} \right)^4 & \text{for Ge}^{72}, \\ 3.7 & \text{for O}^{16}, \end{cases} \quad (5)$$

($m = \text{electron mass}$).

Thus the matrix element $\int r^2$ seems to be nearly independent of mass number A , which is contrast with the expectation ($\int r^2 \propto R^2 \propto A^{2/3}$, $R = \text{nuclear radius}$)⁸⁾. However the similar situation occurs in the nuclear matrix element of a-type transition of the first forbidden β -decay; i.e., the corresponding matrix element B_{ij} , which is expected to be proportional to $A^{1/3}$, is in fact known to be nearly independent of A from the analysis of the so-called ft -values.⁹⁾ And these facts must be explained in parallel way.

Finally we want to add some considerations about S^{32} . It is very likely that the positrons accompanying to the β -decay of P^{32} is originated from the zero-zero transition of S^{32} as was pointed out by Nakano.¹⁰⁾ Assuming the 1.38 MeV level of S^{32} -nucleus with zero spin and even parity, and using

the value of nuclear matrix element :

$$|\int r^2|^2 = 2.3 \sim 3.7 (e^2/2mc^2)^4,$$

the mean life time of pair emission is estimated as

$$1.4 \sim 2.2 \times 10^{-6} \text{ sec.}$$

According to Nakano,¹⁰⁾ the ratio of electron-positron pair emission to orbital electron ejection is 38.0 : 28.6. Furthermore the angular correlation is

$$1 + 1.1 \cos \theta,$$

if we use the plane wave for electron and positron wave functions. Since the estimated life time for pair emission of S^{32} is rather long, it must easily be measured. We hope that these predictions about the zero-zero transition of S^{32} will be soon checked experimentally.

I am deeply indebted to Mr. Nakano for several numerical calculations. The author wishes to thank Professor Tanikawa and Professor Nambu for their kind discussions.

- 1) J. R. Oppenheimer and J. Schwinger, *Phys. Rev.* **56** (1939), 1066.
H. Yukawa and S. Sakata, *Proc. Phys.-Math. Soc. Japan* **17** (1935), 397.
- 2) M. Kojima, *Proc. Imp. Acad. Tokyo*, **19** (1943), 282.
Rasmussen, Hornyak, and Lauritsen, *Phys. Rev.* **77** (1950), 617.
- 3) S. Devons and G. R. Lindsey, *Nature* **164** (1949), 539.
- 4) S. Devons, H. G. Hereward and G. R. Lindsey, *Nature* **164** (1949), 586.
Also the life time of Ge^{72} has been measured by J. C. Bowe et al. (*Phys. Rev.* **73** (1948), 1219 (A)).
- 5) W. W. Havens, L. J. Rainwater and I. I. Rabi, *Phys. Rev.* **82** (1951), 345 (A).
- 6) M. Nomoto, *Sci. Rep. Tohoku Univ., series I.* Vol. XXXIII, No. 3 (1949), 157.
- 7) J. R. Oppenheimer, *Phys. Rev.* **60** (1941), 164 (A).

These calculations are essentially the same as the ones of electron-neutrino angular correlation in β -decay.

- 8) This conclusion is not in agreement with the result of S. D. Drell (*Phys. Rev.* **81** (1951), 656 (A)). The difference seems to come from the following reason: while Drell used the approximate formula (derived by Oppenheimer and Schwinger, ref. 1)) valid at very large excitation energy ($\gg mc^2$) and neglected the Coulomb effect on the electron and positron wave function, we have used the exact expression valid for any value of excitation energy and used the relativistic Coulomb wave functions for electron and positron when we calculated the nuclear matrix element (5) from the observed life times.
- 9) J. P. Davidson, *Phys. Rev.* **82** (1951), 43.
- 10) T. Nakano, *Prog. Theor. Phys.* **6** (1951), in press.

On the Absorption of the Negative π -meson by Deuteron

— Corrigenda —

S. Ogawa, E. Yamada and Y. Nagahara
*Institute of Theoretical Physics,
Nagoya University*

June 26, 1951

Previously, we have analysed the problem under the above subject.¹⁾ There, we have missed some considerations.

In the case of vector (or pseudovector) π -meson, the initial state with the total angular momentum $J=0, 1, 2$ exists in statistical weights, and in the radiative transition, the meson current results in considerable effect, if the perturbation method is performed in vector (or pseudovector) case. On these two points, we have modified our previous analysis and we have attained the same conclusion as Tamor's.²⁾ That is, the pseudo-scalar and pseudovector π -meson is consistent with the experiments, so far as the π -meson capture by Deuterium is concerned.

- 1) S. Ogawa, E. Yamada and Y. Nagahara, *Prog. Theor. Phys.* **6** (1951), 227.
- 2) S. Tamor, *Phys. Rev.* **82** (1951), 38.

Errata

Theory of the Temperature Effect of Electronic Energy Bands in Crystals

T. Muto and S. Ôyama. (Vol. 5, No. 5, 1950, pp. 833-843)

- pp. 835, 30th line, for $\dots + \frac{X_W Y_W}{2\pi\nu M} \cos(wr) \}$, read $\dots + \frac{Y_W}{2\pi\nu M} \cos(wr) \}$.
- pp. 836, equ. (10), for $\dots = N^{-1/3} \dots$, read $\dots = N^{-1/2} \dots$.
- pp. 837, equ. (11), for $\dots = N^{-1/3} \dots$, read $\dots = N^{-1/2} \dots$.
- pp. 838, equ. (19), } for $\frac{1\hbar a^3 C_1^2}{9\pi^2 Mc}$, read $\frac{\hbar a^3 C_1^2}{9\pi^2 Mc}$.
- equ. (19'), }
- „, the last line, for $\left[\dots + \left(\frac{\gamma}{a_1} \right) \log \left(\frac{a_1 w_0}{\gamma} + 1 \right) \right]$, read $\left[\dots + \left(\frac{\gamma}{a_1} \right)^2 \log \left(\frac{a_1 w_0}{\gamma} + 1 \right) \right]$.
- pp. 839, 12th line, for $+m_2 C_2^2 \{T^2 \dots$, read $+m_2^* C_2^2 \{T^2 \dots$.
- pp. 843, 2nd line, for $(C_i \text{ in (ev), } c \text{ cm/sec})$, read $(C_i \text{ in (ev), } c \text{ in cm/sec})$.
- pp. 842, Table 1, 3rd row and 5th column, for 0.043, read 0.047.
- „ „ „ for 0.25, read 0.85..

Note on Dirac's New Quantization Method in the Field Theory

T. Muto and K. Inoue. (Vol. 5, No. 6, 1950, pp. 1033-1044)

- pp. 1035, equ. (5.4), for $\dots = \mathcal{Q}\Psi$, read $\dots = \mathcal{Q}\Psi_2$.
- pp. 1037, 13th line, for $\rightarrow (-k_0', -\mathbf{k})U_{-T}(\mathbf{k}')$, read $\rightarrow (-k_0', -\mathbf{k}')U_{-T}(\mathbf{k}')$.
- pp. 1042, equ. (28), for $s_i p \psi_m(x) = \dots$, read $s_i p^* \psi_m(x) = \dots$.
- „ for $a_n = \int \psi_n^*(x) s_i p \psi_m(x) dx$, read $a_n = \int \psi_n^*(x) s_i p^* \psi_m(x) dx$.
- „ 14th line, for $R_{n,1}(r)$, read $R_{n,l}(r)$.
- „ equ. (32), for $\dots \sum_n (r_{1,0}^{n,1})^2 \dots$, read $\dots \sum_n (r_{2,0}^{n,1})^2 \dots$.
- pp. 1043, equ. (38), for $\dots \frac{k}{3} \left[\frac{4!}{(ik - \frac{Z}{2a})^5} \left(-\frac{ik + Z/2a}{ik - Z/2a} \right)^{-in'-} \left(\dots \right. \right.$
- read $\dots \frac{k}{3} \left[\frac{4!}{(ik - \frac{Z}{2a})^5} \left(-\frac{ik + Z/2a}{ik - Z/2a} \right)^{-in'-3} \left(\dots \right. \right.$

back page of the cover, 23rd line, for Note on Dirac's New Quantized Method ...,
read Notes on Dirac's New Quantization Method ...

Interaction of μ -Meson with Matter, I. — Nuclear Excitation by Electromagnetic Interactions. — T. Muto and M. Tanifuji. (Vol. 6, No. 1, 1951, pp. 27-36)

- pp. 32, 1st line, for $\dots \sum \dots \times$, read $\dots \sum \dots \times'$.
- „, equ. (24)', for $\bar{\epsilon}_l = \frac{\epsilon_l R_0}{\hbar c}$, read $\bar{\epsilon}_l = \frac{R_0}{\hbar c}$.
- „, equ. (27), for $\sigma_l = 12\pi^2 \left(\frac{e^2 Z}{Mc^2} \right)^2 \dots$, read $\sigma_l = 12\pi^2 \left(\frac{e^2 Z}{Mc^2} \right)^2 \dots$.
- „, equ. (28), for $\epsilon_2 = \dots$, read $\epsilon_3 = \dots$.
- pp. 33, the ordinates of the left and right hand sides of Fig. 1,

$$\text{for } 2\pi A \frac{d\pi}{d\pi} (cm^2), \quad \text{read } 2\pi A \frac{d\sigma}{d\Omega} (cm^2).$$

$$\text{for } 2\pi B \frac{dV}{d\pi} (cm^2), \quad \text{read } 2\pi B \frac{d\sigma}{d\Omega} (cm^2).$$

pp. 34, to the ordinate of Fig. 2 $A\sigma(cm^2)$ should be attached.

pp. 35, the ordinates of the left and right hand sides of Fig. 3,

$$\text{for } 2\pi A \frac{d\alpha}{d\pi}, \quad \text{read } 2\pi A \frac{d\sigma}{d\Omega}.$$

$$\text{for } 2\pi B \frac{d\alpha}{d\pi}, \quad \text{read } 2\pi B \frac{d\sigma}{d\Omega}.$$

Theory of the Temperature Effect of Electronic Energy Bands in Crystals, II

T. Muto and S. Ôyama. (Vol. 6, No. 1, 1951, pp. 61-64)

pp. 61, Fig. 1, for $C_1^2=104 \text{ ev}^2$, read when $C_1^2=104 \text{ ev}^2$.

„ „ for when $C_1^2=842 \text{ ev}^2$, read when $C_1^2=84.2 \text{ ev}^2$.

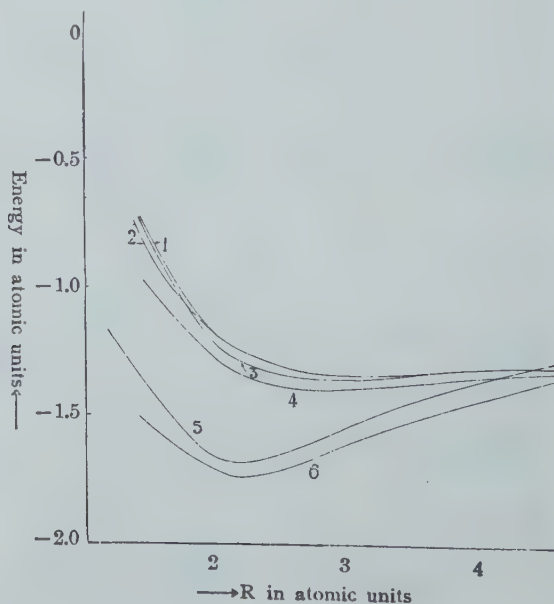
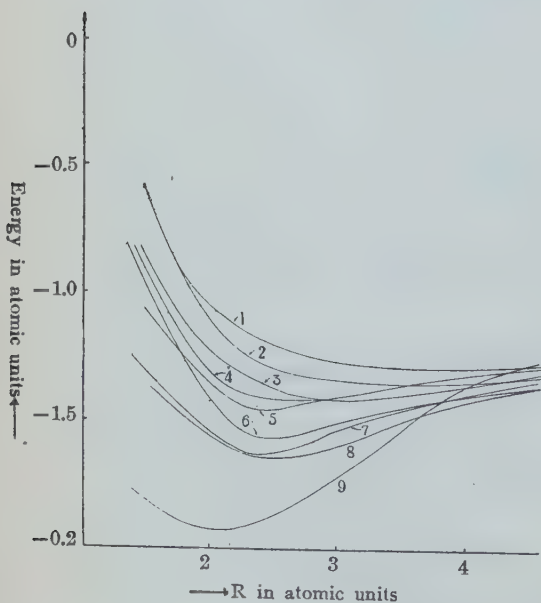
„ „ the parameters attached to the curves should be read $\epsilon_1=1$ and $\epsilon_1=0.65$.

„ the last line for $-2.09 \times 10^5 \frac{\epsilon_1 C_1^2}{\epsilon_1 c^2}$, read $-2.09 \times 10^5 \frac{\epsilon_1 C_1^2}{c^2}$.

Electronic States of C_2 -Molecule, I —Interaction between p -Electrons—G. Araki,

S. Tutihasi and W. Watai. (Vol. 6, No. 2, 1951, pp. 135~153)

Figures 1 and 2 (p. 151) should be replaced by the following figures:



Statistical Thermodynamics of Solutions of Electrolytes and Non-Electrolytes

Syû ONO

*Institute of Applied Science, Faculty of Engineering,
Kyushu University*

(Received April 23, 1951)

In this paper the generalized Born-Green integral equations for solutions of electrolytes and non-electrolytes are derived according to the method previously shown. The assumption of superposition for the potentials of average force of solute molecules or ions is employed throughout the present paper. The integral equations may be used, if the potentials of average force at infinite dilution are known, to compute the deviation from perfect solution behavior.

§ 1. Introduction

Statistical derivation of the thermodynamical functions of multicomponent systems from the potentials of average force of solute molecules or ions at infinite dilution has been performed by McMillan and Mayer.^{1),2)} In the present paper we shall derive the integral equations, based upon the theory of grand partition function, for treating solutions of non-electrolytes, corresponding to McMillan and Mayer's method¹⁾ of power series development, and the one for treating solutions of electrolytes, corresponding to Mayer's²⁾ recent work.

We consider only systems of infinite extent in volume space, the temperatures of which are fixed at T . The systems are multicomponent systems containing $\sigma+1$ types of molecular species, and the subscript s (or t) will be used to refer to species s (or t). The symbols without subscripts refer to a set of quantities:

$$\rho = \rho_0, \rho_1, \dots, \rho_s, \dots, \rho_\sigma \quad (1)$$

$$z = z_0, z_1, \dots, z_s, \dots, z_\sigma \quad (2)$$

The symbol ρ_s is the density of molecules of species s , in molecules per unit volume, and z_s is the fugacity normalized in density unit; the subscript zero indicates the solvent.

The letters k, m, n , etc., when written without subscripts, will be used to indicate sets of integers. And the short hand notations:

$$k! = k_0! k_1! \dots k_s! \dots k_\sigma! \quad (3)$$

$$\rho^k = \rho_0^{k_0} \rho_1^{k_1} \dots \rho_s^{k_s} \dots \rho_\sigma^{k_\sigma} \quad (4)$$

$$z^k = z_0^{k_0} z_1^{k_1} \dots z_s^{k_s} \dots z_\sigma^{k_\sigma} \quad (5)$$

will be used.

The symbol

$$(i_s) = x_{is}, y_{is}, z_{is}, \quad (6)$$

indicates the Cartesian coordinates of the center of mass of the i th molecule of type s . The coordinates of k_s molecules of type s will be represented by

$$\{k_s\} = (0), (1), \dots, (k_s). \quad (7)$$

And we will denote the coordinates of all molecules of such a set k by

$$\{k\} = \{k_0\}, \{k_1\}, \dots, \{k_s\}, \dots, \{k_o\}. \quad (8)$$

The volume elements are similarly denoted by $d(i_s)$, $d\{k_s\}$ and $d\{k\}$. And the notation

$$\left((i_s) \frac{\partial}{\partial (i_s)} \right) = x_{is} \frac{\partial}{\partial x_{is}} + y_{is} \frac{\partial}{\partial y_{is}} + z_{is} \frac{\partial}{\partial z_{is}} \quad (9)$$

will be adopted.

§ 2. Distribution functions and the generalization of the grand partition function

We now define distribution functions $F_n(z, T, \{n\})$, of the coordinates of a set n of molecules by the statement that in a system of infinite extent ($V \gg n_s/\rho_s$ for all s) at the temperature T , and the fugacity set z , the probability that there are an appropriate set n of molecules at the coordinates $\{n\}$ within the volume element $d\{n\}$ is equal to $F_n(z, T, \{n\})d\{n\}$.

And the potential of average force is defined by

$$W_n(z, T, \{n\}) = -kT \ln F_n(z, T, \{n\}). \quad (10)$$

McMillan and Mayer¹⁾ has shown that the distribution functions are given by the equation set

$$\Xi(z, T) \left(\frac{\rho}{z} \right)^n F_n(z, T, \{n\}) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \iint \dots \int \exp \left\{ -\frac{U_{n+m}\{n+m\}}{kT} \right\} d\{m\}, \quad (11)$$

where k is Boltzmann's constant, $U_n\{n\}$ is the potential energy of the set n of molecules at the position $\{n\}$, and $\Xi(T, z)$ is the grand partition function defined by

$$\Xi(z, T) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \iint \dots \int \exp \left\{ -\frac{U_m\{m\}}{kT} \right\} d\{m\}. \quad (12)$$

And here the potential energy of the system is assumed to be a function of the coordinates of the centers of mass of molecules alone. Differentiating (11) with respect to

$$\partial/\partial z^p = \partial^{p_0}/\partial z_0^{p_0} \cdot \partial^{p_1}/\partial z_1^{p_1} \cdots \partial^{p_\sigma}/\partial z_\sigma^{p_\sigma}, \quad (13)$$

we obtain

$$\frac{\partial^p}{\partial z^p} \left\{ \Xi(z, T) \left(\frac{\rho}{z} \right)^n F_n(z, T, \{n\}) \right\} = \sum_{m=p} \frac{z^{m-p}}{(m-p)!} \iint \cdots \int \exp \left\{ -\frac{U_{n+m}\{n+m\}}{kT} \right\} d(m). \quad (14)$$

Then we have, from (11),

$$\begin{aligned} \frac{\partial^m}{\partial z^m} \left\{ \Xi(z, T) \left(\frac{\rho}{z} \right)^n F_n(z, T, \{n\}) \right\} \\ = \Xi(z, T) \left(\frac{\rho}{z} \right)^{n+m} \iint \cdots \int F_{n+m}(z, T, \{n+m\}) d\{m\}, \end{aligned} \quad (15)$$

in which we have replaced p by m . In consequence Taylor's expansion of the function $\Xi(z, T) (\rho/z)^n F_n(z, T, \{n\})$ about the point z is, according to (15),

$$\begin{aligned} \Xi(z, T) \left(\frac{\rho}{z} \right)^n F_n(z, T, \{n\}) \\ = \Xi(z^*, T) \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint \cdots \int \left(\frac{\rho^*}{z^*} \right)^{n+m} F_{n+m}(z^*, T, \{n+m\}) d\{m\}. \end{aligned} \quad (16)$$

This is the generalized form of the grand partition function originally introduced by McMillan and Mayer in the different way from the one described here.

§ 3. Integral equations

As shown by Mayer^{3,4)} (16) can be used to derive a set of integral equations which permit us to calculate the properties of the condensed phases. And in the present paper, we will derive the integral equations with the following assumptions.

It is frequently assumed that for chemically saturated molecules we may write

$$U_n\{n\} = \frac{1}{2} \sum_{s,t} \sum_{i=1}^{n_s} \sum_{j=1}^{n_t} u_{st}(i_s, j_t), \quad (17)$$

where $u_{st}(i_s, j_t)$ is the potential energy of pair of molecules of type s and t , i_s and j_t . Equation (17) is probably very nearly correct for many molecule types. In the case of a fluid, for which single molecule distribution functions are equal to unity, we may similarly assume,

$$W_n\{n\} = \frac{1}{2} \sum_s \sum_t \sum_{i=1}^{n_s} \sum_{j=1}^{n_t} w_{st}(i_s, j_t) \quad (18)$$

in which $w_{st}(i_s, j_t)$ is the potential of average force of a pair of molecules i_s and j_t , the assumption of superposition, which was first introduced by Kirkwood.⁵⁾ This assumption is very useful and appear to be reasonable although it has never been

proved to be even approximately true of liquids.

Introducing (10) and (18) into (16), and differentiating so obtained equation with respect to the coordinate of a certain molecules, i_s , we obtain

$$\begin{aligned} \Xi(z, T) \left(\frac{\rho}{z} \right)^n \frac{\partial F_n(z, T, \{n\})}{\partial(i_s)} \\ = -\frac{1}{kT} \sum_{m=0}^{\infty} \frac{(z-z^*)^m}{m!} \Xi(z^*, T) \iint \dots \iint \left(\frac{\rho}{z} \right)^{n+m} \\ \times \left[\sum_i^{n_t+m_t} \sum_{j=1}^{n_t} \frac{\partial w_{st}^*(i_s, j_t)}{\partial(i_s)} \right] F_{n+m}(z^*, T, \{n+m\}) d\{m\}, \quad (19) \end{aligned}$$

where $w_{st}^*(i_s, j_t)$ corresponds to the fugacity z^* . Replacing $\{m\} + (j_t)$ by $\{m\}$ for the terms in which j_t belongs to m , and making use of (16), we obtain

$$\begin{aligned} \frac{\partial F_n(z, T, \{n\})}{\partial(i_s)} + \sum_i^{n_t} \sum_{j=1}^{n_t} \frac{1}{kT} \frac{\partial w_{st}^*(i_s, j_t)}{\partial(i_s)} F_n(z, T, \{n\}) \\ = \sum_i \frac{(z_t - z_t^*) \rho_t}{kT z_t} \int_V F_{n+1}(z, T, \{n\} + \{j_t\}) \frac{\partial w_{st}^*(i_s, j_t)}{\partial(j_t)} d(j_t). \quad (20) \end{aligned}$$

Here $w_{st}^*(i_s, j_t)$ is assumed to be functions of relative coordinates of i_s and j_t and to satisfy the relation $\partial w_{st}^*(i_s, j_t) / \partial(i_s) = -\partial w_{st}^*(i_s, j_t) / \partial(j_t)$; this condition is valid for monatomic fluids. The principle used in the derivation of (20) is essentially equivalent to that used in the preceding paper.⁶⁾

§ 4. The solutions of the integral equations for pair distribution functions

In the present and subsequent sections let us represent the pair distribution functions by $F_{st}(i_s, j_t)$ and the three molecule distribution functions by $F_{str}(i_s, j_t, k_r)$ for the sake of convenience. Then the integral equation (20) for pair distribution functions can be written as

$$\begin{aligned} \frac{\partial \ln F_{st}(i_s, j_t)}{\partial(i_s)} + \frac{1}{kT} \frac{\partial w_{st}^*(i_s, j_t)}{\partial(i_s)} \\ = \sum_r \frac{(z_r - z_r^*) \rho_r}{kT z_r} \int_V F_{tr}(j_t, k_r) F_{rs}(k_r, j_s) \frac{\partial w_{sr}^*(i_s, k_r)}{\partial(k_r)} d(k_r). \quad (21) \end{aligned}$$

This is a generalization of the Born-Green⁷⁾-Yvon⁸⁾ integral equation. And by means of the same manipulation that adopted by Born and Green⁷⁾ this equation can be brought to the form

$$\begin{aligned} \ln F_{st}(r) + \frac{w_{st}^*(r)}{kT} \\ = \pi \sum_r \frac{(z_r - z_r^*) \rho_r}{kT z_r} \int_0^\infty \int_{-s}^s (s^2 - t^2) \frac{t+r}{r} \{F_{tr}(t+r) - 1\} dt F_{rs}(s) \frac{dw_{rs}^*}{ds} ds, \quad (22) \end{aligned}$$

$F_{rs}(r)$ being defined for negative values to be even functions of r . There are $\sigma(\sigma+1)/2$ types of functions $F_{st}(r)$, each satisfies an equation like (22). And simultaneous solutions of these integral equations will yield values of the functions $F_{rs}(r)$ if $w_{rs}^*(r)$'s are known.

Now, following Green⁽⁹⁾ and Rodriguez,^{(10),(11)} we will linearize equation (22) by writing

$$\psi_{st}(r) = (w_{st}^*(r) - w_{st}(r)) / kT, \quad (23)$$

or

$$F_{st}(r) = \exp \{ (-w_{st}^*(r) / kT) + \psi_{st}(r) \}. \quad (24)$$

Substitution of (23) into (22) leads to

$$\begin{aligned} \psi_{st}(r) = & -\pi \sum_r \frac{(z_r - z_r^*) \rho_r}{z_r} \int_0^\infty \int_{-s}^s (s^2 - t^2) \frac{t+r}{r} \\ & \times [(1 + f_{tr}(t+r)) (\psi_{tr}(t+r) + 1) - 1] dt \{ 1 + \psi_{rs}(s) \} f'_{rs}(s) ds, \end{aligned} \quad (25)$$

where

$$f_{st}(r) = \{ \exp (-w_{st}^*(r) / kT) \} - 1. \quad (26)$$

In the right-hand side of (25), cubes and higher powers of $\psi_{st}(r)$'s are neglected. If we replace $\psi_{st}(r)$ multiplied by the factor, $f_{st}(r)$, by $(\epsilon_{st} - 1)$, its mean values in the neighborhood of the origin, (25) becomes, after integration by parts,

$$r\psi_{st}(r) = 2\pi \sum_r \frac{(z_r - z_r^*) \rho_r}{z_r} \int_0^\infty \int_{-s}^s (t+r) \{ \psi_{tr}(t+r) + \epsilon_{tr} f_{tr}(t+r) \} dt \epsilon_{rs} f_{rs}(s) ds. \quad (27)$$

If we write

$$r\phi_{st}(r) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^\infty s\psi_{st}(s) \sin(rs) ds, \quad (28)$$

and

$$r\mathcal{G}_{st}(r) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^\infty s f_{st}(s) \sin(rs) ds, \quad (29)$$

then (27) becomes

$$\phi_{st}(s) = \sum_r \frac{1}{\lambda_r} \{ \phi_{tr}(s) + \epsilon_{tr} \mathcal{G}_{tr}(s) \} \epsilon_{rs} \mathcal{G}_{rs}(s), \quad (30)$$

where

$$1/\lambda_s = (2\pi)^{2/3} (z_s - z_s^*) \rho_s / z_s. \quad (31)$$

We may determine $\phi_{st}(s)$'s by solving $\sigma(\sigma+1)/2$ equations like (30). Then the distribution functions can be calculated from them by means of (24) and

$$r\phi_{st}(r) = (2\pi)^{-1/2} \int_{-\infty}^\infty s\phi_{st}(s) \sin(rs) ds, \quad (32)$$

which directly follows from (28).

§ 5. Pressure and osmotic pressure

It is possible to compute explicitly the pressure from

$$p - p^* = kT \sum_s \left[\frac{\rho_s(z_s - z_s^*)}{z_s} - \frac{2\pi}{3} \sum_{s,t} \frac{\rho_s(z_s - z_s^*)}{z_s} \frac{\rho_t(z_t - z_t^*)}{z_t} \int_0^\infty r F_{st}(r) \frac{dw_{st}^*(r)}{dr} r^2 dr \right], \quad (33)$$

where p^* is the pressure at the state corresponding to the fugacity set z^* . This formula was justified by Green for the cases of $z^*=0$ and $p^*=0$, and a derivation for the general case is shown in the appendix.

First let us consider the case where the asterisks refer to infinite dilution. Then (33) reduces to

$$p = kT \sum_{s=0}^{\sigma} \rho_s - \frac{2\pi}{3} \sum_{s=0}^{\sigma} \sum_{t=0}^{\sigma} \rho_s \rho_t \int_0^\infty r F_{st}(r) \frac{du_{st}(r)}{dr} r^2 dr, \quad (34)$$

for $w_{rs}^*(r)$ is equal to $u_{rs}(r)$, in the limit of infinite dilution. And in this case equation (31) defining λ_s becomes

$$1/\lambda_s = (2\pi)^{2/3} \rho_s, \quad (35)$$

and $f_{st}(r)$ defined by (25) is written as

$$f_{st}(r) = \exp\left(-\frac{u_{st}(r)}{kT}\right) - 1. \quad (36)$$

Here let us consider the function defined by

$$F(z) = \begin{vmatrix} \lambda_0 - \varepsilon_{00} g_{00}(z) & -\varepsilon_{01} g_{01}(z) \cdots & -\varepsilon_{0\sigma} g_{0\sigma}(z) \\ -\varepsilon_{10} g_{10}(z) & \lambda_1 - \varepsilon_{11} g_{11}(z) \cdots & -\varepsilon_{1\sigma} g_{1\sigma}(z) \\ \cdots & \cdots & \cdots \\ -\varepsilon_{\sigma 0} g_{\sigma 0}(z) & -\varepsilon_{\sigma 1} g_{\sigma 1}(z) \cdots & \lambda_\sigma - \varepsilon_{\sigma\sigma} g_{\sigma\sigma}(z) \end{vmatrix}. \quad (37)$$

It will be seen from (30), (35) and (37) that in the region of sufficiently low densities $F(z)$ is positive for all real z , and consequently the virial expansion of the pressure may be possible. And if we could accept Green and Rodriguez's interpretation of the singularity of the isothermal, the density set for which the upper bound of $F(z)$ is equal to zero would correspond to the metastable state separating the gas from the liquid, at which a real discontinuity in the radial distribution functions and all the thermodynamical quantities appear. Such density sets will form a σ -dimensional hypersurface in the $\sigma+1$ -dimensional space in which the densities are plotted as the coordinates.

The application of the theory to the one or more solutes in a solvent appears to be more important than the one to the gaseous mixture described above. For this case we shall find it convenient to use as the reference fugacity set z^* the one, at which the densities of the solutes are zero and the fugacity of the solvent is such that the pure liquid has some convenient pressure p^* , say one atmosphere, or possibly its orthobaric vapor pressure. And let us consider the case for which

$z_0 - z_0^* = 0$, i.e., the activity of the solvent is held constant. Then the pressure difference $p - p^*$ is the osmotic pressure. For this case (33) is written as

$$\pi = kT \sum_{s=1}^{\sigma} \rho_s - \frac{2\pi}{3} \sum_{s=1}^{\sigma} \sum_{t=1}^{\sigma} \rho_s \rho_t \int_0^{\infty} r F_{st}'(r) \frac{d\psi_{st}^*(r)}{dr} r^2 dr, \quad (38)$$

in which $\psi_{st}^*(r)$'s are the potentials of average force occurring in the pure solvent. In this case λ_s 's are expressible by (35) except for $s=0$ and $\lambda_0^{-1}=0$.

If the pair distribution functions, $F_{st}(r)$, are determined by solving equations (27), the osmotic pressure will be obtained as a function of the temperatures and the densities of the solute. As shown by Green⁹⁾ in the case of one solute the second and the third virial coefficients calculated from the distribution function and (38) agree with the ones obtained from McMillan and Mayer's theory. And the present theory will furnish more accurate results than the one given by Zimm's theory¹²⁾ for polymer solutions in which the molecules assumed to be rigid spheres. And in the case of rigid sphere molecule the solutions of the non-linear integral equation computed by Kirkwood, Maun and Alder¹³⁾ will be utilized in our theory.

§ 6. Solutions of electrolytes

In a infinitely dilute solution of electrolytes the potential of average force between ions of type s and t will be given by

$$w_{st}^*(r) = (z_s z_t \epsilon^2 / D r) + \omega_{st}^*(r), \quad (39)$$

where ϵ is the electronic charge, D the dielectric constant of the solvent, and z_s, z_t are the integer charges of type s and t respectively. $\omega_{st}^*(r)$ will differ from zero only for small values of the distance, and will increase without limit, more rapidly than r^{-1} , as the distance approaches zero. If $w_{st}^*(r)$ is given by (39), the Fourier transformation (29) will not converge, since the absolute value of $f_{st}(r)$ defined by (26) decreases as rapidly as r^{-1} for large distance. In order to avoid this difficulty we assume that the potential of average force between two ions is expressed by

$$w_{st}^*(r) = (z_s z_t \epsilon^2 / D r) e^{-ar} + \omega_{st}^*(r), \quad (40)$$

and let a approach zero. Then (29) converges for all positive values of a . We may assume, as a first approximation

$$\omega_{st}^*(r) = \begin{cases} \infty & r \leq a_s + a_t \\ = 0 & r > a_s + a_t \end{cases}. \quad (41)$$

And here we assume that $w_{st}^*(r)/kT$ is small compared with unity, and may neglect all terms higher than the second for $r > a_s + a_t$. Then $f_{st}(r)$ defined by (26) becomes

$$f_{st}(r) = \begin{cases} -1 & r \leq a_s + a_t \\ = -(Z_s Z_t \epsilon^2 / D r k T) e^{-ar} & r > a_s + a_t \end{cases}. \quad (42)$$

Substituting (42) into (29), we have

$$r g_{st}(r) = -(2/\pi)^{\frac{1}{2}} \int_0^{a_s + a_t} s \sin(rs) ds - (2/\pi)^{\frac{1}{2}} (Z_s Z_t \epsilon^2 / DkT) \int_{a_s + a_t}^{\infty} e^{-as} \sin(rs) ds. \quad (43)$$

If we neglect the first term of equation (43) due to ionic radii, we have

$$r g_{st}(r) = -(2/\pi)^{\frac{1}{2}} (Z_s Z_t \epsilon^2 / DkT) \int_0^{\infty} e^{-as} \sin(rs) ds. \quad (44)$$

Here we use the well-known relation

$$\begin{aligned} e^{-\beta x} &= (2/\pi) \int_0^{\infty} \sin(ux) du \int_0^{\infty} e^{-\beta t} \sin(ut) dt \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{u \sin ux}{\beta^2 + u^2} du. \end{aligned} \quad (45)$$

Then we obtain from (44)

$$g_{st}(r) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Z_s Z_t \epsilon^2}{DkT} \frac{1}{a^2 + r^2}. \quad (46)$$

We assume that all $\phi_{st}(r)$'s can be written in the form

$$\phi_{st}(r) = Z_s Z_t \phi(r). \quad (47)$$

If all ϵ 's are assumed to be unity, substitution of (46) and (47) into (30) leads to

$$\phi(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\epsilon^2}{DkT} \left\{ \frac{1}{a^2 + r^2} - \frac{1}{a^2 + x^2 + r^2} \right\}, \quad (48)$$

where

$$x^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\epsilon^2}{DkT} \sum_r \frac{Z_r^2}{\lambda_r}. \quad (49)$$

Then we have, according to (47),

$$\phi_{st}(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Z_s Z_t \epsilon^2}{DkT} \left\{ \frac{1}{a^2 + r^2} - \frac{1}{a^2 + x^2 + r^2} \right\}. \quad (50)$$

Inserting (50) in (32) and using (45), we obtain

$$r \phi_{st}(r) = (Z_s Z_t \epsilon^2 / DkT) [\exp\{-ar\} - \exp\{-(a^2 + x^2)^{1/2} r\}]. \quad (51)$$

From (23) and (51), we have

$$w_{st}(r) = (Z_s Z_t \epsilon^2 / DkT) (1/r) \exp\{-(a^2 + x^2)^{1/2} r\}. \quad (52)$$

If we let a approach zero, (52) becomes

$$w_{st}(r) = (Z_s Z_t \epsilon^2 / DkT) (1/r) e^{-x r}. \quad (53)$$

And if (35) is used, (49) can be written as

$$\kappa^2 = 4\pi(\epsilon^2/DkT) \sum_r Z_r^2 \rho_r, \quad (54)$$

from which we can see that equation (53) is Debye's solution¹³⁾ of the Poisson-Boltzmann equation. Thus we can conclude by saying that the general integral-equation theory for solutions of electrolytes leads to the Debye-Hückel theory in the limit of vanishing sizes of ions and increasing temperature. And the theory of solutions of electrolytes in which the mutual interactions apart from Coulomb interaction between ions are taken into account will be developed on the basis of the integral equation method.

Appendix

According to the general properties of the grand partition function, we obtain, from (16) for the case $n=0$,

$$\exp \frac{(p-p^*)V}{kT} = \sum_{m \geq 0} \frac{(z-z^*)^m}{m!} \iint_V \cdots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\}. \quad (A1)$$

And then we have

$$p-p^* = kT \frac{\partial}{\partial V} \ln \left[\sum_{m \geq 0} \frac{(z-z^*)^m}{m!} \iint_V \cdots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\} \right]. \quad (A2)$$

For an arbitrary function of the volume V , we have

$$\frac{\partial \Phi(V)}{\partial V} = \frac{1}{3V} \left[\frac{\partial \Phi(\xi^3 V)}{\partial \xi} \right]_{\xi=1}. \quad (A3)$$

And let us consider the function which is expressed in the form

$$\Phi(V) = \iint_V \cdots \int \Psi(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n, \quad (A4)$$

where x is Cartesian coordinates and $dx = dx dy dz$. If we write $x_i = \xi y_i$, we obtain from (A4)

$$\Phi(\xi^3 V) = \xi^{3N} \iint_V \cdots \int \Phi(\xi y_1, \xi y_2, \dots, \xi y_n) dy_1 dy_2 \cdots dy_n. \quad (A5)$$

From (A3) and (A5) we have

$$\begin{aligned} \frac{\partial \Phi(V)}{\partial V} &= \frac{n}{V} \iint_V \cdots \int \Psi(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n \\ &+ \frac{1}{3V} \iint_V \cdots \int \left\{ \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} + z_i \frac{\partial}{\partial z_i} \right) \Psi(x_1, x_2, \dots, x_n) \right\} dx_1 dx_2 \cdots dx_n. \end{aligned} \quad (A6)$$

If we use the above formula, (A2) can be written in the form

$$\begin{aligned}
& \frac{\partial}{\partial V} \ln \left[\sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\} \right] \\
&= \frac{\sum_s \sum_{m \geq 0} m_s \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\}}{V \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\}} \\
&+ \frac{\sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left\{ \sum_s \sum_{i=1}^{m_s} \left((i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) \right\} d\{m\}}{3V \sum_{m=0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\}}, \quad (A7)
\end{aligned}$$

in which the notation given by (9) is utilized. Replacing $m_s - 1$ by m_s , we have

$$\begin{aligned}
& \sum_{m \geq 0} \frac{m_s (z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\} \\
&= (z_s - z_s^*) \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left(\frac{\rho_s^*}{z_s^*} \right) F_{m+1}(z^*, T, \{m\} + (i_s)) d\{m\} d(i_s).
\end{aligned}$$

And according to (16), we have

$$\frac{1}{V} \sum_s \sum_{m \geq 0} \frac{m_s (z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m F_m(z^*, T, \{m\}) d\{m\} = \exp \left(\frac{p - p^*}{kT} \right) \sum_s \frac{\rho_s^* (z_s - z_s^*)}{z_s}, \quad (A.8)$$

$F_s(i_s)$ being unity for a fluid or glassy system. On the other hand, we have

$$\begin{aligned}
& \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left\{ \sum_{i=1}^{m_s} \left((i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) \right\} d\{m\} \\
&= m_s \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left((i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) d\{m\}, \quad (A9)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left(\sum_{i=1}^{m_s} (i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) d\{m\} \\
&= -\frac{1}{2kT} \sum_t \sum_{m \geq 0} \frac{m_s (m_s - \delta_{st})}{m!} (z - z^*)^m \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \\
&\quad \times \left\{ \left((i_s) \frac{\partial}{\partial (i_s)} \right) w_{st}^*(i_s, j_t) \right\} F_m(z^*, T, \{m\}) d\{m\}. \quad (A10)
\end{aligned}$$

And we obtain the following result:

$$\begin{aligned}
& \sum_{m \geq 0} \frac{(z - z^*)^m}{m!} \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m \left\{ \sum_s \sum_{i=1}^{m_s} \left((i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) \right\} d\{m\} \\
&= -\frac{1}{2kT} \sum_s \sum_t \sum_{m \geq 0} \frac{m_s (m_t - \delta_{st})}{m!} (z - z^*)^m \iint_V \dots \int \left(\frac{\rho^*}{z^*} \right)^m
\end{aligned}$$

$$\times \left\{ \left((i_s) \frac{\partial}{\partial (i_s)} \right) w_{st}^* (i_s, j_t) \right\} F_m(z^*, T, \{m\}) d\{m\}. \quad (\text{A11})$$

Replacing $m_s - 2$ by m_s or $m_s - 1$ and $m_t - 1$ by m_s and m_t respectively according as $s = t$ or $s \neq t$, and using (16), we obtain

$$\begin{aligned} & \sum_{m_s \geq 0} \frac{(z_s - z_s^*)^{m_s}}{m_s!} \iint_V \dots \int_V \left(\rho^* \right)^m \left\{ \sum_s \sum_{i=1}^m \left((i_s) \frac{\partial}{\partial (i_s)} \right) F_m(z^*, T, \{m\}) \right\} d\{m\} \\ &= - \frac{2\pi V}{kT} \frac{(z_s - z_s^*)}{z_s} \rho_s \frac{(z_t - z_t^*)}{z_t} \rho_t \exp \frac{(\rho - \rho^*) V}{kT} \int_0^\infty r F_{st}(r) \frac{dw_{st}^*(r)}{dr} r^2 dr. \quad (\text{A12}) \end{aligned}$$

If we substitute (A1), (A8), (A12) into (A7), we have, according to (A2), equation (33).

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Molecular Theory of Liquid Helium

Morikazu TODA

Institute of Physics, Tokyo Bunrika University

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In order to explain the well-known peculiar character of liquid helium an assumption is proposed that as the temperature is raised from 0°K atoms are excited by thermal agitation to form "molecules" of about eight atoms each. Bose statistics is then applied to the system of atoms and molecules. The thermodynamical properties, thermomechanical effect, the so-called second sound, viscosity etc. are quantitatively discussed, giving microscopic interpretation of these phenomena on the line proposed phenomenologically by Tisza.¹⁾

§ I. Introduction

Helium is the only one liquid which does not solidify down to absolute zero temperature under its vapor pressure. It is of no doubt that this peculiar behavior of liquid helium is due to the quantum effect or the large zero point motion of helium atoms owing to their light masses. Liquid helium undergoes a transition, the so-called λ -transition, the reason of which is not yet established. F. London²⁾ and others attributed this transition to the Bose-condensation of helium, because if helium is considered as ideal gas of the same density as liquid helium, Bose-condensation takes place at 3.14°K, which is of the same order to the real λ -point (2.19°K under vapor pressure). This seems to be a strong support to the Bose-condensation hypothesis. On the other hand liquid helium below the λ -point (He II) has extremely low viscosity coefficient and extremely high heat conductivity, both of which cannot be governed by ordinary laws, and the viscosity of liquid helium diminishes as the temperature is lowered and approaches zero at 0°K. This phenomenon, the so-called superfluidity of He II, seems to have some connection to the surface-film (Rollin film) phenomena of He II. Landau³⁾ and others stressed the similarity of the superfluidity of He II to the superconductivity of metals, and according to them the λ -transition is not essential to the Bose-condensation. Landau's argument, however, seems inconsistent, because his treatment on the statistics of "rotons" was at the beginning, for non-restricted number of them, but the temperature of the λ -point was calculated by laying restriction on it.

H. London⁴⁾ developed a phenomenological theory of liquid helium and Tisza¹⁾ extended this idea. He proposed four postulates on the nature of liquid helium, two of which are for the lowest state of energy and two others for the excited states.

His postulates may be summarized as follows: (a) Lowest state E_0 has min. for $V=V_0$, volume of the liquid. (b) The state E_0 is characterized by a vanishing rigidity. (c) Viscosity of liquid helium has a gaseous character. (d) There exists a temperature of transition T_0 . This paper intends to give microscopic foundation on Tisza's phenomenological theory.

Now, energy diagram for liquid and solid helium is illustrated in Fig. 1, in which E_L^0 is the energy of liquid and E_S^0 that of solid both at 0°K . At 0°K the equilibrium state of liquid under zero pressure is L, with increasing pressure up to 25 atm. state L' is reached where transition occurs to solid state S'; the line S'L' represents 25 atm. pressure. The energy and volume of these points are as follow⁶⁾:

Table 1. Energy and volume of solid and liquid helium

	Temperature	Pressure	Volume	Energy	
L'	0°K	25 atm	23.2 cm^3	-13.13 $\frac{\text{cal}}{\text{mol}}$	Difference 1.08 cal/mol
S'	0	25	21.3	-12.05	
L	0	0	27.6	-14.2	
L_λ	2.19	0	27.36	-11	

Peculiarity of condensed helium that the solid curve E_L^0 lies above the liquid curve E_L^0 is seen in Fig. 1, while in case of ordinary substances the liquid state has higher energy than the solid state.

This situation seems to predict that as the temperature is increased some parts of liquid helium II will be excited by thermal fluctuation to crystalline state, but the entropy condition will favour to form large number of these crystallines rather than to grow large crystals, and the number of crystallines formed will increase with in-

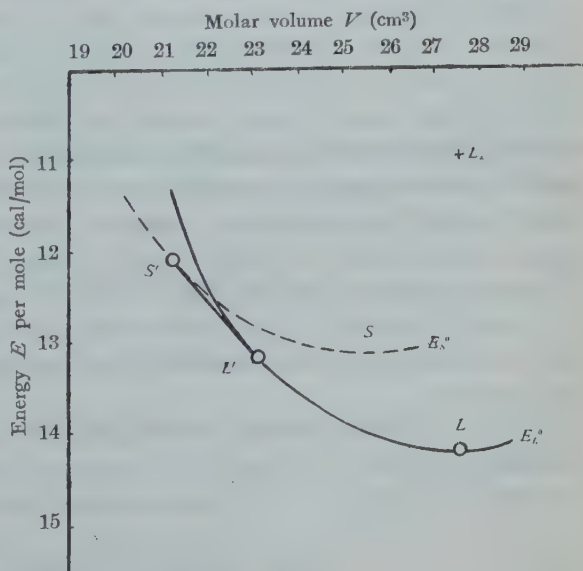


Fig. 1. Energy of liquid and solid helium plotted as a function of volume.

creasing temperature up to the λ -point. This is just the reverse phenomenon to the crystalline-germ formation in ordinary liquid as the temperature is lowered to approach to the freezing point. However, that only one atom is excited to the crystalline state is obviously meaningless; there must exist certain lower limit for the excitation of atoms, and there must be also upper limit being restricted by the energy increment for crystalline formation, and the size of the crystalline will be restricted within a rather narrow range. It will thus suffice to assume that a definite number of atoms are excited to form a "molecule" or a crystalline state. The terminology "molecule" was introduced to speak of the excitation of a group of atoms. Molecular part of the liquid helium will have viscosity of gaseous character and will form the so-called "normal" part.

Ordinary helium atoms with mass-number 4 i. e. He^4 and their aggregate molecules as well obey Bose-statistics. In the next section Bose-statistics of these helium atoms and molecules will be developed with caution as to the lowest state. The lowest or the zero-state is not necessarily the zero-state of one body approximation but rather means many-body approximation, whose wave function will be described by the coordinates of the whole system, that is $\psi_0(1, \dots, N)$, and when a molecule is formed wave function will take the form $\chi\psi_0(1, \dots, N)$, χ being the wave function of the molecule.

§ 2. Bose statistics of atoms and molecules

Let us consider a system of N helium atoms He^4 , Bose-particles, contained in a box of volume V . We assume that by thermal agitation a definite number x of atoms can be simultaneously excited to form a stable molecule, excitation energy of which we shall denote by Δ . By collision or so it will be possible that $2x$ atoms form a larger molecule or globule of atoms with energy about 2Δ . But this globule will immediately separate into two molecules containing x atoms each, as the entropy is more favourable for the latter. Molecules may have rotational motion, but excitation of rotational quantum number will require somewhat larger energy, and we assume that rotation can be neglected. Molecules or excitation quanta Δ have translational motion whose energy levels we shall denote by j , and its energy by ϵ_j . On the other hand the system has Debye phonons of thermal vibration, we shall assume this to be independent of the number of molecules formed.

Let n_j be the number of molecules in translational quantum state j , degeneracy of which is g_j , and let l_i be the number of phonons with energy $\hbar\nu_i$, degeneracy of which is f_i . Thermodynamic probability is then

$$W = \prod_j \frac{(n_j + g_j - 1)!}{n_j! (g_j - 1)!} \prod_i \frac{(l_i + f_i - 1)!}{l_i! (f_i - 1)!} \quad (1)$$

with entropy of the system $S = k \log W$, where k is the Boltzmann constant. We

assume that $g_0=1$, non-degeneracy of the lowest state; then we can omit the atomic state $j=0$ in this expression. We shall denote by N_0 the number of atoms in the lowest state, and by ϵ_0 its energy. Fixing the number of atoms

$$N=N_0+x\sum_j n_j \quad (2)$$

and energy of the system

$$E=\epsilon_0 N_0+\sum_j (\Delta+\epsilon_j)n_j+\sum_i h\nu_i l_i, \quad (3)$$

we maximize the entropy. For brevity's sake we shall put

$$\epsilon_0=0. \quad (3')$$

Then

$$\begin{aligned} & \delta \log W - \frac{a}{x} \delta N - \beta \delta E \\ &= -\frac{a}{x} \delta N_0 + \sum_j \delta n_j \left\{ \log \frac{n_j + g_j - 1}{n_j} - a - \beta (\Delta + \epsilon_j) \right\} \\ &+ \sum_i \delta l_i \left\{ \log \frac{l_i + f_i - 1}{l_i} - \beta h \nu_i \right\} = 0. \end{aligned} \quad (4)$$

Thus we are led to the well-known Bose distribution laws, namely

$$n_j = g_j / [e^{a+\beta(\Delta+\epsilon_j)} - 1], \quad (5)$$

$$l_i = f_i / [e^{\beta h \nu_i} - 1]. \quad (5')$$

Inserting the equilibrium values of n_j we have for the molecular part of thermodynamical quantities namely the number of molecules n^* , energy E and entropy S of the system as follow:

$$n^* = \frac{N - N_0}{x} = \sum_j g_j / [e^{a+\beta(\Delta+\epsilon_j)} - 1], \quad (6)$$

$$E = \sum_j (\Delta + \epsilon_j) g_j / [e^{a+\beta(\Delta+\epsilon_j)} - 1], \quad (7)$$

$$S = \frac{E}{T} + a n^* k T - k \sum_j g_j \log [1 - e^{-a-\beta(\Delta+\epsilon_j)}], \quad (8)$$

where as usual

$$\beta = 1/kT, \quad (9)$$

T being the absolute temperature. Accordingly translational motion separates from vibrational modes, and Bose-condensation will take place with respect to the

former, whose distribution law contains u . However, two circumstances occur as to the variation of δN_0 .

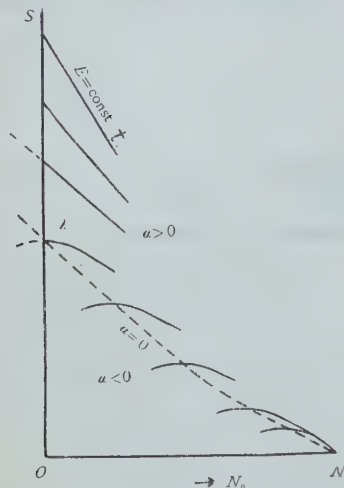


Fig. 2. Entropy versus number of zero atoms N_0 .

($E = \text{const.}$ curves, schema).

It can be shown that

$$(\partial S / \partial n^*)_E = ak.$$

1) If we put

$$\delta N_0 = 0, \quad \text{or } N_0 = \text{const.}$$

we have the equation (6) for the condition determining u , from which S is given as a function of the number of the zero state atoms N_0 , but it is easy to see that above a certain temperature, given by $N_0 = 0, u = 0$ (λ -point), entropy takes its maximum value when

$$N_0 = 0 \quad (T > T_0). \quad (10)$$

This is the case of He I (above the λ -point).

2) On the other hand we may vary $\delta N_0 (\neq 0)$, for which we have

$$u = 0 \quad (T < T_0). \quad (11)$$

In this case N_0 is determined by the condition

$$n^* = \frac{N - N_0}{x} = \sum_j g_j / [e^{\beta(\epsilon_j + \epsilon_1)} - 1]. \quad (11')$$

It must be that $n^* \leq N/x$, which is satisfied for $T \leq T_0$: this is the case of He II (below the λ -point). Probable curves of these two cases are sketched schemati-

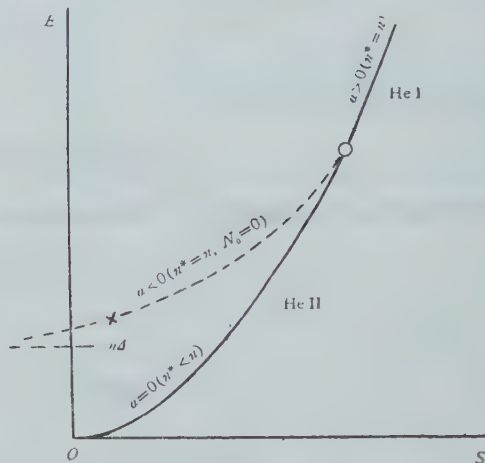


Fig. 3. Energy-entropy diagram (schema) of liquid helium.

$\partial E / \partial S = T$ gives absolute temperature. \times means the temperature $a + \beta(\epsilon_1 + \epsilon_2) = 0$.

cally in Fig. 2. Entropy versus Energy curve will take the form of Fig. 3 (c.f. Eq. (28) below).

β is related to the temperature T , the slope of the curves in the figure, by Eq. (9).

$$n = N/x \quad (12)$$

means the number of molecules when all the atoms are excited (above the λ -point). For He I, $n^* = n$. For He II, $a=0$, and n^* is the number of molecules in Eqs. (6), (7) and (8).

As to the translational motion we assume that excitation quanta Δ behave as if they form ideal gas in an effective volume V_e . Then we have for the density of energy level the expression

$$g_\varepsilon = g(\varepsilon) d\varepsilon = A \varepsilon^{1/2} d\varepsilon, \quad (13)$$

where

$$A = V_e 2\pi (2\mu/k^3)^{3/2}, \quad (13')$$

in which μ is the mass of a molecule

and h the Planck's constant. We have then after some alteration

$$n^* = A e^{-\alpha - \beta \Delta} (kT)^{3/2} f(1/2, b), \quad (14)$$

$$E = n^* \Delta + A e^{-\alpha - \beta \Delta} (kT)^{5/2} f(3/2, b), \quad (15)$$

$$S = \frac{n^* \Delta}{T} + \frac{5}{3} \frac{1}{T} e^{-\alpha - \beta \Delta} (kT)^{5/2} f(3/2, b) + a n^* k, \quad (16)$$

$$F = -a n^* k T - \frac{2}{3} A e^{-\alpha - \beta \Delta} (kT)^{3/2} f(3/2, b). \quad (17)$$

In these formulae

$$b = e^{-\alpha - \beta \Delta}, \quad (18)$$

for He II $a=0$; for He I n^* is to be replaced by $n (=N/x)$, and

$$f(s, b) = \int_0^\infty \frac{\xi^s d\xi}{e^\xi - 1} = \Gamma(s+1) \left(1 + \frac{b}{2^{s+1}} + \frac{b^2}{3^{s+1}} + \dots \right), \quad (19)$$

where $\Gamma(s)$ means Γ -function.

For the range of He II ($a=0$) we may assume that

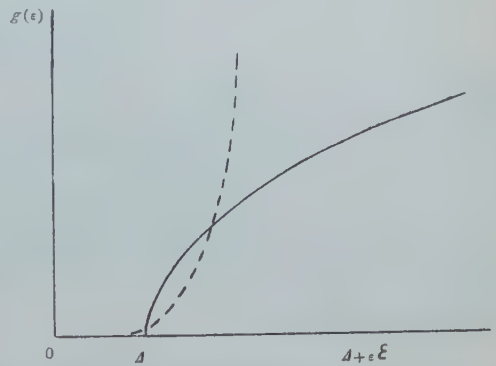


Fig. 4. Density of energy levels (see § 3)

$$\beta\Delta \gg 1, \quad (20)$$

and the above formulae reduce to

$$n^* = J(T)e^{-\beta\Delta}, \quad (21)$$

$$E = J(T)e^{-\beta\Delta} \Delta \left(1 + \frac{3}{2} \frac{kT}{\Delta} \right), \quad (22)$$

$$S = J(T)e^{-\beta\Delta} \frac{\Delta}{T} \left(1 + \frac{5}{2} \frac{kT}{\Delta} \right), \quad (23)$$

$$F = -J(T)e^{-\beta\Delta} kT, \quad (24)$$

where

$$J(T) = A\Gamma(3/2) (kT)^{3/2} = (2\pi\mu kT/h^2)^{3/2} V_e. \quad (25)$$

The last equations are formally identical to these given by Boltzmann statistics, a trivial consequence of the large energy of excitation i. e. $\Delta \gg kT$. These equations are formally identical to those of Landau³⁾. Molar heat is given by $C_v = T\partial S/\partial T$ or, for He II,

$$C_v = J(T)e^{-\beta\Delta} \frac{\Delta^2}{kT^2} \left\{ 1 + 3 \frac{kT}{\Delta} + \frac{15}{4} \left(\frac{kT}{\Delta} \right)^2 \right\}. \quad (26)$$

For He I we have

$$C_v = \frac{3}{2} nk. \quad (27)$$

Curvature of the figures in Fig. 3 may be known by means of the relation

$$\partial^2 E / \partial S^2 = \partial T / \partial S = T / C_v. \quad (28)$$

Dotted curve ($a < 0$) in Fig. 3 corresponds to $C_v = \frac{3}{2} nk$ which is smaller than C_v of the curve $a = 0$, and therefore has larger curvature.

§ 3. Comparison with experiments

We will now compare the preceding results with experiments. As Tisza has pointed out, for liquid helium above 1°K contribution of phonons to entropy can be neglected. Entropy expression of our theory can be written as

$$S = n \frac{\Delta}{T} e^{\Delta/kT_0} e^{-\Delta/kT} \left(\frac{T}{T_0} \right)^{1/2} \left(1 + \frac{5}{2} \frac{kT}{\Delta} \right) \quad (29)$$

where $n=N/x$ is the number of molecules at the λ -point T_0 . Experimental value of Kapitza reveals to be well expressed by this formula with constants

$$\Delta = 4.0kT_0 \quad (\Delta/k = 8.76^\circ K) \quad (30)$$

and

$$n = 0.122N, \quad x = 8.22 \quad (31)$$

as will be seen in Fig. 5.

Energy of He II at $0^\circ K$ is

$$E_0 = -14.2 \text{ cal/mol.}$$

Let us make direct calculation of the energy of liquid helium at the λ -point. For excitation of molecule energy Δ is necessary, and translation energy of this molecule is $(3/2)kT$ as Boltzmann statistics is permissible, therefore at T_0 the energy of liquid of 1 mole atoms is given by (see Eq. (27))

S cal./mol. deg.

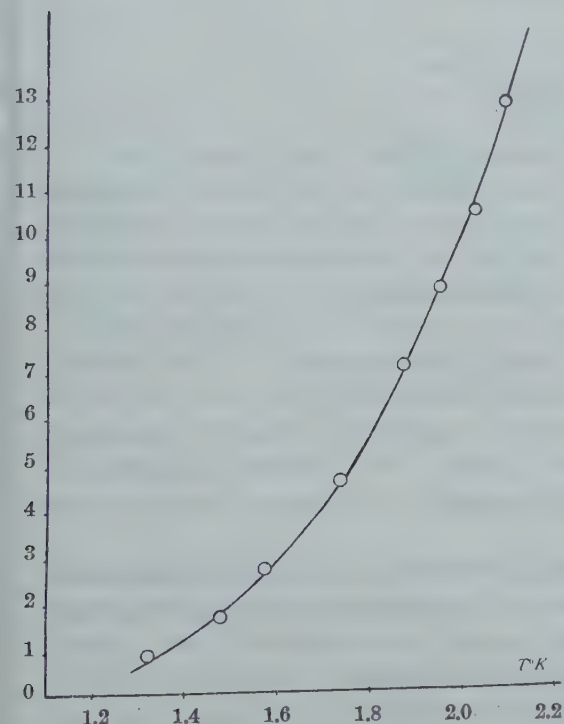


Fig. 5. Entropy-temperature diagram.

○ Experiments. Solid line: theoretical curve with $\Delta = 4.0kT_0$, $x = 8.2$

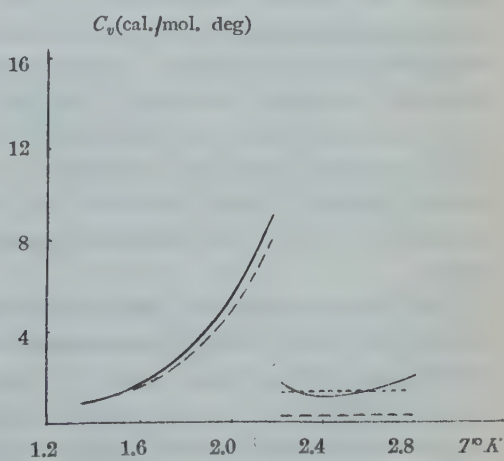


Fig. 6. Molar heat of liquid helium.

— Experiments
 --- theory, with a single Δ
 theory, Eq. (33)

$$\begin{aligned}
 E &= E_0 + n(\Delta + 3kT_0/2) \\
 &= -14.2 + (2.2 + 0.8) \text{ cal/mol.} \\
 &= -11 \text{ cal/mol.}
 \end{aligned}$$

This is the value just given by experiments.

Specific heat given by (26) and (27) has obviously λ -type, and in the He II range coincides with the observed values as is seen in Fig. 6.

Transition temperature T_0 will be given by

$$N/x = (2\pi x m k T / h^2)^{3/2} V_e e^{-\Delta/kT_0}. \quad (32)$$

As a rough approximation we shall assume the effective volume to be identical with real volume,

$$V_e = V = 27.36 \text{ cm}^3,$$

the molar volume at the λ -point. Inserting the above values,

$$\Delta/kT_0 = 4.0, \quad x = 8.2$$

we have

$$T_0 = 2.7^\circ \text{K}$$

which is to be compared with the real λ -point, 2.19°K .

If the temperature of He I is raised sufficiently, energy as well as entropy contribution from atomic oscillation will predominate and the effect of molecular motion will become comparatively small, approaching the so-called quasi-crystalline structure of liquid. But as the critical temperature of He I is only 5.2°K , assumed molecular structure can be expected to remain up to this temperature. Recently ultrasonic absorption in liquid helium was measured.⁶⁾ It was revealed that above 3°K the classical theory of absorption strictly holds, and that below 3°K observed absorption coefficient exceeds considerably classical theoretical values, having a sharp maximum at the λ -point. This anomalous absorption is presumably due to the transition caused by sound waves between molecules and atoms.

We have seen above that our theory gives nearly the same numerical results as was given by Tisza's theory with respect to the temperature dependence of entropy (as well as second sound of liquid helium as we shall see later). Formally speaking our results have exponential temperature dependence $\sim e^{-\Delta/kT}$, and those of Tisza¹⁾ and F. London⁷⁾ have T^r terms as the main factor. But we have already known that these two different forms are equally well to represent the observed values above 1°K ; there must, therefore, be certain formal relationship between these theories.

Indeed it is plausible enough that excitation energy Δ is not restricted to a sharp value but rather broadly distributed. We shall assume therefore that for the sake of internal degree of freedom or so, levels of excitation energy is distributed with density $D(\Delta)$. We shall denote by $f(\epsilon)$ the density of energy levels for a molecule with translation energy ϵ_p (p =momentum) and internal energy Δ . Number of levels of total energy between $\epsilon (= \epsilon_p + \Delta)$ and $\epsilon + d\epsilon$ will be given by

$$\begin{aligned} f(\epsilon) d\epsilon &= A \int \int \epsilon_p^{1/2} D(\Delta) d\epsilon_p d\Delta \\ &\quad \epsilon \leq \epsilon_p + \Delta < \epsilon + d\epsilon \\ &= A d\epsilon \int_0^{\epsilon} \sqrt{\epsilon - \Delta} D(\Delta) d\Delta \end{aligned}$$

where $A = V_e 2\pi (2\mu/k^2)^{3/2}$ as before. If the distribution of D is of the form

$$D(\Delta) = a\Delta^s$$

we have

$$f(\epsilon) = Aa\epsilon^{s+3/2} \int_0^1 \sqrt{1-x} x^s dx = C\epsilon^{s+3/2}. \quad (33)$$

With

$$C = \frac{\Gamma(s+1)\Gamma(3/2)}{\Gamma(s+5/2)} Aa.$$

Experimental formula used by F. London and by Tisza for entropy measurement is

$$S = \text{const. } T^r, \quad r \simeq 5.5.$$

Now, it will be easily seen that entropy can be written for Bose-statistics as

$$S = un^*k + \frac{1}{T} \int_0^\infty \frac{\epsilon f(\epsilon) + F(\epsilon)}{e^{\alpha+\epsilon/kT} - 1} d\epsilon$$

with

$$F(\epsilon) = \int_0^\epsilon f(\epsilon) d\epsilon$$

provided if $F(\epsilon) \log(1 - e^{-\alpha - \beta(\Delta + \epsilon)})$ vanishes for $\epsilon \rightarrow 0$ and for $\epsilon \rightarrow \infty$. To obtain the above experimental formula we must therefore put $f(\epsilon) \sim \epsilon^{r-1}$, or $s = r - 5/2$. Then

$$\begin{aligned} S &= \frac{1+r}{r} C(kT)^r k \int_0^\infty \frac{\zeta^r d\zeta}{e^\zeta - 1} = \frac{\Gamma(r+2)}{r} C(kT)^r k, \\ n^* &= C(kT)^r \int_0^\infty \frac{\zeta^{r-1} d\zeta}{e^\zeta - 1} = \Gamma(r) C(kT)^r, \end{aligned}$$

or

$$S = (r+1)n^*k.$$

At the λ -point $n^*=n$, and for the value of C we have

$$C = \frac{r}{\Gamma(r+2)} \frac{S_0}{(kT_0)^r k} = \frac{n}{\Gamma(r)} \frac{1}{(kT_0)^r}.$$

Therefore for $s=3$ ($r=5.5$) we have

$$a = \frac{\Gamma(5.5)}{\Gamma(1.5)\Gamma(4)} \frac{C}{A} = \frac{1}{6} \frac{n}{V_e} \left(\frac{k^2}{2\pi\mu k T_0} \right)^{3/2} \frac{1}{(kT_0)^4} \cong 0.005/(kT_0)^4.$$

If we write

$$a\mathcal{A}^3 d\mathcal{A} = 4\mathcal{A}^3 d\mathcal{A}/\mathcal{A}_m^4$$

so that

$$\int_0^{\mathcal{A}_m} D(\mathcal{A}) d\mathcal{A} = 1,$$

$$\mathcal{A}_m/k = \left\{ 24(V_e/n) (2\pi\mu k T_0/k^2)^{3/2} \right\}^{1/4} T_0 \cong 5T_0 \cong 11^\circ\text{K}.$$

Since distribution $D(\mathcal{A})$ for large value of \mathcal{A} is insignificant, we may cut off $D(\mathcal{A})$ for $\mathcal{A} \geq \mathcal{A}_m$. We see that \mathcal{A}_m is the same in order to that given assuming single value of \mathcal{A} , Eq. (30). Density of energy levels $f(\epsilon)$ is given in Fig. 4 by the broken curve. Since by Eqs. (13) and (21)

$$g(\epsilon) = n e^{\mathcal{A}/kT_0} (\epsilon/kT_0)^{1/2} / (\Gamma(3/2) kT_0),$$

it is convenient to compare the density of energy levels by taking the ratio, that is

$$f(\epsilon)/g(\epsilon) = \Gamma(3/2) (\epsilon/kT_0)^{r-1} e^{-\mathcal{A}/kT_0} / \Gamma(r) (\epsilon/kT_0)^{1/2},$$

putting

$$\epsilon = \mathcal{A} + \epsilon (\epsilon \geq 0), \quad \mathcal{A}/kT_0 = 4.0, \quad r = 5.5,$$

we have

$$f(\epsilon)/g(\epsilon) = 0.0003 (\epsilon/kT_0)^{4.5} / (\epsilon/kT_0)^{0.5}.$$

With this modified density of energy levels we have the following entropy expression:

$$S = S_0 (T/T_0)^r, \quad r = 5.5,$$

$$S_0 = 1.62 \text{ cal/mol. deg.},$$

namely that given experimentally by Tisza and by F. London. This modification has little effect on theoretical values in the range of He II, where, therefore, the expressions in the preceding section with single value of Δ are quite applicable.

However, the difference between (13) and (33) becomes significant when the He I range is concerned. Eq. (13) gives for the molar heat of He I the value $C_v = (3/2)nk \cong 0.37$ cal/mol., which is obviously too small compared with the observed values. On the other hand the density of levels of (33) increases much steeper than that of (13) and hence has larger specific heat. Indeed it is easy to see that using (33) we have for He I

$$C_v = 5.5nk \cong 1.4 \text{ cal/mol. deg.}$$

which coincides with the observed values as is seen in Fig. 6 (dotted curve). In addition, it is obvious that phonon contribution on the specific heat is negligible both below and above the λ -point.

§ 4. Conjecture

The excitation energy Δ is of course due to the change in kinetic as well as potential energy of atoms when they rearrange themselves, say, from liquid to solid-like aggregation state. This seems to be a kind of transition in lattice structure; however, as will be seen soon later, this energy change Δ seems to be rather surface effect than body effect. We mean by surface effect the energy increment at the boundary of the globule of atoms or molecule contacting with zero-state atoms. Body effect, on the contrary, is the contribution from interior of the molecule,

To clarify the circumstances, we shall consider that the molecule forms solid lattice designated by S in Fig. 1, and shall make use of the energy difference between states S and L as the body effect, which is about 1 cal per mole of atoms. If a molecule contains eight atoms, energy difference is about eight cal per mole of molecules or corresponds to 4°K of temperature which is to be compared with $\Delta/k \cong 9^\circ\text{K}$. The difference of about 5°K shall be attributed to the surface effect owing to the distortion of atomic arrangement near the surface of molecules, where the atomic distance will be elongated and potential energy between atoms will increase. But this effect will be accompanied by a decrease in zero-point motion and therefore above estimation of body effect seems to be over-estimating. Thus surface effect will exceed 5°K and main contribution to the excitation energy Δ will come from surface effect.

Somewhat different consideration will be afforded: indeed if the eight atoms in a molecule form a regular cube, half of the atomic bonds will go out of the molecular surface and the increment will come from these. Assuming simple cubic arrangement and nearest neighbour approximation throughout, the energy

within a molecule can be estimated as energy of twelve bonds. Since three bonds are to be attributed to each atom, the excitation energy due to bonds is roughly $12 \cdot (1 \text{ cal}/3) = 4 \text{ cal}$ per mole of molecules. The rest energy, corresponding to temperature of about 7°K , will be of surface effect.

Atomic globules with smaller as well as larger number by one or two of atoms will have irregular form and therefore larger excitation energy; excitation of these atomic globules will be impossible. But globules with several eights atoms or so may have regular form and may be formed.

Indeed by collision of two molecules larger atomic globule containing about sixteen atoms will be formed, resulting instantaneous association. This effects the non-ideality of molecular gas, and improvement of our theory will be afforded by taking into account the effect of molecular association, or, in other words, its effect on the effective volume V_e characterizing the free space of molecules. As the first approximation we may put $V_e = V$, from which we have obtained the transition temperature as $T_0 = 2.7^\circ\text{K}$ (see Eq. (33)).

That this value is rather too high means that in a matter of fact

$$V_e > V. \quad (34)$$

Second approximation will be accomplished by taking into account the binary collision of molecules. It seems worth noticing that nearly the same circumstances to Eq. (Eq. (34)) occurs as to the case of gaseous helium: that is, the second virial coefficient $B(T)$ becomes negative at the temperature of liquid helium, Boyle point $T_B (B(T_B) = 0)$ being about 15°K for helium. So that if we put the equation of state of gaseous helium as $pV_e' = NkT$, we have for sufficiently low temperatures the relation

$$V_e' = V - B(T) > V.$$

We have to consider the association or collision of molecules in more detail. As we have noticed before, during the collision process of two molecules the atomic globules will interpenetrate each other and a kind of association takes place. The collision will therefore be schematically represented as is seen in Fig. 7. During the processes A, B and C, or the reverse, the whole volume is maintained same for it contains $2x$ atoms always.

To estimate the variation of potential energy, we shall assume that excitation energy \mathcal{A} is a kind of surface energy. When two molecules are in contact (A in Fig. 7) their surface area A_A is larger than the area A_C when their centres of gravity coincide and form a large globule (C in the figure). The volume being the same, we have

$$(A_A - A_C)/A_A = 1 - 2^{-1/3} = 0.206.$$

Therefore, according to our assumption, the energy difference between situations A and C will be given as

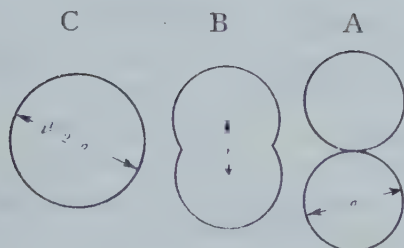


Fig. 7. Collision of two molecules.
A; molecules contact, B; interpenetrate,
C: forming a larger globule.

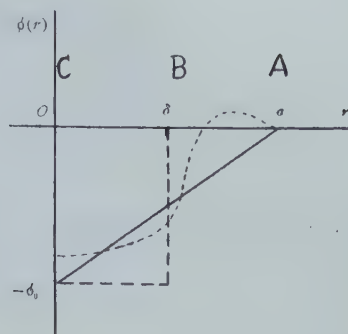


Fig. 8. Potential energy of colliding molecules as a function of the separation r .

$$\phi_0 = (1 - 2^{-1/3})\Delta \approx 0.214\Delta. \quad (35)$$

As an approximation we may assume that potential energy is a linear function of the distance r of the centres of gravity of the two molecules in consideration, then potential energy will be given by the solid line in Fig. 8 as a function of the distance r of the centres of the two molecules in consideration, then potential energy will be given by the solid line in Fig. 8 as function of molecular separation. As another approximation broken line in the figure will also be used.

When the mutual interaction between molecules is present, we have for the energy of the system, instead of Eq. (3), the expression

$$E = \sum_j (\Delta + \epsilon_j) n_j + U, \quad (36)$$

where

$$U = \sum_{\mu > \nu} n_\mu n_\nu \phi_{\mu\nu} \quad (36')$$

and $\phi_{\mu\nu}$ represents the interaction energy between two molecules at μ and ν respectively in position-space, being a function of the distance $r_{\mu\nu}$. In place of Eqs. (6), (7) and (8), we have then

$$n_j = g_j / [\exp \{a + \beta(\Delta + \epsilon_j + \partial U / \partial n_j)\} - 1], \quad (37)$$

$$S = ak \sum_j n_j + \frac{\Delta}{T} \sum_j n_j + \frac{5}{3} \frac{1}{T} \sum_j n_j \epsilon_j + \frac{1}{T} \sum_j n_j \frac{\partial U}{\partial n_j}, \quad (38)$$

$$F = -ak \sum_j n_j - \frac{2}{3} \sum_j n_j \epsilon_j + (U - \sum_j n_j \partial U / \partial n_j). \quad (39)$$

We see that from Eq. (35)

$$U = \frac{1}{2} \sum_{\mu} n_{\mu} \partial U / \partial n_{\mu}, \quad U - \sum_{\mu} n_{\mu} \partial U / \partial n_{\mu} = -U$$

and as to the position-space summation can be replaced approximately by

$$n_{\mu} = (n^* - 1) d\tau_{\mu} / V \quad (n^* = \sum_j n_j), \quad (40)$$

when $\beta A \gg 1$ we have from Eqs. (37) and (40)

$$n^* = \sum_j n_j \cong n_0^* \left[1 + \frac{n^* - 1}{V} \int_0^{\infty} 4\pi r^2 dr \{-\phi(r)\} \right] \quad (41)$$

where

$$n_0^* = \sum_j g_j e^{-a - \beta(A + \epsilon_j)} \quad (41')$$

means the number of excited molecules in the first approximation (see Eq. (6)). We notice the so-called virial coefficient

$$B(T) = 2\pi n^* \int_0^{\infty} r^2 dr (1 - e^{-\phi(r)/kT}) \cong \frac{2\pi n^*}{kT} \int_0^{\infty} r^2 dr \phi(r)$$

is appearing in the above expressions.

Now that $\phi(r)$ is given as in Fig. 8, we have

$$\begin{aligned} 4\pi \int_0^{\infty} \phi(r) r^2 dr &= -(\pi/3) \phi_0 \sigma^3 (\sim - (4\pi/3) \phi_0 \delta^3) \\ &= -0.42(\pi/3) A \sigma^3 (\sim -0.42(4\pi/3) A \delta^3) \end{aligned}$$

(the values in the bracket correspond to the potential energy of broken line in Fig. 8). From Eq. (41) we have then for the effective volume

$$V = V + 0.22(A/kT) n^* \sigma^3 (\sim V + 0.88(A/kT) n^* \delta^3). \quad (42)$$

If we insert the observed value $T_0 = 2.19^\circ\text{K}$ in Eq. (33), we have

$$V_e = 34.77 \text{ cm}^3 (T = T_0)$$

at the λ -point. And since $A/kT_0 = 4.0$, $n = N/8.2$, we have the following value for the dimension of a molecule ($\sigma = 2\delta$)

$$\sigma = 4.9 \text{ \AA} (\sim 6.2 \text{ \AA}) \quad (43)$$

which is a plausible size for a molecule containing about eight atoms as we shall see below. But it must be remarked that above value of ϕ_0 is somewhat overestimated, since we have neglected the body effect of interaction and attributed whole effect to the surface energy. If the body effect is taken into consideration associa-

tion energy will be diminished and we will have the potential curve dotted in Fig. 8. Another small effect that molecules will deform when they contact is also included; for this effect potential curve will have a small maximum. Now that ϕ_0 is overestimated, actual diameter of a molecule will be somewhat larger than that estimated above, say $6 \sim 7 \text{ \AA}$.

We can estimate the molecular size by another method: we shall assume that in a molecule eight atoms arrange themselves in regular cubic form simple cubic. If the molar volume of solid helium $V = 21.3 \text{ cm}^3$ is used we obtain $a = 3.28 \text{ \AA}$ for the atomic spacing, and for the diagonal of the cube $\sqrt{3}a = 5.67 \text{ \AA}$. Therefore the molecular diameter will be

$$\sigma \cong 5.88 \text{ \AA} \sim 8.27 \text{ \AA}. \quad (43')$$

We shall employ $\sigma \cong 7 \text{ \AA}$ for the interaction diameter.

§ 5. Viscosity of liquid helium.

Extremely low viscosity of He II has been interpreted by Tisza to be the consequence of diminishing number of the so-called normal component as the temperature is lowered. After the present consideration normal component is of molecules. This component resists shearing force and if η_n and ρ_n are the coefficient of viscosity and density respectively of this component, and ρ is the density of liquid helium, the apparent viscosity η will be

$$\eta = (\rho_n / \rho) \eta_n \quad (44)$$

which diminished sharply below the λ -point. Above the λ -point η_n will be the viscosity of He I, which is, as Tisza has pointed out, of same order as that of gaseous helium, but is about two times greater than the latter.

Viscosity of He I follows the $T^{1/2}$ -law of kinetic theory of gases, and its nature will be, therefore, a kinetic one as Tisza has proposed. This argument is certainly correct, and normal component of He II as well as He I must be considered to be gas-like with respect to viscosity character.

Tisza assumed that super-fluid component of He II has vanishing microscopic rigidity. The most spectacular effect of the microscopic rigidity is freezing. Large zero-point motion of helium atoms prevents liquid from solidification. In general, viscosity of liquids consists of two parts, as Born and Green⁹⁾ have pointed out, one of which is kinematical and the other comes from interatomic potential energy. Rigidity of solid against shear is typical of potential part. This potential part of rigidity seems to vanish in case of liquid helium, and for this reason lattice structure, if it were formed, will be unstable against strain or even against thermal agitation, and solid cannot be formed as can be predicted by Born's theory of fusion⁹⁾.

We shall now calculate the potential part of rigidity by the so-called potential-cage method¹⁰⁾. We notice a central atom enclosed by its nearest neighbours, and assume they form lattice. Let $\phi(r)$ be the potential energy between atoms r apart and a be the atomic distance when distortion is absent. If the number of nearest neighbours is Z , potential energy with respect to the central atom is given by

$$u_0 = Z\phi(a).$$

Now pure shearing distortion vertical to z -axis and along x -direction displaces atoms on the sphere S_0 of radius a with coordinates (x_0, y_0, z_0) to (x, y, z) , where

$$x = x_0 + g z, \quad y = y_0, \quad z = z_0.$$

Then the potential energy can be calculated by the method of potential cage, smoothing the nearest neighbours on the sphere S_0 and make distortion. The result is

$$u = Z \iint \phi \sqrt{x^2 + y^2 + z^2} dS_0 / 4\pi a^2$$

If we expand the integrand for small distortion we have

$$u - u_0 = Z(1/30)g^2 \{4a\phi'(a) + a^2\phi''(a)\}.$$

Now making use of the Lennard-Jones¹⁰⁾ type of potential,

$$\phi(r) = \phi_0 \{ (r_0/r)^{12} - 2(r_0/r)^6 \},$$

we have

$$u - u_0 = Z\phi_0(1/30)g^2 \{108(r_0/a)^{12} - 36(r_0/a)^6\}. \quad (45)$$

From which rigidity G is given as

$$G = \frac{N}{V} \frac{6}{5} Z \{3(r_0/a)^{12} - (r_0/a)^6\} \phi_0.$$

For helium $r_0 = 2.05 \text{ \AA}$; and for $a = 3.94 \text{ \AA}$ assuming face centred lattice we have

$$G < 0, \quad \text{or} \quad u - u_0 < 0,$$

and for $a = 3.58 \text{ \AA}$ (simple cubic) G is scarcely different from zero ($|u - u_0|/g^2 \ll \phi_0$). It will be safe, therefore, to assert that rigidity vanishes in case of liquid helium. Zero-point motion of atoms makes atomic distance so far, that lattice formation seems to be impossible under these circumstances, and helium cannot solidify if sufficient pressure is absent.

As to the kinetic part of viscosity of He I we shall make calculation using

the well known formula

$$\eta = 0.499 \mu \bar{v} \nu l, \quad (46)$$

for Boltzmann statistics will suffice in dealing with molecules. In the above formula \bar{v} is the mean velocity, ν the number of molecules per unit volume and l the mean free path:

$$\left. \begin{aligned} \bar{v} &= \sqrt{8kT/\pi\mu}, \\ \nu &= n^*/V, \\ l &= 1/\sqrt{2} \pi \delta^2 \nu \end{aligned} \right\} \quad (47)$$

where δ means the effective diameter of molecules. We have argued that colliding molecules interpenetrate each other, potential energy being given as Fig. 8. Effective diameter will be given by the broken line in the figure, that is $\delta \cong \sigma/2$. We have shown, however, in the preceding section that molecular diameter σ is about 7\AA . Therefore effective diameter will be

$$\delta \cong \sigma/2 = 3 \sim 4\text{\AA}. \quad (48)$$

Data on the viscosity of He I is well represented by the formula

$$\eta = 14 \times 10^{-6} \sqrt{T} \text{ poises} \quad (49)$$

as is seen in Fig. 9. Inserting this formula and obtained value of $\mu = 8.2m_{\text{He}}$ in Eq. (47), we have

$$\begin{aligned} \delta^2 &= 166 \times 10^{-22} \sqrt{T} / \eta, \text{ c.g.s.} \\ &= 12 \times 10^{-16} \text{ cm}^2. \end{aligned} \quad (50)$$

We obtain thus from viscosity data

$$\delta = 3.5\text{\AA} \quad (50')$$

which coincides with the roughly estimated value of Eq. (48).

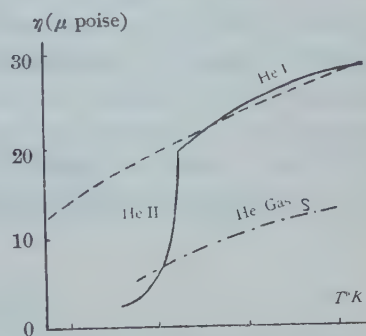


Fig. 9. Viscosity of Helium
---- Eq.(49)

§ 6. Hydrodynamics of He II

He II can be regarded as a mixture of two components, namely of superfluid component and normal component after Tisza¹¹. Superfluid component

consists of atoms, normal component of molecules. Therefore normal component is solute in superfluid component as solvent and its osmotic pressure p_n may be calculated by thermodynamical considerations, with the results analogous to the Clapeyron-Clausius Eq. i. e.

$$(\partial p / \partial T) = \rho s_n \quad (51)$$

where s_n means entropy per unit mass of normal component, and ρ the density. Neglecting entropy contribution of Debye phonons, which is correct for $T > 1^\circ\text{K}$, s_n can be replaced by the total entropy s per unit mass. With the help of experimental value of entropy we can obtain p_n , the osmotic pressure of normal component, as a function of temperature. As entropy is the derivative of free energy with respect to temperature the results may be written in terms of free energy or, using the 1st approximation (25), as an analogue of the van't Hoff's equation;

$$p = -F/V = n^* kT/V. \quad (52)$$

We see that molecules exert pressure as if they form an ideal gas.

Now, total pressure p in liquid helium is written as a sum of ordinary hydrodynamical pressure p_0 and osmotic pressure p_n :

$$p = p_0 + p_n. \quad (53)$$

As p_0 exerts on super ρ_s as well as on normal component ρ_n , but p_n only on normal component, if we write their velocity by v_s and v_n respectively, we have, neglecting viscosity, the following equations of motion

$$\rho_s dv_s/dt = -(\rho_s/\rho) \text{grad } p_0, \quad (54)$$

$$\rho_n dv_n/dt = -(\rho_n/\rho) \text{grad } p_0 - \text{grad } p_n, \quad (54')$$

with

$$\left. \begin{aligned} \rho &= \rho_s + \rho_n, \\ d\rho &= -d\rho/\alpha\rho, \\ dp_n &= \rho s_n dT \end{aligned} \right\} \quad (55)$$

where α means the compressibility, together with the equation of continuity:

$$\partial\rho/\partial t + \text{div } j = 0, \quad (56)$$

$$j = \rho_s v_s + \rho_n v_n. \quad (56')$$

These sets of equations, if an equation governing the transition between ρ_s and ρ_n and the equation of state are given, will describe the motion of fluid completely.

Now we will derive the so-called first approximation neglecting nonadiabatic transition from one component to the other. So far as dissipation effect is ignored this approximation seems to suffice. Then separate equations of continuity are given for the two components:

$$\partial \rho_s / \partial t + \operatorname{div}(\rho_s v_s) = 0, \quad (57)$$

$$\partial \rho_n / \partial t + \operatorname{div}(\rho_n v_n) = 0. \quad (57')$$

New variables, mean velocity v and relative velocity v' of normal component, will be introduced:

$$v = (\rho_s v_s + \rho_n v_n) / \rho, \quad v' = v_n - v_s. \quad (58)$$

When the velocity of flow is small, we neglect the terms containing square of velocities and obtain

$$\rho \, dv/dt = -\operatorname{grad} p, \quad (58')$$

$$\rho_n dv'/dt = -\operatorname{grad} p_n. \quad (58'')$$

Inserting Eq. (55) one has

$$dv/dt = \kappa^{-1} \operatorname{grad} \rho, \quad (59)$$

$$dv'/dt = (\rho/\rho_n) s \operatorname{grad} T. \quad (59')$$

For small velocities continuity equations become

$$\partial \rho / \partial t + \rho \operatorname{div} v = 0, \quad \partial s / \partial t + (\rho_s / \rho) s \operatorname{div} v = 0; \quad (60)$$

and equations (59) of motion take the following forms

$$\partial^2 \rho / \partial t^2 = (\rho / \kappa) \nabla^2 v, \quad (61)$$

$$\partial^2 s / \partial t^2 = (s^2 \rho_s / \rho_n) \nabla^2 T. \quad (61')$$

Since $T ds = c_2 dT$ the second equation may be written as

$$\partial^2 T / \partial t^2 = c_2^2 \nabla^2 T, \quad (62)$$

where

$$c_2 = \sqrt{\frac{T s^2}{c_v} \frac{\rho_s}{\rho_n}} \quad (63)$$

means the velocity of the so-called second sound. This result is the same as was given by Tisza and Landau.

In fact it will be shown that above equations are identical to those given by Tisza, and to Landau's equations if the square of velocities is neglected. Indeed

since $dg = -sdT + vdp$ (g = thermodynamic potential), $dv_s/dt = (1/\rho)\text{grad}(p - p_n) = -\text{grad}g$. And the merit of the present method lies rather in its intuitive method of derivation: it permits further calculations, for instance, taking into account the effect of viscosity. We have thus a set of hydrodynamic equations

$$\rho_s dv_s/dt = \rho_s F_s - (\rho_s/\rho)\text{grad } p_0, \quad (64)$$

$$\rho_n dv_n/dt = \rho_n F_n - (\rho_n/\rho)\text{grad } p_0 - \text{grad } p_n + \eta_n F^2 v_n + (1/3)\text{grad } \theta_n \quad (64')$$

with $\theta_n = \text{div } v_n$, and F means the body forces.

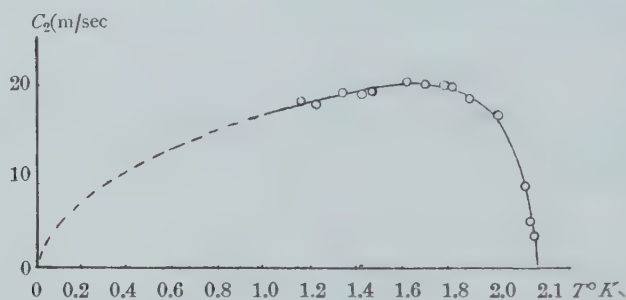


Fig. 10. Velocity of second sound
○ Measurements — calculated by Eq (65)

Inserting s , ρ_n , $\rho_s = \rho - \rho_n$ and c_v from (23), (21) and (26) into the above equation we have

$$c_2^2 = \frac{\left\{1 + \frac{5}{2}(kT/\Delta)\right\}^2}{1 + 3(kT/\Delta) + \frac{15}{4}(kT/\Delta)^2} \left(1 - \frac{n^*}{n}\right) \frac{nkT}{M}. \quad (65)$$

This is compared with experiments using the value $\Delta/kT_0 = 4.0$ in Fig. 10; the agreement is satisfactory. For sufficiently low temperatures we have ($x = 8.2$)

$$c_2 \sim \sqrt{nkT/M} = \sqrt{NkT/xM} = 16.1\sqrt{T} \text{ m/sec}, \quad (65')$$

the velocity of second sound proportional to the square root of the absolute temperature. But this does not hold, because phonon contribution, which we have neglected, becomes predominant below 1°K.

§ 7. Summary

Bose statistics of atoms and associated molecules is discussed and the λ -transition of liquid helium is assumed to be the phenomenon of the Bose-condensation. In He I all the atoms are in molecular state, each containing about eight atoms; as the temperature is lowered passing the λ -point certain molecules melt into the

lowest state which is atomic. The present theory is analogous to that of Tisza who proposed the possibility of excitation of groups of atoms into the normal state, but formally speaking rather to the equations of Landau's microscopic theory. In the present approximation the thermal oscillation of atoms or the Debye phonon is treated independent of molecular excitation, giving no influence on the occurrence of the Bose-condensation, and the terminology "normal" does not imply phonon excitation. Below 1°K this approximation seems to fail.

Molecules, whose excitation quanta behave gas-like, are those forming the normal component in the Tisza's macroscopic theory, having viscosity of gaseous character. Molecules interpenetrate each other when collision takes place. Action diameter of a molecule is estimated from absolute value of the λ -transition and from viscosity of He I to be about 6 or 7 Å.

He II consists of molecules dissolved in atomic helium of lowest state with vanishing viscosity. Molecules exert osmotic pressure. Intuitive treatment of this two components-system yields the thermo-hydrodynamics of He II with the results same to those of Tisza and of Landau as far as the velocity of flow is small and the transition between normal (molecular) and super (atomic) component is neglected.

The phenomenological theory of Tisza is interpreted from microscopic point of view. It is hoped that the assumption of molecule-forming character of helium atoms will be supported by experiments and by theoretical treatment thereof.

It is of course desired to see if the isotope He^3 has no λ -transition. Solution of He^3 and He^4 seems to deviate strongly from ideal law. In the present paper nothing is said with respect to the solution of He^3 or to the behavior of pure He^3 , which will be treated in the next paper.

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On the Liquid He^3 and its Mixture with He^4

Morikazu TODA and Akira ISIHARA

*Institutes of Physics, Tokyo Bunrika University
and Tokyo University*

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Recent experiments have shown that the λ -point of liquid helium is lowered by addition of the isotope He^3 . This depression of the λ -point is discussed theoretically by thermodynamics as well as by statistical mechanics.

§ 1. Introduction

The liquid of pure He^4 undergoes a transition at the so-called λ -point 2.19°K . Higher temperature phase is named as He I and the lower as He II , both are liquid phases and the transition is of second order. It was recently revealed¹⁾ that the λ -point is depressed if the isotope He^3 is mixed; for instance the λ -point of the solution containing 20.3% of He^3 (in mole fraction) is 1.63°K , about 26% lower than that of pure He^4 . When the concentration is raised to 80% the λ -point becomes 1.0°K , and with increasing the concentration it falls to 0°K as the pure liquid He^3 , whose normal boiling point is about 3°K , has no λ -transition.

The liquid He^4 , as is well known, consists of two parts, namely the normal (excited) part and the super (ground) part. The solution of He^4 and He^3 is therefore to be considered as mixture of three components, the concentration of the normal being the internal parameter determined by the condition of thermodynamical equilibrium. Generally speaking the characteristic function, free energy or so, of the solution is thus function of four variables, which are ξ the molar fraction of He^3 , y fraction of the atoms excited to the normal state, and two thermodynamic parameters such as the absolute temperature T and the pressure P or the volume V of the whole system. Though this functional form is not yet established, as to the entropy of mixing something is known since the experiment due to Taconis and others²⁾ on the vapor pressure of dilute solutions of He^3 in He^4 , from which they concluded that entropy of mixing of this solution is given as

$$S_{\text{mix}} = k \{ N_4^n \log N_4^n / (N_4^n + N_3) + N_3 \log N_3 / (N_4^n + N_3) \}, \quad (1)$$

where N_4^n represents the number of normal atoms and N_3 that of He^3 atoms; that is to say He^3 atoms mixes only with normal part of He^4 . When the concentration of He^3 is dilute the above mentioned hypothesis of Taconis suffices to treat the thermodynamics of the solution.

But for arbitrary concentration we need the knowledge of the chemical

potentials³⁾ of both He^3 and He^4 , which are known only in case of pure substances. It is known that the super part of He^4 has vanishing entropy, while the entropy of the normal part is approximately proportional to T^σ with $\sigma \cong 5.5$. The entropy S_4^0 of pure He^4 is given approximately by

$$S_4^0 = S_\lambda (T/T_\lambda^0)^\sigma, \quad S_\lambda = 1.62 \text{ cal/mol. deg.} \quad (2)$$

where S_λ is the entropy at the λ -point T_λ^0 of pure He^4 . The thermodynamic potential G_4^0 and the fugacity μ_4^0 or the chemical potential per mole for n_4 moles of pure He^4 are given as

$$G_4^0 = n_4 \mu_4^0 = - \int S_4^0 dT, \quad (3)$$

$$\mu_4^0 = E_0 - \frac{1}{\sigma + 1} S_\lambda T (T/T_\lambda^0)^\sigma. \quad (4)$$

In this approximation it is generally accepted, though not strictly correct, that the ratio y of the normal to the total number of atoms is given by

$$y = (T/T_\lambda^0)^\sigma \quad (5)$$

in case of HeII of pure He^4 .

§ 2. Thermodynamical treatment

For He II we choose expressions containing a parameter ν :

$$S = S_\lambda y \left\{ 1 + \frac{\sigma}{\nu(\sigma + 1)} \left[1 + \left(\frac{T_\lambda^0}{T} y^{1/\sigma} \right)^\nu \right] \right\} \quad (6)$$

with the enthalpy linear in y :

$$H = E + PV = H_0 + y \frac{\sigma}{\sigma + 1} S_\lambda T \quad (7)$$

and

$$\mu_4^0 = H_0 - \frac{\sigma + \nu}{\nu(\sigma + 1)} S_\lambda T y + \frac{\sigma}{\nu(\sigma + 1)} S_\lambda y^{(\sigma + \nu)/\sigma} T (T_\lambda^0/T)^\nu. \quad (8)$$

These expressions coincide with Kilpatrick⁴⁾ when $\nu = 1$, and with the well known expressions given by Eq. (2), (3) and (4) for pure He^4 .

Now the thermodynamic potential G of the solution is given by

$$G = n_3 \mu_3^0 + n_4 \mu_4^0 + RT \left\{ n_3 \log \frac{v_3 n_3}{v_3 n_3 + v_4 n_4} + n_4 \log \frac{v_4 n_4}{v_3 n_3 + v_4 n_4} \right\}, \quad (9)$$

where v_3 and v_4 denote molar volume of pure He^3 and He^4 respectively, and the last term is the entropy of mixing in which we have modified the Taconis' hypothesis by replacing the molar concentration in Eq. (1) by volume concentration as the latter is more suitable in case of solutions of molecules different in size.

We denote by y the fraction of excited atom, which is determined by the condition of thermodynamic equilibrium

$$\partial G / \partial y = 0. \quad (10)$$

Thus we have

$$-\frac{1}{RT} \partial \mu_4^0 / \partial y = \frac{(a-1)\xi_e}{(a-1)\xi_e + 1} + \log \frac{1-\xi_e}{(a-1)\xi_e + 1} \quad (11)$$

where ξ_e denotes effective concentration given by

$$\xi_e = n_3 / (n_3 + n_4^n) \quad (12)$$

and

$$a = v_3 / v_4. \quad (13)$$

Inserting the above expression for μ_4^0 , we get the following equation determining the concentration of the normal as a function of T and ξ :

$$\left\{ \frac{T_\lambda^0}{T} y^{1/\sigma} \right\}^\nu = 1 - \frac{\nu(\sigma+1)}{\sigma+\nu} \frac{R}{S_\lambda} \left\{ \frac{(a-1)\xi_e}{(a-1)\xi_e + 1} + \log \frac{1-\xi_e}{(a-1)\xi_e + 1} \right\} \quad (14)$$

which gives Eq. (5) in case of pure He⁴.

Assuming second order transition we put $y=1$ for $T=T_\lambda$ and obtain

$$\frac{T_\lambda}{T_\lambda^0} = 1 / \left\{ 1 - \frac{\nu(\sigma+1)}{\sigma+\nu} \frac{R}{S_\lambda} \left[\frac{(a-1)\xi}{(a-1)\xi + 1} + \log \frac{1-\xi}{(a-1)\xi + 1} \right] \right\}^{1/\nu}. \quad (15)$$

This gives the λ -point as a function of the concentration ξ of He³.

The actual value of a is about 1.5. To see the effect of a we have calculated T_λ/T_λ^0 for $a=1$ and 1.5 both for $\nu=1$. From the fact that the curves coincide almost completely we see that the effect of a is quite small. We shall take $a=1$ hereafter. Then we have

$$T_\lambda/T_\lambda^0 = 1 / \left\{ 1 - \frac{\nu(\sigma+1)}{\sigma+\nu} \frac{R}{S_\lambda} \log (1-\xi) \right\}^{1/\nu}. \quad (16)$$

For $\nu=\infty$, $T_\lambda=T_\lambda^0$ and for $\nu=0$ it is easily shown that

$$T_\lambda/T_\lambda^0 \rightarrow (1-\xi)^{\frac{\sigma+1}{\sigma} R/S_\lambda}. \quad (17)$$

Fig. 1 represents these curves, we see that the curve $\nu=0.5$ reproduces experimental values very well.

If the transition is assumed first order equations determining the transition are

$$(\partial G / \partial n_3)_I = (\partial G / \partial n_3)_{II}$$

$$(\partial G / \partial n_4)_I = (\partial G / \partial n_4)_{II}$$

where suffixes I and II represent He I and He II phases respectively. The first of these gives

$$\xi_e^I = \xi_e^{II}$$

and the second gives

$$RT_\lambda \left\{ \frac{(a-1)\xi_e}{(a-1)\xi_e + 1} + \log \frac{1-\xi_e}{(a-1)\xi_e + 1} \right\}_\lambda = - \left(\frac{\mu_{II}^0 - \mu_{4II}^0}{y_I - y_{II}} \right)_\lambda$$

from which T_λ is seen as a function of y_I , y_{II} , ξ , μ_{4I}^0 , and μ_{4II}^0 . These two equations give consequently T_λ as a function of y_I , ξ , μ_{4I}^0 and μ_{4II}^0 . Of these μ_{4II}^0 and y_{II} are given by Eq. (8) and (14), μ_{4I}^0 remains unknown.

For second order transition the right-hand side of the last equation must be replaced by $-(\partial\mu_{4II}^0/\partial y_{II})_\lambda$ and thus yields Eq. (11).

§ 3. Molecular theory

One of the authors developed in the preceding paper a hypothesis on the atomic structure of the "normal". According to it, when atoms in the super (ground) state are violated by thermal energy of the amount Δ , the excitation quantum, about eight atoms form an instantaneous "molecule" of the normal part. This formation of molecules will come to play from the unharmonicity of the vibrational modes of the thermal wave, which, for small frequencies giving rise to Debye phonons, propagate with constant velocity of the sound but for large frequencies, where the wave length is comparable with the atomic distance, unharmonicity will come to play maybe to hook together eight atoms to form a molecule.

If x atoms form a molecule by excitation quantum Δ , the free energy of He II is given by

$$F_4^0 = - \frac{yN_4}{x} kT \log \left[\left(\frac{2\pi \bar{m} kT}{h^2} \right)^{3/2} e^{-\Delta/kT} \frac{V}{yN_4/x} e \right] \quad (18)$$

with $\bar{m} = xm$, $y = N_4^n/N_4$ and V is the whole volume of the system.

For solution of He³ in He⁴ we assume this to be ideal and get

$$F = F_{40}(V) + n_3 \mu_3^0 + RT \left\{ n_3 \log \frac{y n_4}{a n_3 + y n_4} + n_4 \log \frac{y n_4}{a n_3 + y n_4} \right\}. \quad (19)$$

We shall put $a=1$ as the effect of this parameter is small. Then equilibrium condition

$$\partial F / \partial y = 0 \quad (20)$$

gives

$$(2\pi \bar{m} kT / h^2)^{3/2} e^{-\Delta/kT} V / (yN_4/x) = (1 - \xi_e) e^{-(x-1)\xi_e}, \quad (21)$$

where ξ_e is given by Eq. (12).

Assuming second order transition we can obtain the λ -point of the solution putting $y=1$, $T=T_\lambda$, $\xi_e=\xi$.

For pure He⁴ this gives

$$(2\pi\overline{m}kT_{\lambda}^0/k^2)^{3/2} \exp(-\Delta/kT_{\lambda}^0) = N_4/V_0x, \quad (22)$$

and for the solution

$$(2\pi\overline{m}kT_{\lambda}/k^2)^{3/2} \exp(-\Delta/kT_{\lambda}) = (1-\xi)e^{-(x-1)\xi}N_4/Vx. \quad (23)$$

The volume of the solution is

$$V = V_0(an_3 + n_4)/n_4 = V_0/(1-\xi) \quad (\text{for } a=1). \quad (24)$$

Thus we get

$$\begin{aligned} (T_{\lambda}/T_{\lambda}^0)^{3/4} \exp[-(T_{\lambda}^0/T_{\lambda}-1)\Delta/2kT_{\lambda}^0] \\ = (1-\xi) \exp[-(x-1)\xi/2]. \end{aligned} \quad (25)$$

The last equation gives $T_{\lambda}/T_{\lambda}^0$ as a function of the concentration ξ and this is illustrated by the curve II in Fig. 1 using the value

$$\Delta/kT_{\lambda}^0 = 4.0. \quad (26)$$

Agreement with experiment is quite satisfactory excluding the immediate neighbourhood of 0°K. This small discrepancy will be attributed to the contribution from the phonon which we have neglected throughout.

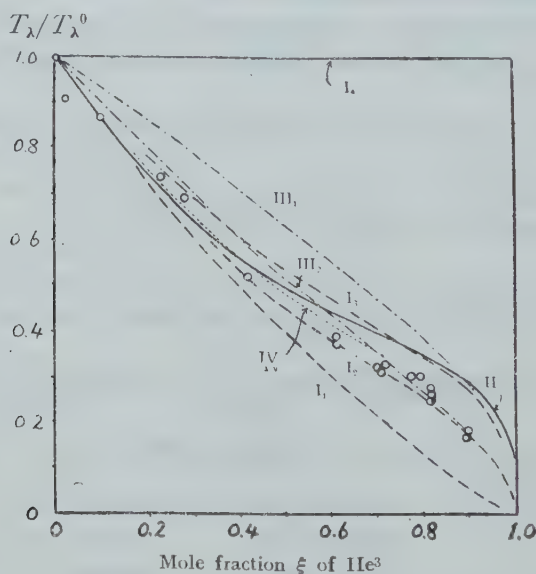


Fig. 1

- I: Eq. (16). $I_1: \nu=0$. $I_2: \nu=0.5$. $I_3: \nu=1.0$. $I_4: \nu=\infty$,
 II: Eq. (25). Molecular Theory,
 III: Eq. (28). Ideal gas. $III_1: a=1$. $III_2: a=1.5$,
 IV: by A. Harashima (Journ. Phys. Soc. Japan, in press),
 O: Experimental values.

For comparison ideal-gas approximation will also be given. The λ -point is, in this approximation, given by the equation

$$2.612(2\pi mkT_\lambda/\hbar^2)^{3/2} = N_4/V. \quad (27)$$

Then in place of Eq. (24) we get

$$T_\lambda/T_\lambda^0 = \{ (1-\xi)/[1+(a-1)\xi] \}^{2/3}. \quad (28)$$

In this ideal case the λ -point curve depends on the value of a . If we choose $a=1.5$ the experimental values are well represented as is seen in the figure.

§ 4. Pure liquid He^3

That the liquid of He^3 contrary to He^4 suffers no λ -transition and remains normal down to 0°K is a fact which can be regarded to be established by experiments. The reason of this has obviously connection to the statistics obeyed by the atoms: He^4 obeys Bose-statistics and He^3 Fermi-statistics. Up to now, however, no theory could clarify these circumstances.

Above cited theory proposed by one of the authors imagine the normal part of He^4 as the molecules formed by the excited atoms. This molecule obeys also Bose-statistics.

This tendency of forming molecule seems to be destroyed in case of He^3 atoms by their large zero-point energies, and at the same time as He^3 atoms cannot fall into the ground state because of Fermi-statistics, the liquid of He^3 will be able to have no super part at all.

It can be tested easily that He^3 obeys the Cailletet-Mathias' law of rectilinear diameter, the sum of the densities of liquid and vapor being almost independent of temperature, and that the hole theory of liquid is applicable to the liquid of He^3 in this connection. The fact that viscosity of liquid He^3 decreases with increasing temperature⁵⁾ may be also regarded as an evidence of the normal character of He^3 .

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On the Theory of Chemically Reacting Gas

Kazuo TAKAYANAGI

Department of Physics, Saitama University

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The method to determine the molecular velocity distribution in steady reaction in homogeneous gas phase with negligible reaction heat is formulated, and especially the case where one of the reacting molecules is very light is studied in detail. In this case the perturbation of the distribution function by the chemical reaction becomes most remarkable, and the expansion by the Sonine's polynomials is not always useful. Even then, the correction to the reaction rate is very small except for the most unfavorable case and therefore we may use the Maxwellian distribution to calculate the reaction rate.

§ 1. Introduction

Before we can get the theoretical rate of the gaseous reaction, we have to know in the first place the distribution functions (f 's) about the molecular velocities (and their internal energies) for each component of gas, and next the reaction probability per encounter for various relative velocities of colliding particles. Since the forms of f 's depend on the reaction probability, it is natural to consider the latter at the outset and this may be calculated (at least in principle) by means of the ordinary quantum mechanics. For this sake, however, we must solve a many-body problem which is very hard to handle on account of its mathematical complexity.¹⁾ It is usual, therefore, to assume the Maxwell-Boltzmann distributions for f 's and to use the relations in the equilibrium state to obtain the reaction probability²⁾, which are not allowed a priori*.

Recently, Prof. Prigogine and his collaborators studied the perturbation of Maxwell distribution by the chemical reaction from the viewpoint of the kinetic theory of gases⁵⁾. The main purpose of their investigation, however, is to show the possibility of application of the Chapman-Cowling's method⁶⁾ for this problem, and many points remain unsolved.

In the following, one of these remaining problems (i. e., the bimolecular reaction involving very light molecules) is considered. Owing to the lack of detailed information about the reaction probability, the results obtained are more or less qualitative.

* Some authors tried to get the foundation for the use of the equilibrium relations.³⁾⁴⁾

We start from the qualitative discussions about f 's in the steady gaseous reaction (§2). Then, the mathematical formulation is presented (§3), and this is solved for various cases (§§4, 5 and 6). Finally, the correction to the reaction rate is given (§7).

§ 2. Qualitative discussions

The kinetic theory of the chemically reacting gases gives, in general, the problem to solve the following set of Boltzmann-equations one for each component A, B, C, \dots , under the given boundary condition in space and time:

$$\frac{\partial f_A}{\partial t} + \mathbf{c}_A \cdot \frac{\partial f_A}{\partial \mathbf{r}} + \mathbf{F}_A \cdot \frac{\partial f_A}{\partial \mathbf{c}_A} = \left(\frac{df_A}{dt} \right)_{\text{scatt}} + \left(\frac{df_A}{dt} \right)_{\text{react}} \quad (1)$$

and the similar equations for B, C, \dots , where f 's are the distribution functions such that $f(\mathbf{c}, \mathbf{r}, t) d\mathbf{c} d\mathbf{r}$ represents the number of molecules in the volume element $d\mathbf{c}$ of the velocity space and $d\mathbf{r}$ of the ordinary space at the given instant t . More generally, it is desired that the f 's include the information about the internal states of molecules.* In the following, however, we neglect the effect of internal degrees of freedom for simplicity. \mathbf{F} means the acceleration due to the external forces.**

If the reaction probability is very small, the f 's will be almost Maxwellian. On the contrary, if the reaction probability is large and the activation energy is not high compared with mean kinetic energy of molecules at the temperature considered, the reaction proceeds very fast and the form of f 's may be quite different from the Maxwellian. In the intermediate region, however, we may expect the case where f 's are perturbed slightly from the Maxwellian and are yet represented by functions containing only two kinds of parameter, i. e., the concentrations n_A, n_B, n_C, \dots (molecule per unit volume), and the temperature T (defined by putting the mean kinetic energy as $3kT/2$)***, and their time dependence appear only through these parameters:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial n} \cdot \frac{dn}{dt} + \frac{\partial f}{\partial T} \cdot \frac{dT}{dt} \quad (2)$$

* As was shown by Prof. Careri,⁷⁾ "steric" factor may become larger than unity if one neglect the internal energy in the collision theory of reaction.

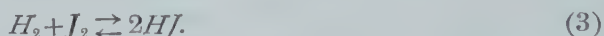
** The equation as (1) are applied not only to the bimolecular reactions considered below but also to the unimolecular reactions and the photochemical reactions, where it will be most interested to investigate the deviation from the Boltzmann distribution about the internal energy.

*** In general the temperature may change from one component to the other and the internal energy distribution may follow the different temperature also. Moreover, it is questionable whether the temperature measured with suitable thermometer equals to the one defined by the mean kinetic energy though it will be a small effect.

Such reaction may be called "steady reaction" because of the invariance of the form of distribution functions. Substituting (2) into the Eq. (1) we have the equations containing no time explicitly.

If the reaction heat is not small, we must take account of the nature of the wall containing the reacting gas. For the wall with very small heat conductivity, the temperature of gas will rise accompanying the reaction to a considerably high value, while if the wall is good conductor of heat, the boundary of gas remains at the same temperature with the external and the temperature gradient will appear in the gas giving rise a problem of heat conduction. The distribution functions in the exo-and endothermic reactions was investigated by Prigogine and his coworkers with the assumption that the perturbation from the Maxwellian is same form for all kinds of molecule in the gas which is not good approximation because the f 's for the reactants may be very different from the f 's for the products.

In the following, we consider the bimolecular reaction $A+B \rightleftharpoons C+D$ where the heat of reaction is very small and may be neglected. Furthermore, we assume that the mass m_A of molecule A is very small compared with that of the other molecules (m_B , m_C and m_D), for the perturbation of the distribution function may become most remarkable in this case because of the smallness of the energy-exchange by scattering between the molecules with very different masses. A typical example is given by the famous reaction



The high energy components of A and B are lost by the reaction, the produced molecules C and D have, on the other hand, large energy and this asymmetry will disappear immediately by the elastic scattering between the reactants and the products. This disappearance of the asymmetry is prevented if one of the molecule considered is very light. In the following, therefore, we consider the f_B , f_C and f_D to be Maxwellian with temperature T and only the f_A is perturbed appreciably from the Maxwellian. Furthermore, the effect of C and D on the f_A is that to make the perturbation from the Maxwellian small. Thus the perturbation of the f_A in the absence of C and D gives the upper limit of this. For this reason we assume for a while that the n_C and n_D is negligibly small (cf. § 6).

§ 3. The mathematical formulations

The equation to be solved is given as follows:

$$\frac{\partial f_A}{\partial n_A} \int \left(\frac{df_A}{dt} \right)_r d\mathbf{c}_A = \left(\frac{df_A}{dt} \right)_s + \left(\frac{df_A}{dt} \right)_r. \quad (4)$$

The small deviation from the Maxwellian distribution is calculated by putting

the f_A as a series

$$f_A = f_A^0 + f_A^1 + f_A^2 + \dots \quad (5)$$

and dividing the equation (4) into infinite number of equation and finally solving them successively. Usually, there is no definite parameter of this expansion. In the case considered, however, we may use e. g. the reaction probability a as the parameter of expansion, considering that the collision between A and B with the kinetic energy of relative motion larger than a definite value E^* (called activation energy) have the average reaction probability a . Substituting (5) into (4) and picking out the terms without a , we have

$$\left(\frac{df_A^0}{dt} \right)_s = 0, \quad (4a)$$

i. e., the first term of the expansion (5) must be Maxwellian type. The integration constants are fixed by the requirements that

$$\begin{aligned} \int f_A^0 d\mathbf{c}_A &= \int f_A d\mathbf{c}_A = n_A, \\ \int \frac{1}{2} m_A c_A^2 f_A^0 d\mathbf{c}_A &= \frac{3}{2} kT \left(\neq \int \frac{1}{2} m_A c_A^2 f_A d\mathbf{c}_A \right). \end{aligned} \quad (4b)$$

Then the terms linear in a from (4) give the following equation:

$$\frac{\partial f_A^0}{\partial n_A} \int \left(\frac{df_A^0}{dt} \right)_r d\mathbf{c}_A = \left(\frac{df_A}{dt} \right)_s^{(1)} + \left(\frac{df_A^0}{dt} \right)_r. \quad (4c)$$

The scattering term in the Eq. (4) is given as the form of sum

$$\left(\frac{df_A}{dt} \right)_s = \left(\frac{df_A}{dt} \right)_{s,A} + \left(\frac{df_A}{dt} \right)_{s,B}. \quad (6)$$

The last term, for example, is due to the scattering of A by B -molecules. In general, the terms expressing the scattering of A by C , D , and other molecules present must be added in the right hand side of this equation. If we treat the scattering as the elastic scattering between the rigid sphere, we have the following forms for each terms in the Eq. (6)*

$$\begin{aligned} (df_A/dt)_{s,A} &= \iint (f'_A f'_{A1} - f_A f_{A1}) k_A d\mathbf{k}_A d\mathbf{c}_{A1}, \\ (df_A/dt)_{s,B} &= \iint (f'_A f'_B - f_A f'_B) k_{AB} d\mathbf{k}_{AB} d\mathbf{c}_B. \end{aligned} \quad (7)$$

* As regards the notation unless otherwise specified, we follow the ref. 6.

In the first approximation, we put (as stated above) $f_h = f_B^0$, $f_c = f_C^0$, $f_D = f_D^0$, and $f_A = f_A^0 + f_A^1 \equiv f_A^0(1 + G)$. Therefore,

$$\begin{aligned} (df_A/dt)_{s,A}^{(1)} &= \iint (f_A^{0'} f_{A1}^{1'} - f_A^0 f_{A1}^1) k_A d\mathbf{k}_A d\mathbf{c}_{A1} + \iint (f_A^{1'} f_{A1}^{0'} - f_A^1 f_{A1}^0) k_A d\mathbf{k}_A d\mathbf{c}_{A1} \\ &= \iint f_A^0 f_{A1}^0 (G'_1 - G_1 + G'_1 - G) k_A d\mathbf{k}_A d\mathbf{c}_{A1}, \end{aligned} \quad (8)$$

$$\begin{aligned} (df_A/dt)_{s,B}^{(1)} &= \iint (f_A^{1'} f_B^{0'} - f_A^1 f_B^0) k_{AB} d\mathbf{k}_{AB} d\mathbf{c}_B \\ &= \iint f_A^0 f_B^0 (G' - G) k_{AB} d\mathbf{k}_{AB} d\mathbf{c}_B. \end{aligned}$$

In the next, the reaction term $(df_A/dt)_r$ means that obtained by replacing f_A in the expression $(df_A/dt)_r$ with f_A^0 . It is allowed because of the smallness of the perturbation. It must be noted here that the smallness of the reaction rate does not mean the smallness of the perturbation. If we are only concerned about the rough degree of deformation of the distribution function through the entire region of the velocity space and *not* about the change of reaction rate, chemical reaction can always be treated as a small perturbation and in such case we may adopt kT/E^* as the parameter of expansion (6). When the reaction rate is to be investigated, however, only the deformation of the distribution function *above* E^* is essential and the degree of this deformation may become large even if kT/E^* is very small. Thus the expansion by kT/E^* is excluded.

If the reaction probability α is not small enough compared with unity, chemical reaction give a large perturbation and it is dangerous to use the zeroth order function f_A^0 in the reaction term $(df_A/dt)_r$. Namely, since the number of molecules A above E^* is reduced by a factor*

$$\epsilon = \frac{\int_{E^*}^{\infty} W^0(1+G)dE}{\int_{E^*}^{\infty} W^0 dE}.$$

If we neglect this reduction we have as the correction function not the true function G but some function G^0 which is order of magnitude of G/ϵ . When the factor ϵ becomes smaller and smaller, $|G^0|$ become larger and larger and therefore the correction factor $(1+G^0)$ for the distribution function may become negative (G is of course negative at large energy) which is inconsistent with the physical meaning. This difficulty is removed to a certain extent by proce-

* Here and elsewhere in the following, $W(E)$ means the distribution function about the energy range $(E, E+dE)$. If the distribution is isotropic we have $W(E) = (4\pi/m)f(c)c$, $c = |\mathbf{c}|$.

ding as follows: To use ϵG^0 instead of G^0 where ϵ is given by

$$\epsilon = \frac{\int_{E^*}^{\infty} W^0(1 + \epsilon G^0) dE}{\int_{E^*}^{\infty} W^0 dE} \quad \text{or} \quad \epsilon = \frac{\int_{E^*}^{\infty} W^0 dE}{\int_{E^*}^{\infty} W^0(1 - G^0) dE}. \quad (9)$$

We call this renormalization " ϵ -process" for brevity.

Now, we desire the concrete expression for $(df_A/dt)_r$. We shall take two models for the reaction.

(i) If we assume that the reaction probability is a function $P(g_{||})$ of the relative velocity $g_{||}$ along the line of centres at collision, $(df_A/dt)_r$ may be written as

$$(df_A/dt)_r = f_A \cdot \int_0^{\infty} P(g_{||}) F(g_{||}; c_A) dg_{||}. \quad (10)$$

Here, $F(g_{||}, c_A) dg_{||}$ means the number of $g_{||}$ -collisions with B per unit time for a A -molecule having velocity c_A . The most simple choice for $P(g_{||})$ is:

$$P(g_{||}) = \begin{cases} 0, & g_{||} < g^*, \\ \beta, & g_{||} \geq g^*, \quad \beta: \text{const.} \end{cases} \quad (11)$$

where g^* is related to the activation energy E^* by

$$1/2(m_A m_B / m_A + m_B) g^{*2} = E^*.$$

The function F , on the other hand, has the following form

$$\begin{aligned} F(g_{||}; c_A) dg_{||} & \quad (12) \\ &= n_B \pi D_{AB}^2 \frac{1}{c_A} \left\{ \phi \left[\left(\frac{m_B}{2kT} \right)^{1/2} (g_{||} + c_A) \right] - \phi \left[\left(\frac{m_B}{2kT} \right)^{1/2} (g_{||} - c_A) \right] \right\} g_{||} dg_{||}, \end{aligned}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx,$$

and D_{AB} is half the sum of the radii of A and B .

Now, for the reacting collisions, $g_{||}$ must be very large. The large value of mass ratio m_B/m_A , therefore, leads the reasonable approximation that $c_A \gg c_B$ for the reacting collisions. Then the function $F dg_{||}$ becomes (putting $c_B = 0$)

$$F dg_{||} = n_B \pi D_{AB}^2 dg_{||}, \quad 0 \leq g_{||} \leq c_A$$

and finally $(df_A/dt)_r = f_A \beta n_B \pi D_{AB}^2 \int_{g^*}^{c_A} dg_{||}$

$$= \begin{cases} \beta n_B \pi D_{AB}^2 (c_A - g^*) f_A, & c_A \geq g^* \\ 0, & c_A \leq g^* \end{cases} \quad (13)$$

(ii) when we take the model that the collisions between A and B with the kinetic energy of relative motion larger than E^* lead reaction with probability α , similar equations are obtained as follows:

$$D(g) = \begin{cases} 0, & g < g^* \\ \alpha, & g \geq g^* \end{cases} \quad (11')$$

$$\begin{aligned} F(g; c_A) dg &= n_B \pi D_{AB}^2 \frac{1}{c_A} \left\{ \exp \left[-\frac{m_B}{2kT} (c_A - g)^2 \right] - \exp \left[-\frac{m_B}{2kT} (c_A + g)^2 \right] \right\} \left(\frac{m_B}{2\pi kT} \right)^{1/2} g^2 dg, \end{aligned} \quad (12')$$

$$\left(\frac{df_A}{dt} \right)_r = \begin{cases} \alpha n_B \pi D_{AB}^2 c_A f_A, & c_A \geq g^* \\ 0, & c_A < g^*. \end{cases} \quad (13)$$

§ 4. Expansion by the Sonine's polynomials

The most familiar device to solve the problem is the expansion of the correction function G by the Sonine's polynomials* (and the determination of its coefficients). Namely we substitute the expansion

$$G = \sum_{n=1}^{\infty} a_n S_{1/2}^{(n)}(V^2), \quad V = (m_A/2kT)^{1/2} \mathbf{c}_A, \quad (14)$$

into the equation (4c) and after multiplying $S_{1/2}^{(r)}(V^2)$ ($r=1, 2, 3, \dots$) to the both sides integrate over the whole \mathbf{c} -space. When this procedure is done, we have a set of infinite number of simultaneous equations for a_1, a_2, a_3, \dots

$$\sum_{n=1}^{\infty} a_n ([m, n] + [m, n]') = [m], \quad m=1, 2, \dots \quad (15)$$

where

$$\begin{aligned} [r, s] &= \iiint S_{1/2}^{(r)}(V^2) f_A^0 f_{A1}^0 (S_{1/2}^{(s)}(V^2) + S_{1/2}^{(s)}(V_1^2) - S_{1/2}^{(s)}(V'^2) - S_{1/2}^{(s)}(V'^2_1)) k_A d\mathbf{k}_A d\mathbf{c}_A d\mathbf{c}_{A1} \\ &= [s, r] \end{aligned}$$

$$[r, s]' = \iiint S_{1/2}^{(r)}(V^2) f_A^0 f_B^0 (S_{1/2}^{(s)}(V^2) - S_{1/2}^{(s)}(V'^2)) k_{AB} d\mathbf{k}_{AB} d\mathbf{c}_A d\mathbf{c}_B = [s, r]'$$

$$[r] = \int S_{1/2}^{(r)}(V^2) (df_A^0/dt)_r d\mathbf{c}_A.$$

* As is well known Sonine's polynomials are defined as

$$S_m^{(n)}(x) = \sum_{p=0}^n (-x)^p (m+n)(m+n-1) \cdots (m+p+1) / p! (n-p)!,$$

the orthogonality of which are $\int_0^{\infty} e^{-x} S_m^{(p)}(x) S_m^{(q)}(x) x^m dx = \begin{cases} 0 & (p \neq q) \\ \Gamma(m+p+1) / p! & (p=q). \end{cases}$

By the conditions (4b), we have $a_0=0$ in the expansion (14) while a_1 is not zero,

Practically, it is impossible to solve this set of equations, and usually one takes the first few terms only. This is permissible at least for the slowly varying perturbation from the Maxwellian. But it is not allowed for a sudden change of the distribution function. In the chemical reactions, $(df_A/dt)_r$ is appreciable only in a small region (near the activation energy). Thus it is hopeless that the right hand sides of (15) converge to zero quickly, because the polynomials $S_{1/2}^{(r)}(V^2)$ is no more oscillating near the activation energy ($V^* \gg 1$), at least for the small r (see Figs. 1a and 1b). Even then, the approximation by the first term may be useful if the concentration of A is considerably large. In this case A - A -scattering will occur frequently and, therefore, the distribution function will be slowly varying, or rather Maxwellian with a temperature $T - \Delta T$ (T is the temperature of B -component and $-\Delta T$ is a small temperature fall). Observing the relation

$$f^\circ(T - \Delta T) = f^\circ(T) \sum (\Delta T/T)^n S_{1/2}^{(n)}(mc^2/2kT),$$

we may expect that even the first term alone gives the good approximation. It is confirmed in another way as follows: $[m, n]$, $[m, n]'$ are proportional to $n_A^2 D_A^2 / \sqrt{m_A}$, $(n_A n_B D_{AB}^2 / \sqrt{m_A}) (m_A/m_B)$ respectively and their coefficients are roughly of the same order of magnitude, except $[m, 1] = [1, n] = 0$. Then if $n_A/n_B \gg (m_A/m_B) (D_{AB}/D_A)^2$, we may conclude* $a_1 \gg a_2, a_3, \dots$ as expected. On the other hand, if $n_A/n_B \lesssim (m_A/m_B) (D_{AB}/D_A)^2$, the coefficients a_1, a_2, a_3, \dots have values of the same

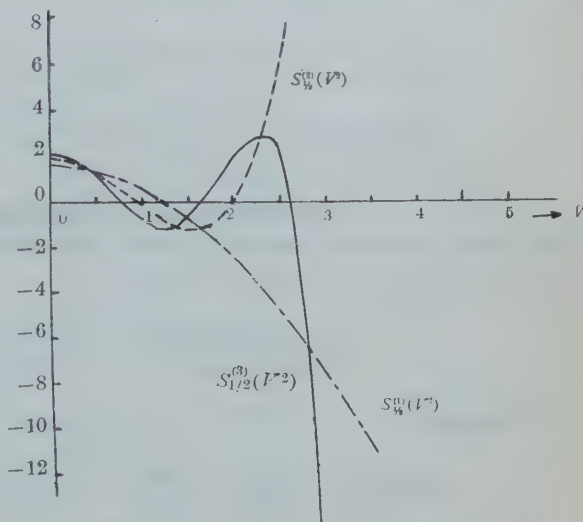


Fig. 1a Sonine's polynomials $S_{1/2}^{(r)}(V^2)$.

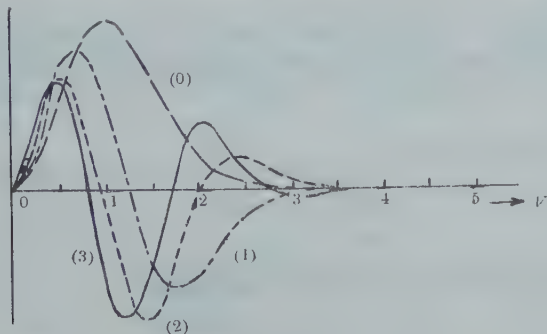


Fig. 1b $f^{(r)} = \text{Maxwellian} \times S_{1/2}^{(r)}(V^2)$

order. Thus the use of Sonine's polynomials is impractical.

* The authors confirmed this by taking the first three terms of (14).

§ 5 Method of differential equation

Fortunately, we can solve this problem by another method. Utilizing the smallness of the velocity-(or energy-) change of the molecule A at a scattering with B , we can approximate the integral

$$\iint (f'_A f'_B - f_A f_B) k_{AB} d\mathbf{k}_{AB} d\mathbf{c}_B$$

by the differential expression⁸⁾

$$\frac{kT}{m_B} \frac{1}{c_A^2} \frac{d}{dc_A} \left(\frac{c_A^3}{l} \frac{df_A}{dc_A} \right) + \frac{m_A}{m_B} \frac{1}{c_A^2} \frac{d}{dc_A} \left(\frac{c_A^4}{l} f_A \right); \quad l = \frac{1}{n_B \pi D_{AB}^2}.$$

Changing the variable from the velocity to the energy, we obtain the differential equation (for $n_A \ll n_B$):

$$2\sqrt{\frac{2}{m_A}} \frac{m_A}{m_B} \frac{kT}{l} E^{3/2} \left\{ W(E) \left(\frac{1}{2E} - \frac{1}{kT} \right) - \frac{dW(E)}{dE} \right\} \quad (16)$$

$$= \bar{v} \int_0^E W(E) dE - \frac{\beta}{l} \sqrt{\frac{2}{m_A}} \int_{E^*}^E (E^{1/2} - E^{*1/2}) W(E) dE; \quad \bar{v} = \frac{1}{n_A} \int \left(\frac{df_A}{dl} \right)_r d\mathbf{c}_A$$

(for $E < E^*$, the last integral has to be replaced by zero), which is similar to the Smit's equation⁹⁾ for the velocity distribution function of electrons in the case of gas discharge. Putting for brevity,

$$2\sqrt{\frac{2}{m_A}} \frac{m_A}{m_B} \frac{kT}{l} \frac{1}{\bar{v}} = \frac{1}{A}, \quad 2 \frac{m_A}{m_B} \frac{kT}{\beta} = \frac{1}{B},$$

(16) becomes

$$E^{3/2} \left\{ \left(\frac{1}{2E} - \frac{1}{kT} \right) W - \frac{dW}{dE} \right\} = A \int_0^E W dE - B \int_{E^*}^E (E^{1/2} - E^{*1/2}) W dE. \quad (17)$$

Observing that the Maxwell-distribution $W^0 dE = (2/\sqrt{\pi}) (E/kT)^{1/2} \exp(-E/kT) dE/kT$ is the solution of the equation obtained from (17) replacing A and B by zero, we put

$$W = W^0 (1 + W_1). \quad (18)$$

Then we have

$$-\frac{dW_1}{dE} = \frac{1}{E^{3/2} W^0} \left\{ A \int_0^E W dE - B \int_{E^*}^E (E^{1/2} - E^{*1/2}) W dE \right\},$$

which gives

$$W_1 = - \int_0^E dE \frac{1}{E^{3/2} W^0} \left\{ A \int_0^E W dE - B \int_{E^*}^E (E^{1/2} - E^{*1/2}) W dE \right\} + \text{const.} \quad (19)$$

integration constant being fixed by the condition

$$\int_0^{\infty} W^0 W^1 dE = 0. \quad (20)$$

As the right hand side of (19) is a small perturbation we may use the Maxwellian W^0 in this expression, and, therefore, (19) gives the solution directly (with ϵ -process). The general feature of this solution can be seen from Fig. 2 which is calculated numerically by (19) putting $E^* = 10kT$. At first stage $|W_1|$ is very small and suddenly increases near the activation energy. Such behavior will be represented by the function with following form,

$$\xi(e^{\eta E} - \zeta) \quad (21)$$

though the gradient of W_1 above E^* is considerably smaller than (21).

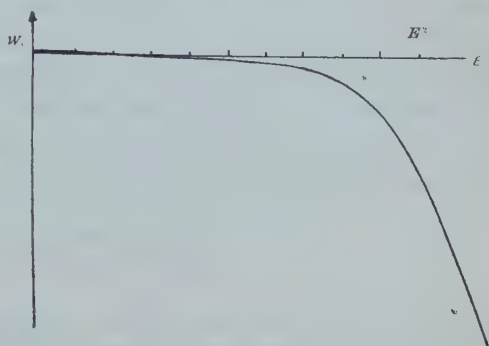


Fig. 2 General feature of the function $W_1(E)$

Above E^* , by taking the main terms of (19), we have

$$W_1 = W_1(E = E^*) + \beta \frac{m_B I}{m_A} \\ I = -\frac{kT}{4} \frac{E - E^*}{E^{*2}} \left(\frac{E - E^*}{2kT} + 1 - \frac{E - E^*}{E^*} \right), \quad E \geq E^*.$$

The values $W_1(E = E^*)$ for various E^* is given in Table I.

In the same way, we have for the model (ii) (§ 3)

$$W_1' = W_1'(E = E^*) - \frac{a}{2} \frac{m_B}{m_A} \log \frac{E}{E^*}, \quad E \geq E^*. \quad (22')$$

$W_1'(E = E^*)$ in this case is given by Table II. The difference of these two tables is due to the different expression of \bar{v} for each case.

Table I.

E^*/kT	$W_1(E = E^*), \text{ model (i)}$
10	$-3.39 \times 10^{-3} \beta m_B/m_A$
14	-1.56×10^{-3}
18	-8.97×10^{-4}
22	-5.83×10^{-4}
26	-4.09×10^{-4}
30	-3.04×10^{-4}

Table II.

E^*/kT	$W_1'(E = E^*), \text{ model (ii)}$
10	$-7.18 \times 10^{-2} a m_B/m_A$
14	-4.56×10^{-2}
18	-3.34×10^{-2}
22	-2.64×10^{-2}
26	-2.18×10^{-2}
30	-1.86×10^{-2}

§ 6 Effects of other molecules

As the reaction proceeds, the concentrations of C and D become more and more large. At first we must take into account the reverse reaction and this may be represented in the first approximation by replacing $n_A n_B$ in the $(df_A/dt)_r$ by $(n_A n_B - n_C n_D/K)$. K is the equilibrium constant in the temperature considered. In the next approximation, the perturbations of the distribution functions must be accounted. For C and D the high energy part of the distribution function increases by the chemical reaction, while for A and B decreases. Thus the reaction rate drops by the decrease of the normal reaction and the increase of the inverse reaction. Such effect will be pictured by the resistance for the moving body in fluid rising from the pressure increase before it and the decrease behind. Moreover, existence of C and D make the distribution function of A close to the Maxwellian. The perturbation of f_A is reduced roughly by the factor

$$\left(\frac{n_B}{m_B}\right) \bigg/ \left(\frac{n_B}{m_B} + \frac{n_C}{m_C} + \frac{n_D}{m_D}\right).$$

If there are molecules other than C and D , they give the same effect on the distribution function of A . If the experiment starts from the initial conditions of large n_B , very small n_A , and other n 's=0, the effects due to the products C , D will be very small throughout the measurement.

§ 7 Correction to the reaction rate

We return to the case where the concentrations of C and D is negligible and compute the correction to the reaction rate. The reaction rate before the correction is given by (see Eqs. (13), (13'))

$$v^{(0)} = \beta n_A n_B \pi D_{AB}^2 \int_{V^*}^{\infty} \left(\frac{2kT}{m_A}\right)^{1/2} (V - V^*) \frac{4}{\sqrt{\pi}} \exp(-V^2) V^2 dV \quad (23)$$

$$\text{or} \quad v^{(0)} = \alpha n_A n_B \pi D_{AB}^2 \int_{V^*}^{\infty} \left(\frac{2kT}{m_A}\right)^{1/2} V \frac{4}{\sqrt{\pi}} \exp(-V^2) V^2 dV, \quad (23')$$

while the corrected expression is

$$v = \beta n_A n_B \pi D_{AB}^2 \int_{V^*}^{\infty} \left(\frac{2kT}{m_A}\right)^{1/2} (V - V^*) \frac{4}{\sqrt{\pi}} \exp(-V^2) V^2 (1 + W_1) dV, \quad (24)$$

$$\text{or} \quad v = \alpha n_A n_B \pi D_{AB}^2 \int_{V^*}^{\infty} \left(\frac{2kT}{m_A}\right)^{1/2} V \frac{4}{\sqrt{\pi}} \exp(-V^2) V^2 (1 + W_1') dV. \quad (24')$$

Using the expression (22) or (22') for W_1 and W_1' we can calculate the eqs. (24), (24'). Here β or α is fixed by equating $v^{(0)}$ with the experimental reaction

rate. Owing to this semi-empirical determination, it is no more needed to perform the ϵ -process (§ 3).

As a example, we shall consider the typical reaction (3), for $T=E^*/26k$; it is close to the highest temperature in which Bodenstein¹⁰ measured this reaction rate (781°K). From his experimental value we obtain $\beta=0.26$ (or $\alpha=0.005$). Using this value, $(1+W_1)$ is calculated and roughly given as Table III (model (i)), and this reduces the reaction rate about 7%. If we use the model (ii), this correction is approximately 2%.

Table III.

E/kT	$1+W_1$
26	0.99
27	0.97
28	0.94
29	0.90
30	0.85
31	0.78
32	0.71

Besides, if the concentration of A is not small enough i. e., if $n_A/n_B \gtrsim m_A/m_B (D_{AB}/D_A)^2$ the correction to the reaction rate is negligibly small. Thus we may expect that this effect can be tested experimentally, by varying the concentration ratio n_A/n_B .

§ 3 Conclusion

The perturbation of Maxwell distribution in the steady reaction with small heat of reaction is found to be very small except for the special case (large mass ratio and large concentration ratio). In the latter case, however, experimental check will give a large contribution to the theory of elementary process of chemical reaction.

The author is much indebted to Prof. M. Kotani for his kind advices throughout this work.

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Point of Condensation and the Volume Dependency of the Cluster Integrals

Shigetoshi KATSURA* and Hisaaki FUJITA**

*Department of Applied Science, Tôhoku University

**Department of Physics, Tôhoku University

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In the theory of imperfect gas, the cluster integrals b_l depend on V (total volume of the system). The double limit process $V \rightarrow \infty$ and $l \rightarrow \infty$ defines two different functions $\chi(z)$ and $\chi_\infty(z)$ (or, $\zeta(z)$ and $\zeta_\infty(z)$). The real behaviour of partition function is not given by $\chi(z)$, but by $\chi_\infty(z)$, $\zeta_\infty(z)$. Instead of the singularity of $\chi(z)$, the upper limit of the regular interval of $\chi_\infty(z)$ is important. Inadequateness of the method of generating function is discussed.

§ 1. Introduction

The method of the cluster integrals was developed by Mayer¹⁾, Kahn and Uhlenbeck,²⁾ Born and Fuchs³⁾, and others to discuss the condensation of the imperfect gas. They paid little attention to the volume dependency of the cluster integrals and were led to several defects. For example, treatment for pure liquid phase was unsatisfactory, for the pressure variation in the liquid phase relates to the dependency of the cluster integrals directly.

From our point of view, the volume dependency of cluster integrals plays a dominant role also in determining the point of condensation. This paper aims thus to give some remarks to the former theories, and to offer a few examples on the mechanism of condensation.

Kahn and Uhlenbeck took $b_l(V)$ —cluster integrals varying with the volume—into their consideration but by very short way, and Born and Fuchs expected that the consideration of $b_l(V)$ would lead to the second singularity which was related to the appearance of the liquid phase.

§ 2. General treatment

Let N particles of the same kind be in a vessel whose volume is V , then the partition function $Q(T, V, N)$ of this system is

$$Q(T, V, N) = (2\pi mkT/h^2)^{3N/2} \mathcal{Q}(T, V, N),$$

$$\mathcal{Q}(T, V, N) = (1/N!) \int_{V,N} \dots \int \exp[-U(q)/kT] dq_1 \dots dq_{3N}, \quad (1)$$

where q represents each coordinate of molecules, $U(q)$ is the potential energy of the system, and $\mathcal{Q}(T, V, N)$ is the configurational partition function. Denoting the distance between i th and j th molecules by r_{ij} , $U(q)$ is represented by

$$U(q) = \sum_i \sum_j u(r_{ij}).$$

If $u(r)$ is a short range force, $\mathcal{Q}(T, V, N)$ is reduced to

$$\mathcal{Q}(T, V, N) = \int \dots \int \sum \prod f_{ij} d\tau_1 \dots d\tau_N = \sum_{m_i} \prod_i \frac{(V \cdot b_i)^{m_i}}{m_i!}, \quad (\sum_i l m_i = N), \quad (2)$$

where

$$f_{ij} = \exp[-u(r_{ij})/kT] - 1, \quad b_i(V) = \frac{1}{l! V} \int \dots \int \sum_{\substack{\text{cluster} \\ l \geq i}} \prod_{j \geq l} f_{ij} d\tau_1 \dots d\tau_j. \quad (3)$$

Cluster integral b_l depends on V , but when V is far greater than the volume of the molecular interaction, variation of $b_l(V)$ becomes small as V increases, and $\lim_{V \rightarrow \infty} b_l(V)$ is definite and finite. This convergence, however, is not uniform*. We denote this limit by b_l^0 , that is,

$$b_l^0 = \lim_{V \rightarrow \infty} b_l(V). \quad (4)$$

Pressure p , specific volume v , of the system, are both represented by $b_l(V)$, that is, with the aid of the fugacity z ,

$$p(V)/kT = \sum_{l=1}^{\infty} b_l(V) z^l, \quad N/V = 1/v(V) = \sum_{l=1}^{\infty} l b_l(V) z^l. \quad (5)$$

We are interested in the pressure and the specific volume in the case $V \rightarrow \infty$. We denote them simply by p and v henceforth.

Let us define the following four functions:

$$\chi(z) = \sum_{l=1}^{\infty} b_l^0 z^l = \sum_{l=1}^{\infty} \lim_{V \rightarrow \infty} b_l(V) z^l, \quad (6)$$

$$\zeta(z) = \sum_{l=1}^{\infty} l b_l^0 z^l = \sum_{l=1}^{\infty} \lim_{V \rightarrow \infty} l b_l(V) z^l = z \chi'(z), \quad (7)$$

$$\chi_{\infty}(z) = \lim_{V \rightarrow \infty} \sum_{l=1}^{\infty} b_l(V) z^l, \quad (8)$$

$$\zeta_{\infty}(z) = \lim_{V \rightarrow \infty} \sum_{l=1}^{\infty} l b_l(V) z^l. \quad (9)$$

Where we understand that the former two, i.e., $\chi(z)$ and $\zeta(z)$ represent the two analytic functions, whose functional elements around $z=0$ are $\sum_{l=1}^{\infty} b_l^0 z^l$ and

* Snyder's proof⁽⁴⁾ for the uniform convergence of $b_l(V)$ seems to be doubtful.

$\sum_{l=1}^{\infty} l b_l^0 z^l$ respectively. Their singularity is denoted by z_s . On the other hand, $\chi_{\infty}(z)$ and $\zeta_{\infty}(z)$ are defined in the circle of convergence of $\sum_{l=1}^{\infty} b_l(V) z^l$ or $\sum_{l=1}^{\infty} l b_l(V) z^l$.

Mayer and others¹⁾²⁾³⁾ state that

$$p/kT = \chi(z) \quad \text{and} \quad 1/v = \zeta(z) \quad (10)$$

and that their common singularity z_s (which is on the circle of convergence of $\sum_{l=1}^{\infty} b_l^0 z^l$) would give the point of condensation.

This reasoning is, however, not logical. (5) leads us to the statement that

$$p/kT = \chi_{\infty}(z), \quad 1/v = \zeta_{\infty}(z), \quad (11)$$

and that the point of condensation should be given by the irregular point of $\chi_{\infty}(z)$ or $\zeta_{\infty}(z)$.

$\chi(z)$ or $\zeta(z)$ is not necessarily equal to $\chi_{\infty}(z)$ or $\zeta_{\infty}(z)$. These functions (6)–(9) are defined by double limit process, and careful treatment is always necessary for commutation of the order whenever double limit appears.

The function $\chi_r(z) \equiv \sum_{l=1}^{\infty} b_l(V) z^l$ converges in the total complex plane as far as V is finite, while the behaviour of the limit function $\chi_{\infty}(z) = \lim_{V \rightarrow \infty} \chi_r(z)$ is rather complicated. We can only say as follows: if we take the domain where $\chi_r(z)$ uniformly converges to $\chi_{\infty}(z)$, then $\chi_{\infty}(z)$ is regular in that domain. The set of the values of z to which $\chi_{\infty}(z)$ is regular, has an upper limit. We denote this by z^* .

Now we can prove* that $\chi(z)$ is equal to $\chi_{\infty}(z)$ in some range where z is sufficiently small, namely, $\chi(z)$ is equal to $\chi_{\infty}(z)$ when $0 \leq z < \text{Min}(z^*, z_s) = z^*$. (It is clear that $z^* \leq z_s$.) In the range $z > z^*$, however, $\chi_{\infty}(z)$ behaves like the bamboo grafted on the tree (the fox's skin sewed to the lion's), that is, $\chi_{\infty}(z)$ has an angle or a jump, or becomes infinite at $z = z^*$. $\chi(z)$ is regular when $z < z_s$, so that the true point of condensation z^* , the graft point of the tree and the bamboo, differs from z_s the joint of the tree, in general.

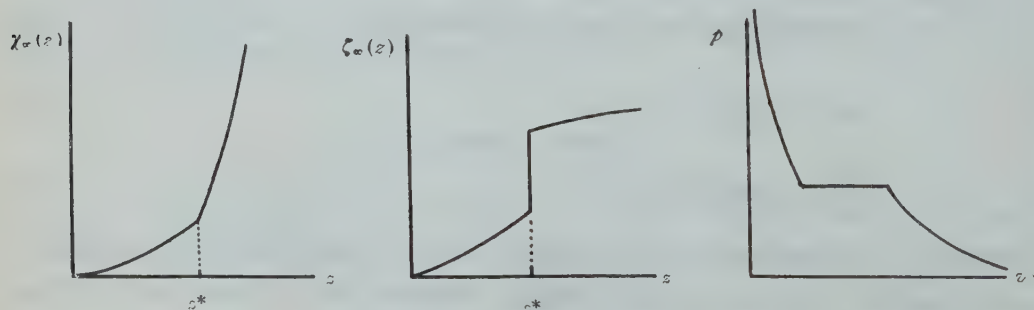


Fig. 1

* We are indebted to Prof. S. Izumi. (Mr. Matsubara gave another proof.)

It is clear that if $\chi_\infty(z)$ consists of two smooth functions combined at z^* and its gradient discontinuously increases there (Fig. 1a), then $\zeta_\infty(z)$ has a jump (Fig. 1b) and consequently the isotherm has a horizontal part (Fig. 1c). On the other hand, the theory of Kahn and Uhlenbeck cannot give the isotherm of the liquid phase even if $z^* = z_s$.

In fact, we can offer examples for $\zeta_r(z)$ which shows such a behaviour.

Example 1 (Fig. 2): Let

$$b_l(a) = \exp(-a^2/2) a^l H_{l-1}(a) / (l-1)! \quad (12)$$

where $H_l(a)$ is Hermite polynomial of l th degree. Adopting such $b_l(a)$,

$$\zeta_a(z) \equiv \sum_{l=1}^{\infty} l b_l(a) z^l = a \int_0^z \exp[-a^2(t-1)^2/2] dt, \quad (13)$$

hence

$$\zeta_\infty(z) = \lim_{a \rightarrow \infty} \zeta_a(z) = \begin{cases} 0 & (z < 1) \\ \sqrt{2\pi} & (z > 1) \end{cases} \quad (14)$$

hence $z^* = 1$. On the other hand $\lim_{a \rightarrow \infty} b_l(a) = 0$. Hence

$$\zeta(z) = \sum_{l=1}^{\infty} l b_l^0 z^l \equiv 0. \quad (0 < z < \infty) \quad (15)$$

We have no singularity of $\zeta(z)$.

Properties of $b_l(a)$: When a is large, (the dominant term of $H_l(a)$ is a^l),

$$\frac{d}{da} b_l(a) \cong \frac{d}{da} \frac{\exp(-a^2/2) a^{2l}}{(l-1)!} \cong \frac{\exp(-a^2/2) a^{2l-1} (2l - a^2)}{(l-1)!}. \quad (16)$$

That is, $b_l(a)$ has a maximum at $a \cong \sqrt{2l}$ approximately (not monotonously increasing). Therefore the convergence $\lim_{a \rightarrow \infty} b_l(a) = b_l^0 = 0$ is not uniform.

Another example will be given in the next section.

We can show that if $b_l(V)$ is always positive and increasing monotonously with V , its limit being b_l^0 , then $z^* = z_s$. Proof*: For an arbitrary z , which satisfies $z < z_s$, $\chi(z)$ and $\zeta(z)$ are the superior series to $\chi_r(z)$ and $\zeta_r(z)$ respectively, so that $\chi(z) = \chi_\infty(z)$, $\zeta(z) = \zeta_\infty(z)$ ($z < z_s$), hence, $\chi_\infty(z)$ is regular for $z < z_s$. Therefore, $z^* \geq z_s$. From $z^* \leq z'_s$ and $z^* \geq z_s$, we have $z^* = z_s$.

The real behaviour of $b_l(V)$, however, is very complicated (negative or greater than b_l^0), when $V \leq lv_0$ (v_0 represents the volume of the molecular interaction). For instance, $b_2(V) = \frac{1}{2V} \int_0^{(3V/4\pi)^{1/3}} \exp(-u(r)/kT) 4\pi r^2 dr$ for molecules of imperfect gas, takes negative values when $V \cong 2v_0$, even if the temperature is:

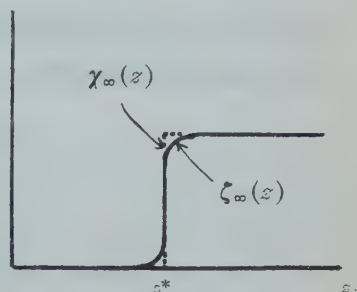


Fig. 2

* Due to Mr. Matsubara.

low (Fig. 3). We expect that the behaviour of $b_l(V)$ is such one as shown in Fig. 3 or alike, and that this complicated dependency of $b_l(V)$ on $V(V \lesssim l\nu_0)$ is the essential factor of the condensation. Cf. Example 1 and 2, too. So that we must consider as $z^* < z_s$ in general.

§ 3. Note on the method of the generating function

Let $\zeta(z)$ be such an analytic function whose inverse function is shown in Fig. 4b. (For example, $\zeta(z)$ is the inverse function of the appropriate polynomial of the third degree.) The value of the proper branch of $\zeta(z)$ on the real axis of z is such one as shown in Fig. 4a. The singularities of $\zeta(z)$ are denoted by $z_s^{(1)}$ and $z_s^{(2)}$. Speaking in detail, $z_s^{(2)}$ is the branch point connecting the first and the second surfaces, and $z_s^{(1)}$ is the one connecting the second and the third. $z_s^{(2)}$ is not a singular point on the third surface and $z_s^{(1)}$ is not a singular point on the first surface.

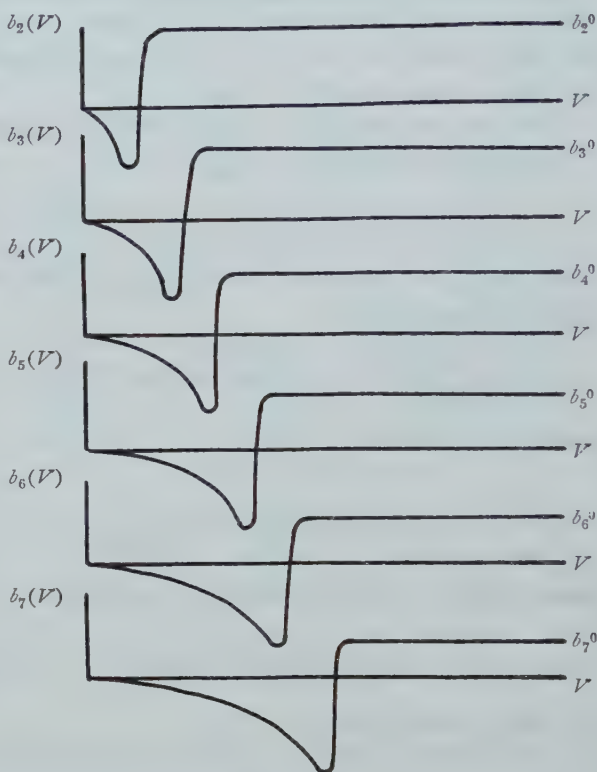


Fig. 3

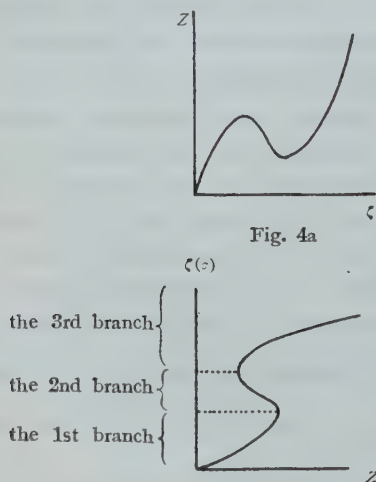


Fig. 4a

Fig. 4b

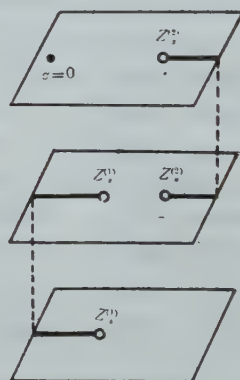


Fig. 5

Riemannian surface of this $\zeta(z)$ is such one that Fig. 5 shows. Cuts are just on the real axis in this figure. Following Kahn and others, let us try to adopt the method of the generating function to treat such $\zeta(z)$.

The limit

$$\lim_{N \rightarrow \infty} [\mathcal{Q}(T, V, N)]^{1/N} \quad (17)$$

is given by the radius of convergence R of the following generating function,

$$F_v(x) = \sum_{N=1}^{\infty} \mathcal{Q}(T, Nv, N) x^N = \frac{1}{2\pi i} \oint_{\xi} \frac{d\xi}{\xi} \sum_{N=1}^{\infty} \left(\frac{x}{\xi} \exp[Nv\chi(\xi)] \right)^N, \quad (18)$$

where $\chi(\xi) = \sum_{l=1}^{\infty} b_l^0 \xi^l$.

Order of the integration and the summation can be exchanged when $|(x/\xi) \exp[\chi(\xi)]| < 1$, so that

$$\begin{aligned} F_v(x) &= \frac{1}{2\pi i} \oint d\xi \left\{ \xi - x \exp[v\chi(\xi)] \right\} \\ &= 1 / \{1 - zv\chi'(z)\} = 1 / \{1 - v\zeta(z)\}, \end{aligned} \quad (19)$$

where $z = z(x, v)$ is the value of ξ which satisfies

$$\xi - x \exp[v\chi(\xi)] = 0. \quad (20)$$

Singular point (radius of convergence) of $F_v(x)$ is given by either $z(v)$ which satisfies $1/v = \zeta(z)$, or $z_s^{(1)}$, $z_s^{(2)}$, which are the singular points of $\zeta(z)$. We denote them by $R(v)$, $R_s^{(1)}(v)$ and $R_s^{(2)}(v)$ respectively, then,

$$R(v) = z(v) \exp[-v\chi\{z(v)\}], \quad (21)$$

$$R_s^{(1)}(v) = z_s^{(1)} \exp[-v\chi(z_s^{(1)})], \quad (22)$$

$$R_s^{(2)}(v) = z_s^{(2)} \exp[-v\chi(z_s^{(2)})]. \quad (23)$$

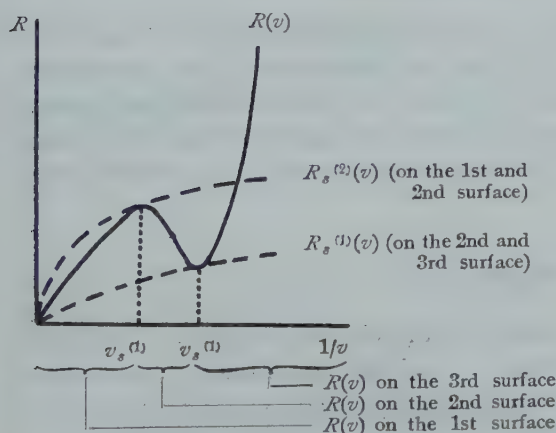


Fig. 6a

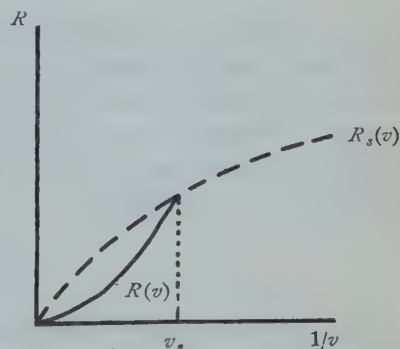


Fig. 6b

Fig. 6a shows their behaviours. When V is large, we consider $\zeta(z)$ on the first Riemannian surface, and then the smallest x giving singular point of $F_v(x)$ is $R(v)$. At the value of $v=v_s^{(2)}$ where $R(v_s^{(2)})=R_s^{(2)}(v_s^{(2)})$, $R_s^{(2)}(v)$ curve osculates with $R(v)$ curve and so the behaviour (gradient) of the partition function shows no discontinuity. If we hold to consider the first Riemannian surface to discuss the behaviour of R , then the isotherm would be horizontal where $v < v_s^{(2)}$. But its derivative is continuous at $v=v_s^{(2)}$. As we discussed in our former paper, $z_s^{(2)}$ is considered to be the end point of the supersaturation, and the real point of the condensation is represented by a smaller value of z , that is, $z=z^*$. The method of the generating function is not adequate to treat such a state of matters. In order to get the result that the partition function has an angle point by the generating function method, $R(v)$ and $R_s^{(2)}(v)$ curves must be assumed to behave as Fig. 6b shows. But it is not plausible. If we pass to the second Riemannian surface at the singular point $z_s^{(2)}$, and to the third surface at $z_s^{(1)}$, we get the isotherm of van der Waals type by the method of the generating function. $z_s^{(1)}$ represents the point where the superheat of the liquid ends.

In fact, we can show the example that the value of $\zeta_\infty(z)$ shows a transition from the value of the first branch of $\zeta(z)$ to the one of the third branch at the point $z=z^*$.

Example 2: Take the following $b_i(\tau)$,

$$b_i(\tau) = \frac{1}{l} \frac{1}{2\pi i} \oint \frac{1}{z^{i+1}} \left[\sum_{i=1}^{\infty} \frac{\tau^i}{(i-1)!} \exp[-if(i/\tau)] z^i / \sum_{i=1}^{\infty} \frac{\tau^i}{i!} \exp[-if(i/\tau)] z^i \right] dz, \quad (24)$$

where $f(\rho) = A\rho^4 - B\rho^2$, $A > 0$, $B > 0$, or alike, then,

$$\zeta_\tau(z) = \frac{1}{\tau} \frac{\sum_{i=1}^{\infty} i(\tau^i/i!) \exp[-if(i/\tau)] z^i}{\sum_{i=1}^{\infty} (\tau^i/i!) \exp[-if(i/\tau)] z^i}. \quad (25)$$

The behaviour of this $\zeta_\tau(z)$ and $\zeta_\infty(z)$ derived therefrom were discussed in the former paper⁵⁾: when $\tau < \infty$, $\zeta_\tau(z)$ has an infinite radius of convergence but has a steep ascent at $z \cong z^*$, and it becomes to have a rectangular part as $\tau \rightarrow \infty$; and when $\tau \rightarrow \infty$, $\zeta_\infty(z)$ is equal to the first branch or the third branch of $\zeta(z)$ corresponding to $z < z^*$ or $z > z^*$ respectively, where $\zeta(z)$ is the inverse function of $z = \zeta \exp[f(\zeta)]$.

§ 4. Conclusion

The authors have discussed the method of the cluster integrals. The cluster integrals b_i depend on V , the total volume V , and the condensation point should not be given by the singular point z_s of the analytic function $\zeta(z) = \sum_{l=1}^{\infty} \lim_{V \rightarrow \infty} l b_l(V) z^l$, but be given by the irregular point z^* of the (non-analytic) function $\zeta_\infty(z)$.

$= \lim_{V \rightarrow \infty} \sum_{l=1}^{\infty} l b_l(V) z^l$. We have got the possibility of finding the isotherm of the liquid phase in this way. The method of the generating function is not adequate for such a state of matters. The essential factor of the condensation may be the complicated dependency of $b_l(V)$ on V where $V \lesssim l v_0$.

What we expected above, is hoped to be proved exactly by the aid of the detailed knowledge of $b_l(V)$ about real imperfect gas. This is the problem of the future. In any case, the importance of $\chi_{\infty}(z)$ and $\zeta_{\infty}(z)$ (not $\chi(z)$ and $\zeta(z)$) is clear.

It is the authors' pleasure to thank Mr. T. Matsubara in Ôsaka University for his valuable discussions.

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Note added in proof Mr. S. Ono (in Kyushu University) contributed an important paper, "Statistical Theory of Phase Change" (Busseiron Kenkyû, No. 38 1951 May, in Japanese). He approached the problem whether z^* is equal to z_g or not, from the direction of method of crystal statistics. Though some of his reasoning contradicts to ours apparently, our conclusion seems to us to be unchanged even when the method of crystal statistics would be applied.

The Effect of Helium 3 Ingredient on the Wave Propagation in Liquid Helium II^{*)}

Shoichiro KOIDE and Tunemaru USUI

Physics Department, Faculty of General Culture, University of Tokyo

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The effect of He^3 ingredient on the wave propagation in liquid helium II is discussed phenomenologically on the basis of the two fluid model. The velocity of second sound numerically calculated is compared with experimental data. The comparison reveals the peculiarity of the normal and supercomponents essentially different from the He^3 component.

§ 1. Introduction

In a previous paper¹⁾ (henceforth referred to as I) we have developed a phenomenological theory of liquid helium II on the basis of two fluid model, intending to purify its footing as far as possible. The fundamental thermodynamic equations of motion thus obtained on the assumption of thermostatical local equilibrium, are found to agree with those of Gorter²⁾ in their essential character.

As a result of freeing the theory from Tisza's relation, it is found to have ability to explain the rise of second sound velocity towards lower temperatures. In fact, Nakajima and Shimizu³⁾ have shown that the T^3 -entropy predominant at temperatures lower than 1°K (the so-called phonon part) gives the desired effect, if the extrapolated temperature dependency of the concentration of normal fluid is adopted. This assumption is of course provisional, and the recent measurement of the second sound velocity down to 0.1°K⁴⁾ shows that the concentration of normal fluid near 0°K will be proportional to the fourth power of temperature. The consistent description of the temperature dependence of second sound velocity is one of difficult problems to be deferred for the future.

Recently, Lynton and Fairbank⁵⁾ have reported the data of second sound velocity in the case of admixture of helium 3 isotope. Their investigation covers He^3 concentrations between 0.09 percent and 0.80 percent in the temperature range between 1.24°K and 1.95°K. The velocity of second sound was found to increase markedly with increasing concentration and also the above-mentioned low temperature rise seems to begin to appear at higher temperature with increasing concentration. However, the range in which the above-mentioned rise does not appear is sufficiently wide and it seems possible to extend the theory to the case of He^4 - He^3 mixture without abandoning Tisza's relation.

^{*)} Read at the Minute Meeting of the Physical Society of Japan, April, 1951.

In this paper we discuss phenomenologically the effect of helium 3 admixture on the propagation velocity of second sound, restricting ourselves in the temperature region between the lambda-point and 1.5°K. The investigation of this phenomenon is expected to clarify the motion of helium 3 component in the wave motion and then to help ourselves to understand the peculiar character of helium 3 components.

A theory of this phenomenon seems to have been given by Pomeranchuk⁶⁾. Although his paper is not available here, we believe that his starting point is probably different from our purely phenomenological one. It may, therefore, be allowed to publish our extension of the thermodynamical theory based on the two fluid model.

We shall treat the thermostatics of the mixture in Part I, and then, using the results, discuss in Part II the wave phenomena in the mixture, especially the propagation velocity of second sound.

Part I. Thermostatics

§ 2. Pressure dependence of the lambda-point of pure He⁴

As an introductory discussion we shall first treat the pressure dependence of the lambda-point of pure helium 4 on the basis of the two fluid model. As stated in I, the condition of thermodynamic equilibrium is given by

$$\bar{\mu}_n - \bar{\mu}_s = \left(\frac{\partial \mu}{\partial \hat{\epsilon}} \right)_{pT} = 0, \quad (2.1)$$

where μ is the Gibbs' free energy per unit mass, $\bar{\mu}_n$ and $\bar{\mu}_s$ are partial free energies (referred to unit mass) of normal and super-fluid, respectively, and $\hat{\epsilon}$ is the mass concentration of normal fluid. Along the states of constant $\hat{\epsilon}$, therefore, we get

$$\frac{\partial^2 \mu}{\partial T \partial \hat{\epsilon}} dT + \frac{\partial^2 \mu}{\partial p \partial \hat{\epsilon}} dp = 0,$$

or

$$\left(\frac{\partial p}{\partial T} \right)_{\hat{\epsilon}} = \left(\frac{\partial s}{\partial \hat{\epsilon}} \right)_{pT} \left/ \left(\frac{\partial v}{\partial \hat{\epsilon}} \right)_{pT} \right., \quad (2.2)$$

with the relations

$$\left(\frac{\partial \mu}{\partial T} \right)_{p\hat{\epsilon}} = -s, \text{ the entropy per unit mass,}$$

and

$$\left(\frac{\partial \mu}{\partial p} \right)_{T\hat{\epsilon}} = v, \text{ the specific volume.}$$

Making use of the equalities

$$\left(\frac{\partial s}{\partial T} \right)_p = \left(\frac{\partial s}{\partial \hat{\epsilon}} \right)_{pT} \left(\frac{\partial \hat{\epsilon}}{\partial T} \right)_p + \left(\frac{\partial s}{\partial T} \right)_{p\hat{\epsilon}}$$

and

$$\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial v}{\partial \tilde{\epsilon}}\right)_{pT} \left(\frac{\partial \tilde{\epsilon}}{\partial T}\right)_p + \left(\frac{\partial v}{\partial T}\right)_{p\tilde{\epsilon}},$$

we can transform further Eq. (2.2) into the following form:

$$\left(\frac{\partial p}{\partial T}\right)_{\tilde{\epsilon}} = \frac{\left(\frac{\partial s}{\partial T}\right)_p - \left(\frac{\partial s}{\partial T}\right)_{p\tilde{\epsilon}}}{\left(\frac{\partial v}{\partial T}\right)_p - \left(\frac{\partial v}{\partial T}\right)_{p\tilde{\epsilon}}} = \frac{C_{pII} - T\left(\frac{\partial s}{\partial T}\right)_{p\tilde{\epsilon}}}{a_{II} - \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_{p\tilde{\epsilon}}} \cdot \frac{1}{T\tilde{v}} \quad (2.2')$$

where C_p and α are specific heat at constant pressure and thermal expansion coefficient, respectively.

$$\lim_{\tilde{\epsilon} \rightarrow 1} T \left(\frac{\partial s}{\partial T}\right)_{p\tilde{\epsilon}} = C_{pI\lambda}^* \quad (2.3)$$

and

$$\lim_{\tilde{\epsilon} \rightarrow 1} \frac{1}{v} \left(\frac{\partial v}{\partial T}\right)_{p\tilde{\epsilon}} = \alpha_{I\lambda}, \quad (2.4)$$

we obtain

$$\frac{dp}{dT_\lambda} = \lim_{\tilde{\epsilon} \rightarrow 1} \left(\frac{\partial p}{\partial T}\right)_{\tilde{\epsilon}} = \frac{\Delta C_{p\lambda}}{\Delta \alpha_\lambda} \frac{1}{T_\lambda v_\lambda}, \quad (2.5)$$

the well-known Ehrenfest's equation. Accordingly we can interpret consistently the lambda-point as the temperature where $\tilde{\epsilon}$ equals to unity, *i. e.*, the superfluid vanishes.

We have assumed above the relation

$$C_{pII\lambda} - C_{pI\lambda} = \lim_{\tilde{\epsilon} \rightarrow 1} T \left(\frac{\partial s}{\partial \tilde{\epsilon}}\right)_{pT} \left(\frac{\partial \tilde{\epsilon}}{\partial T}\right)_p.$$

Now if we take up as usual the approximation

$$\tilde{\epsilon} = \left(\frac{T}{T_\lambda}\right)^\sigma,$$

we get

$$\left(\frac{\partial \tilde{\epsilon}}{\partial T}\right)_p = \frac{\sigma \tilde{\epsilon}}{T},$$

and so

$$\Delta C_{p\lambda} = \lim_{\tilde{\epsilon} \rightarrow 1} \sigma \tilde{\epsilon} \left(\frac{\partial s}{\partial \tilde{\epsilon}}\right)_{pT}.$$

As stated in I, if H. London's relation is valid it holds that

$$\tilde{\epsilon} \left(\frac{\partial s}{\partial \tilde{\epsilon}}\right)_{pT} = s,$$

*) Cf. reference 1.

by the strength of which the above equation reduces to

$$\Delta C_{v\lambda} = \sigma s_{\lambda}.$$

According to Keesom⁷, the experimental value of the jump of specific heat is

$$\Delta C_{p\lambda} = 1.90 \text{ cal. deg}^{-1} \text{ gr}^{-1}.$$

Inserting this value and also $s_{\lambda} = 0.405 \text{ cal. deg}^{-1} \text{ gr}^{-1}$, we obtain for the value of σ ,

$$\sigma = 4.7,$$

a reasonable value. This may be regarded as one of justifications of the assumption made above.

§ 3. Effect of He³ ingredient

We shall write for the densities of the normal, super- and helium 3 components ρ_n , ρ_s and ρ_3 , respectively, so that the total density ρ is given by

$$\rho = \rho_n + \rho_s + \rho_3, \quad (3.1)$$

and the density of helium 4, ρ_4 ,

$$\rho_4 = \rho_n + \rho_s. \quad (3.2)$$

The mass concentrations of each component are defined as

$$\hat{\xi}_\gamma = \frac{\rho_\gamma}{\rho}, \quad (\gamma = n, s, 3). \quad (3.3)$$

It is also convenient to introduce another definition for the relative concentrations of helium 4 components through

$$\tilde{\xi} = \frac{\rho_n}{\rho_4} \quad \text{and} \quad \bar{\xi} = \frac{\rho_s}{\rho_4}. \quad (3.4)$$

Then the following identities are selfevident:

$$\hat{\xi}_n + \hat{\xi}_s + \hat{\xi}_3 = \hat{\xi}_4 + \hat{\xi}_3 = 1, \quad \tilde{\xi}_n + \bar{\xi}_s = 1,$$

and

$$\tilde{\xi}_n = \frac{\hat{\xi}_n}{\hat{\xi}_4}, \quad \bar{\xi}_s = \frac{\hat{\xi}_s}{\hat{\xi}_4}.$$

Using partial quantities we can write for the Gibbs' free energy per unit mass of the mixture

$$\mu = \hat{\xi}_n \bar{\mu}_n + \hat{\xi}_s \bar{\mu}_s + \hat{\xi}_3 \bar{\mu}_3. \quad (3.5)$$

Under the condition of constant p , T , and $\hat{\xi}_3$, therefore,

$$\partial \mu = (\bar{\mu}_n - \bar{\mu}_s) \partial \tilde{\xi}_n + \left[\hat{\xi}_n \left(\frac{\partial \bar{\mu}_n}{\partial \hat{\xi}_n} \right)_{p, T, \hat{\xi}_3} + \hat{\xi}_s \left(\frac{\partial \bar{\mu}_s}{\partial \hat{\xi}_n} \right)_{p, T, \hat{\xi}_3} + \hat{\xi}_3 \left(\frac{\partial \bar{\mu}_3}{\partial \hat{\xi}_n} \right)_{p, T, \hat{\xi}_3} \right] \partial \hat{\xi}_n.$$

Gibbs-Duhem relation state that the sum under the square brackets vanishes, and hence the themostatic equilibrium condition $(\partial\mu)_{pT_3}=0$ gives

$$\bar{\mu}_n - \bar{\mu}_s = 0. \quad (3.6)$$

From the above expression for $\partial\mu$, we can also write for this condition

$$\bar{\mu}_n - \bar{\mu}_s = \left(\frac{\partial\mu}{\partial\hat{\xi}_n} \right)_{pT_3} = 0, \quad (3.7)$$

or

$$\left(\frac{\partial\mu}{\partial\hat{\xi}_n} \right)_{pT_3} = 0. \quad (3.7')$$

Along the equilibrium states of constant p and $\hat{\xi}_n$, we obtain hereupon the following equation:

$$\frac{\partial^2\mu}{\partial\hat{\xi}_3\partial\hat{\xi}_n}d\hat{\xi}_3 + \frac{\partial^2\mu}{\partial T\partial\hat{\xi}_n}dT = 0,$$

or

$$\left(\frac{\partial T}{\partial\hat{\xi}_3} \right)_{p\hat{\xi}_n} = \left(\frac{\partial^2\mu}{\partial\hat{\xi}_3\partial\hat{\xi}_n} \right)_{pT} \left/ \left(\frac{\partial s}{\partial\hat{\xi}_n} \right)_{pT_3} \right. \quad (3.8)$$

In the case of $\text{He}^3\text{-He}^4$ mixture, we shall also interpret the lambda-point as the temperature, where the superfluid component vanishes: $\hat{\xi}_n=1$. Then, the $\hat{\xi}_3$ dependence of the lambda-point at constant pressure is given by

$$\left(\frac{\partial T}{\partial\hat{\xi}_3} \right)_p = \lim_{\hat{\xi}_n \rightarrow 1} \left(\frac{\partial T}{\partial\hat{\xi}_3} \right)_{p\hat{\xi}_n} = \lim_{\hat{\xi}_n \rightarrow 1} \frac{\left(\frac{\partial^2\mu}{\partial\hat{\xi}_3\partial\hat{\xi}_n} \right)_{pT}}{\left(\frac{\partial s}{\partial\hat{\xi}_n} \right)_{pT_3}}.$$

§ 4. Approximation as a dilute solution

We shall write the mole concentrations as N_n , N_s and N_3 . Therefore,

$$N_n + N_s + N_3 = 1.$$

We may take the atomic weights of He^4 and He^3 with sufficient accuracy as 4 and 3, respectively. Then the relations between N 's and $\hat{\xi}$'s are

$$\hat{\xi}_n = 4N_n / [4(N_n + N_s) + 3N_3],$$

$$\hat{\xi}_s = 4N_s / [4(N_n + N_s) + 3N_3],$$

and

$$\hat{\xi}_3 = 3N_3 / [4(N_n + N_s) + 3N_3].$$

On the basis of the fact that at temperatures below the lambda-point the

fugacity of helium 3 in the solution $\text{He}^3\text{-He}^4$ is really given by Raoult's law, counting only the normal component as the solvent⁽⁸⁾, we can give at least as an approximation, the partial molal Gibbs' free energies the following expressions:

$$\bar{G}_n = \bar{G}_n^0 + RT \ln \frac{N_n}{N_n + N_s},$$

$$\bar{G}_s = \bar{G}_s^0$$

and

$$\bar{G}_3 = G_3^0 + RT \ln \frac{N_3}{N_n + N_3}.$$

Therefore, the molal Gibbs' free energy of the mixture can be expressed with sufficient accuracy by

$$\begin{aligned} G &= N_n \bar{G}_n + N_s \bar{G}_s + N_3 \bar{G}_3 \\ &= N_n \bar{G}_n^0 + N_s \bar{G}_s^0 + N_3 G_3^0 + RT \left[N_n \ln \frac{N_n}{N_n + N_3} + N_3 \ln \frac{N_3}{N_n + N_3} \right]. \end{aligned}$$

Expressing N 's in terms of ξ 's, we finally obtain the relevant expression for the Gibbs' free energy per unit mass as follows:

$$\begin{aligned} \mu &= G/[4(N_n + N_s) + 3N_3] \\ &= \hat{\xi}_4(\bar{\xi}_n \bar{\mu}_n^0 + \bar{\xi}_s \bar{\mu}_s^0) + \hat{\xi}_3 \mu_3^0 + RT \left[\frac{\hat{\xi}_4 \bar{\xi}_n}{4} \ln \frac{\bar{\xi}_n}{\bar{\xi}_n + \frac{4\bar{\xi}_3}{3\hat{\xi}_4}} + \frac{\hat{\xi}_3}{3} \ln \frac{\bar{\xi}_3/\hat{\xi}_4}{\frac{3\bar{\xi}_n}{4} + \frac{\bar{\xi}_3}{\hat{\xi}_4}} \right]. \end{aligned} \quad (4.1)$$

Using this expression we can derive by straightforward calculations the following formulae:

$$\begin{aligned} s &= - \left(\frac{\partial \mu}{\partial T} \right)_{p, \bar{\xi}_n, \bar{\xi}_3} \\ &= \hat{\xi}_4(\bar{\xi}_n \bar{s}_n^0 + \bar{\xi}_s \bar{s}_s^0) + \hat{\xi}_3 s_3^0 - R \left[\frac{\hat{\xi}_4 \bar{\xi}_n}{4} \ln \frac{\bar{\xi}_n}{\bar{\xi}_n + \frac{4\bar{\xi}_3}{3\hat{\xi}_4}} + \frac{\hat{\xi}_3}{3} \ln \frac{\bar{\xi}_3/\hat{\xi}_4}{\frac{3\bar{\xi}_n}{4} + \frac{\bar{\xi}_3}{\hat{\xi}_4}} \right], \end{aligned} \quad (4.2)$$

$$\left(\frac{\partial \mu}{\partial \bar{\xi}_n} \right)_{p, T, \bar{\xi}_3} = \hat{\xi}_4(\bar{\mu}_n^0 - \bar{\mu}_s^0) + \frac{RT}{4} \hat{\xi}_4 \ln \frac{\bar{\xi}_n}{\bar{\xi}_n + \frac{4}{3} \frac{\bar{\xi}_3}{\hat{\xi}_4}}, \quad (4.3)$$

$$\left(\frac{\partial^2 \mu}{\partial \bar{\xi}_n^2} \right)_{p, T, \bar{\xi}_3} = \hat{\xi}_4 \left[\left(\frac{\partial^2 \mu_4^0}{\partial \bar{\xi}_n^2} \right)_{p, T} + \frac{RT}{\hat{\xi}_n} \cdot \frac{\bar{\xi}_3/\hat{\xi}_4}{3\bar{\xi}_n + 4\bar{\xi}_3/\hat{\xi}_4} \right],$$

$$\left(\frac{\partial^2 \mu}{\partial \xi_3 \partial \xi_n}\right)_{pT} = -\frac{RT}{3} \cdot \frac{1}{\xi_4} \cdot \frac{1}{\xi_n + \frac{4}{3} \frac{\xi_3}{\xi_4}}$$

and

$$\left(\frac{\partial s}{\partial \xi_n}\right)_{pT\xi_3} = \xi_4 \left[\left(\frac{\partial s_4^0}{\partial \xi_n}\right)_{pT} - \frac{R}{4} \ln \frac{\xi_n}{\xi_n + \frac{4}{3} \frac{\xi_3}{\xi_4}} \right] \quad (4.4)$$

In the calculation of Eq. (4.3) we have used the Gibbs-Duhem relation:

$$\bar{\xi}_n \cdot \frac{\partial \bar{\mu}_n^0}{\partial \bar{\xi}_n} + \bar{\xi}_s \cdot \frac{\partial \bar{\mu}_s^0}{\partial \bar{\xi}_n} = 0. \quad (3.8)$$

Insertion of these formulae into Eq. (3.8) leads to

$$\left(\frac{\partial T}{\partial \xi_3}\right)_{p\xi_n} = -\frac{RT}{\left[\left(\frac{\partial s_4^0}{\partial \xi_n}\right)_{pT} + \frac{R}{4} \ln \left(1 + \frac{4}{3} \frac{\xi_3}{(1-\xi_3)\bar{\xi}_n}\right)\right] (1-\xi_3)^2 \left(3\bar{\xi}_n + 4 \frac{\xi_3}{1-\xi_3}\right)}. \quad (4.5)$$

Making use of this expression we shall perform some numerical computation in the next section.

§ 5. Numerical values of $\bar{\xi}_n$ and $\left(\frac{\partial T}{\partial \xi_n}\right)_{p\xi_3}$

For the sake of convenience we restrict ourselves in the temperature region where H. London's equation holds and this in turn states that¹⁾

$$\left(\frac{\partial s_4^0}{\partial \xi_n}\right)_{pT} = s_0(p, T) \div s_{\lambda 0}(p). \quad (5.1)$$

The dependence of s_0 on T was shown to be negligible in the paper I on the basis of Tisza's idea that the normal component can be considered in a sense as a continuation of He I. In order to obtain a safe conclusion, we shall treat concretely the temperature region: $T_\lambda \geq T \geq 1.5^\circ \text{K}$. Then, for the range of helium 3 concentration: $\xi_3 \lesssim 0.008$, $\ln \left(1 + (4/3) \frac{1}{\xi_n} \cdot \frac{\xi_3}{1-\xi_3}\right)$ can be consistently replaced with $(4/3) \frac{\xi_3}{\xi_n}$, and also the factor $3\bar{\xi}_n + 4 \frac{\xi_3}{1-\xi_3}$, with $3\bar{\xi}_n + 4\bar{\xi}_3$.

In these approximations we can integrate the above differential equation for $T(\xi_3, \bar{\xi}_n, p)$, Eq. (4.5), and obtain under the condition of constant pressure

$$T(\xi_3, \bar{\xi}_n) = T(0, \bar{\xi}_n) \left[(1-\xi_3)^{\frac{2+\frac{3}{4} \frac{\alpha+1}{\alpha} \bar{\xi}_n}{(1+\frac{3}{4} \bar{\xi}_n)(1+\frac{4}{4\alpha} \bar{\xi}_n)}} \exp\left(-\frac{\bar{\xi}_3}{1-\xi_3}\right) \right]^{\frac{\frac{3}{4} \bar{\xi}_n}{(1+\frac{3}{4} \bar{\xi}_n)(1+\frac{4}{4\alpha} \bar{\xi}_n)}}$$

$$\times \frac{\left[\left(1 + \frac{4}{3} \frac{\xi_3}{\bar{\xi}_n} \right)^{\left(1 + \frac{3}{4} \frac{\bar{\xi}_n}{\xi_3} \right)^{-2}} \right]^{\frac{\alpha}{\alpha-1}}}{\left[\left(1 + \frac{4\alpha}{3} \frac{\xi_3}{\bar{\xi}_n} \right)^{\left(1 + \frac{2}{4\alpha} \frac{\bar{\xi}_n}{\xi_3} \right)^{-2}} \right]}, \quad (5.2)$$

where $R/4s_{\lambda 0}$ has been represented by α .

We have computed values of the above function for several values of ξ_3 and $\bar{\xi}_n$, the results being listed in Table I. In the table are also listed the values of $(\partial T / \partial \bar{\xi}_n)_{p, \xi_3}$. To the values of this differential coefficient the last factor of the

ξ_3	$T(^{\circ}\text{K})$	$\bar{\xi}_n$	$(\partial T / \partial \bar{\xi}_n)_{p, \xi_3}$
0	2.186	1.0000	—
	2.000	0.6078	0.5876
	1.900	0.4560	0.7440
	1.700	0.2446	1.2411
	1.500	0.1214	2.2071
0.001	2.183	1.0000	—
	2.000	0.617	0.588
	1.900	0.465	0.744
	1.700	0.2535	1.240
	1.500	0.1301	2.201
0.003	2.176	1.0000	—
0.005	2.169	1.0000	—
	2.000	0.652	0.585
	1.900	0.499	0.740
	1.700	0.2861	1.224
	1.505	0.1619	2.089
0.008	2.158	1.0000	—
	2.000	0.677	0.583
	1.900	0.523	0.735
	1.693	0.3094	1.192
	1.525	0.1909	1.912

Table I.

expression (5.2) contributes rather much and therefore a fairly accurate value of α is indispensable. We have taken the following experimental values:

$$T(0, \bar{\xi}_n) = T_{\lambda 0}(\bar{\xi}_n)^{\frac{1}{\sigma}}, \text{ with } \sigma = 5.60 \text{ and } T_{\lambda 0} = 2.186^{\circ}\text{K},$$

$$s_{\lambda 0} = 0.405 \text{ cal. gr}^{-1} \text{ deg}^{-1} \text{ and hence } \alpha = 1.220.$$

The values of $T_{\lambda}(\xi_3)$ are obtained, as stated before, on the assumption that the lambda-point is the temperature at which $\bar{\xi}_n$ is equal to unity.

Part II. Thermodynamics (wave propagation)

§ 6. Irreversible production of entropy

Appealing to a heuristic method, we first formulate the expression for the irreversible production of entropy, treating the helium 3 component as though it behaved like the other two, *i. e.*, normal and super-components.

The following mass conservation law is invariably valid. Writing the velocity fields of three components, \mathbf{v}_n , \mathbf{v}_s , and \mathbf{v}_3 , respectively, we define the velocity of the centre of gravity, \mathbf{v}_1 , through the relation:

$$\rho \mathbf{v}_1 = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s + \rho_3 \mathbf{v}_3. \quad (6.1)$$

Then the mass conservation law is given by

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \rho \mathbf{v}_1, \quad (6.2)$$

or separated for the three components,

$$\frac{\partial \rho_3}{\partial t} = -\operatorname{div} \rho_3 \mathbf{v}_3, \quad (6.2')$$

$$\frac{\partial \rho_n}{\partial t} = -\operatorname{div} \rho_n \mathbf{v}_n + \Gamma \quad (6.2'')$$

and

$$\frac{\partial \rho_s}{\partial t} = -\operatorname{div} \rho_s \mathbf{v}_s - \Gamma \quad (6.2''')$$

where Γ means the rate of production of the normal fluid per unit volume which is always accompanied by annihilation of an equal amount of the superfluid.

Using the above equations, the momentum conservation law can be easily obtained as follows:

$$\rho_n \frac{D\mathbf{v}_n}{Dt} + \rho_s \frac{D\mathbf{v}_s}{Dt} + \Gamma(\mathbf{v}_n - \mathbf{v}_s) + \rho_3 \frac{D\mathbf{v}_3}{Dt} = \nabla \cdot \Pi \equiv -\operatorname{grad} p + \nabla \cdot \Pi', \quad (6.3)$$

where the third term $\Gamma(\mathbf{v}_n - \mathbf{v}_s)$ appears as a result of transferring the mass Γ per unit time from the velocity \mathbf{v}_s to \mathbf{v}_n . Π' represents the viscosity tensor.

In order to formulate the energy conservation law, it is convenient first to introduce another set of velocity fields, *i. e.*, velocities relative to the centre of gravity, by the following equations:

$$\mathbf{U}_\tau = \mathbf{v}_\tau - \mathbf{v}_1, \quad (\tau = n, s, 3). \quad (6.4)$$

It is clear that these three velocities are not independent of each other but satisfy the identity:

$$\rho_n \mathbf{U}_n + \rho_s \mathbf{U}_s + \rho_3 \mathbf{U}_3 = 0.$$

Corresponding to the thermostatical equation under constant volume,

$$\delta U = T(\delta S)_{\delta M_T} + \sum_{\tau} \bar{h}_{\tau} \delta M_{\tau}, \quad (6.5)$$

we can write down the energy conservation law as follows:

$$\frac{\partial}{\partial t} \left[\rho u + \sum_{\tau} \frac{1}{2} \rho_{\tau} v_{\tau}^2 \right] = -\operatorname{div} \left[\mathbf{q}_0 + \sum_{\tau} \left(\bar{h}_{\tau} \rho_{\tau} \mathbf{v}_{\tau} + \frac{1}{2} v_{\tau}^2 \rho_{\tau} \mathbf{v}_{\tau} \right) \right] + \{ \nabla \cdot \Pi \}, \quad (6.6)$$

where \mathbf{q}_0 means an ordinary heat current density, and $\{ \nabla \cdot \Pi' \}$ represents the work done by the viscosity stress. The term $-\operatorname{div} \mathbf{q}_0$ corresponds to $T(\delta S)_{\delta M_T}$ but the terms $-\frac{\partial}{\partial t} \sum \frac{1}{2} \rho_{\tau} v_{\tau}^2$ and $\{ \nabla \cdot \Pi' \}$ do appear only in the latter equation because of their essentially kinetic character. It may be noted in this connection that

$$\begin{aligned} & \frac{\partial}{\partial t} \sum \frac{1}{2} \rho_{\tau} v_{\tau}^2 + \operatorname{div} \sum \frac{1}{2} v_{\tau}^2 \rho_{\tau} \mathbf{v}_{\tau} \\ &= \rho_n \cdot \mathbf{v}_n \frac{D\mathbf{v}_n}{Dt} + \rho_s \mathbf{v}_s \frac{D\mathbf{v}_s}{Dt} + \frac{\Gamma}{2} (v_n^2 - v_s^2) + \rho_s \mathbf{v}_s \cdot \frac{D\mathbf{v}_s}{Dt}. \end{aligned} \quad (6.7)$$

We shall finally formulate the expression for the irreversible production of entropy. First, the equation expressing the entropy balance may be written as follows:

$$\frac{\partial \rho s}{\partial t} = -\operatorname{div} \left[\frac{\mathbf{q}_0}{T} + \sum \bar{S}_{\tau} \rho_{\tau} \mathbf{v}_{\tau} \right] + \left(\frac{dS}{dt} \right)_{\text{irr}}. \quad (6.8)$$

This corresponds to the thermostatical equation under constant volume:

$$\delta S = (\delta S)_{\delta M_T} + \sum \bar{S}_{\tau} \delta M_{\tau}. \quad (6.9)$$

Secondly, if we assume that Gibbs' fundamental formula:

$$Tds = du + p dv - \sum \bar{\mu}_{\tau} \cdot d\zeta_{\tau}$$

is always valid in the baricentric system even when the continuous medium is in motion, we get the following equation⁹⁾.

$$\begin{aligned} & T \left[\frac{\partial \rho s}{\partial t} + \operatorname{div} \rho u \mathbf{v}_1 \right] \\ &= \frac{\partial \rho u}{\partial t} + \operatorname{div} \rho u \mathbf{v}_1 + p \operatorname{div} \mathbf{v}_1 - (\bar{\mu}_n - \bar{\mu}_s) \Gamma + \sum \bar{\mu}_{\tau} \operatorname{div} \rho_{\tau} \mathbf{U}_{\tau}. \end{aligned} \quad (6.10)$$

Elimination of the term $(\partial \rho s / \partial t + \operatorname{div} \rho s \mathbf{v}_1)$ from Eq. (6.8) and Eq. (6.10) leads to

$$\begin{aligned} T \left(\frac{\partial S}{\partial t} \right)_{\text{irr}} &= T \operatorname{div} \frac{\mathbf{q}_0}{T} + \sum \bar{h}_{\tau} \operatorname{div} \rho_{\tau} \mathbf{U}_{\tau} + T \sum \rho_{\tau} \mathbf{U}_{\tau} \operatorname{grad} \bar{S}_{\tau} \\ &+ \frac{\partial \rho u}{\partial t} + \operatorname{div} \rho h \mathbf{v}_1 - \mathbf{v}_1 \operatorname{grad} p - (\bar{\mu}_n - \bar{\mu}_s) \Gamma. \end{aligned} \quad (6.11)$$

Using Eq. (6.6) for eliminating $\partial \rho / \partial t$, we can reduce the above expression with the relation $\bar{\mu}_r = \bar{h}_r - T \bar{s}_r$ into

$$\begin{aligned}
 T \left(\frac{dS}{dt} \right)_{\text{irr}} = & -\mathbf{v}_r \cdot \left[\sum \rho_r \frac{D\mathbf{v}_r}{Dt} + \text{grad } p \right] \\
 & - \sum \rho_r \mathbf{U}_r \cdot \left[\frac{D\mathbf{v}_r}{Dt} + \text{grad } \bar{\mu}_r + \bar{s}_r \text{ grad } T \right] \\
 & - \left[\frac{1}{2} (v_n^2 - v_s^2) + \bar{\mu}_n - \bar{\mu}_s \right] \Gamma \\
 & + \{ \nabla \cdot \Pi' \} \\
 & + T \mathbf{q}_0 \cdot \text{grad } \frac{1}{T}.
 \end{aligned} \tag{6.11'}$$

Transforming again the above equation with the use of Eq. (6.3),

$$\begin{aligned}
 T \left(\frac{\partial S}{\partial t} \right)_{\text{irr}} = & - \sum \rho_r \mathbf{U}_r \cdot \left[\frac{D\mathbf{v}_r}{Dt} + \text{grad } \bar{\mu}_r + \bar{s}_r \text{ grad } T \right] \\
 & - \left[\frac{1}{2} (U_n^2 - U_s^2) + \bar{\mu}_n - \bar{\mu}_s \right] \Gamma \\
 & + \{ \nabla \cdot \Pi' \} - \mathbf{v}_1 \cdot (\nabla \cdot \Pi') \\
 & + T \mathbf{q}_0 \cdot \text{grad } \frac{1}{T}.
 \end{aligned} \tag{6.12}$$

By the way, it is to be noted that, putting $\rho_3=0$ and $\Pi'=\Pi'_n$ the viscosity tensor of normal fluid, one gets for the case of pure helium 4 the same result as in the paper I.

§ 7. The equation of motion for super-component and the velocity of sounds

If we restrict ourselves to the discussion of infinitesimal motion of the mixture, we are allowed to make a simple assumption concerning the velocity of helium 3 component. Our final aim is to obtain the equations of reversible, infinitesimal motion. We shall treat in this section only the case in which the helium 3 component moves with the centre of gravity, *i. e.*,

$$\mathbf{U}_3=0, \quad \text{or} \quad \mathbf{v}_3=\mathbf{v}_1. \tag{7.1}$$

Another case in which the helium 3 component moves with the normal component will be treated in Appendix.

In the present case

$$\rho_4 \mathbf{v}_1 = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s,$$

or

$$\rho_n \mathbf{U}_n + \rho_s \mathbf{U}_s = 0.$$

This specification of U_s reduces Eq. (6.12) for the irreversible entropy production to

$$\begin{aligned} T \left(\frac{dS}{dt} \right)_{\text{irr}} &= U_s \cdot \left[\rho_s \left(\frac{\partial \mathbf{v}_n}{\partial t} - \frac{\partial \mathbf{v}_s}{\partial t} \right) + \rho_s \text{grad} (\bar{\mu}_n - \bar{\mu}_s) + \rho_s (\bar{S}_n - \bar{S}_s) \text{grad } T \right] \\ &\quad - \left[\frac{1}{2} (U_n^2 - U_s^2) + \bar{\mu}_n - \bar{\mu}_s \right] I' \\ &\quad + \{ \nabla \cdot \Pi' \} - \mathbf{v}_1 \cdot (\nabla \cdot \Pi') + \Gamma \mathbf{q}_0 \cdot \text{grad } \frac{1}{T} \\ &= \rho_s (U_n - U_s) \left[\frac{\partial \mathbf{v}_s}{\partial t} + \frac{1}{\rho} \text{grad } p - \frac{\rho_n}{\rho_4} \text{grad} (\bar{\mu}_n - \bar{\mu}_s) \right. \\ &\quad \left. - \frac{\rho_n}{\rho_4} (\bar{S}_n - \bar{S}_s) \text{grad } T \right] + \dots \dots \end{aligned} \quad (7.2)$$

From this result we can extract the desired equation of reversible motion of super-component with the assumption of local equilibrium¹⁾, $\bar{\mu}_n = \bar{\mu}_s$, as follows:

$$\frac{\partial \mathbf{v}_s}{\partial t} = - \frac{1}{\rho} \text{grad } p + \frac{\rho_n}{\rho_4} (\bar{S}_n - \bar{S}_s) \text{grad } T, \quad (7.3)$$

Once the equation of motion for super-component is obtained, the equations for the other components can easily be obtained by making use of the above-mentioned assumption and the momentum conservation law, but we shall refrain here from writing down explicitly the results.

For the plane wave motion of the type: $\star + \star' \exp(i\omega t - ikx)$, we obtain from the equations of motion and mass conservation laws the following set of equations for $x = c^2 = (\omega/k)^2$:

$$\begin{cases} x \frac{\rho'_s}{\rho_s} = \frac{p'}{\rho} - \bar{\xi}_n (\bar{S}_n - \bar{S}_s) T', \\ x \rho' = p', \\ \frac{\rho'_s}{\rho_s} = \frac{\rho'}{\rho}, \end{cases}$$

or, after some manipulations of thermostatical transformation, we get the secular equation for x :

$$\begin{aligned} x^2 - \left\{ \left(\frac{\partial p}{\partial \rho} \right)_{\bar{\xi}_n \bar{\xi}_s} + \left[\frac{(\partial p / \partial \rho)_{T \rho_s}}{(\partial p / \partial \rho)_{\bar{\xi}_n \bar{\xi}_s}} \right] \bar{\xi}_n \bar{\xi}_s (\bar{S}_n - \bar{S}_s) \left(\frac{\partial T}{\partial \bar{\xi}_n} \right)_{\rho \rho_s} \right\} x \\ + \left(\frac{\partial p}{\partial \rho} \right)_{\bar{\xi}_n \bar{\xi}_s} \cdot \bar{\xi}_n \bar{\xi}_s (\bar{S}_n - \bar{S}_s) \left(\frac{\partial T}{\partial \bar{\xi}_n} \right)_{\rho \rho_s} \left[1 - \frac{\rho_s}{\rho} \left(\frac{\partial \rho}{\partial \rho_s} \right)_{pT} \right] \left[1 + \frac{\bar{\xi}_s}{\rho} \left(\frac{\partial \rho}{\partial \bar{\xi}_s} \right)_{p \bar{\xi}_n} \right] = 0. \quad (7.4) \end{aligned}$$

From this, with the following reasonable assumptions :

$$\left(\frac{\partial \rho}{\partial \rho}\right)_{T, p_3} \left/\left(\frac{\partial \rho}{\partial \rho}\right)_{\xi_n, p_3}\right. \doteq 1, \quad (7.5)$$

and
$$\left|\frac{\rho_3}{\rho}\left(\frac{\partial \rho}{\partial \rho_3}\right)_{pT}\right|, \quad \left|\frac{\xi_3}{\rho}\left(\frac{\partial \rho}{\partial \xi_3}\right)_{p\xi_n}\right| \ll 1, \quad (7.5')$$

we can easily obtain for the velocities of first and second sound

$$c_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\xi_n, p_3}, \quad (7.6)$$

and

$$c_2^2 = \bar{\xi}_n \hat{\xi}_s (\bar{s}_n - \bar{s}_s) \left(\frac{\partial T}{\partial \hat{\xi}_n}\right)_{p, p_3} \\ \doteq \bar{\xi}_n \hat{\xi}_s \left(\frac{\partial s}{\partial \hat{\xi}_n}\right)_{pT, \xi_3} \left(\frac{\partial T}{\partial \hat{\xi}_n}\right)_{p, p_3}, \quad (7.7)$$

respectively.

It may be noted that c_2^2 can also be written as

$$\frac{\bar{\xi}_n \hat{\xi}_s}{\hat{\xi}_4} \left(\frac{\partial^2 \mu}{\partial \hat{\xi}_n^2}\right)_{pT, \xi_3}. \quad (7.7')$$

Using the approximation described in Section 4, we finally get the following expression :

$$c_2^2 = \bar{\xi}_n (1 - \bar{\xi}_n) \left[\left(\frac{\partial s_4^0}{\partial \hat{\xi}_n}\right)_{pT} + \frac{R}{4} \ln \left(1 + \frac{4}{3} \frac{1}{\bar{\xi}_n} \frac{\bar{\xi}_n}{1 - \bar{\xi}_n}\right) \right] \left(\frac{\partial T}{\partial \hat{\xi}_n}\right)_{p, p_3}. \quad (7.8)$$

Restriction ourselves in the region where H. London's equation is valid, we calculate numerically c_2 from the above obtained expression. The results are listed in Table II, and also are plotted in Figure I. The experimental points in the figure are taken from the data of Lynton and Fairbank⁵⁾.

Temperature (°K)	Velocity (m/sec.)	Temperature (°K)	Velocity (m/sec.)
Pure He ⁴		0.1% He ³	
2.000	15.4	2.000	15.36
1.900	17.7	1.900	17.74
1.700	19.7	1.700	20.01
1.500	20.0	1.500	20.67
0.5% He ³		0.8% He ³	
2.000	15.10	2.000	14.85
1.900	17.85	1.900	17.85
1.700	20.87	1.693	21.20
1.505	22.45	1.525	23.10

Table II. Velocity of second sound c_2 (calculated).

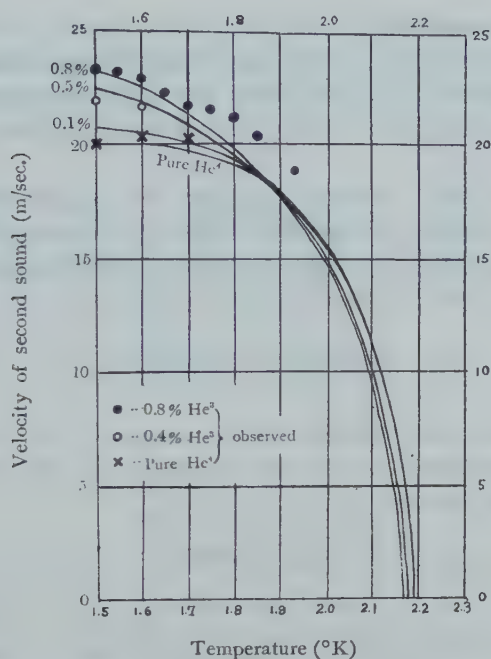


Fig. 1. Velocity of second sound.

—: calculated.

§ 8. Discussion

Taking advantage of the reversible character of the propagation of the second sound, where the characteristic "second motion" plays an essential role, we have investigated the propagation velocity in two reasonable cases. It may be tempting to assume that the helium 3 component should always move in company with the normal component. This case is treated in Appendix. The result, however, is definitely contradictory to the experimental fact, *i. e.*, the velocity of second sound decreases with the increase of helium 3 content, at least in the range where H. London's relation holds. The origin of this tendency consists essentially in the factor, $1/(\xi_n + \xi_3/\xi_4)$, involved in the expression for c_2^2 , and this in turn, comes from the assumption that the helium 3 component should move with the normal component.

Another case, in which the helium 3 component moves with the centre of gravity is treated in Section 7. In case of the propagation of second sound, therefore, the helium 3 component is assumed to be essentially at rest. The results, mentioned in the preceding section, shows that the calculated velocity is in rather good agreement with the experimental results. The curvature of the c_2-T curve is seen to be too small just below each lambda-point, but the same deviation already exists in the case of pure helium 4. We may, therefore, consider the discrepancy optimistically as being attributable to imperfection of the empirical formula for the con-

centration of normal component.

We may allowed to conclude that the helium 3 component is actually at rest in case of the propagation of second sound, while the other components are in wave motion. It is to be noted that the assumption of local equilibrium is not inconsistent with the above picture.

In this connection, the thermal Rayleigh disc experiment; as proposed by Pellam and Morse¹⁰⁾, may be considered valuable for testing the above conclusion. The torque acting on a Rayleigh disc of radius r , set in the second sound field at the angle of attack $\pi/4$, is given by¹¹⁾

$$N = \frac{4}{3} \frac{\rho_4 \rho_n r^3}{\rho_3} \bar{U}_n^2 \left[1 + \frac{1}{5} \left(\frac{\omega r}{c_2} \right)^2 + \cdots \right], \quad (8.1)$$

where \bar{U}_n^2 is the mean square velocity in the absence of the disc. As the centre of gravity may be considered at rest and hence so also the helium 3 component, the corresponding heat flow density \dot{H} is

$$\begin{aligned} \dot{H} &= \bar{h}_n \rho_n \mathbf{U}_n + \bar{h}_s \rho_s \mathbf{U}_s \\ &= \rho_n \mathbf{U}_n (\bar{h}_n - \bar{h}_s). \end{aligned} \quad (8.2)$$

Taking into account the assumption of local equilibrium:

$$0 = \bar{\mu}_n - \bar{\mu}_s \equiv (\bar{h}_n - \bar{h}_s) - T(\bar{s}_n - \bar{s}_s), \quad (8.3)$$

we can write in a more familiar form,

$$\begin{aligned} \dot{H} &= T \rho_n (\bar{s}_n - \bar{s}_s) \mathbf{U}_n \\ &= T \rho_n \tilde{\xi}_n \left(\frac{\partial s}{\partial \tilde{\xi}_n} \right)_{p, T, \tilde{\xi}_3} \cdot \mathbf{U}_n. \end{aligned} \quad (8.4)$$

If we take this relation into consideration, we find that at 1.700°K, for instance, the acting torque in the case of $\tilde{\xi}_3 = 0.005$ will reduce to *ca.* 86 percent of the value attained in the case of pure helium 4 under the condition that the heat flow density should be kept the same. Hence this type of experiment may be considered practicable and will afford an interesting check.

The process that the helium 3 component are enticed by the normal component may, then, be considered irreversible. One of the possible processes is the diffusion of helium 3 component, which may be expected to occur toward the direction of larger $\tilde{\xi}_n/\tilde{\xi}_s$, as may be suggested by the expression for entropy. Another possible process is the viscosity, which may be expected to have the tendency to equalize the paces of normal and helium 3 components. In any case, a more prudent investigation is necessary in order to formulate the proper equations governing the phenomena of finite motion. We are now investigating to formulate this idea, but it is highly desirable to have experimental data of the phenomena where finite motion of the components occurs, to determine the constants which will necessarily appear.

In conclusion, it may also be noted that by testing this extension the formalism developed in the paper I is provided with a new support. The calculations described in Part I of the present paper are, in fact, fatal steps of that treatment of pure helium 4.

Appendix

The case in which the helium 3 component moves with the normal component will be treated here. In the present case,

$$U_3 = U_n, \quad \text{or} \quad v_3 = v_n, \quad (\text{A.1})$$

and hence

$$\rho v_1 = (\rho_n + \rho_3) v_n + \rho_s v_s,$$

or

$$(\rho_n + \rho_3) U_n + \rho_s U_s = 0.$$

This specification of U_3 reduces Eq. (6.12) for the irreversible entropy production to

$$T \left(\frac{dS}{dt} \right)_{\text{irr}} = \rho_s (U_n - U_s) \cdot \left[\frac{\partial v_s}{\partial t} + \text{grad } \bar{\mu}_s + \bar{s}_s \text{ grad } T \right] \quad (\text{A.2})$$

$$+ \dots\dots.$$

Here the relations :

$$\sum_{\tau} \rho_{\tau} d\bar{\mu}_{\tau} = dp - \rho_s dT,$$

and

$$\rho_s = \sum_{\tau} \rho_{\tau} \bar{s}_{\tau}$$

and the momentum conservation law have been taken into account. Further transformation with the relations

$$\mu_s = \bar{\mu} - \xi_n \left(\frac{\partial \mu}{\partial \xi_n} \right)_{pT\xi_3} - \xi_3 \left(\frac{\partial \mu}{\partial \xi_3} \right)_{pT\xi_n} \quad (\text{A.3})$$

and

$$\bar{s} = s - \xi_n \left(\frac{\partial s}{\partial \xi_n} \right) - \xi_3 \left(\frac{\partial s}{\partial \xi_3} \right) \quad (\text{A.3}')$$

give us the following expression :

$$T \left(\frac{dS}{dt} \right)_{\text{irr}} = \rho_s (U_n - U_s) \cdot \left[\frac{\partial v_s}{\partial t} + \frac{1}{\rho} \text{grad } p - \left\{ \xi_n \left(\frac{\partial s}{\partial \xi_n} \right) + \xi_3 \left(\frac{\partial s}{\partial \xi_3} \right) \right\} \text{grad } T \right.$$

$$\left. - \xi_n \text{grad} \left(\frac{\partial \mu}{\partial \xi_n} \right) - \xi_3 \text{grad} \left(\frac{\partial \mu}{\partial \xi_3} \right) \right] \quad (\text{A.4})$$

$$+ \dots\dots.$$

As in Section 7, the assumption of local equilibrium model, $(\partial\mu/\partial\hat{\xi}_n)=0$, enables us to extract the equation of reversible motion of super-component as follows:

$$\frac{\partial v_s}{\partial t} = -\frac{1}{\rho} \text{grad } p + \left[\hat{\xi}_n \left(\frac{\partial s}{\partial \hat{\xi}_n} \right) + \hat{\xi}_3 \left(\frac{\partial s}{\partial \hat{\xi}_3} \right) \right] \text{grad } T + \hat{\xi}_3 \text{grad} \left(\frac{\partial \mu}{\partial \hat{\xi}_3} \right)_{pT\hat{\xi}_n}. \quad (\text{A} \cdot 5)$$

Therefore, the equations for x of plane wave motion are given by

$$\begin{cases} x'_{\rho n} = \frac{p'}{\rho} + \frac{\hat{\xi}_s}{1-\hat{\xi}_s} \left[\hat{\xi}_n \left(\frac{\partial s}{\partial \hat{\xi}_n} \right) + \hat{\xi}_3 \left(\frac{\partial s}{\partial \hat{\xi}_3} \right) \right] T' + \frac{\hat{\xi}_s \hat{\xi}_3}{1-\hat{\xi}_s} \left(\frac{\partial \mu}{\partial \hat{\xi}_3} \right)', \\ x_{\rho'} = \rho', \\ \frac{\rho'_3}{\rho_3} = \frac{\rho'_n}{\rho_n}, \end{cases}$$

or, after somewhat tedious manipulations of thermostatical transformation, we get the secular equation for x :

$$\begin{aligned} x^2 - & \left[\left(\frac{\partial p}{\partial \rho} \right)_{\hat{\xi}_n \hat{\xi}_3} + \frac{\hat{\xi}_s}{1-\hat{\xi}_s} \left(H \left\{ \left(\frac{\partial p}{\partial \rho} \right)_{p\rho_3} / \left(\frac{\partial p}{\partial \rho} \right)_{\hat{\xi}_n \hat{\xi}_3} \right\} \hat{\xi}_n \left(\frac{\partial T}{\partial \hat{\xi}_n} \right)_{p\rho_3} + \hat{\xi}_3 \left(\frac{\partial T}{\partial \hat{\xi}_3} \right)_{\hat{\xi}_n \rho} \right) \right. \\ & + \hat{\xi}_3 \left\{ \hat{\xi}_n \left(\frac{\partial K}{\partial \hat{\xi}_n} \right)_{\hat{\xi}_3 \rho} + \hat{\xi}_3 \left(\frac{\partial K}{\partial \hat{\xi}_3} \right)_{\hat{\xi}_n \rho} \right\} \left. \right] x + \left(\frac{\partial p}{\partial \rho} \right)_{\hat{\xi}_n \hat{\xi}_3} \frac{\hat{\xi}_s}{1-\hat{\xi}_s} \left(H \left[\hat{\xi}_n \left(\frac{\partial T}{\partial \hat{\xi}_n} \right)_{p\rho_3} \right. \right. \\ & + \hat{\xi}_3 \left(\frac{\partial T}{\partial \hat{\xi}_3} \right)_{\hat{\xi}_n \rho} \left\{ \left(\frac{\partial T}{\partial \hat{\xi}_n} \right)_{p\rho_3} \left(\frac{\partial \rho_n}{\partial \rho_3} \right)_{pT} / \left(\frac{\partial T}{\partial \hat{\xi}_n} \right)_{\rho \rho_3} \left(\frac{\partial \rho_n}{\partial \rho_3} \right)_{pT} \right\} \left. \right] \left[1 + \frac{\hat{\xi}_3}{\rho} \left(\frac{\partial \rho}{\partial \hat{\xi}_3} \right)_{p\hat{\xi}_n} \right] \\ & + \hat{\xi}_3 \left[\hat{\xi}_n \left(\frac{\partial K}{\partial \hat{\xi}_n} \right)_{\hat{\xi}_3 \rho} \left\{ \left(\frac{\partial K}{\partial \rho_n} \right)_{p\rho_3} / \left(\frac{\partial K}{\partial \rho_n} \right)_{\rho \rho_3} \right\} + \hat{\xi}_3 \left(\frac{\partial K}{\partial \hat{\xi}_3} \right)_{\hat{\xi}_n \rho} \left\{ \left(\frac{\partial K}{\partial \rho_3} \right)_{p\rho_n} / \left(\frac{\partial K}{\partial \rho_3} \right)_{\rho \rho_n} \right\} \right] \\ & \times \left[1 + \frac{\hat{\xi}_n}{\rho} \left(\frac{\partial \rho}{\partial \hat{\xi}_n} \right)_{p\hat{\xi}_3} + \frac{\hat{\xi}_3}{\rho} \left(\frac{\partial \rho}{\partial \hat{\xi}_3} \right)_{p\hat{\xi}_n} \right] = 0. \end{aligned} \quad (\text{A} \cdot 6)$$

This gives in a reasonable degree of approximation the velocities of both the waves as follows:

$$c_1^2 = \left(\frac{\partial p}{\partial \rho} \right)_{\hat{\xi}_n \hat{\xi}_3}, \quad (\text{A} \cdot 7)$$

$$c_2^2 = \frac{\hat{\xi}_s}{1-\hat{\xi}_s} \left(H \left[\hat{\xi}_n \left(\frac{\partial T}{\partial \hat{\xi}_n} \right)_{p\rho_3} + \hat{\xi}_3 \left(\frac{\partial T}{\partial \hat{\xi}_3} \right)_{p\hat{\xi}_n} \right] + \hat{\xi}_3 \left(\hat{\xi}_n \left(\frac{\partial K}{\partial \hat{\xi}_n} \right)_{p\rho_3} + \hat{\xi}_3 \left(\frac{\partial K}{\partial \hat{\xi}_3} \right)_{p\hat{\xi}_n} \right) \right), \quad (\text{A} \cdot 8)$$

with

$$H \equiv \hat{\xi}_n \left(\frac{\partial s}{\partial \hat{\xi}_n} \right)_{pT\hat{\xi}_3} + \hat{\xi}_3 \left(\frac{\partial s}{\partial \hat{\xi}_3} \right)_{pT\hat{\xi}_n} \quad (\text{A} \cdot 8')$$

and

$$K \equiv \left(\frac{\partial \mu}{\partial \hat{\xi}_3} \right)_{pT\hat{\xi}_n}. \quad (\text{A} \cdot 8'')$$

If we make use of the dilute solution approximation, the velocity of the second sound reduces to a simple form :

$$c_2^2 = \frac{\xi_s}{1 - \xi_s} \frac{\bar{\xi}_n^2}{\xi_4} \left(\frac{\partial^2 \mu_4^0}{\partial \xi_n^2} \right) = \frac{\bar{\xi}_s}{\xi_n + \xi_3 / \xi_4} \frac{\bar{\xi}_n^2}{\xi_4} \left(\frac{\partial^2 \mu_4^0}{\partial \xi_n^2} \right)_{pT}. \quad (\text{A.9})$$

Under the condition of constant p and $\bar{\xi}_n$, the variation of c_2^2 with ξ_3 can easily be studied. As can easily be seen from

$$\left(\frac{\partial c_2^2}{\partial \xi_3} \right)_{p\bar{\xi}_n} = - \frac{\xi_s}{\xi_n + \xi_3} \frac{c_2^2}{\xi_4} - \frac{\bar{\xi}_s}{\xi_n + \xi_3 / \xi_4} \frac{\bar{\xi}_n^2}{\xi_4} \left(\frac{\partial^2 \mu_4^0}{\partial \xi_n^2} \right) \left(\frac{\partial T}{\partial \xi_3} \right)_{p\bar{\xi}_n},$$

the velocity of second sound decreases with the increase of helium 3 content, at least in the range where H, London's relation holds, in contradiction to the experimental results (cf. Eq. (5.1)).

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Transformation Functions in the Complex Domain

Giiti IWATA

Department of Physics, Tokyo University

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If we consider transformation functions in the complex domain, they represent clear features that could not be expected in the real domain.

§ 1. Transformation function ($z|z'$)

When an operator x has only real eigenvalues ranging from $-\infty$ to ∞ , $\delta(x-x')$ represents the transformation function ($x|x'$) which has the properties

$$\int_{-\infty}^{\infty} f(x) dx (x|x') = f(x'), \quad \int_{-\infty}^{\infty} (x|x') dx' g(x') = g(x).$$

When an operator z has all complex numbers as eigenvalues, we consider the transformation function ($z|z'$) should have the following properties

$$\frac{1}{2\pi i} \int_C f(z) dz (z|z') = f(z'), \quad \frac{1}{2\pi i} \int_C (z|z') dz' g(z') = g(z)$$

where $f(z)$ is regular outside C , $g(z)$ inside C , while C being a circle with its center at origin. So we can represent ($z|z'$) by

$$(z|z') = \frac{1}{z' - z}, \quad |z| < |z'|.$$

We assume hereafter implicitly that in the expression $(z|*|z')$ z' is greater than z in their absolute values. Contrarily to this case we assume that in the expression $(l|*|l')$ l is greater than l' in their absolute values, l, l' being eigenvalues of an operator L .

§ 2. Discrete spectra

Main features of our previous paper (this journal, 6. 2) may be recapitulated in matrix form as follows.

When an operator L has discrete spectrum, there exist eigenvalues l_n and eigenfunctions $(z|n)$, $(n|z)$, $n=0, 1, 2, \dots$ which satisfy the conditions

$$\begin{aligned} (z|n) &\text{ regular inside } C, \quad (n|z) \text{ regular outside } C \\ (z|l|z') (z'|n) &= (z|n) l_n, \quad (n|z) (z|L|z') = l_n (n|z'), \end{aligned} \quad (1)$$

$$(n|z)(z|n') = (n|n') = \delta_{nn'},$$

$$(z|n)(n|z') = (z|z') = \frac{1}{z' - z}$$

where we use the convention

$$(*|z)(z|**) = \frac{1}{2\pi i} \int_C (*|z) dz (z|**),$$

$$(*|n)(n|**) = \sum_{n=0}^{\infty} (*|n)(n|**).$$

When L represents $p_0(z)\bar{z}^2 + p_1(z)\bar{z} + p_2(z)$, \bar{z}, z being related by $\bar{z}z - z\bar{z} = 1$, we have in the z -diagonal representation

$$(z|L|z') = p_0(z) \frac{2}{(z' - z)^3} + p_1(z) \frac{1}{(z' - z)^2} + p_2(z) \frac{1}{z' - z},$$

since we get from $\bar{z}z - z\bar{z} = 1$ and its consequence $\bar{z}^2z - z\bar{z}^2 = 2\bar{z}$

$$(z|\bar{z}|z')z' - z(z|\bar{z}|z') = (z|z'), \quad (z|\bar{z}|z') = \frac{1}{(z' - z)^2},$$

$$(z|\bar{z}^2|z')z' - z(z|\bar{z}^2|z') = 2(z|\bar{z}|z'), \quad (z|\bar{z}^2|z') = \frac{2}{(z' - z)^3}.$$

In case $p_0(z), p_1(z), p_2(z)$ have no singular point other than $z = \infty$, the equations (1) may be transformed into differential equations as follows

$$L_z(z|n) = (z|n)l_n,$$

$$\bar{L}_{z'}(n|z') + \frac{1}{2\pi i} \int_{\infty} (n|z) dz (z|L|z') = (n|z')l_n.$$

When $(z|L|z')$ is not zero at $z = \infty$, there remains the part that spoils the symmetry of equations.

As an application of the precedings we calculate here eigenvalues of the operator $\bar{z}z, z, \bar{z}$ being related by the commutation relation $\bar{z}z - \rho z\bar{z} = 1$, ρ a constant. The case when ρ is equal to 1, is familiar to us.

From $\bar{z}z - \rho z\bar{z} = 1$, we get

$$(z|\bar{z}|z')z' - \rho z(z|\bar{z}|z') = (z' - \rho z)(z|\bar{z}|z') = (z|z')$$

whence we have ($\rho \neq 1$)

$$(z|\bar{z}|z') = \frac{z}{(z' - \rho z)(z' - z)} = \frac{1}{\rho - 1} \left(\frac{1}{z' - \rho z} - \frac{1}{z' - z} \right).$$

Substituting this expression in (1), we have

$$\frac{1}{\rho - 1} \left\{ (\rho z|n) - (z|n) \right\} = l_n(z|n),$$

$$\frac{1}{\rho - 1} \left\{ \frac{1}{\rho} \left(n \left| \frac{z}{\rho} \right. \right) - (n|z) \right\} = l_n(n|z)$$

whose solutions are

$$(z|n) = z^n, \quad (n|z) = z^{-n-1}, \quad l_n = \frac{\rho^n - 1}{\rho - 1}.$$

When $\rho = 1$,

$$\begin{aligned} (z|z\bar{z}|z') &= \frac{z}{(z' - z)^2}, \\ z \frac{d}{dz} (z|n) &= l_n (z|n), \quad - \frac{d}{dz} z (n|z) = l_n (n|z), \\ l_n &= n, \quad (z|n) = z^n, \quad (n|z) = z^{-n-1}. \end{aligned}$$

§ 3. Continuous spectra

When two operators q, p have all real numbers as eigenvalues, there exist transformation functions $(q|p)$, $(p|q)$ which have the following properties

$$\begin{aligned} (q|p)(p|q') &= (q|q'), & (q|p)p(p|q') &= (q|q'), \\ (p|q)(q|p') &= (p|p'), & (p|q)q(q|p') &= (p|p') \end{aligned}$$

where $(*|q)(q|**)$ stands for

$$\int_{-\infty}^{\infty} (*|q) dq (q|**).$$

In the complex domain there exist also similar things.

When two operators z, L allow all complex numbers as eigenvalues, the equations that determine transformation functions $(z|l)$ are in the z -diagonal representation as follows

$$(z|L|z')(z'|l) = (z|l)L. \quad (2)$$

As a first step we consider the case when there exist for each l only one $(z|l)$ regular in both z and l except for $z = \infty$ and $l = \infty$.

Transformation function $(l|z)$ adjoint to $(z|l)$ is determined by the following conditions

1. $(l|z)$ is regular at $z = \infty$, $l = \infty$,
2. $(l|z)(z|l') = (l|l')$,

$$(l|l') = \frac{1}{l - l'}, \quad |l| > |l'|.$$

Expanding $(z|l)$, $(l|z)$ in power series in z, l as follows

$$(z|l) = \sum_{n,r=0}^{\infty} a_{nr} z^n l^r, \quad a_{00} = 1,$$

$$(l|z) = \sum_{m,n=0}^{\infty} b_{mn} l^{-m-1} z^{-n-1}$$

we have

$$(l|z)(z|l') = \sum_{m,r} \left(\sum_n b_{mn} a_{nr} \right) l'^r l^{-m-1} = \sum_r l'^r l^{-r-1}.$$

Therefore we have

$$\sum_n b_{mn} a_{nr} = \delta_{mr}$$

which may be written in matrix form as

$$BA=1, \quad B=(b_{mn}), \quad A=(a_{nr}).$$

From this relation we have

$$AB=1,$$

which deserves more careful attentions.

Assuming the validity of this latter relation we have

$$\begin{aligned} (z|l)(l|z') &= \sum_{n,m} \left(\sum_r a_{nr} b_{rm} \right) z^n l'^{m-1} \\ &= \sum_{n,m} \delta_{nm} z^n l'^{m-1} \\ &= 1/(z'-z) = (z|z'). \end{aligned}$$

Multiplying (2) by $(l|z'')$ from the right, we have

$$\begin{aligned} (z|L|z')(z'|l)(l|z'') &= (z|L|z')(z'|z'') = (z|L|z'') \\ &= (z|l)l(l|z''), \end{aligned}$$

$$i. e. \quad (z|L|z') = (z|l)l(l|z').$$

Multiplying this equation by $(l'|z)$ from the left, we have the equation to be satisfied by $(l|z)$,

$$\begin{aligned} (l'|z)(z|L|z') &= (l'|z)(z|l)l(l|z') = (l'|l)l(l|z') \\ &= \frac{1}{2\pi i} \int_c \frac{ldl}{cl'-l} (l|z') \\ &= l'(l'|z') - \frac{1}{2\pi i} \int_c dl(l|z'). \end{aligned}$$

This equation is slightly different from the equation (2) satisfied by $(z|l)$.

An example of the preceding is given here.

Let $L=\bar{z}$, $\bar{z}z-\rho z\bar{z}=1$, then we have

$$\begin{aligned} (z|\bar{z}|z') &= \frac{1}{(z'-\rho\bar{z})(z'-z)} = \frac{1}{(\rho-1)z} \left\{ \frac{1}{z'-\rho\bar{z}} - \frac{1}{z'-z} \right\} \\ \frac{1}{(\rho-1)z} \left\{ (\rho\bar{z}|l) - (z|l) \right\} &= (z|l)l. \end{aligned}$$

Expanding $(z|l)$ in Taylor series in z , we have

$$(z|l) = 1 + lz + \frac{(lz)^2}{1+\rho} + \frac{(lz)^3}{(1+\rho)(1+\rho+\rho^2)} + \dots$$

whence we get

$$(l|z) = \frac{1}{lz} + \frac{1}{(lz)^2} + \frac{1+\rho}{(lz)^3} + \frac{(1+\rho)(1+\rho+\rho^2)}{(lz)^4} + \dots$$

In the case when ρ is 1, we have

$$(z|\bar{z}|z') = \frac{1}{(z'-z)^2},$$

$$\frac{d}{dz}(z|l) = (z|l)l, \quad -\frac{d}{dz}(l|z) = l(l|z) - \frac{1}{z},$$

$$(z|l) = e^{lz},$$

$$(l|z) = \frac{1}{lz} + \frac{1!}{(lz)^2} + \frac{2!}{(lz)^3} + \frac{3!}{(lz)^4} + \dots$$

Concerning the case when there exist two solutions $(z|l)$ for each l , we make a cursory mention of the possibility that we may have $(z|l)_i$ and $(l|z)_i$ ($i=1, 2$) satisfying the following relations

$$(l|z)_i (z|l')_k = (l|l') \delta_{ik}, \quad i, k=1, 2.$$

$$(z|l)_1 (l|z')_1 + (z|l)_2 (l|z')_2 = (z|z'),$$

e. g.

$$L = \bar{z}^2 - z\bar{z}, \quad \bar{z}z - z\bar{z} = 1.$$

$$(z|l)_1 = 1 + \frac{l}{2!} z^2 + \frac{l(l+2)}{4!} z^4 + \frac{l(l+2)(l+4)}{6!} z^6 + \dots,$$

$$(z|l)_2 = z + \frac{l+1}{3!} z^3 + \frac{(l+1)(l+3)}{5!} z^5 + \dots,$$

$$(l|z)_1 = \frac{1}{lz} + \frac{2!}{l(l+2)z^3} + \frac{4!}{l(l+2)(l+4)z^5} + \dots,$$

$$(l|z)_2 = \frac{1}{(l+1)z^2} + \frac{3!}{(l+1)(l+3)z^4} + \frac{5!}{(l+1)(l+3)(l+5)z^6} + \dots,$$

$$(z|l)_1 (l|z')_1 = \frac{z'}{z'^2 - z^2}, \quad (z|l)_2 (l|z')_2 = \frac{z}{z'^2 - z^2},$$

$$(z|l)_1 (l|z')_1 + (z|l)_2 (l|z')_2 = \frac{z' + z}{z'^2 - z^2} = \frac{1}{z' - z}.$$

Note on Very Large Cosmic-Ray Stars, II

Yoshio YAMAGUCHI

Department of Physics, Osaka City University

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It is proposed and discussed a new mechanism of emission of nuclear fragments from highly excited nuclei upon the basis of liquid drop model.

§ 1. Introduction

In the previous paper¹⁾ we showed that the nuclear surface tension decreases with increasing excitation energy or nuclear temperature. While the free particle model of a nucleus was adopted there, it seems more reasonable to use the liquid drop model as is readily seen from the nature of the phenomenon. We want to discuss here the emission of nuclear fragments from very large cosmic-ray stars, which have been analyzed by Harding, Lattimore and Perkins²⁾ and Perkins³⁾, throughout upon the basis of liquid drop model.

§ 2. The surface tension in the liquid drop model

Bethe⁴⁾ gave the expression of nuclear surface energy in the liquid drop model as follows:

$$U(T) = 0.113 \left(\frac{O(0)}{O(T)} \right)^{\frac{2}{3}} T^{\frac{7}{3}} + U_0 \quad (1)$$

(U and T measured in MeV),

where we use new values of parameters listed in Table I. As was stated in I,

Table I

	unit	Value adopted by Bethe	Value used in this paper
r_0	10^{-13} cm	2.05	1.37 a
P	MeV	10	19 a
ζ_0	MeV	11.5	22 a (Fermi energy)
Γ	MeV	9.6	14 b
K	MeV	4	7.55 c

In this table we use Bethe's notation. (cf. ref. 4)).

a Fernbach, Serber and Taylor, *Phys. Rev.* **75** (1940), 1352.

b N. Bohr and J. A. Wheeler, *Phys. Rev.* **56** (1939), 194.

c L. Rosenfeld, *Nuclear Forces*, Vol. II.

the surface tension $O(T)$ is connected with U by the well-known thermodynamical relation :

$$U = O - T \frac{\partial O}{\partial T}. \quad (2)$$

From the Weizsäcker-Bethe's semi-empirical formula we find

$$U(0) = U_0 = O(0) = 14 \text{ MeV for } T=0, \quad (3)$$

while the third law of thermodynamics demands another initial condition :

$$\frac{\partial O}{\partial T} = 0 \quad \text{at } T=0. \quad (4)$$

The inhomogeneous differential equation (2) with (1) is solved by numerical integration under the initial condition (3) and (4). The result is represented by the curve *A* in Fig. 1. For comparison, if we put $O(T) = 14 \text{ MeV}$ in the expression (1) (i.e., we use $U = 0.113 T^{\frac{7}{3}}$), (2) becomes a linear differential equation, whose solution obeying (3) and (4) is a curve *B* in the same figure. A curve *C* represents the "quadratic law" derived in I :

$$O(T) = \frac{T_c^2 - T^2}{T_c^2} \cdot 14 \text{ MeV}, \quad (T_c \approx 9 \text{ MeV}).$$

As is seen from Fig. 1 the temperature dependence of nuclear surface tension is not sensitive to the adopted nuclear model and this fact has already been remarked in I.

The relation of temperature to excitation energy depends of course on the adopted nuclear model. HLP showed that the cosmic-ray star data are consistent with the liquid drop model only when volume waves do not contribute to the nuclear excitation. Therefore we want to adopt this rather curious hypothesis without any proof, though this fact may concern the "central core model" of a nucleus suggested from the electron-nucleus scattering experiments.⁵⁾ For heavy nuclei

(mass number ~ 100) the surface energy (1) (which is equal to the total energy, according to our assumption) gives roughly the same numerical results as given from the free particle model, if we take into account the temperature change of surface tension in (1).

Thus it turns out that our previous consideration based on the free particle model are not essentially altered even if we adopt the liquid drop model. In other words, our previous results were justified also from the liquid drop model.

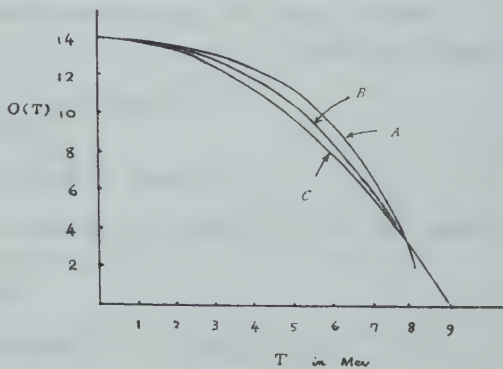


Fig. 1

Furthermore the width Γ_f for symmetrical fission is of the same order of magnitude as the neutron width Γ_n in highly excited nuclei (with excitation energy $\sim AE_b$; E_b =average binding energy of a nucleon, A =mass number) as was shown in I. Since the width corresponding to two extreme cases of (large and small) particle emission are of the same order of magnitude, the width for a particle with arbitrary mass number may not be greatly different from Γ_f or from Γ_n , as long as we stand on the statistical point of view.

§ 3. Emission of large fragments due to dynamical process

The statistical theory predicts that particles with arbitrary mass number can evaporate roughly with the same emission probability. This result is contradicted with the observation of HLP and of Perkins. We are, therefore, compelled to abandon the statistical treatment, in which the statistical equilibrium or quasi-static characters of the process concerned are assumed, and to search a new mechanism of emission of fragments. It is very likely that some new events occur before the statistical equilibrium is reached because of very high excitation energy.

In this connection, let us again refer to Bagge's old work.⁶⁾ Formerly he showed that the Coulomb barrier height decreases with increasing excitation energy due to the increase of average amplitude of nuclear surface waves. However, he did not take into account the change of surface tension with temperature. Thus we correct this effect and recalculate the mean square root λ of amplitude of surface wave for various temperatures (see Table II). The infinity at $T=T_c$ is merely originated from the assumption of infinitesimal vibration which is no more valid at very high temperature. In any way we can regard $\frac{\lambda}{R}$ as ~ 0.5 at $T \sim T_b$. (R =nuclear radius).

It is not unreasonable that a fragment with linear dimension (corresponding mass number is $(\lambda/r_0)^3 \sim 10$) is occasionally ejected from the nucleus, through the appropriate surface waves excited by incident cosmic-ray particle with great energy, considering the results concerning widths obtained in § 2. In this connection we also refer to Mayer and Teller's mechanism of origin of chemical element ("fission of cold polynutron").⁷⁾ Since the wave packet (with linear dimension $\sim \lambda$) of surface wave has the total energy $AE_b \left(\frac{\lambda}{R}\right)^2 \sim 150$ MeV on the average, the mean

Table II
Calculated values of λ/R for ${}_{35}\text{Br}^{80}$ nucleus.

temperature	excitation energy	λ/R
1.5 MeV	18 MeV	0.054
3	72	0.12
6	288	0.24
8	512	0.40
9	640	∞

R =nuclear radius

kinetic energy of emitted fragments will be $\sim (1/2) 150 \text{ MeV} = 75 \text{ MeV}$. Thus we may expect that in very large cosmic-ray stars we frequently observe nuclear fragments with average mass number ~ 10 and average kinetic energy $\sim 75 \text{ MeV}$. These two figures are fairly in agreement with observed values.³⁾

Unfortunately this model can not explain the angular distribution of fragments. A possible explanation may be concerned with some "impulse" action, which is often met with in break-down phenomena of elastic bodies, but no definite conclusions can be stated.

After possible fragment emission the special modes of the surface wave (favourable to fragment emission) will soon damp and thermal equilibrium will be reached, and then the normal evaporation process starts. The latter process is responsible for the major parts of prongs in large star.

The author is very much indebted to Dr. Perkins for information of new experimental results before publication.

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On the Crystal Statistics of Two-Dimensional Ising Ferromagnets

Tunenobu YAMAMOTO

Department of Physics, Kyoto University

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Statistical problems of two-dimensional Ising ferromagnets of various lattice types are reduced to those of square nets so that direct applications of Onsager-Kaufman's method are allowed. Most of the physically interesting lattices can be treated systematically, but lattices with longer period of translational symmetry could hardly be handled. The triangular and the honeycomb lattices are treated simultaneously, and particularly their dual relation is given in the most general form.

§ 1. Introduction

Since the usefulness of the method of eigenvalue problem in crystal statistics of Ising ferromagnets was made evident by Kramers and Wannier,¹⁾ Montroll,²⁾ Lassetre and Howe,³⁾ and others, remarkable progresses have been achieved in the two-dimensional cases of no magnetic field. Namely, Onsager⁴⁾ at first established a theory which allowed him to obtain an exact solution of the two-dimensional square net. But because of its very complicated structure, it was not possible to attack other lattice types until Nambu,⁵⁾ Husimi and Syoji,⁶⁾ and especially Kaufman⁷⁾ contracted the formalism considerably. Now three typical lattice types are in our hand. As already mentioned, the problems of the square net were investigated in full detail by Onsager himself and by Kaufman. Husimi and Syoji calculated the exact eigenvalues of the honeycomb problem, and Wannier⁸⁾ also treated the latter, reducing it to a kind of square net and applying the Onsager-Kaufman's method. Furthermore, using the so-called dual transformation they went to the case of triangular lattice and discussed its thermodynamic behavior. Although the dual transformation is useful when the spin-spin couplings are isotropic in three directions, and although it can give the partition function of the entire crystal, all results obtained through the transformation from the honeycomb lattice cannot be said to describe the whole aspects of the lattice under consideration, for not only the case of anisotropic couplings but also the problems regarding the propagation of order inside the crystal cannot be treated at all. But the author does not know anyone ever has attacked directly the triangular problems on the Onsager-Kaufman's line.

On the other hand, Nambu⁵⁾ and Newell⁹⁾ formulated another method, in which spins are added one by one on screwed lattice points. Their results were exactly the same as those of Onsager in the case of square net. A merit of this method is that it is easily applicable to the triangular lattice, which indeed Newell¹⁰⁾

solved in the anisotropic case. In this method, however, there are defects that the eigenfunctions are hardly obtainable and that therefore the propagation of order is unknown. Moreover, it is difficult to apply the method to the honeycomb lattice.

At the present status of our problems, it is desired that statistical problems of two-dimensional lattices in general are investigated on the way of Onsager and Kaufman. So we attempted the generalization of Onsager-Kaufman's method so as to be applicable to various lattice types, and succeeded to solve the triangular and a few other lattices straightforwardly. Lattices with rather longer periods of translational symmetry are difficult to be handled by our method.

§ 2. Transformation of lattice types; Method of imaginary couplings

Onsager-Kaufman's formalism is in itself proper for the square net. It is in principle applicable to all kinds of square nets, but to other lattice types its direct application is not always allowed. As shown by Wannier, the honeycomb can be considered as a kind of square net with respect to its statistical character, therefore it is in the range of Onsager-Kaufman's method. In the case of triangular lattice, however, no topological deformations are effective as in the honeycomb, in order to bring the former into a square net. Nevertheless, as described immediately below, this lattice can also be considered as a limiting case of a square net, so that Onsager-Kaufman's method is still of use. All other types of two-dimensional lattices could be shown as well to be special cases of certain square nets.

We assume that a triangular lattice has anisotropic couplings as shown in Fig. 1. Now suppose that we replace virtually every spin by a pair of spins, and that spins in one and the same pair interact with each other with imaginary coupling parameter K . Real couplings are redistributed as shown in Fig. 2. Now this new lattice is unquestionably a square one, therefore there are no difficulties to use the Onsager-Kaufman's method. Then, if we proceed to the limit $K \rightarrow \infty$, two members of every pair coupled with K have to orient themselves parallel to each other. As a result, every pair behaves just as a single spin, i.e., we come back again to the original triangular one.

On the contrary, the limiting process $K \rightarrow 0$ leads us to a honeycomb, as already shown by Wannier. Accordingly, the honeycomb and triangular lattices should be supposed as two opposing limits of square nets. At this point we could find the reason of their dual relation. We shall return to discussions on this relation once more in § 4.

In general, we can transform all the two-dimensional lattices into square nets by introducing zero and infinite couplings. When the transforms have the translational symmetry of lower than three steps in two directions, their statistical problems can easily be solved by Onsager-Kaufman's method, so are their limiting cases, i.e., the original ones.

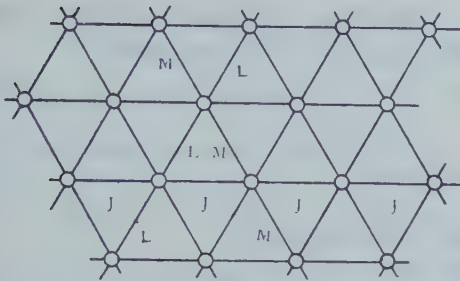


Fig. 1

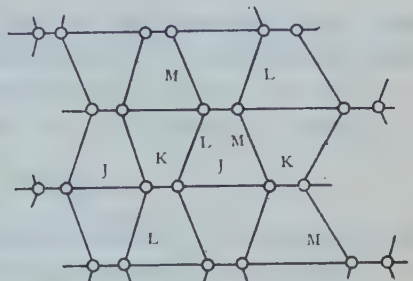


Fig. 2

§ 3. Mathematical considerations of the limiting processes $K \rightarrow \infty$ or 0

The limiting processes $K \rightarrow \infty$ or 0 have definite physical meanings. But since the formalism of Onsager and Kaufman is of rather abstract character, it will be worth-while to note when and how we should perform the limiting processes in the course of calculations.

For the brevity of description, we shall call the bonds along which the crystal are built up by the operators \mathbf{C} 's as longitudinal, and the others through which the operators \mathbf{s} 's connect spins with each other as transversal.

(i) Coupling K is transversal and tends to infinity.

In our formalism the central problem is to solve the eigenvalue problem of a certain operator \mathbf{V} which is defined in the spin space and characteristic of the lattice under consideration. The cooperation of the coupling K is expressed in \mathbf{V} through the following exponential factor for every pair of spins connected by that coupling:

$$\exp K \mathbf{s} \mathbf{s}' = \exp (-i K \mathbf{P}' \mathbf{Q}),$$

where \mathbf{s} , \mathbf{s}' and \mathbf{P}' , \mathbf{Q} have the same meanings as in reference 7. If we suppose K as sufficiently large, the above expression is easily transformed into the following form:

$$(2 \sinh 2K^*)^{\frac{1}{2}} \{1 - i \mathbf{P}' \mathbf{Q}\}$$

where

$$\sinh 2K \cdot \sinh 2K^* = 1.$$

Now the operator $\frac{1}{2} \{1 - i \mathbf{P}' \mathbf{Q}\}$ is a projection operator because of the following identity

$$\frac{1}{2} \{1 - i \mathbf{P}' \mathbf{Q}\} \cdot \frac{1}{2} \{1 - i \mathbf{P}' \mathbf{Q}\} = \frac{1}{2} \{1 - i \mathbf{P}' \mathbf{Q}\},$$

and its eigenvalues equal to 1, or 0 for $-i \mathbf{P}' \mathbf{Q} = \mathbf{s} \mathbf{s}' = +1$ or -1 respectively. Accordingly, the factor selects all the states in which two spins of every pair connected by the coupling K are parallel. Evidently it is the very thing that we expect of the infinite coupling. Unfortunately, since the projection operator is

not a spin representation, we are forced to solve the eigenvalue problem with finite K and then let K tend to infinity at the final stage of calculations.

(ii) K is transversal and tends to zero.

There is nothing to say of this case, as

$$\lim_{K \rightarrow 0} \exp(-iK \mathbf{P}' \mathbf{Q}) = 1.$$

We are allowed to proceed with zero coupling from the start.

(iii) K is longitudinal and tends to infinity.

A new step coupled through K with the previous one is described by the following operator

$$\exp K + \exp(-K) \cdot \mathbf{C} = (2 \sinh 2K)^{\frac{1}{2}} \exp K^* \mathbf{C} = (2 \sinh 2K)^{\frac{1}{2}} \exp iK^* \mathbf{PQ}.$$

As K tends to infinity, K^* does to zero, and thus the operator does to $\exp K$, as is easily expected. Evidently this exponential is nothing but a Boltzmann factor due to our imaginary coupling, so we can omit it after all. Thus we may safely assume that:

$$\lim_{K \rightarrow \infty} (2 \sinh 2K)^{\frac{1}{2}} \exp(iK^* \mathbf{PQ}) = 1.$$

As a result we can ignore all the factors including the coupling K in \mathbf{V} and solve the eigenvalue problem of the rest.

(iv) K is longitudinal and tends to zero.

In this case,

$$\lim_{K \rightarrow 0} (2 \sinh 2K)^{\frac{1}{2}} \exp(iK^* \mathbf{PQ}) = 1 + i\mathbf{PQ},$$

which is not a spin representation. Therefore we must postpone the limiting process until the application of the theory of spin representation is performed and eigenvalues are revealed.

§ 4. The triangular and the honeycomb lattices

As a starting point we take a square net shown in Fig. 1, where couplings J and K alternate in one direction, and couplings L and M do in another direction. As usual we close the lattice by $2m$ steps in the former direction and by $2n$ steps in the latter one.

According to the formalism of Onsager and Kaufman, the eigenvalue problem characteristic to our lattice is related to the following operator \mathbf{V} , if we construct the lattice step by step in the direction along which the couplings J and K operate:

$$\mathbf{V} = \mathbf{V}'_2 \mathbf{V}'_1 \mathbf{V}_2 \mathbf{V}_1 \dots \quad (1)$$

where

$$\mathbf{V}_1 = (2 \sinh 2J)^{\frac{n}{2}} (2 \sinh 2K)^{\frac{n}{2}} \prod_{r=1}^n \exp(J^* \mathbf{C}_{2r-1}) \prod_{r=1}^n \exp(K^* \mathbf{C}_{2r})$$

$$\begin{aligned}
 &= (2 \sinh 2J)^{\frac{n}{2}} (2 \sinh 2K)^{\frac{n}{2}} \prod_1^n \exp (iJ^* \mathbf{P}_{2r-1} \mathbf{Q}_{2r-1}) \prod_1^n \exp (iK^* \mathbf{P}_{2r} \mathbf{Q}_{2r}), \\
 &\mathbf{V}'_1 = J \text{ and } K \text{ interchanged in } \mathbf{V}_1, \\
 &\mathbf{V}_2 = \prod_1^n \exp L \mathbf{s}_{2r-1} \mathbf{s}_{2r} \prod_1^n \exp M \mathbf{s}_{2r} \mathbf{s}_{2r+1} \\
 &= \prod_1^n \exp (-iL \mathbf{P}_{2r} \mathbf{Q}_{2r-1}) \prod_1^{n-1} \exp (-iM \mathbf{P}_{2r+1} \mathbf{Q}_{2r}) \cdot \exp (iM \mathbf{P}_1 \mathbf{Q}_{2n} \mathbf{U}), \\
 &\mathbf{V}'_2 = L \text{ and } M \text{ interchanged in } \mathbf{V}_2
 \end{aligned} \tag{2}$$

$$\mathbf{U} = \mathbf{C}_1 \mathbf{C}_2 \mathbf{C}_3 \cdots \mathbf{C}_{2n},$$

$$\text{and} \quad J \equiv J/kT, \quad \text{etc.}$$

As already discussed in detail in the last section, now we can go at once to the triangular case. Most easily we can reach there by putting $K \rightarrow \infty$, $K^* \rightarrow 0$. Then we should remove all the factors dependent on K and K^* in \mathbf{V} . As a result the operator $\mathbf{V}^{(t)}$ of our eigenvalue problem takes the following form:

$$\mathbf{V}^{(t)} = \mathbf{V}'_2 \mathbf{V}'_1{}^{(t)} \mathbf{V}_2 \mathbf{V}_1{}^{(t)} \tag{3}$$

where

$$\begin{aligned}
 \mathbf{V}_1{}^{(t)} &= (2 \sinh 2J)^{\frac{n}{2}} \prod_1^n (iJ^* \mathbf{P}_{2r-1} \mathbf{Q}_{2r-1}), \\
 \mathbf{V}'_1{}^{(t)} &= (2 \sinh 2J)^{\frac{n}{2}} \prod_1^n \exp (iJ^* \mathbf{P}_{2r} \mathbf{Q}_{2r}).
 \end{aligned} \tag{4}$$

Now we are ready for the triangular problems and the straightforward calculations analogous to Kaufman's will lead us to the partition function and other properties of this lattice.

On the other hand, if we put $K=0$ in Eq. (2), the resulting operator $\mathbf{V}^{(h)}$ should correspond to the honeycomb lattice, as was often stated. The result is

$$\mathbf{V}^{(h)} = \mathbf{V}'_2{}^{(h)} \mathbf{V}'_1{}^{(h)} \mathbf{V}_2{}^{(h)} \mathbf{V}_1{}^{(h)} \tag{5}$$

where

$$\begin{aligned}
 \mathbf{V}_2{}^{(h)} &= \prod_1^n \exp (-iL \mathbf{P}_{2r} \mathbf{Q}_{2r-1}) \\
 \mathbf{V}'_2{}^{(h)} &= \prod_1^{n-1} \exp (-iL \mathbf{P}_{2r+1} \mathbf{Q}_{2r}) \cdot \exp (iL \mathbf{P}_1 \mathbf{Q}_{2n} \mathbf{U}).
 \end{aligned} \tag{6}$$

Here we can have an insight into the origin of the dual relation between the triangular and the honeycomb lattices. For this purpose we may carry out the following transformation \mathbf{D} on $\mathbf{V}^{(h)}$, which is the same as Onsager used previously in order to prove the self-dual character of square nets:

$$\mathbf{D}: \mathbf{P}_r \rightarrow \mathbf{Q}_r, \quad \mathbf{Q}_r \rightarrow \mathbf{P}_{r+1}.$$

(It should be noted here that the dual relation between triangular and the honey-

comb lattices is half due to the self-dual property of their common progenitor.) Then we have

$$\mathbf{D}: \mathbf{V}^{(h)} \rightarrow \tilde{\mathbf{V}}^{(h)} = \tilde{\mathbf{V}}_2'^{(h)} \tilde{\mathbf{V}}_1' \tilde{\mathbf{V}}_2^{(h)} \tilde{\mathbf{V}}_1 \quad (7)$$

where

$$\tilde{\mathbf{V}}_1 = (2 \sinh 2J)^{\frac{n}{2}} (2 \sinh 2K)^{\frac{n}{2}} \prod_{r=1}^n \exp(-iJ^* \mathbf{P}_{2r} \mathbf{Q}_{2r-1}) \cdot \prod_1^n \exp(-iK^* \mathbf{P}_{2r+1} \mathbf{Q}_{2r}),$$

$$\tilde{\mathbf{V}}_1' = J \text{ and } K \text{ interchanged in } \tilde{\mathbf{V}}_1, \quad (8)$$

$$\tilde{\mathbf{V}}_2^{(h)} = \prod_1^n \exp(iL \mathbf{P}_{2r} \mathbf{Q}_{2r}),$$

$$\tilde{\mathbf{V}}_2'^{(h)} = \prod_1^{n-1} \exp(iL \mathbf{P}_{2r+1} \mathbf{Q}_{2r+1}) \cdot \exp(-iL \mathbf{P}_1 \mathbf{Q}_1 U).$$

If we compare Eqs. (3), (4), (7) and (8), we can recognize with ease that there exists a close parallelism between $\mathbf{V}^{(t)}$ and $\tilde{\mathbf{V}}^{(h)}$. So as to avoid unnecessary confusions and make correspondence clearer, we alter the notations of couplings on the honeycomb side, i.e.,

$$L \rightarrow J, \quad K \rightarrow L, \quad J \rightarrow M.$$

Then, we may tabulate the correspondence as follows:

$\mathbf{V}^{(t)}$	J	L	M	$\mathbf{V}_1^{(t)}$	\mathbf{V}_2	$\mathbf{V}_1'^{(t)}$	\mathbf{V}_2'
$\tilde{\mathbf{V}}^{(h)}$	J^*	L^*	M^*	$\tilde{\mathbf{V}}_2'^{(h)}$	$\tilde{\mathbf{V}}_1'$	$\tilde{\mathbf{V}}_2^{(h)}$	$\tilde{\mathbf{V}}_1$

Except for differences which will be described just below, $\mathbf{V}^{(t)}$ equals to the Hermitian conjugate of $\tilde{\mathbf{V}}^{(h)}$ because of Hermitian character of their components \mathbf{V} 's. The differences in question are as follows: First $\mathbf{V}^{(t)}$ has a factor $(2 \sinh 2J)^{\frac{n}{2}}$ whereas $\tilde{\mathbf{V}}^{(h)}$ does $(2 \sinh 2L)^{\frac{n}{2}} \cdot (2 \sinh 2M)^{\frac{n}{2}}$, secondly the parameters J , L , and M on one side are just replaced by its asterisks on the other side, and thirdly a few factors including U seem to disturb the above correspondence. However, it can be proved that the last mentioned unpleasant feature is only apparent and we are allowed to disregard it entirely. (See Appendix.) Accordingly, if we denote one of eigenvalues of $\mathbf{V}^{(t)}$ by $\lambda^{(t)}(J, L, M)$, then $\lambda^{(h)}(J, L, M)$ is also a member of those of $\tilde{\mathbf{V}}^{(h)}$, i.e., of $\mathbf{V}^{(h)}$, provided the following condition is satisfied:

$$\lambda^{(t)}(J, L, M) = \lambda^{(h)}(J^*, L^*, M^*) / [2 \sinh 2J^* \cdot \sinh 2L^* \cdot \sinh 2M^*]^{\frac{n}{2}}. \quad (9)$$

Remembering the relation between the partition function of a lattice per spin f and the eigenvalues of the \mathbf{V} , we can conclude without difficulty the following equation:

$$f^{(t)}(J, L, M) = f^{(h)}(J^*, L^*, M^*) / [2 \sinh 2J^* \cdot \sinh 2L^* \cdot \sinh 2M^*]^{\frac{1}{2}}. \quad (10)$$

where $f^{(t)}$ and $f^{(h)}$ refer to the triangular and the honeycomb lattices, respectively. Eq. (10) reduces to the so-called dual relation first pointed out by Wannier¹⁾ when we put $J=L=M$. Thus this equation may be called the generalized dual relation between the two anisotropic lattices under consideration.

§ 5. Conclusion

We proposed a method which enables us to reduce statistical problems of general two-dimensional lattices to those of square ones so that Onsager-Kaufman's method can be used straightforwardly. By this method, indeed, most of the physically interesting lattices can be treated systematically, but lattices with longer period of translational symmetry could hardly be handled.

The triangular and the honeycomb lattices were treated simultaneously, and particularly their dual relation was given in the most general form.

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Appendix

In case of square nets, it could be easily seen from Kaufman's article that the self-dual character is not disturbed by the presence of factors including U , although he did not take care of them explicitly. Since our two lattices are two limiting cases of one and the same square net, namely, the one is derived by $K \rightarrow \infty$ and the other by $K \rightarrow 0$, it is almost self-evident that also their dual relation will not be influenced by such factors.

In order to show this vividly, we should go from our spin space to a linear space spanned by P 's and Q 's, and investigate rotations in that space which induce spin representations expressed by V 's. For this purpose it is more convenient to compare $V^{(t)}$ not with $\tilde{V}^{(h)}$ but with $V^{(h)}$ directly.

Before entering into this space, we must eliminate U from V 's after Kaufman. As a result our problem splits into two parts which can be treated separately.

Namely :

$$V^{(t)} = \frac{1}{2} (1+U) V^{(t)+} + \frac{1}{2} (1-U) V^{(t)-}$$

$$V^{(t)\pm} = V_2'^{\pm} V_1'^{(t)} V_2^{\pm} V_1^{(t)}$$

$$V_2^{\pm} = \prod_1^n \exp(-iL P_{2r} Q_{2r-1}) \cdot \prod_1^{n-1} \exp(-iM P_{2r+1} Q_{2r}) \cdot \exp(\pm iM P_1 Q_{2n}),$$

$$V_2'^{\pm} = L \text{ and } M \text{ interchanged in } V_2^{\pm}$$

$$\text{and } V^{(h)} = \frac{1}{2} (1+U) V^{(h)+} + \frac{1}{2} (1-U) V^{(h)-}$$

$$V^{(h)\pm} = V_2'^{(h)\pm} V_1' V_2^{(h)} V_1$$

$$V_1 = \prod_1^n \exp(iL^* P_{2r} Q_{2r}) \cdot \prod_1^n \exp(iM^* P_{2r+1} Q_{2r+1}),$$

$$V_1' = L \text{ and } M \text{ interchanged in } V_1,$$

$$V_2^{(h)} = \prod_1^n \exp(-iJP_{2r} Q_{2r-1}),$$

$$V_2'^{(h)\pm} = \prod_1^{n-1} \exp(-iJP_{2r+1} Q_{2r}) \cdot \exp(\pm iJP_1 Q_{2n}).$$

where double signs should be read in order, and the notation of coupling parameters in $V^{(h)}$ are changed as in the text.

Now we consider rotations in the generator space which induce representations $V^{(t)\pm}$ and $V^{(h)\pm}$ in the spin space. We denote these by $R^{(t)\pm}$ and $R^{(h)\pm}$ respectively. First we take up the triangular case. Rotations represented by $V_1^{(t)}$, $V_1'^{(t)}$ and V_2^\pm are easily found in the form:

$$R_1^{(t)} = \begin{pmatrix} c(J^*) & is(J^*) & & & \\ -is(J^*) & c(J^*) & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix},$$

$$R_1' = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & c(J^*) & is(J^*) & & \\ & -is(J^*) & c(J^*) & & \\ & & & \ddots & \end{pmatrix},$$

$$R_2^\pm = \begin{pmatrix} c(M) & & & & \pm is(M) \\ & c(L) & is(L) & & \\ & -is(L) & c(L) & & \\ & & c(M) & is(M) & \\ & & -is(M) & c(M) & \\ & & & \ddots & \\ \mp is(M) & & & & c(M) \end{pmatrix},$$

where $c(J^*) = \cosh 2J^*$, $s(J^*) = \sinh 2J^*$, etc.

Finally the rotation of $V_2'^\pm$ is obtained from that of V_2^\pm through interchange of L and M . The resultant rotation is given by the following matrix:

$$R^{(i)\pm} = \begin{pmatrix} a & b & \mp c \\ c & a & b \\ & c & a & b \\ & & \ddots & \\ \mp b & & & c & a \end{pmatrix}$$

where

$$a = \begin{pmatrix} c(L)c(M)c(J^*) & ic(L)c(M)s(J^*) & 0 & 0 \\ +s(L)s(M)c(J^*)^2 & +is(L)s(M)s(J^*)c(J^*) & & \\ -ic(L)c(M)s(J^*) & c(L)c(M)c(J^*) & is(L)c(M) & -s(M)c(M)s(J^*) \\ -is(L)s(M)s(J^*)c(J^*) & +s(L)s(M)c(J^*)^2 & +ic(L)s(M)c(J^*) & \\ -c(L)s(M)s(J^*) & -ic(L)s(M)c(J^*) & s(L)s(M) & ic(M)^2s(J^*) \\ -s(L)c(M)s(J^*)c(J^*) & -is(L)c(M)c(J^*)^2 & +c(L)c(M)c(J^*) & \\ is(L)c(L)s(J^*)^2 & -s(L)c(L)s(J^*)c(J^*) & -ic(L)^2s(J^*) & c(L)c(M)c(J^*) \\ & & & +s(L)s(M) \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -is(M)^2s(J^*)c(J^*) & s(M)^2s(J^*)^2 & 0 & 0 \\ -s(M)c(M)s(J^*)c(J^*) & -is(M)c(M)s(J^*)^2 & 0 & 0 \\ ic(L)s(M)c(J^*)^2 & -c(L)s(M)s(J^*)c(J^*) & 0 & 0 \\ +is(L)c(M)c(J^*) & -s(L)c(M)s(J^*) & & \end{pmatrix},$$

and

$$c = \begin{pmatrix} s(L)^2s(J^*)^2 & is(L)^2s(J^*)c(J^*) & -s(L)c(L)s(J^*) & -is(L)c(M)c(J^*) \\ & & & -ic(L)s(M) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next we go to the honeycomb case. As before we can find rotations corresponding to V_1 , V_1' , $V_2^{(h)}$, and $V_2^{(h)\pm}$ in the form

$$R_1 = \begin{pmatrix} c(M^*) & is(M^*) \\ -is(M^*) & c(M^*) \\ & c(L^*) & is(L^*) \\ & -is(L^*) & c(L^*) \\ & & \ddots & \end{pmatrix},$$

R_1' = matrix obtained from the above through interchange of L^* and M^* ,

$$\mathbf{R}_2^{(h)} = \begin{pmatrix} 1 & & & \\ & c(J) & is(J) & \\ & -is(J) & c(J) & \\ & & & 1 \\ & & & & \ddots \end{pmatrix},$$

and finally

$$\mathbf{R}_2'^{(h)\pm} = \begin{pmatrix} & c(J) & & & & \pm is(J) \\ & & 1 & & & \\ & & & 1 & & \\ & & & & c(J) & is(J) \\ & & & & -is(J) & c(J) \\ & & & & & \ddots \\ \mp is(J) & & & & & & c(J) \end{pmatrix}.$$

In order to make comparison with $\mathbf{R}^{(i)\pm}$ easier, we prefer to calculate the Hermitian conjugate of $\mathbf{V}^{(h)}$ which has evidently the same eigenvalues as the original one. Then after multiplication of matrices in the reversed order as in $\mathbf{V}^{(h)}$, we arrive at just the same form as $\mathbf{R}^{(i)\pm}$, i.e.,

$$\mathbf{R}^{(i)\pm} = (\mathbf{R}^{(h)\pm})^*$$

where asterisk means Hermitian conjugate of matrix and changes of coupling parameters into its asterisks simultaneously. Thus we have proved the dual relation between the triangular and the honeycomb lattices exactly.

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The Vacuum in Quantum Electrodynamics

Hiroomi UMEZAWA and Susumu KAMEFUCHI

Institute of Theoretical Physics, Nagoya University

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Without use of perturbation calculation a general formula is derived for the vacuum induced current due to external electromagnetic field. This formula is valid for any charged field with arbitrary spin and interaction type. The terms corresponding to self-charge and observable current are shown to be of negative signs. The condition for the finiteness of the self-charge is also considered. These circumstances give rise to the serious difficulties in solving the problem of divergence of self-charge etc. from the standpoint of the realistic mixed field theory. An application of this formula to induced current in the c^2 -approximation is stated in detail. In order to settle these difficulties entirely it is expected that the further material property may be introduced for the vacuum.

§ 1. Introduction

It may be said that any problem in the quantum field theory depends on the property of the vacuum, because external effects always influence the vacuum fluctuation and what we can observe experimentally is nothing else but the deviation from the vacuum induced by the external disturbance. In this respect it is an important problem to investigate the property of the vacuum fluctuation in connection with the definition of the vacuum.

The phenomenon of vacuum polarization in quantum electrodynamics is the most simple and typical problem relating to vacuum fluctuation, and has the realistic effect, for example, in the problem of the energy level of the electron in hydrogen atom. Moreover, the vacuum polarization has the characteristic that it always contains the effects of all sorts of charged field coexisting in the nature.

In the discussion of this phenomenon there arise the following two important problems, namely, the problem of the gauge invariancy of the theory (self-energy of photon) and that of the gauge invariant part of the induced current.

The former has been investigated by many authors¹⁾ taking account of various charged fields. As is well known, the photon self-energy can be made vanishing in view of the realistic standpoint. Moreover, we know that this problem relates closely to the properties of the zero point oscillation of the charged fields and so to the definition of the vacuum.²⁾

Now, it is important to research the more detailed contents of the materiality of the vacuum in connection with the investigation of the gauge invariant part of the induced current. This part has been also calculated by various authors for

the charged fields of spin 0, $\frac{1}{2}$, $1^{(1)}$ and $\frac{3}{2}^{(3)}$ in the e^2 -approximation. These results can be written as follows:

$$\delta J_\mu = \sum_{n=0}^{\infty} a_{n+1} \square^n J_\mu^e(X),$$

where δJ_μ is the gauge invariant part of the induced current due to the external current $J_\mu^e(X)$. a_1 , a_2 and a_3 diverge for the cases of spin $(0, \frac{1}{2}, 1, \frac{3}{2})$, $(1, \frac{3}{2})$ and $(\frac{3}{2})$, respectively.

a_1 is usually managed by the renormalization procedure, which corresponds, in the Dirac-Heisenberg-Weisskopf's⁴⁾ subtraction theory, to the subtraction of the field independent polarizability of the vacuum.

Terms a_2, a_3, \dots give the observable effects. In the case of electron they correspond to the Uehling term,⁵⁾ which affords a few percent contribution to the level shift of the electron of the hydrogen atom in the opposite direction to the electromagnetic shift. When we take account of other charged fields, that is, proton, meson etc., there arises a possibility that the above effect is more strengthened because of the same signs of a_n for various charged fields as is seen below. However, when a_n -terms are convergent, a_n 's of heavy charged particles are very small because of their large masses ($a_2 \propto 1/(\text{mass})^2$) and so their effects on the level shift will be small compared with that of electron. But, in fact, a_2 diverges in the case of spin 1 and this divergence can not be avoided by the charge renormalization procedure¹⁾ in the framework of the present quantum field theory. In the e^2 -approximation it was shown that a_n 's have the negative sign.¹⁾ Moreover, this property of a_n 's was, in this approximation, shown to be independent of the spin property of charged fields.^{1a)}

Because of this property of a_n the cancellation of the divergences of a_n of various charged fields is impossible. The remaining possibilities to solve these difficulties of a_n within the framework of the present theory have been considered to taking into account the higher approximations of the perturbation calculation and the tensor interaction between charged fields and electromagnetic field. In the e^4 -approximation for spin $\frac{1}{2}^{(6)}$ it was shown that a_1 has the negative sign, too. The introduction of the tensor coupling in the cases of spin $\frac{1}{2}$ and $1^{(1b)}$ could not solve the difficulties either, because of the negative sign of a_1 and the appearance of new divergences in a_2 and a_3 . This property of the tensor coupling was also anticipated by one of the authors (H.U.)^{1a)} and will be proved in this paper more strictly irrespective of the spin property of charged fields.

In this paper the general discussions of the gauge invariant part of the induced current (taking into account the above two possibilities) are developed independently of the spin properties of charged fields and the perturbation approximation. In §2, without use of any approximation method, we shall derive the general formula, being valid for any kind of charged field with arbitrary spin and coupling type. In §3, based on this general formula, we shall develop the general discussion on

the properties of gauge invariant induced current. In § 4, these general considerations are further developed in the e^2 -approximation.

It can be expected from the intuitive consideration that a_1 is of negative sign, because in the vicinity of a charged source the induced particles with the charge of the same sign as that of the source are repelled and the induced particles with the charge of the opposite sign are attracted and so, as a result, the apparent charge of the source will be found to be smaller than its initial charge. This property of a_1 is explained as a result of the usual feature in the electrodynamics that the external disturbance is weakened by some internal effects in the matter, when we regard the vacuum as a polarizable matter. This feature of a_1 and, moreover, the fact that all a_n 's have negative signs are proved in the general form of the present quantum field theory (§ 2).

§ 2. Derivation of general formula^{*)}

(i) For the general consideration of self-charge (and observable current) it is sufficient to compute the induced current in vacuum due to a given external potential $A_\mu^e(X)$ and examine its first order term with respect to $A_\mu^e(X)$. In quantum electrodynamics of electron, for instance, such a circumstance is easily understood by the following considerations due to Dyson⁷⁾; Between the observable charge e_1 and the initial charge e there exists the following relation

$$e_1^2 = Z_3 e^2, \quad (1)$$

since $Z_1 = Z_2$ as shown by Ward^{7a)}. The radiative correction $A_\mu^{e'}(k)$ for the external potential $A_\mu^e(k)$ is given by

$$\begin{aligned} A_\mu^{e'}(k) &= (2\pi i) Z_3 D_{F1}'(e_1) k^2 A_\mu^e(k) \\ &= Z_3 \left[1 + \frac{D_c}{(2\pi i)} + \frac{D_c^2}{(2\pi i)^2} + \dots \right] A_\mu^e(k), \end{aligned} \quad (2)$$

where the definitions of e_1 , Z_3 , D_{F1}' and D_c are the same as those given by Dyson.⁷⁾ In Eq. (2) the first term on the right hand side corresponds to the "charge renormalization term" and its factor is equal to the factor in (1). Thus we find that when $A_\mu^{e'}$ is known the factor of its charge renormalization term can be regarded as the factor giving the observable charge e_1 . Now, it is evident from these considerations that for our present purpose we have only to calculate the induced current δJ_μ , which is linear in the external field A_μ^e . In the following we shall study $A_\mu^{e'}$ by calculating the current operator in the Heisenberg representation.^{**)}

*) The natural unit $c = \hbar = 1$ and the Heaviside unit are used throughout this paper.

**) Although the Eqs. (1) and (2) are the relations obtained from the discussion in the mixed representation and the quantities which we are now interested in are those in the Heisenberg representation, the similar relation also holds in the latter representation when the real particles are absent in the remote past and distant future.

In general, the Heisenberg operator $F(X)$ is given by

$$F(X) = (S^{-1}[\sigma] F(X) S[\sigma])_{X/\sigma}, \quad (3)$$

$$S[\sigma] = 1 - i \int_{-\infty}^{\sigma} H(X') S[\sigma'] d^4 X', \quad (3')$$

where $F(X)$ denotes the corresponding quantity in the infinite past and X/σ means the point X lying on surface σ . When the interaction is switched off in the remote past, $F(X)$ satisfies the free field equation of motion. As shown by Yang and Feldman,⁹⁾ the relations of the quantities of the infinite past are just the same as those in the interaction representation and so henceforth the former is identified with the latter.

The interaction Hamiltonian $H(X)$ of the system of the charged fields and the electromagnetic field is decomposed into two parts:

$$H(X) = H^i(X) + H^e(X), \quad (3'')$$

where H^i and H^e denote the interactions of charged fields with the quantized (internal) and external electromagnetic fields, respectively.

As we are concerned with the contributions which are linear in $A_{\mu}^e(X)$, it is convenient to rewrite $S[\sigma]$ as given by (3') in the following form;

$$S[\sigma] = S_1[\sigma] S_2[\sigma], \quad (4)$$

with

$$S_1[\sigma] = 1 - i \int_{-\infty}^{\sigma} H^i(X') S[\sigma'] d^4 X', \quad (4')$$

$$S_2[\sigma] = 1 - i \int_{-\infty}^{\sigma} (S_1^{-1}[\sigma'] H^e(X') S_1[\sigma']) S_2[\sigma'] d^4 X'. \quad (4'')$$

In (4''), $S_1^{-1}[\sigma] H^e(X) S_1[\sigma]$ denotes the interaction between the external field and the charged particles clothed in the cloud of electromagnetic field. In the following discussion we shall expand $S_2[\sigma]$ in power of A_{μ}^e and fix our attention to its first order term. As for $S_1[\sigma]$, the calculations are carried out without use of any approximation method, and so the results thus obtained are regarded as including all the higher order radiative corrections.

(ii) In general, the Lagrangian of the system consisting of the electromagnetic field and field of charged particle with arbitrary spin and coupling type can be written in the following form:

$$L = L^{\circ} + L^{\text{int}}, \quad (5)$$

$$L^{\circ} = X_{\alpha\beta}^{\circ} Q_{\beta;\mu}^* Q_{\alpha;\mu} + Y_{\alpha\mu}^{\circ} Q_{\alpha;\mu} + Q_{\alpha;\mu}^* Y_{\alpha\mu}^{\circ*}, \quad (5')$$

$$L^{\text{int}} = Z_{\alpha\mu} Q_{\alpha;\mu} + Q_{\alpha;\mu}^* Z_{\alpha\mu}^* + V. \quad (5'')$$

In the above expressions Q_{α} 's represent the field quantities of charged field and electromagnetic field and $Q_{\alpha;\mu} = \partial Q_{\alpha} / \partial X_{\mu}$. X° , Y° , Z , V are functions of Q_{α} 's

and do not contain $Q_{\alpha\mu}$'s. X° , Y° , contain only the field variables of the corresponding individual field and Z , V contain the variables of both the charged field and electromagnetic field A_μ . Z contains A_μ linearly and V at most quadratically. Further, we can assume, without loss of generality, that L^{int} does not contain the variables $Q_{\alpha\mu}$ of electromagnetic field (i.e. corresponding $Z_{\alpha\mu} = Z_{\alpha\mu}^* = 0$). Because, although in the tensor coupling case there appear the terms such as $\partial_\nu A_\mu$ in (5), we can eliminate these terms without modifying the equations of motion of field quantities by adding a 4-dimensional divergence to the initial Lagrangian. (5), (5') and (5'') are the most general Lagrangian which is admissible within the framework of the canonical formalism of quantum field theory.

The current operator $S_\mu(X)$ in the Heisenberg representation⁹⁾ is defined by

$$S_\mu(X) = \frac{\partial L^{\text{int}}}{\partial A_\mu} = \frac{\partial Z_{\alpha\lambda}}{\partial A_\mu} Q_{\alpha\lambda} + Q_{\alpha\lambda}^* \frac{\partial Z_{\alpha\lambda}^*}{\partial A_\mu} + \frac{\partial V}{\partial A_\mu}. \quad (6)^*)$$

On going into the interaction representation, the interaction Hamiltonian $H(X, \sigma)$ and the current operator $j_\mu(X, \sigma)$ take the following forms:

$$H(X, \sigma) = -L^{\text{int}} - Z_{\alpha\mu} A_{\alpha\beta}^{\circ*} Z_{\beta\nu}^* N_\mu N_\nu, \quad (7)^{**})$$

with

$$X_{\alpha\beta}^{\circ} A_{\beta\gamma}^{\circ} = \delta_{\alpha\gamma}, \quad A_{\alpha\beta}^{\circ} = A_{\beta\alpha}^{\circ*},$$

and

$$\begin{aligned} j_\mu(X, \sigma) = S_\mu(X) + \frac{\partial Z_{\alpha\lambda}}{\partial A_\mu} A_{\alpha\beta}^{\circ*} Z_{\beta\lambda'}^* N_\lambda N_{\lambda'} \\ + Z_{\alpha\lambda} A_{\alpha\beta}^{\circ*} \frac{\partial Z_{\beta\lambda'}}{\partial A_\mu} N_\lambda N_{\lambda'}, \end{aligned} \quad (8)$$

namely

$$j_\mu(X, \sigma) = - \frac{\partial H(X, \sigma)}{\partial A_\mu}. \quad (8')$$

The so called normal dependent parts of (7) and (8) are found to be quadratic functions of A_μ 's.

Noticing the relation (8') between the Hamiltonian and the current operator, we can rewrite (7) into a more conventional form:

$$H = H^i + H^e, \quad (9)$$

*) Our definition of S_μ above coincides with the usual definition⁹⁾ $\delta \int L dX = \int S_\mu \delta A_\mu dX$, since in our case the derivative of A_μ does not appear in the Lagrangian.

**) For the tensor coupling of spinor field the Lagrangian is not of the form (5), and so the surface dependent term in $H(X, \sigma)$ must be determined from the integrability condition as in the Kanesawa-Koba theory. (Prog. Theor. Phys. 4 (1949), 297) For the tensor coupling of vector field, such an exceptional situation does not appear.

$$H^i = -j_\mu^{(1)} A_\mu^i - \frac{1}{2} j_\mu^{(2)} [A_\nu^i] A_\mu^i, \quad (9')$$

$$H^e = -j_\mu^e A_\mu^e, \quad (9'')$$

where we have put $A_\mu = A_\mu^i + A_\mu^e$. The current j_μ has been decomposed into the following forms:

$$j_\mu = j_\mu^{(1)} + j_\mu^{(2)} [A_\nu], \quad j_\mu^{(1)} \sim e, \quad j_\mu^{(2)} \sim e^2, \quad (10)$$

or

$$j_\mu = j_\mu^i + j_\mu^e, \quad (10')$$

with

$$j_\mu^i = j_\mu^{(1)} + j_\mu^{(2)} [A_\nu^i], \quad j_\mu^e = j_\mu^{(2)} [A_\nu^e]. \quad (10'')$$

$j_\mu^{(2)} [A_\nu^i]$ and $j_\mu^{(2)} [A_\nu^e]$ mean those parts of j_μ which are linear in A_μ^i , A_μ^e . In (9) we have already omitted the terms quadratic in $A_\mu^{(e)}$.

(iii) The vacuum induced current $\delta j_\mu(X)$ is afforded by

$$\delta j_\mu(X) = \langle j_\mu^i + j_\mu^e \rangle = S_2^{-1} [\sigma] (j_\mu^i(X) + j_\mu^e(X)) S_2 [\sigma] \quad (11)^{*}$$

- (vacuum value),

where

$$j_\mu^{i \text{ or } e} = S_1^{-1} [\sigma] j_\mu^{i \text{ or } e}(X) S_1 [\sigma], \quad (12)$$

i.e. j_μ represents the current due to the particles clothed in the cloud of electromagnetic field. As far as the first order contribution of j_μ^e with respect to A_μ^e is concerned, we can take $S_2=1$ for the second term on the right hand side of (11), since j_μ^e itself contains A_μ^e linearly. Then, we find that $\langle j_\mu^e \rangle \propto A_\mu^e$. As we are now interested in the gauge invariant current alone, we shall omit the non-gauge invariant term such as $\langle j_\mu^e \rangle$ from the following considerations and pay our attention only to the contribution coming from j_μ^i in (11).

Understanding these circumstances, we may write δj_μ as follows:

$$\delta j_\mu(X) = \langle S_2^{-1} [\sigma] j_\mu^i(X) S_2 [\sigma] \rangle_0 - \langle j_\mu^i(X) \rangle_0 \quad (11')$$

Now, expanding S_2 in power of A_μ^e we find that the first order term of δj_μ turns out to be

$$\delta j_\mu(X) = i \int_{-\infty}^{\infty} \langle [j_\mu^i(X), j_\nu^i(X')] \rangle_0 A_\nu^e(X') d^4 X', \quad (13)$$

$$= \int_{-\infty}^{\infty} K_{\mu\nu}(X-X') A_\nu^e(X') d^4 X', \quad (14)$$

*) Källen's treatment⁽¹⁰⁾ is equivalent to putting $S_1=1$ and taking into account only the higher order correction arising from S_2 . The induced current thus obtained, however, does not give the genuine charge renormalization term.

where

$$K_{\mu\nu}(X-X')=i\langle [\mathbf{j}_{\mu}^s(X), \mathbf{j}_{\nu}^s(X')] \rangle_0. \quad (14')^*)$$

As is easily seen in (12), the current \mathbf{j}_{μ} includes all the higher order radiative corrections. In this sense, Eq. (13) can be regarded as the generalization of the Schwinger's formula¹⁰⁾ for the induced current.^{**) (Henceforth we shall omit the superscript "i".)}

The Fourier transforms of \mathbf{j}_{μ} and A_{μ}^e are introduced as follows:

$$\left. \begin{aligned} \mathbf{j}_{\mu}(X) &= \int \mathbf{j}_{\mu}(k) e^{ikX} d^4k, \\ A_{\mu}^e(X) &= \int A_{\mu}^e(l) e^{ilX} d^4l. \end{aligned} \right\} \quad (15)$$

Substituting (15) into (13) and performing the integral $\int_{-\infty}^t dt' \int d\mathbf{X}'$ (making the surface σ flat), (13) reads the following form:

$$\begin{aligned} \delta J_{\mu}(X) &= (2\pi)^3 \int d^4l \int_0^{\infty} dk_0 \left[\sum \langle \mathbf{l}, k_0 | \mathbf{j}_{\mu}(-\mathbf{l}, -k_0) | 0 \rangle^* \right. \\ &\quad \times \langle \mathbf{l}, k_0 | \mathbf{j}_{\nu}(-\mathbf{l}, -k_0) | 0 \rangle \frac{1}{k_0 - l_0} \\ &\quad + \sum \langle -\mathbf{l}, k_0 | \mathbf{j}_{\mu}(\mathbf{l}, -k_0) | 0 \rangle \\ &\quad \times \langle -\mathbf{l}, k_0 | \mathbf{j}_{\nu}(\mathbf{l}, -k_0) | 0 \rangle^* \left. \frac{1}{k_0 + l_0} \right] A_{\nu}^e(l) e^{ilX}, \end{aligned} \quad (16)$$

where \sum means the summation over all the possible intermediate states with energy k_0 and total momentum \mathbf{l} (or $-\mathbf{l}$). As k_0 is the energy of the intermediate state induced from the vacuum, it is evident that $k_0 > 0$. Here, we have also used the Hermitian property of the current operator \mathbf{j} .

Next we shall examine the relation between the first and second terms on the right hand side of Eq. (16). Let $u_{\pm}(\mathbf{P})$ be the C -number wave function of the charged particle with the momentum \mathbf{P} and charge $\pm e$, and let $v(\mathbf{k})$ be the wave function of the photon with momentum \mathbf{k} . In general, the matrix element of \mathbf{j} appearing in (16) will consist of the energy denominator and the numerator which contains wave functions u_{\pm} 's and v 's. Since we have the relations $v(\mathbf{k}) = v(-\mathbf{k}) = v^*(\mathbf{k})$ and $u_{\pm}(\mathbf{P})^* = u_{\mp}(-\mathbf{P})$ (if necessary, taking the appropriate representations of fundamental matrices γ or β), the operation of taking the complex conjugate of a wave function can be regarded as the operation of reversing the signs

*) On the physical (invariancy of the whole theory for the translation of coordinate) and mathematical (commutation relation of field operators and the vacuum value of their quadratic expressions being functions of the distance of two world points) grounds it is evident that $K_{\mu\nu}(X, X') = K_{\mu\nu}(X - X')$.

**) It is to be noticed that the exact formula (13) can be obtained by substituting \mathbf{j}_{μ} in the Schwinger's formula by the current \mathbf{j}_{μ} arising from the particles clothed in the cloud of electromagnetic field. In this connection, see reference (12).

of both charges and directions of momenta of all the particles present in this state. When performing this operation on $\langle \mathbf{l}, k_0 | \mathbf{j}_\mu(-\mathbf{l}, -k_0) | 0 \rangle$, the numerator turns into that of $\langle -\mathbf{l}, k_0 | \mathbf{j}_\mu(\mathbf{l}, -k_0) | 0 \rangle$, while the energy denominator of the former, being equal to that of the latter from the beginning, does not change its form. From the consideration of invariance of the whole theory for the charge conjugation, it is evident that for a state (\mathbf{l}, k_0) among the states in the first summation of (16), there always exists, in the second summation, a corresponding state $(-\mathbf{l}, k_0)$ as can be obtained by the above operation. Therefore, we see that

$$\begin{aligned} & \sum \langle \mathbf{l}, k_0 | \mathbf{j}_\mu(-\mathbf{l}, -k_0) | 0 \rangle^* \langle \mathbf{l}, k_0 | \mathbf{j}_\nu(-\mathbf{l}, -k_0) | 0 \rangle \\ &= \sum \langle -\mathbf{l}, k_0 | \mathbf{j}_\mu(\mathbf{l}, -k_0) | 0 \rangle \langle -\mathbf{l}, k_0 | \mathbf{j}_\nu(\mathbf{l}, -k_0) | 0 \rangle^*. \end{aligned} \quad (17)$$

Inserting (17) into (16), the induced current $\delta J_\mu(X)$ is reduced to

$$\begin{aligned} \delta J_\mu(X) &= 2^4 \pi^3 \sum \int_{-\infty}^{\infty} d^4 l \int_0^{\infty} dk_0 \langle \mathbf{l}, k_0 | \mathbf{j}_\mu(-\mathbf{l}, -k_0) | 0 \rangle^* \\ &\quad \times \langle \mathbf{l}, k_0 | \mathbf{j}_\nu(-\mathbf{l}, -k_0) | 0 \rangle \frac{k_0}{k_0^2 - l_0^2} A_\nu^e(l) e^{i l X}. \end{aligned} \quad (18)$$

(iv) Now, we shall see the property of the kernel $K_{\mu\nu}^e(X-X')$. Introducing its Fourier transforms $K_{\mu\nu}(X) = \int K_{\mu\nu}(l) e^{i l X} d^4 l$, we obtain

$$\delta J_\mu(X) = \int K_{\mu\nu}(l) \mathcal{A}_\nu^e(l) e^{i l X} d^4 l. \quad (19)$$

As we are concerned about the gauge invariant part of $\delta J_\mu(X)$, (19) can be further rewritten in the form

$$\delta J_\mu(X) = \int G_\nu(l) F_{\mu\nu}^e(l) e^{i l X} d^4 l. \quad (19')$$

$G_\nu(l)$ is a 4-vector composed of the 4-vector l_μ , so it must, in general, take the form

$$G_\nu(l) = h(l^2) l_\nu. \quad (20)$$

By use of (19'), (20), we obtain

$$\begin{aligned} \delta J_\mu(X) &= h(-\square) \frac{\partial F_{\mu\nu}^e(X)}{\partial X_\nu} = h(-\square) \square A_\mu^e(X) \\ &= \sum_{n=1}^{\infty} C_n \square^n A_\mu^e(X), \end{aligned} \quad (19'')$$

$$\text{i.e.} \quad \delta J_\mu(X) = \int L(l^2 - l_0^2) A_\mu^e(l) e^{i l X} d^4 l. \quad (19''')$$

Here we have used the Lorentz condition for the external potential A_μ^e . Now, let us consider $\delta J_1(X)$. As A_1^e , A_2^e and $aA_3^e + bA_4^e$ are linearly independent, by means of (19), (19''), and (19''') we get

$$K_{1\nu} = L(l^2 - l_0^2) \delta_{1\nu}. \quad (21)$$

(A_3 and A_4 are not linearly independent, but connected with each other by the

Lorentz condition. Therefore, in the above consideration we must not use $K_{3\nu}$ or $K_{4\nu}$. Since $K_{\mu\nu}$ is a tensor of second rank and the relation (21) holds in any coordinate system, we get finally the result

$$K_{\mu\nu} = K_{11} \delta_{\mu\nu} = K \delta_{\mu\nu}. \quad (21')$$

Among the components of $K_{\mu\nu}$, therefore, we have only to calculate the component K_{11} .

(v) Using (18), (21') and expanding $1/(k_0^2 - l_0^2)$ in power of $\square = l_0^2 - l^2$, we find

$$\begin{aligned} \delta J_\mu(X) = & 2^4 \pi^3 \int d^4 l \sum \int_0^\infty dk_0 |\langle l, k_0 | j_1(-l, -k_0) | 0 \rangle|^2 k_0 \\ & \times \sum_{n=1}^\infty \frac{\square^n}{(k_0^2 - l^2)^{n+1}} A_\mu^e(l) e^{iX}. \end{aligned} \quad (22)$$

When we take into account the covariancy of (22) for the Lorentz transformation (both sides of (22) must be 4-vectors) and the linear independence of each term on the right hand side expanded in \square , the expansion coefficients must be world scalars if the Lorentz invariant integration $\int_0^\infty dk_0$ is performed. Moreover, as these coefficients do not contain l_0 , it is evident that they do not contain l , and so \square either. Now, as is easily seen, $|\langle l, k_0 | j_1(-l, -k_0) | 0 \rangle|^2 > 0$ (In deriving this expression, we have used the Hermitian property of j , $k_0 > 0$ and $(k_0^2 - l^2) > 0$ (the energy of the intermediate state is generally larger than the magnitude of the corresponding total momentum of the state). Therefore, *all the expansion coefficients are found to be positive.*

Introducing the external current given by

$$J_\mu^e(X) = -\square A_\mu^e(X),$$

we thus obtain the *general formula*

$$\delta J_\mu(X) = \sum_{n=0}^\infty a_{n+1} \square^n J_\mu^e(X) \quad (a_n < 0, \quad n=1, 2, \dots) \quad (23)$$

with

$$a_n = 2^4 \pi^3 \sum \int_0^\infty dk_0 |\langle l, k_0 | j_1(-l, -k_0) | 0 \rangle|^2 \frac{k_0}{(k_0^2 - l^2)^{n+1}}. \quad (24)$$

Eq. (23) corresponds to the generalization of the formula previously given by one of the authors (H.U.) and R. Kawabe.¹⁾ It is noteworthy that this general formula (23) has been derived without use of any approximation method and is valid for any charged particle irrespective of its spin property and coupling type.

In the following sections we shall discuss the general properties derived from this formula and show its application to the second order induced current.

§ 3. General results of the formula

The formula obtained in the preceding section is of very general features. For in its derivation, starting from the general Lagrangian we have only used the general properties and requirements of the theory which are to be allowed within the framework of the present quantum field theory, i.e., the Hermitian property of current operator \mathbf{j} , the invariances of the theory for the charge conjugation, gauge and Lorentz transformations and Lorentz condition for external field. This formula is, therefore, valid for any system consisting of charged particles and electromagnetic field, being independent on its spin property and coupling type. Moreover, this has been obtained without using any approximation. In the following, we shall discuss the general results which are directly derivable from this formula.

As was already mentioned in the foregoing section, the expansion coefficients a_n 's (in (23)) are of *negative signs*. Since, in particular, between the coefficient of the first (renormalization) term a_1 and (renormalized) observable charge e_1 there is the relation

$$a_1 = \frac{e_1^2 - e^2}{e^2}, \quad (25)$$

we find the self charge, inclusive of all the higher order radiative corrections, is of negative sign. This result may be regarded as due to the general feature of electrodynamics that an external disturbance is always weakened by the effect of some internal phenomena (Lenz's law). (I)

If the coefficients a_n given by (24) lead to diverging results in the actual calculations, by applying the momentum conservation law to the intermediate state and representing the integrand in terms of the variable k_0 it is easily found in (24) that the order of divergence of a_{n+1} is lower than that of a_n by order 2. (II)

The current operator \mathbf{j} can be expanded in usual way in power of e , i.e., $\mathbf{j}_\mu = \sum_{n=1}^{\infty} \mathbf{j}_\mu^{(n)}$. Substituting $\mathbf{j}_\mu = \sum_i \mathbf{j}_\mu^{(n_i)}$ into (22), we get the same formula as (23), for $\mathbf{j}_\mu^{(n)}$ is Hermite operator.* When the various charged particles a 's are co-existing in vacuum the same formula is also obtained by inserting $\mathbf{j}_\mu = \sum_a \mathbf{j}_{\mu a}$ into (22). (III)

When the approximate expression $\mathbf{j} = \sum_{n=1}^{2m} \mathbf{j}^{(n)}$ (or $= \sum_{n=1}^{2m+1} \mathbf{j}^{(n)}$) is inserted into (22), the induced current ∂J_μ can be obtained, which includes all the contributions to the order $2m$ (or $2(m+1)$) and a part of higher order contributions of the order

*) $\mathbf{j}_\mu^{(n)} = \int_{-\infty}^t dX_1 \int_{-\infty}^t dX_2 \cdots \int_{-\infty}^t dX_n i [H(X_n), i [H(X_{n-1}), i [\cdots i [H(X_1), j_\mu(X)] \cdots]]]$,

which is easily found to be Hermite when remembering the fact that the commutator $i[A, B]$ with Hermite operators A, B is Hermite.

$2(m+1), \dots, 4m$ (or $2(m+2), \dots, 2(m+1)$). Now, for example, let us consider the self charge inclusive of the corrections to the 4-th order of e . In view of (III) its e^2 and e^4 corrections arising from $|\langle |j^{(1)}| \rangle|^2$ and $|\langle |j^{(2)}| \rangle|^2$, respectively, are found to be of negative sign. But, there is another e^4 -contribution coming from the cross term $\langle |j^{(3)}| \rangle \langle |j^{(1)}| \rangle$, which appears when $j = \sum_{n=1}^3 j^{(n)}$ are inserted into (22). The sign of this part can not be determined by the above consideration only. There may be, therefore, a possibility that if the contribution $\langle |j^{(3)}| \rangle \times \langle |j^{(1)}| \rangle$ contains a positively diverging part, the total self charge becomes finite as a result of cancellation of diverging parts. (If, a part of e^6 -contribution coming from $\langle |j^{(3)}| \rangle \langle |j^{(3)}| \rangle$ is added to the e^2 and e^4 -self charges, the total becomes negative in virtue of (III).)

When the higher order contributions of j are in this way taken into account successively, there may generally appear various types of divergences in the expression of self-charge. In order that the self-charge becomes finite as a whole, the sum of the coefficients of the terms with the same order of divergence must be vanishing. Now, let $f(e^2)$ be the sum of the coefficients of the highest order diverging terms. Then, in order to obtain the finite self-charge, it must be satisfied at least that $f(e^2)=0$. Now suppose that the algebraic equation $f(e^2)=0$ has the roots of odd multiplicity $e^2=A$. Then, it will become possible that the self-charge has a positive value for such value of e^2 as $e^2 > A$ or $e^2 < A$. Such a result is evidently inconsistent with the above general conclusions (I), (III), which are true for any magnitude of coupling constant e . Therefore, we can say that $f(e^2)=0$ has always roots of even multiplicity only. Moreover, in order to cancel all kinds of divergences in the self-charge the coefficients of lower order divergences must also vanish for the same value of e determined by $f(e^2)=0$, which has made the highest order divergence vanishing.

In quantum electrodynamics, in general, the similar consideration as Dyson's⁷⁾ shows the following fact: The renormalization factor of the effective potential A_{μ}^{el} is given by $1/2\pi i c(e)$, while after the charge renormalization is consistently carried out this factor becomes $1/2\pi i \cdot c(e_1)$. (It is easily verified that the result is obtained by simply substituting e_1 for e). In this case, therefore, we may state the above result as follows: *The condition for obtaining the finite self-charge in quantum electrodynamics is that one of the roots (with even multiplicity) of the equation $f(e^2)=0$ must be equal to $\hbar c/137$.* (IV)^{*})

In the e^2 -approximation of perturbation calculation it was shown by many authors¹⁾³⁾ that the self-charges are negatively diverging for the spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$

*) From this result we see the following fact: Even if in the 4-th order perturbation the condition for cancellation of divergences of self-charge is given by $e_1^2(e_1^2-A)=0$, the root $e_1^2=A$ can not be regarded as the correct value which guarantees the theory from divergence, since it is the root of odd multiplicity. Therefore, at least the sixth order contribution must be taken into account.

particles. In view of (I), (III) the divergence difficulties appearing in the self-charge and observable current can never be removed by introducing any charged particle or any coupling types. In the actual calculations, however, there frequently appear the ambiguities resulting from the infinite integrals,¹¹ which lead to the result inconsistent with the above general property. It seems probable in our treatment that some of the ambiguities have been settled by first performing the time integral $\int_{-\infty}^t dt'$. Even if we can eliminate some of the divergences appearing in a_n by suitably defining the ambiguous integrals, it is evident by (II) that another divergence, the order of divergence of which is lower than a_n by 2, will inevitably appear in the next term a_{n+1} . Moreover, it will be seen that use of these ambiguities can not remove all the difficulties. (V) More detailed discussions in the ϵ^2 -approximation relating to this point will be developed in the next section.

§ 4. Application of general formula - ϵ^2 -approximation -

According to (III) in the preceding section, we can discuss the induced current δJ_μ in ϵ^2 -approximation by means of (23). As in the lowest approximation cross terms such as $\langle |j_a| \rangle \langle |j_a| \rangle$ do not appear, the discussions become very simple.

Various attempts to settle the divergence difficulties encountered in this problem of self-charge and observable current in ϵ^2 -approximation, i.e., introducing the charged particle with higher spin or assuming the tensor coupling, etc. have been suggested by many authors. But, in view of (I), (III), it is easily seen that these attempts can never remove these difficulties.

As for the coefficients a_n 's the result (III) is also valid when they are divergent. Further it will be found from the following consideration that the order of divergence of a_n is always even: In (24), the factor $k_0/(k_0^2 - l^2)^{n+1}$ is proportional to $k_0^{-(2n+1)}$ for large value of k_0 . Since the current operator is the quadratic function of the field operators and their derivative (e.g. $kU^*(k)U^*(-k)$) and further the commutators or the vacuum expectation values of the quadratic expressions of their Fourier amplitudes are rational functions of (E_k, k) , we find that $|\langle |j| \rangle|^2 \propto k^{2m}$. For large k_0 , therefore, a_n behaves as $k_0^{2(m-n)}$, namely its order of divergence is even. For spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ particles these properties were confirmed by many authors.¹³⁾ Considering the results obtained, it seems almost impossible to find a clue to overcome the divergence difficulties.

Recently, Katayama and McConnell¹⁴⁾ have shown that by use of convenient definition of the infinite ambiguous integrals there arises a possibility of saving the difficult situations. In the following we shall criticize their method.

In the self-charge problem for the vector charged particle, adopting the convenient definitions of ambiguities McConnell eliminated second order divergence with negative sign and obtained the logarithmically diverging self-charge with positive sign, being possible to compensate the logarithmically diverging term with negative sign of another particle. His starting point is

$$(\epsilon_1^2 - \epsilon^2)/\epsilon^2 = (a/4\pi) \{K_1 - (1/2)K_2\}, \quad (26)$$

with

$$K_1 = (1/im^2) \int_{-\infty}^{\infty} \exp(im^2 Z)/Z^2 \cdot \epsilon(Z) dZ, \quad (27)$$

$$K_2 = \int_{-\infty}^{\infty} \exp(im^2 Z)/Z \cdot \epsilon(Z) dZ. \quad (27')$$

Further, using the relation

$$K_1 = 2 + K_2 + 1/im^2 (1/Z_0 - 1/Z_0')_{Z_0, Z_0' \rightarrow 0} \quad (27'')$$

and taking its principal value, he obtained

$$(\epsilon_1^2 - \epsilon^2)/\epsilon^2 = (2 - \log \gamma m^2 Z_0)_{Z_0 \rightarrow 0} (a/4\pi). \quad (26')$$

As the self-charges due to scalar and spinor fields diverge logarithmically and have negative signs, Katayama proposed, as the condition for finite self-charge, the relation

$$4N_s + N_\sigma - 3N_v = 0, \quad (28)$$

where N_s , N_σ and N_v are numbers of scalar, spinor and vector fields, respectively. McConnell has adopted his definition (27'') for the physical reason that otherwise there appears imaginary diverging term in the final expression of the self-charge (26). Examining the process of calculation we see, however, that before he has omitted the negative imaginary part in question he has unconsciously overlooked or consciously eliminated by definition a real diverging contribution. K_1 comes from $\mathcal{A}^{(1)}(0)$ in the expression of self-charge. For $\mathcal{A}^{(1)}(0)$, we have following two representations:

$$\int dp \delta(p^2 + m^2), \quad (29)$$

$$\int dZ \int dp \exp[iZ(p^2 + m^2)]. \quad (30)$$

The former gives the real second order divergence with negative sign. On the other hand his result can be obtained by the latter definition (30) and

$$\int dp \exp(iZ p^2) = (i\pi^2/Z) \epsilon(Z). \quad (31)$$

The above relation is correct only for $Z \neq 0$, but is ambiguous and requires a further definition at $Z=0$. In fact, if taking into account the contribution at $Z=0^*)$, we find again the real second order diverging self-charge with negative

*) We can introduce the contribution at $Z=0$ in the following form

$$\lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} dZ \int dp = 2\epsilon \int dp,$$

which gives the real second order contribution, since ϵ is an infinitesimal of the order $(\frac{1}{p})^2$. (p : cut off momentum)

sign. We have no reason to omit this term. As regards this ambiguity at $Z=0$, McConnell has not given any physical reason. We may say, therefore, that his method has no consistent basis throughout his argument. Even if some of the difficulties are eliminated in a similar way as McConnell has attempted, as will be seen from the following consideration it may be very difficult to deal with all the difficulties encountered in this problem.

In treating the divergences by exclusive use of the ambiguous properties of infinite integrals, at first sight the following two ways seem to be possible, i.e., (denoting the a_n terms of fields A, B, \dots by $a_n(A), a_n(B), \dots$).

- (i) to change the sign of the highest divergence in $a_n(B)$ and make this term compensate the corresponding divergence in $a_n(A)$, or
- (ii) to eliminate the highest order divergence in $a_n(B)$ and make the remaining divergence compensate $a_n(A)$.

By the method (i) we must change the highest order diverging term $-f_n(f_n > 0)$ of $a_n(B)$ into the same order divergence but with positive sign $f_n'(f_n' > 0)$. However, when considering $f_n, f_n' > 0$, it seems to be very difficult in most cases to lower the order of divergence $f_n + f_n'$ without lowering those of f_n and f_n' , respectively. The latter manipulation is equivalent to (ii).

Next let us consider the method (ii). In order that the divergence in $a_n(A)$ due to a charged field A can be compensated by the corresponding term $a_n(B)$ due to a field B , it is required that before eliminating some divergence from $a_n(B)$ by a convenient definition the order of divergence of $a_n(B)$ is higher than that of $a_n(A)$. In other words, for our present purpose we must introduce such a charged field B as has the divergence of same order as $a_n(A)$ and with positive sign after the elimination of the highest order one in $a_n(B)$. (There is a possibility that the divergent term of lower order can have positive value, for the sign of $a_n(B)$ is generally determined by its highest order divergence.) Now, let $a_n'(B)$ be the remaining divergent term of $a_n(B)$ after our manipulation. Then, there holds the following relation:

$$\text{order of divergence of } a_n(A) = \text{that of } a_n'(B) < \text{that of } a_n(B).$$

According to the general result (II), it is found that

$$\text{order of divergence of } a_m(A) < \text{that of } a_m(B). \quad (m > n)$$

Thus, the introduction of a new charged field B to compensate the divergence in $a_n(A)$ gives rise to the appearance of new divergences in $a_m(B)$ ($m > n$) while $a_m(A)$ is finite. If we successively apply the above method to the divergence which has appeared in $a_m(B)$, we must finally eliminate logarithmic divergences in $a_m'(B')$ by means of our manipulation only. This is, however, often very difficult. Thus, in this case (ii) it becomes necessary to eliminate the highest order divergence appearing in each term a_m inclusive of the logarithmic divergence. But, this requirement is in most cases equivalent to the elimination of divergences

of all a_m' of a charged field A by means of defining ambiguities without introducing another field B .

From the above considerations, we can say that it may be almost impossible to find an entire solution for the difficulties of all terms a_n 's of induced current in the e^2 -approximation within the framework of the present quantum field theory.

§ 5. Conclusion

In this paper general discussions of the gauge invariant part of induced current have been developed independently of the spin property of charged fields and types of interactions between charged fields and electromagnetic field. In the e^2 -approximation the introduction of the tensor coupling^{1b)} cannot solve the difficulties of the charge renormalization term a_1 and observable current a_2, a_3, \dots . (§ 4) The utilization of convenient definitions of ambiguous integrals,^{1b)} which seem to be inadequate definitions, occasionally eliminates the divergence of the charge renormalization term a_1 , but in this case new divergence difficulties appear in observable current. Thus, it is impossible to solve all difficulties of the gauge invariant induced current in the e^2 -approximation within the framework of the present quantum field theory.

This difficult circumstance seems to be not amended when we take into account the higher order effect in the perturbation approximation (§ 3), although we can not definitely conclude it since there remains an improbable possibility (§ 3, IV). (It is noteworthy, however, that if the possibility (IV) is successful in avoiding divergencies, the magnitude of the coupling constant, e , of any charged field is uniquely determined when we require the whole theory to be a consistent closed system.*))

The origin of the above difficulties is the fact that *any a_n has always negative sign* (§ 2, § 3, I). This is the usual feature in electrodynamics that an external effect is weakened by some internal effects in the matter, when we regard the vacuum as a polarizable matter. Therefore, it is necessary to change this property by means of certain methods.

One of such methods is to regard the vacuum as a diamagnetic matter, the structure of which is to be conveniently assumed.

The other method, which is beyond the limits of the present quantum field theory, is to apply the renormalization procedure to the divergences of all a_n -terms ($n=1, 2, \dots$). In the case of spin 0 and $\frac{1}{2}$ only the a_1 -term diverges and so the renormalization procedure is successful. In the case in which spin 1 field is also

*) The fact that the observable charge of every charged field has the same value $|e|$ may be understood in view of the charge renormalization procedure, for the renormalization of charge of any charged field is to be carried out in taking account simultaneously of the effect of all charged fields existing in the nature.

considered, it is required to introduce an interaction with the form $(A_\mu \square J_\mu)$ in order to cancel the divergence of the a_2 -term, but this new interaction seems to cause a further divergence in the a_3 -term*) because of its higher derivative. And so the renormalizations of all a_n -terms will become successively necessary. But such an interaction will be equivalent to non-local interactions. Therefore, it becomes an interesting problem to clarify the relation between the consistency of renormalization procedure and the non-local interaction. Thus, as stressed by S. Sakata^{1b)} in connection with the problem of the "structure of interactions," the possibility of the existence of charged fields of spin 1 becomes very serious for the problem whether the renormalization procedure can furnish the closed quantum electrodynamics in which only finite times of renormalization procedures (in the above sense) are sufficient to construct a consistent theory, and also for the problem of non-local interaction in quantum field theory.

At any rate, in order to settle these difficulties entirely it is expected that a more wealthy material property may be introduced for the vacuum.

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This expectation has been confirmed by further investigation relating to the problem of applicability of the renormalization theory.

On the Production of Negative Protons

Mitsuo TAKETANI, *Tokyo*

and

Shigeru MACHIDA, *Physics Department, University of Tokyo*

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Cross sections for the production of negative protons in various processes was calculated assuming the pseudoscalar meson theory. Numerical results are summarized in Table II, § 5. In the lack of satisfactory theory for the high energy processes considered here, we have employed the Weizsäcker-Williams method. Taking account of the reactions and the non-linearities of the meson field phenomenologically to some extent, we have treated the case of modified meson spectrum as suggested by Heisenberg. Obtained cross sections for the production of negative protons by the nucleon-nucleon collisions are smaller by the factor of about 10^{-4} in the case of modified meson spectrum than those in the case of neglecting the reactions and non-linearities of meson field completely. Also the cross section for the production of negative protons by proton-proton collisions near threshold is calculated by the method of Feynman's relativistic perturbation.

§ 1. Introduction

Recently Fermi and Yang¹⁾ and Heisenberg²⁾ have proposed composite theories of bosons by fermions. Such an attempt in company with the others might be suggestive to clarify the concept of "elementary particles" and to inquire into the correct future theory as discussed by Heisenberg.³⁾ But it seems necessary in the present stage of knowledge, together with the above mentioned attempts, to know more precisely the substantialistic properties of the "mesons" and the "nucleons" by experimental procedures.

From this point of view, one of us (M. T.) has proposed⁴⁾ to examine experimentally the "elementarity" of a particle by its creation process, if the interactions between the particles were not too strong. Although a strong support was obtained for the existence of the anti-nucleon by the fact that the γ -decay of the neutral π -meson⁵⁾ was established experimentally,⁶⁾ it is of much importance, for the sake of clarifying the substantialistic properties of nucleons, whether or not the anti-nucleon exists really, and whether or not the nucleon-anti-nucleon field can be described by the Dirac equation. In this paper we calculate the cross sections for the production of nucleon pairs by various processes, and discuss the results.

§ 2. Various processes and the threshold energies for the production of negative protons

Reactions producing the negative protons are considered to be the following:
 (a) $\gamma + \gamma \rightarrow P^- + P$, (b) $\gamma + \pi^- \rightarrow P^- + N$, $\gamma + \pi^0 \rightarrow P^- + P$, (c) $\pi^- + \pi^+ \rightarrow P^- + P$,

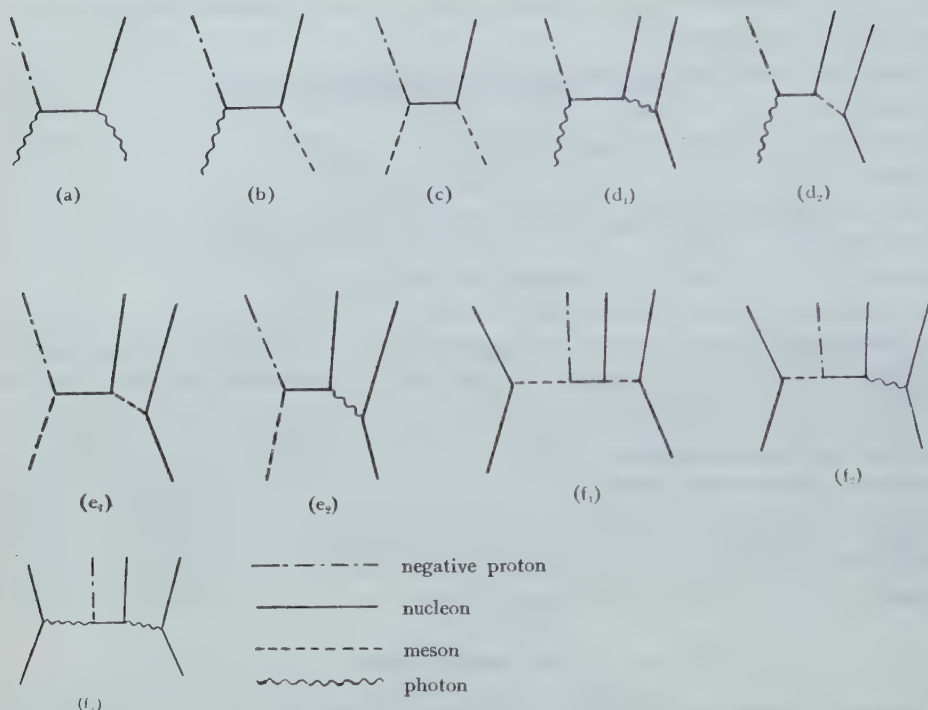


Fig. 1. Feynman diagrams for the production of negative protons

$\pi^- + \pi^0 \rightarrow P^- + N$, $\pi^0 + \pi^0 \rightarrow P^- + P$, (d) $\gamma + P \rightarrow P^- + P + P$, $\gamma + N \rightarrow P^- + P + N$, (e) $\pi^- + P \rightarrow P^- + P + N$, $\pi^- + N \rightarrow P^- + N + N$, $\pi^0 + P \rightarrow P^- + P + P$, $\pi^0 + N \rightarrow P^- + P + N$, (f) $P + P \rightarrow P^- + P + P + P$, $P + N \rightarrow P^- + P + P + N$, (γ : photon, $\pi^{\pm,0}$: charged or neutral π -mesons, P : proton, N : neutron, P^- : negative proton). Anti-neutrons would be produced by the similar reactions. We show in Fig. 1 the Feynman diagrams of these reactions of the lowest order in the perturbation calculation. Although both for the production by cosmic rays or by the bevatron, which will be constructed in the near future, the processes (a), (b) and (c) would be completely negligible, we consider them too, because we utilize the method of Weizsäcker and Williams.

Now, we calculate the threshold kinetic energies for the above mentioned reactions, for they are necessary in the calculations of next section. When an incident particle with rest mass m_1 bombarded a target particle with rest mass m_2 at rest and rest mass m_0 was produced, the threshold kinetic energy E which the incident particle should have, is, according to the relativistic kinematics,⁷⁾ given by

$$E = (m_0/m_2)(m_1 + m_2 + m_0/2). \quad (1)$$

The values of E for the various processes are given in Table 1. (We choose $\hbar=c=1$. M and μ are the masses of a nucleon and a π -meson, respectively.)

Table I.

Process	Reference system	Threshold kinetic energy
(a)	c.m.	$2M=1.9$ BeV
(b)	meson at rest	$(2M/\mu)(M+\mu)=14$
(c)	meson at rest	$(2M/\mu)(M+2\mu)=16$
(d)	nucleon at rest	$4M=3.7$
(e)	nucleon at rest	$2(2M+\mu)=4.0$
(f)	nucleon at rest	$6M=5.6$

§ 3. Method of calculation. On the non-linear characters of the meson field and the use of the Weizsäcker-Williams method

If we consider the case of production of negative protons by the cosmic radiation, we must evaluate the cross sections for the processes (d), (e), and (f) discussed in the preceding section. As the cross section for the process (a) can be obtained from the cross section for the production of an electron-pair⁹⁾ by altering the masses of the produced particles, we begin with the calculation of the cross sections for the processes (b) and (c) using the Feynman's relativistic perturbation method. Then we can evaluate the cross sections for the processes (d), (e) and (f) by the use of the Weizsäcker-Williams method.^{9),10)*} Considering the energy spectrum of the cosmic radiation, the energy regions of the incident particles which is of mainly importance will be the regions of about 10~30 BeV. In this energy region the multiple production of nucleons would not be of much importance.

Now, it would be considerably difficult to give convincing evidences of whether or not the negative protons exist among the cosmic ray particles, the future experiment using the bevatron would be very important for this purpose. In the latter case we must consider the production by the nucleon-nucleon collisions near the threshold. In this energy region the approximation of the Weizsäcker-Williams method being not good, we calculate the cross section for this case by the method of Feynman's relativistic perturbation.

If we consider the interaction between the meson and the nucleon with differentiation, the non-linear characters of the system, such as the multiple processes, would appear markedly in high energy phenomena. As the critical energy for these phenomena would be, according to the considerations of Heisenberg,¹²⁾ about the rest energy of the meson, it would be of great importance to take into account the reactions of the meson field and the correlations between the field

* We calculated the cross sections for the processes (c₂) using the pseudoscalar and vector meson theory with both couplings respectively and reported the results at the annual meeting of the Physical Society of Japan, in April 1949.¹¹⁾

components. On the other hand the method of Weizsäcker and Williams holds good only when we can neglect the field reactions and the correlations between the field components. Still the short range character of the interactions considered here makes the approximations of the Weizsäcker-Williams method worse than in the case of electromagnetic interactions. Unfortunately there being no theory which can take into account satisfactorily the field reactions and the correlations of the field components in the present, we calculate the cross sections in the following manner: In the calculation of the cross sections for the "elementary" processes by the Feynman method, we adopt the pseudoscalar meson theory with pseudoscalar coupling, which does not include the differentiation, and use the Weizsäcker-Williams method, with the meson spectrum derived from the pseudoscalar meson theory with both couplings and to take into account the effects of non-linear characters of the system we will treat also the case of modified meson spectrum as suggested by Heisenberg¹³⁾ from the analogy between the meson production and the turbulent motions. By including a parameter, which represents the above mentioned effects, we might be able to take into account them to some considerable extent phenomenologically.

Discussions on the possible method for the detection of negative protons have been given by Ashkin, Auerbach, and Marshak¹⁴⁾ in their work on the annihilation processes for negative protons. Earlier results on the production and annihilation cross sections of negative protons obtained by McConnell¹⁵⁾ are not in agreement with the results of Ashkin et al and ours. The discrepancy cannot be a consequence of the neglect of radiation damping by Ashkin et al and by us and of the Heitler method used by McConnell, because it remains at the low energy limit where the effect of radiation damping should become unimportant as was pointed out by Ashkin et al. Our results on the transition matrix element of process (c) is in agreement with the result of Ashkin et al. So we can conclude that McConnell's results are incorrect.

§ 4. Cross section for various production processes

(i) Meson-Photon Collision

Process (b): $\pi^- + \gamma \rightarrow P^- + N$

The Feynman diagrams (Fig. 2 (a), (b)) lead to matrix elements proportional to

$$H_a = -igc \bar{u}_1 \gamma_5 (p_1 - k - M)^{-1} e u_2$$

and

$$H_b = -igc \bar{u}_1 \gamma_5 u_2 [(k + l)^2 - \mu^2]^{-1} e$$

respectively, where u_1 and u_2 are the Dirac spinors representing neutron and negative proton respectively, g is the "mesic" charge of the nucleon, M is the nucleon mass, $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$, $p = p_\mu\gamma_\mu$, $e = e_\mu\gamma_\mu$ and $\bar{u}_1 = u_1^*\gamma_4$. p_μ , k_μ , and l_μ are the four dimensional momenta of nucleon, meson and photon respectively, and e_μ

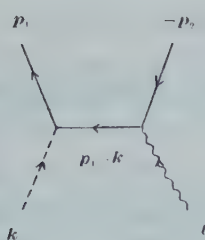


Fig. 2 (a)



Fig. 2 (b)

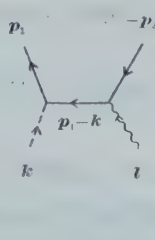


Fig. 2' (a)



Fig. 2' (b)

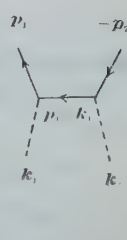


Fig. 3

is the polarization vector of the photon. (We use Feynman's notations and choose $\hbar=c=1$.) Following the well-known procedure we obtain for the differential cross section for production of a negative proton and a neutron with oppositely directed momenta in the solid angle $d\Omega$:

$$d\sigma(\pi, \gamma) = d\Omega \frac{g^2 e^2 p}{2kE(\epsilon + k)} \{A(\theta) + B(\theta) + C(\theta)\}, \quad (2)$$

$$A(\theta) = [2(E\epsilon - pk \cos \theta) \{E(2\epsilon - k) + \epsilon p(\sin \theta + \cos \theta) - pk \sin \theta\} - \mu^2 \{M^2 + 2p^2(1 - \sin \theta \cos \theta)\}] \cdot (2E\epsilon - 2pk \cos \theta - \mu^2)^{-2}, \quad (3)$$

where θ is the angle between the momentum of the negative proton and the momentum of the photon, and (p, E) and (k, E) denote the magnitude of the three-dimensional momentum and energy of each of the produced nucleon or anti-nucleon and of incident meson respectively. $A(\theta)$, $B(\theta)$, and $C(\theta)$ are the terms due to the diagrams (a), (b), and their interference respectively. Here we give only $A(\theta)$ explicitly. If we consider M to be large in comparison with p and μ , then it can be easily seen that $B(\theta)/A(\theta) \sim (p/M)^2$ and $C(\theta)/A(\theta) \sim p/M$. In the case of the production of negative protons by photon-nucleon collisions or by meson-nucleon collisions the main contributions from eq. (2) will come from the energy region near the threshold, because the threshold energy for the process (b) is considerably high as can be seen from the Table 1, and the number of photons or mesons equivalent to the field generated by a moving nucleon is smaller in high energy region. As the energy in the *c.m.* system is considerably smaller than the corresponding one in the laboratory system, we can consider M to be large compared to p and μ , if the energy of an incident particle to be about 20~40 BeV. Then we get

$$A(\theta) \approx (1/2)(k + p \cos \theta)/(k - p \cos \theta) \approx 1/2 \gg B(\theta), C(\theta), \quad (3')$$

and the total cross section becomes

$$\sigma(\pi, \gamma) = \pi g^2 e^2 p / kE(\epsilon + k). \quad (4)$$

For the process $\pi^0 + \gamma \rightarrow p^- + p$ (π^0 means a neutral π -meson) there are two diagrams as shown in Fig. 2', and the total cross section becomes zero in our approximation by their interference effects.

(ii) *Meson-Meson Collision**Process (c)* $\pi^- + \pi^+ \rightarrow p^- + p$

The associated Feynman diagram (Fig. 3) leads to a matrix element proportional to

$$H = -ig^2 \bar{u}_1 \gamma_5 (\mathbf{p}_1 - \mathbf{k}_1 - M)^{-1} \gamma_5 u_2,$$

where k_1 and k_2 are four-dimensional momenta of the positive and negative mesons respectively. We find for the differential cross section (*c.m.*-system)

$$d\sigma(\pi^-, \pi^+) = d\Omega g^4 p / (2kE^2) D(\theta), \quad (5)$$

$$D(\theta) = \frac{k^2 p^2 (1 - \cos^2 \theta) + M^2 (k^2 + p^2)}{[p^2 + k^2 + M^2 - 2pk \cos \theta]^2}, \quad (6)$$

where θ is the angle between the momentum of the negative proton and the momentum of the negative meson, and k and p denote the magnitude of the three-dimensional momenta of the meson and the nucleon respectively, and E is the energy of the proton. Since we can consider $M \gg p, \mu$ as in the case (i), the total cross section becomes

$$\sigma(\pi^-, \pi^+) \approx \pi g^4 p / (2kE^2). \quad (7.1)$$

For the processes which include neutral mesons, such as $\pi^0 + \pi^0 \rightarrow p^- + p$ and $\pi^0 + \pi^- \rightarrow P^- + N$, there are another Feynman diagram besides Fig. 3, and interference effect occurs between them. In the "symmetrical" meson theory of nuclear forces, the coupling constant of the neutral meson to the nucleons has opposite sign for neutron and proton, on the other hand in the "neutral" theory it has a same sign. Such a possible difference in the interaction between the neutral meson and the nucleon has no effect for the process $\pi^0 + \pi^0 \rightarrow P^- + P$, but does make a difference for the process $\pi^- + \pi^0 \rightarrow P^- + N$, as was pointed out by Ashkin et al in the reverse processes.¹⁴⁾ The total cross sections for these processes are

$$\sigma(\pi^0, \pi^0) = \pi g^4 p / (kE^2) (\mu^4 / 8k^3 M) \sim (1/4) (\mu/M)^4 \cdot \sigma(\pi^-, \pi^+), \quad (7.2)$$

$$\sigma(\pi^0, \pi^-) = (2\pi g^4 p) / (kE^2) \times \begin{cases} 1 & \sim 4\sigma(\pi^-, \pi^+), \\ \mu^4 / 8k^3 M & \sim (1/2) (\mu/M)^4 \sigma(\pi^-, \pi^+), \end{cases} \quad (7.3a)$$

$$\quad \quad \quad \mu^4 / 8k^3 M \sim (1/2) (\mu/M)^4 \sigma(\pi^-, \pi^+), \quad (7.3b)$$

where (7.3a) is to be used if neutron and proton have the opposite neutral mesic charge, and (7.3b) if the neutral mesic charge are the same. In these formulae we have chosen the absolute value of the neutral mesic charge equal to that of the charged mesic charge.

(iii) *Photon-Nucleon Collisions**Process (d₁)* : $\gamma + P \rightarrow P^- + P + P$

The cross section for the process (*d₁*) of Fig. 1 can be obtained by altering the electron mass to the nucleon mass from that for the electron-pair creation due to photon-electron collision.¹⁶⁾

$$\sigma^{(1)}(\gamma, P) = 56e^6 / (9M^2) \ln(2\nu/M) = 1.1 \times 10^{-30} \ln(2\nu/M) \text{ cm}^2, \quad (8)$$

where ν denotes the frequency of the incident photon in the system in which the initial proton is at rest.

Process (d_2): $\gamma + N \rightarrow P^- + P + N$

The cross section for the process (d_2) of Fig. 1 can be evaluated according to the method of Weizsäcker and Williams, if we know the cross section for the production of negative protons by the photon-meson collisions, $\sigma(\gamma, \pi)$, and the meson spectrum of a moving nucleon.

We denote the energy of the incident photon in the laboratory system in which the initial nucleon is at rest by $W = \eta M$. We consider the events in the reference system in which the initial nucleon moves with the velocity v in the opposite direction with the photon, where v is given by the equation $(1-v^2)^{-\frac{1}{2}} = \eta/2$ (we call this system the S -system). Since v is very near to one in the energy region considered here, the energy of the photon in the S -system becomes M^* . Transforming the $\sigma(\gamma, \pi^-)$ in the $c.m.$ -system, eq. (4), into the S -system, we obtain

$$\sigma(\gamma, \pi^-) = \frac{\pi g^2 e^2}{2(M\epsilon')^{3/2}} \sqrt{M(\epsilon' - M)}, \quad (4')$$

$\epsilon' = \epsilon^2/M$ denotes the energy of the meson in the S -system. If we substitute the meson field of a neutron moving with the velocity v by a equivalent free meson field, we obtain the number of negative pseudoscalar mesons with pseudoscalar and pseudovector couplings in the energy region $\epsilon' \sim \epsilon' + d\epsilon'$ as follows,**

$$q(\epsilon') d\epsilon' = \frac{10^2}{2} \frac{g^2 \mu^a}{\pi} \frac{d\epsilon'}{(\epsilon')^{1+a}} \sqrt{\frac{2\epsilon'}{\eta M}}, \quad (9)$$

where a is a parameter introduced to represent the effects of the reactions and non-linear characters of the field to some extent phenomenologically. In the case of complete neglect of all these effects, a becomes zero and we obtain the result given by Heitler and Peng.¹⁷⁾ According to Heisenberg,¹³⁾ in the energy region considered here, the value of a which is slightly smaller than one seems to agree well with the experiments on the meson production by Powell et al.¹⁸⁾ But this estimation of a is not so certain, because of the uncertainties in the experiments and the difficulties lying in the current theories. We will calculate the cross sections for the two cases $a=0$ and $a=1$. The conditions $\eta \gg 1$ and $\eta M \gg \epsilon' \gg \mu$, under which the equation (9) holds, are satisfied approximately in the energy region considered here.

* Exactly $(M/2)\eta^2 \cdot [1 - \{1 - (4/\eta^2)\}^{1/2}]$.

** η is twice times ξ used by Heitler and Peng and by us afterwards. In deriving the equation (9), we have put the minimum impact parameter to be the Compton wave-length of the nucleon. but the results do not much affected by this choice. Especially for the case of $a=1$ considered after, the dependence of the results on this value is quite negligible.

Since the cross section for the process $\gamma + \pi^0 \rightarrow P^- + P$ is very near to zero shown in the case (i), we need not to consider the neutral meson field attached to the nucleon. Then the cross section for the reaction (d_2) of Fig. 1 is given by

$$\sigma^{(2)}(\gamma, N) = \int_{\epsilon'_0}^{\epsilon'_m} \sigma(\pi, \gamma) q(\epsilon') d\epsilon',$$

where $\epsilon'_m \sim W/2$, $\epsilon'_0 \sim M$ and W is the energy of the incident photon in the laboratory system. Performing the integration we get

$$\begin{aligned} \sigma^{(2)}(\gamma, N) &= \frac{10^2}{2\sqrt{2}} \frac{g^4 \epsilon^2}{M^2} \sqrt{\frac{M}{W}} I = 1.12 \times 10^{-28} g^4 \sqrt{\frac{M}{W}} I \text{ cm}^2, \\ I &= \begin{cases} \tan^{-1} \sqrt{\frac{W}{2M} - 1} - \frac{2M}{W} \sqrt{\frac{W}{2M} - 1}, & \text{for } a=0, \\ \frac{M}{W} \left[\frac{1}{4} \tan^{-1} \sqrt{\frac{W}{2M} - 1} + \left(\frac{1}{2} - \frac{2M}{W} \right) \frac{M}{W} \sqrt{\frac{W}{2M} - 1} \right], & \text{for } a=1. \end{cases} \end{aligned} \quad (10)$$

So

$$[\sigma]_{a=1} \sim [\sigma]_{a=0} \cdot \frac{\mu}{M} \sim 0.15 [\sigma]_{a=0}. \quad (10')$$

The values of eq. (10) for the various values of W are given in Table II, § 5.

The cross section for the process $\gamma + P \rightarrow P^- + P + P$, which occurs by the intermediary of the meson field, becomes very small compared to the equation (10) due to the interference effects, because for the former process only the neutral meson can contribute in the lowest order of perturbation theory.

(iv) *Meson-Nucleon Collisions.*: $\pi^- + P \rightarrow P^- + P + N$

Process (e_1)

We calculate the cross section for this case from $\sigma(\pi^-, \pi^+)$, eq. (7.1), and the meson-spectrum of a moving nucleon. Transforming into the reference system in which π^- is at rest, we get

$$\sigma(\pi^-, \pi^+) = (\pi g^4 / \mu \epsilon') \sqrt{1 - 2M^2 / \epsilon' \mu}, \quad (7')$$

where $\epsilon' = (2E^2 / \mu) - \mu$ is the energy of the π^+ -meson in this reference system.

The spectrum of the free positive mesons equivalent to the meson field of a proton moving with the velocity v is given by

$$q(\epsilon') d\epsilon' = \frac{10^2}{2} g^2 \frac{\mu^\alpha}{\pi} \frac{d\epsilon'}{(\epsilon')^{1+\alpha}} \sqrt{\frac{\epsilon'}{\xi M}} \quad (11)$$

where $\xi = (1 - v^2)^{-1/2}$.

The cross section for the production of a negative proton by a collision between a proton at rest and a negative π -meson is

$$\sigma^{(1)}(\pi^-, P) = \int_{\epsilon'_0}^{\epsilon'_m} \sigma(\pi^-, \pi^+) q(\epsilon') d\epsilon',$$

where $\epsilon' \sim M\xi$ and $\epsilon'_0 = (2/\mu)(M+2\mu)$ (cf. Table I). Performing the integration, we get

$$\sigma^{(1)}(\pi^-, P) = \frac{10^2}{2\sqrt{2}} \frac{g^6}{M\mu} \sqrt{\frac{\mu}{M}} \sqrt{\frac{\mu}{W}} \cdot K = 4.2 \times 10^{-26} \cdot g^6 \sqrt{\frac{\mu}{W}} \cdot K \text{ cm}^2,$$

$$K = \sec^{-1} \sqrt{\frac{W}{2M}} - \sec^{-1} \sqrt{\frac{M+2\mu}{M} - \frac{2M}{W}} \sqrt{\frac{W}{2M}} - 1 + \sqrt{\frac{M}{M+2\mu}} - 1, \text{ for } a=0,$$

$$K = \frac{1}{8} \left(\frac{\mu}{M} \right)^2 \left\{ \sec^{-1} \sqrt{\frac{W}{2M}} - \sec^{-1} \sqrt{\frac{M+2\mu}{M} - \frac{2M}{W}} - \sqrt{\frac{W}{2M}} - 1 \left(1 - \frac{2M}{W} \right) + \sqrt{\frac{2\mu}{M}} \frac{2\mu}{M+2\mu} \right\},$$

for $a=1$, (12)

$$[\sigma]_{a=1} \approx [\sigma]_{a=0} \cdot \frac{1}{8} \left(\frac{\mu}{M} \right)^2 \sim 3 \times 10^{-3} [\sigma]_{a=0}, \quad (12')$$

where $W = \xi\mu$ is the energy of the incident meson in the reference system in which the incident proton is at rest. The values of (12) for various incident energies are given in Table II, § 5.

Eq. (12) is the cross section in the charged theory. In the symmetrical theory the cross section becomes about five times the equation (12) according to eq. (7.3a), and in the case of adding the neutral meson field of the neutral theory the cross section is not altered according to eq. (7.3b). The cross section for the process $\pi^0 + P \rightarrow P^- + P + P$ becomes smaller than (12) by a factor of order 10^{-4} according to the eq. (7.2).

Process (e_2)

Transforming the cross section for the production of a negative proton by a photon-meson collision, eq. (4), into the reference system in which the meson is at rest, we obtain

$$\sigma(\pi^-, \gamma) = \frac{\pi g^2 e^2}{\mu(\mu + \nu)} \sqrt{\frac{\mu(\mu + \nu) - 2M^2}{\mu(\mu + \nu)}} \quad (4'')$$

where $\nu = (\hat{\epsilon}/\mu)(\epsilon + k)$ is the energy of the photon in this reference system.

The spectrum of the photons in the frequency region $\nu \sim \nu + d\nu$ of the equivalent radiation field of a proton moving with velocity v is given by^{(9), (10)}

$$N(\nu) d\nu = e^2 (2/\pi) (d\nu/\nu) (1/v^2) \ln(vM\xi/\nu), \quad (13)$$

where $\hat{\epsilon} = (1 - v^2)^{-1/2}$. This expression holds if the condition $M\xi \gg \nu$ is satisfied. We can put $v \sim 1$ in the energy region considered here.

We obtain following expression for the cross section for the production of a negative proton by a collision between a negative meson with the energy $W = \mu\xi$ and a proton at rest through the intermediary of the electromagnetic field:

$$\sigma^{(2)}(\pi^-, P) = \int_{\nu_0}^{\nu_m} \sigma(\pi^-, \gamma) N(\nu) d\nu,$$

$$\nu_m \sim M\xi, \quad \nu_0 = (2M/\mu)(M + \mu) \quad (\text{cf. Table 1}),$$

$$\sigma^{(2)}(\pi^-, P) = (2g^2\epsilon^4/3M^2)L = 1.5 \times 10^{-32}g^2L \text{ cm}^2,$$

$$L = \frac{1}{2} \ln \left\{ \frac{W}{M} \left(\sqrt{1 - \frac{2M}{W}} - 1 \right) - 1 \right\} + \left(\frac{1}{2} - \frac{\mu}{M} \right) \sqrt{\frac{\mu}{M}} \ln \frac{W}{2(M+\mu)} - 1 \left\{ \right. \\ \left. - \frac{2}{3} \left(\frac{4M}{W} + 1 \right) \sqrt{1 - \frac{2M}{W}} + \sqrt{\frac{\mu}{M}} \left(\frac{3}{2} - \frac{1}{3} \frac{\mu}{M} \right) - \frac{1}{2} \ln \left\{ \left(1 + \sqrt{\frac{\mu}{M}} \right) / \left(1 - \sqrt{\frac{\mu}{M}} \right) \right\} \right\}. \quad (14)$$

Values of $\sigma^{(2)}(\pi^-, P)$ for the various incident energies are given in Table II, §5.

The cross section for the process $\pi^0 + P \rightarrow P^- + P + P$ through the intermediary of electromagnetic field is much smaller than that given by eq. (14) by the effects of interference as was stated in the case (i).

(v) Nucleon-Nucleon Collisions

Since in the case of production of negative protons by proton-proton collisions only the neutral meson field contributes in the lowest order of the perturbation theory, the cross section is about 10^{-4} times smaller than that by proton-neutron collisions according to equation (7.2). So we treat the case of proton-neutron collisions only in this section.

Process (f_1)

The cross section for the production of a negative proton by a meson-proton collision, eq. (12), varies only slightly with the energy, and can be written as follows (cf. Table II, §5):

$$\sigma^{(1)}(\pi^-, P) \approx \frac{10^2}{2\sqrt{2}} \frac{g^6}{M\mu} \sqrt{\frac{\mu}{M}} \left(\frac{1}{8} \frac{\mu^2}{M^2} \right)^a \times 0.07 \\ = 3.1 g^6 \left(\frac{1}{8} \frac{\mu^2}{M^2} \right)^a \times 10^{-27} \text{ cm}^2 \text{ (charged theory)}. \quad (12'')$$

Then the cross section for the process (f_1) of Fig. 1 is given by

$$\sigma^{(1)}(P, N) = \int_{\epsilon_0}^{\epsilon_m} \sigma^{(1)}(\pi^-, P) q(\epsilon) d\epsilon,$$

where $\epsilon_m \sim M\zeta = W$, $\epsilon_0 = 2(2M + \mu)$ (cf. Table I), W is the energy of an incident nucleon in the laboratory system;

$$\sigma^{(1)}(P, N) \approx \frac{7 \times 10^2}{2\sqrt{2}\pi} \frac{g^8}{M\mu} \sqrt{\frac{\mu}{M}} Q = 9.3 \times g^8 \times 10^{-26} Q \text{ cm}^2, \\ Q = \begin{cases} 1 - \sqrt{\frac{2(2M+\mu)}{W}}, & \text{for } a=0, \\ \frac{1}{8} \left(\frac{\mu}{M} \right)^2 \left\{ \sqrt{\frac{W}{2(2M+\mu)}} - 1 \right\} \frac{\mu}{W}, & \text{for } a=1. \end{cases} \quad (15)$$

The results obtained depend very sensitively on the value of g^2 , and besides the result for $a=1$ is smaller by a factor of about 10^{-4} than that for $a=0$, in the limit of very high energy the former approaches to zero and the latter to a finite

Table II. Cross section for the production of a negative proton

Reaction	γ -nucleon			Meson-nucleon			$P-N$				$P-P$
	(d_1)	(d_2)		(e_1)	(e_2)		(f_1)		(f_2)		(f_3)
Order	e^6	g^4e^2		g^6		g^2e^4	g^8		g^4e^4		e^8
		$a=0$	$a=1$	$a=0$	$a=1$		$a=0$	$a=1$	$a=0$	$a=1$	
Unit											
Energy	10^{-30}	$10^{-29} \cdot g^4$	$10^{-30} \cdot g^4$	$10^{-27} \cdot g^6$	$10^{-29} \cdot g^6$	$10^{-32} \cdot g^2$	$10^{-26} \cdot g^8$	$10^{-30} \cdot g^8$	$10^{-30} \cdot g^4$	$10^{-31} \cdot g^4$	10^{-38}
$\frac{W}{M}=4.4$	2.4	1.8	1.8	1.9	0.56	0.18	—	—	—	—	—
6	2.7	2.1	1.9	2.5	0.75	0.60	1.6	1.8	0.08	0.13	0.74
8	3.1	3.1	1.8	3.9	1.2	1.3	2.7	2.5	0.81	1.3	1.0
10	3.2	3.1	1.8	3.9	1.2	1.5	2.7	2.2	1.0	1.8	1.1
14	3.7	2.7	1.6	3.6	1.1	1.8	4.1	2.8	1.4	2.1	1.2
16	3.8	2.7	1.6	3.4	1.0	2.0	4.1	2.7	1.6	2.9	1.2
20	4.1	2.4	1.4	3.1	0.94	2.1	4.9	2.8	1.8	2.6	1.3
30	4.5	1.9	1.4	2.3	0.70	1.8	5.6	2.6	2.8	4.3	1.4
Equation	(8)	(10)		(12)	(12')	(14)	(15)		(16)		(17)

maximum value. The cross section in the symmetrical theory is about five times the one given by eq. (15) according to (7.3a).

Process (f_2)

As in the case of (f_1) we can put

$$\sigma^{(2)}(\gamma, N) \approx \frac{10^2}{2\sqrt{2}} \frac{g^4 e^2}{M^2} \left(\frac{\mu}{M}\right)^a \times 0.21 = 2.5 \times \left(\frac{\mu}{M}\right)^a g^4 \times 10^{-29} \text{ cm}^2. \quad (10')$$

Then the cross section for the process (f_2) of Fig. 1 is given by

$$\sigma^{(2)}(P, N) = \int_{\nu_0}^{\nu_m} \sigma^{(2)}(\gamma, N) N(\nu) d\nu,$$

where $\nu_0 = 4M$, and W is the energy of an incident nucleon in the laboratory system ;

$$\sigma^{(2)}(P, N) \sim \frac{11}{\sqrt{2}\pi} \frac{g^4 e^2}{M^2} \left(\frac{\mu}{M}\right)^a \left(\ln \frac{W}{4M}\right)^2 = 7.5 g^4 \cdot 10^{-31} \left(\frac{\mu}{M}\right)^a \left(\ln \frac{W}{4M}\right)^2. \quad (16)$$

Process (f_3)

The cross section for the production of a negative proton by a proton-proton collision through the intermediary of electromagnetic field alone can be obtained from that for the production of an electron pair by an electron-electron collision:^{(10), (19)}

$$\sigma^{(2)}(P, P) = \frac{28}{27\pi} \frac{e^8}{M^2} \left(\ln \frac{W}{M}\right)^3 = 4.3 \times 10^{-37} \left(\ln \frac{W}{M}\right)^3 \text{ cm}^2. \quad (17)$$

(vi) Production of Negative Protons by Proton-Proton Collisions near Threshold

If we consider the case of the experiment on the production of negative

protons by a bevatron, which is hoped to be constructed in the near future, it will be of primary importance to analyze the one by proton-proton collisions for the sake of its physical simplicity. Since the calculations in (v) do not hold good in this case, we will estimate the cross section for the process (f_1) of Fig. 1

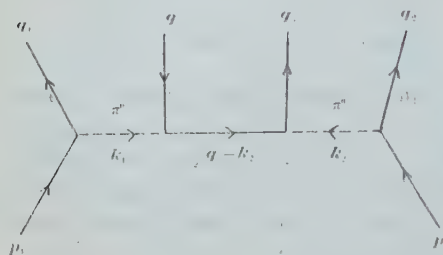


Fig. 4

using the Feynman's method. The Born approximation is not a good approximation if the kinetic energies of the nucleons in the final state is not large compared with the potential energies, but such an energy region is small in comparison with that considered here. So we evaluate the cross section using the Born approximation.

In the lowest order approximation of perturbation theory, only the neutral meson field can contribute to the production of negative protons by proton-proton collisions. In this approximation neither negative protons nor antineutrons can be produced by proton-proton collision in the charged meson theory.

The results obtained are the same whether the coupling of the neutral meson to the nucleons has the same sign for neutron and proton or the opposite sign. There are other Feynman diagrams besides Fig. 4 obtained by interchanging q_1 , q_2 , q_3 and q from it.* In the *c.m.*-system the differential cross section is

$$d\sigma = \frac{4\pi^2}{2pE} \frac{dq_1}{E(q_1)} \frac{dq_2}{E(q_2)} \frac{dq_3}{E(q_3)} \frac{dq}{E(q)} \delta_p \delta_E \frac{1}{2^2} \sum_{\text{spin}} |I|^2,$$

where p and E denote the three-dimensional momentum and energy of each of the approaching protons, δ_p and δ_E are the δ -functions representing the conservation of momentum and energy respectively, and $E(q) = (M^2 + q^2)^{1/2}$ etc.

$$I = (-i/2) (2\pi)^{-3} M^3 [\bar{u}(q_1) \gamma_5 \tau_4 u(p_1)] [\bar{u}(q_2) \gamma_5 \tau_3 u(p_2)] \cdot \\ \cdot [\bar{u}(q_3) \gamma_5 \tau_j (q_3 - k_2 - M)^{-1} \gamma_5 \tau_i u(-q)] \cdot (k_1^2 - \mu^2)^{-1} (k_2^2 - \mu^2)^{-1} \\ + (\text{Interchange of } q_1, q_2, q_3 \text{ and } q).$$

Considering the magnitudes of q_1 , q_2 , q_3 and q to be small compared with M , we obtain

$$\sum_{\text{spin}} |I|^2 \sim 2^3 \cdot 9 \cdot g^8 (4\pi)^{-6} \{M^2(p-M)^2\}^{-1}.$$

Since main contributions to the total cross sections come from the parts of momentum space where the magnitudes of momenta of final negative proton and protons are comparable, we can evaluate the total cross section letting them nearly equal:

* In reality there are diagrams other than that mentioned above. But we have omitted them for the sake of comparison with the preceding calculations using the Weizsäcker-Williams method.

$$\sigma \sim (3/2^3 \cdot 10\pi) (g^8/M^2) (\epsilon/M)^4 \{1 - (\epsilon/M)\} \approx 5.2 \times 10^{-30} g^8 (\epsilon/M)^4 \text{ cm}^2,$$

where $\epsilon = 2E - 4M = \sqrt{2MIV} - 4M$ denotes the sum of kinetic energies in the final state in the *c.m.*-system, and IV is the incident energy in the laboratory system.

§ 5. Conclusions

Cross sections calculated in the preceding section are summarized in Table II for various incident energies IV . The cross section for the production of negative protons by proton-proton collisions through the intermediary of meson field alone is about 10^{-4} times smaller than the value (f_1) in Table II. In the case of collisions with nuclei other than hydrogen, the value of (d_1) , (e_2) , (f_2) and (f_3) must be multiplied by Z^2 . The effect of screening for the electric field of a nucleus is very small in the energy regions considered here. The cross sections for the case (f_1) , which are the largest among those by nucleon-nucleon collisions, vary as much as a factor of about 10^4 when a varies from zero to one. So we might be able to examine whether or not the value of a , consistent with that obtained in the case of meson production, would be obtained by the analysis of the experimental results on the negative protons in the cosmic radiation.

In order to detect anti-nucleons, we will have to investigate the phenomena accompanying with the annihilation of anti-nucleons. These have been discussed by Ashkin et al.¹⁴⁾ In conclusion we express our cordial thanks to Prof. S. Nakamura for his kind interest on this work.

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Photo-Meson Production from Deuteron

Shigeru MACHIDA and Taro TAMURA

Department of Physics, University of Tokyo

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Photo-meson production from deuteron is treated phenomenologically, using the "impulse" approximation. It is found that the distortion of the nuclear wave function is essential and that its effect is taken into account adequately by "shape independent approximation". Production of negative mesons is diminished by the effects of Coulomb interaction between two protons in the final state.

§ 1. Introduction

The experiment of photo-meson production is one of the most powerful methods to clarify the features of the interaction among nucleons, mesons, and electromagnetic fields. But in the early stage of this experiment,¹⁾ carbon was used as target, so the true nature of photo-meson production from nucleon were masked by the effect of nuclear binding on nucleon. In recent experiments by Berkeley group,²⁾ liquid hydrogen was used as target, and many important features of electromagnetic nature of proton were revealed. For the photo-production of negative mesons we cannot use free neutrons as target. Since the deuteron is the simplest nucleus, which contains neutron, the photo-meson production from it is the most direct method for this purpose. The analysis of this experiment may also yield some light on the high frequency Fourier component of the deuteron ground state, and of the exchange current in the deuteron.

Now, as the energies of the participating nucleons are not high, in our case, the distortion of the nucleon wave function will be most essential and the detailed nature of the meson will be of minor importance.³⁾ So we have treated the photo-meson production from deuteron phenomenologically, leaving at first the matrix elements for photo-production of mesons from free nucleon undetermined, and lately determining it semi-empirically. Furthermore, it is found sufficient to use the method of distorted waves with the shape independent approximation.⁴⁾

The main part of our calculation follows the so-called "impulse approximation" formulated by Chew⁵⁾ for the analysis of n - d scattering. Though the validity of this approximation is not established yet, the wave length of γ -rays at 330 MeV ($\approx 6 \times 10^{-14}$ cm) is smaller by one order of magnitude than the extension of deuteron ($\approx 1/a \approx 4 \times 10^{-13}$ cm), so it may be justified to consider that one nucleon in the deuteron does not suffer any disturbance, when the other is bombarded by γ -ray, and produces a meson, and to consider that the effect of the latter to the

former is only to bind the former so as to give the momentum distribution corresponding to the deuteron ground state, before bombardment, and to restrict the possible final two nucleon states according to the Pauli principle. On the other hand, if our approximation were justified, this method would fit to our purpose, since the π -production from deuteron reflects directly the feature of π -production from free neutron.

§ 2. Use of impulse approximation

In this section we treat only π^+ -production. The calculation for π^- -production is very similar. The matrix element of the present reaction can be written as follows:

$$(\xi_1, \xi_2 | R_d | \xi_p^0, \xi_n^0) = \sum_{\xi_1', \xi_2'} \sum_{\xi_p^0, \xi_n^0} G_{\xi_1, \xi_2}(\xi_1', \xi_2') (\xi_1' | R_p | \xi_p^0) \delta_{\xi_2', \xi_n^0} G_D(\xi_p^0, \xi_n^0) \quad (1)$$

here $G_{\xi_1 \xi_2}$ is the Fourier transform of the two neutron continuum state wave function, and G_D is the one for the deuteron ground state wave function. ξ 's mean the momentum of nucleons in their respective states, as well as their spins. In equation (1), as is clearly seen, already the impulse approximation is used, because the R_p , which means the matrix element of positive photomeson production from free proton, whose initial and final momentum and spin are characterized by ξ_p^0 and ξ_1' respectively, is separated out. δ_{ξ_2', ξ_n^0} shows the fact that the neutron in the deuteron does not suffer any interaction during the reaction.

If we forsake Pauli principle and spin for a moment and factor out the δ -function which represents the momentum conservation, R_p is written as

$$(\mathbf{k}_1' | R_p | \mathbf{k}_p^0) = \delta(\mu + \mathbf{n}_1' - \mathbf{p}_0 - \nu) (\mathbf{k}' | r_p | \mathbf{k}) \quad (2)$$

where

$$\mathbf{k} = \frac{1}{2}(\nu - \mathbf{p}_0), \quad \mathbf{k}' = \frac{1}{2}(\mu - \mathbf{n}_1) \quad (3)$$

and μ , \mathbf{n}_1 , \mathbf{p}_0 , ν are momentum of meson, final neutron, initial proton and photon, respectively. Substituting (2) into (1), we obtain

$$(\mathbf{k}_1, \mathbf{k}_2 | R_d | \mathbf{k}_p^0, \mathbf{k}_n^0) = \sum_{\mathbf{k}_1', \mathbf{k}_p^0} \sum_{\mathbf{k}_n^0} G_{\mathbf{k}_1, \mathbf{k}_2}(\mathbf{k}_1', \mathbf{k}_n^0) \delta(\mu + \mathbf{n}_1' - \mathbf{p}_0 - \nu) (\mathbf{k}' | r_p | \mathbf{k}) G_D(\mathbf{k}_p^0, \mathbf{k}_n^0). \quad (4)$$

If we take for $G_{\mathbf{k}_1, \mathbf{k}_2}$ a plane wave, then

$$\sum_{\mathbf{k}_1'} G_{\mathbf{k}_1, \mathbf{k}_2}(\mathbf{k}_1', \mathbf{k}_n^0) \delta(\mu + \mathbf{n}_1 - \mathbf{p}_0 - \nu) = \delta(\mathbf{p}_0 + \nu - \mu - \mathbf{n}_1) \delta(\mathbf{n}_2 - \mathbf{n}_0). \quad (5)$$

Therefore (4) becomes

$$(\mathbf{k}_1, \mathbf{k}_2 | R_d | \mathbf{k}_p^0, \mathbf{k}_n^0) = \sum_{\mathbf{p}_0, \mathbf{n}_0} \delta(\mathbf{p}_0 + \nu - \mu - \mathbf{n}_1) \delta(\mathbf{n}_1 - \mathbf{n}_0) (\mathbf{k}' | r_p | \mathbf{k}) G_D(\mathbf{p}_0, \mathbf{n}_0) \quad (6)$$

$$= \left(\frac{\mu - \mathbf{n}_1}{2} | r_p | \frac{2\nu - \mu - \mathbf{n}_1}{2} \right) G_D(-\nu + \mu + \mathbf{n}_1, \mathbf{n}_2) \quad (7)$$

So in this case, free proton photo-meson production matrix element can be taken out of integral sign, without making any further assumption.

If we take for G_{k_1, k_2} the continuum 1S -state wave function, and take into account Pauli principle and spin, then it is clear that the final spin state must be antisymmetric between two neutrons, so the formula corresponding to (7) is written as

$$\begin{aligned}
 & (\xi_1, \xi_2 | R_d | \xi_p^0, \xi_n^0) \\
 &= \frac{1}{\sqrt{2}} \sum_{n_2', p_0} \left[\left(\frac{\mu - n_1}{2} | r_p | \frac{\nu - p_0}{2} \right) G_{n_1, n_2} (p_0 + \nu - \mu, n_2') G_D (n_1 + \mu - \nu, n_2') \right. \\
 & \quad \left. + \left(\frac{\mu - n_2'}{2} | r_p | \frac{\nu - p_0}{2} \right) G_{n_2, n_1} (p_0 + \nu - \mu, n_2') G_D (n_2 + \mu - \nu, n_2) \right]. \quad (8)
 \end{aligned}$$

In (8)

$$\begin{cases} G_D (n_1 + \mu - \nu, n_2) = G_D (p_0, n_2') = \delta(p_0 + n_2') g_D \left(\frac{p_0 - n_2'}{2} \right), \\ G_{n_1, n_2} (p_0 + \nu - \mu, n_2') = \delta(p_0 + \nu - \mu + n_2' - n_1 - n_2) g_{n_1 - n_2} \left(\frac{p_0 + \nu - \mu - n_2'}{2} \right) \end{cases}$$

where g_D and $g_{n_1 - n_2}$ mean momentum wave function corresponding to deuteron ground state and two neutron continuum state in the center of gravity system respectively. Therefore

$$\begin{aligned}
 \sum_{n_2', p_0} G_{n_1, n_2} (p_0 + \nu - \mu, n_2') G_D (p_0, n_2') &= \int d n_2' \int d p_0 G_{n_1, n_2} (p_0 + \nu - \mu, n_2') G_D (p_0, n_2') \\
 &= \delta(\nu - \mu - n_1 - n_2) \int d p_0 g_{n_1 - n_2} \left(\frac{2p_0 + \nu - \mu}{2} \right) g_D (p_0). \quad (10)
 \end{aligned}$$

Then (8) turns out to be

$$\begin{aligned}
 & (\xi_1, \xi_2 | R_d | \xi_p^0, \xi_n^0) \\
 & \approx \frac{1}{\sqrt{2}} \left[\left(\frac{\mu - n_1}{2} | r_p | \frac{\nu - p_0}{2} \right) \delta(\nu - \mu - n_1 - n_2) \int d p_0 g_{n_1 - n_2} \left(\frac{2p_0 + \nu - \mu}{2} \right) g_D (p_0) \right. \\
 & \quad \left. + \left(\frac{\mu - n_2}{2} | r_p | \frac{\nu - p_0}{2} \right) \delta(\nu - \mu - n_1 - n_2) \int d p_0 g_{n_2 - n_1} \left(\frac{2p_0 + \nu - \mu}{2} \right) g_D (p_0) \right]. \quad (11)
 \end{aligned}$$

In (11) we used another approximation to take r_p out of the integral sign on p_0 . Such could be done exactly, when plane wave was assumed for final two neutrons' wave function as was shown above. Also in the present case such a procedure will be justified because p_0 is very much smaller than ν , and as was pointed out by Lax and Feshbach⁽⁶⁾ and by Brueckner⁽⁷⁾ r_p depends little on the momentum of nucleon. Another reason to justify this procedure is the fact, that, if we assume scalar, pseudoscalar, or vector meson theory and calculate the matrix element using Born approximation and treating nucleons non-relativistically, then r_p does

not depend on the momenta of nucleons.^{(6),(7)} Considering the latter fact it will be also justified to put

$$\left(\frac{\mu - \mathbf{n}_1}{2} |r_p| \frac{\nu - \mathbf{p}_0}{2}\right) \approx \left(\frac{\mu - \mathbf{n}_2}{2} |r_p| \frac{\nu - \mathbf{p}_0}{2}\right).$$

Therefore from (11) we finally obtain for R_d^2

$$|(\xi_1, \xi_2 | R_d | \xi_p^*, \xi_n^*)|^2 = \frac{1}{2} \delta(\nu - \mu - \mathbf{n}_1 - \mathbf{n}_2) \left| \left(\frac{\mu - \mathbf{n}_1}{2} |r_p| \frac{\nu - \mathbf{p}_0}{2} \right)^2 (I_n + I_{-n})^2 \right| \quad (12)$$

where

$$I_n = \int g_n \left(\frac{2\mathbf{p}_0 + \nu - \mu}{2} \right) g_D(\mathbf{p}_0) d\mathbf{p}_0 = \int d\mathbf{r} \varphi_n(\mathbf{r}) \varphi_D(\mathbf{r}) e^{-i \frac{\nu - \mu}{2} \mathbf{r}} \quad (13)$$

and $\mathbf{n} = (\mathbf{n}_1 - \mathbf{n}_2)/2$.

§ 3. Calculation of the cross section

The cross section is written as (in the following we use the unit $\hbar=c=1$)

$$d\sigma = 2\pi |R_d|^2 \delta(E_\nu - E_1 - E_2 - E_\mu - \varepsilon) \rho_F \quad (14)$$

where E_1, E_2, E_μ, E_ν are energies of final two neutrons, meson and photon respectively, and ε is the binding energy of deuteron, and $\rho_F = (2\pi)^{-9} d\mathbf{n}_1 d\mathbf{n}_2 d\mu$. Then from (12)

$$d\sigma = \frac{2\pi}{2} |r_p|^2 \frac{d\mu}{(2\pi)^3} \iint (I_n + I_{-n})^2 \delta(\nu - \mu - \mathbf{n}_1 - \mathbf{n}_2) \delta(E_1 + E_2 + E_\mu - E_\nu + \varepsilon) \frac{d\mathbf{n}_1 d\mathbf{n}_2}{(2\pi)^6}. \quad (15)$$

If we take for φ_D the Hulthen's wave function and restrict φ_n only to S-state, then (13) is written

$$I_n = 4\pi A \int_0^\infty r^2 dr \frac{\sin lr}{lr} \cdot \frac{\sin(nr + \delta_0) - \exp(-\eta r) \sin \delta_0}{nr} \cdot \frac{e^{-ar} - e^{-\beta r}}{r}, \quad (16)$$

where $A = \sqrt{\frac{a\beta(a+\beta)}{2\pi(a-\beta)^2}}$ and $l = |\nu - \mu|/2$. The term $\exp(-\eta r) \sin \delta_0$ in S-state wave function was introduced for the correction at the origin, but the contribution of this term can be shown to be small, and we neglect it. Then Eq. (16) and similar one for I_{-n} give

$$I_n + I_{-n} = \frac{2\pi A}{nl} \left\{ \cos \delta_0 \log \left(\frac{\beta^2 + (l-n)^2}{a^2 + (l-n)^2} \cdot \frac{a^2 + (l+n)^2}{\beta^2 + (l+n)^2} \right) + 2 \left(\tan^{-1} \left(\frac{2al}{a^2 + n^2 - l^2} \right) - \tan^{-1} \left(\frac{2\beta l}{\beta^2 + n^2 - l^2} \right) \right) \right\}. \quad (17)$$

Substituting (17) into (15), and changing variables from \mathbf{n}_1 and \mathbf{n}_2 to $\mathbf{N} = \mathbf{n} + \mathbf{n}_2$, and $\mathbf{n} = (\mathbf{n}_1 - \mathbf{n}_2)/2$ and performing integration on \mathbf{N} , we obtain

$$d\sigma_a = \frac{2\pi}{2.3} |r_p|^2 (2\pi A)^2 \frac{d\mu}{(2\pi)^3} \int \frac{1}{n^2 l^2} \left\{ \cos^2 \delta_0 \log \left(\frac{\beta^2 + (l-n)^2}{\alpha^2 + (l-n)^2} \cdot \frac{\alpha^2 + (l+n)^2}{\beta^2 + (l+n)^2} \right) \right. \\ \left. + 2 \sin \delta_0 \left(\tan^{-1} \left(\frac{2\alpha l}{\alpha^2 + n^2 - l^2} \right) - \tan^{-1} \left(\frac{2\beta l}{\beta^2 + n^2 - l^2} \right) \right) \right\} \frac{4\pi n^2 dn}{(2\pi)^3} \delta \left(\frac{n^2}{M} + \frac{l^2}{M} + \mu_0 - \nu + \epsilon \right). \quad (18)$$

In (18) we treated nucleons non-relativistically. If we assume shape independent approximation,

$$n \cos \delta_0 = -\frac{1}{a_s} + \frac{1}{2} n^2 r_0 \quad (19)$$

where a_s , and r_0^* are 1S scattering length and effective range respectively. Using well-known relation $\int f(n) \delta(E) dn = (f(n) (dn/dE))_{E=0}$, we finally obtain as cross section

$$d\sigma_a = \frac{2\pi}{3} |r_p|^2 A^2 \frac{\mu \mu_0 d\mu_0 d\Omega}{(2\pi)^3} \cdot J \quad (20)$$

where

$$J = \left[\frac{\left\{ \left(-\frac{1}{a_s} + \frac{1}{2} n^2 r_0 \right) \log \left(\frac{\beta^2 + (l-n)^2}{\alpha^2 + (l-n)^2} \cdot \frac{\alpha^2 + (l+n)^2}{\beta^2 + (l+n)^2} \right) \right.}{l^2 \left[n^2 + \left(-\frac{1}{a_s} + \frac{1}{2} n^2 r_0 \right)^2 \right]} \right. \\ \left. + 2n \left(\tan^{-1} \left(\frac{2\alpha l}{\alpha^2 + n^2 - l^2} \right) - \tan^{-1} \left(\frac{2\beta l}{\beta^2 + n^2 - l^2} \right) \right) \right]^2 \cdot \frac{M}{2n} \Bigg|_{n=\sqrt{M(\nu-\mu_0-\epsilon)-l^2}} \quad (20a)$$

and M is the mass of nucleon. In the following calculations, we put

$$|r_p|^2 = C/\mu_0 \nu \quad (20b)$$

where C is a constant. This assumption may be justified by the fact, that the results derived using (20b) was shown to agree well with experiments.^{2),6) 7)}

§ 4. Analysis of the results and comparison with experiments

(i) Energy spectrum of meson, when bombarded by 340 MeV monochromatic γ -ray

The energy spectrum at 90 degree from the direction of incident γ -ray, calculated from (20) assuming γ -ray to be a monochromatic beam of 340 MeV, is shown in Fig. 1. There appears a rather broad peak at high energy end. This peak is the result, that, when meson energy is high, n becomes very small and the denominator of (20) decreases rapidly. This is the effect of the well known

* We took $\alpha = 0.231 \times 10^{13} \text{ cm}^{-1}$, $\beta = 1.410 \times 10^{13} \text{ cm}^{-1}$, $1/a_s = -0.0422 \times 10^{13} \text{ cm}^{-1}$, and $r_0 = 2.6 \times 10^{-13} \text{ cm}$.

resonance at 1S virtual level. The lump at the low energy side of this peak comes from the maximum of the numerator which occurs when the logarithmic term takes its largest value, i.e. when $l \approx n$. The 1S resonance peak is more marked at zero degree, when very sharp peak appears at high energy end as is shown in Fig. 2. This peak comes mainly from the denominator of (20), because in this energy region the logarithmic factor does not vary appreciably. This fact, that the effect of 1S resonance is more pronounced at smaller angles, is the same

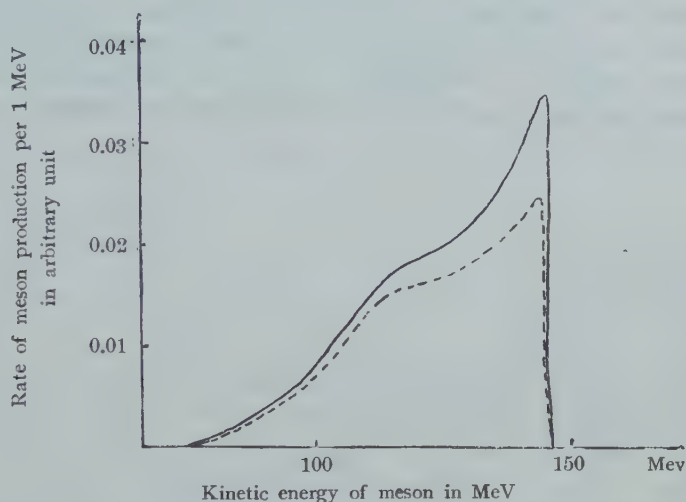


Fig. 1. Energy spectra of mesons produced at 90° by monochromatic 340 MeV γ -ray, — π^+ , - - - π^- .

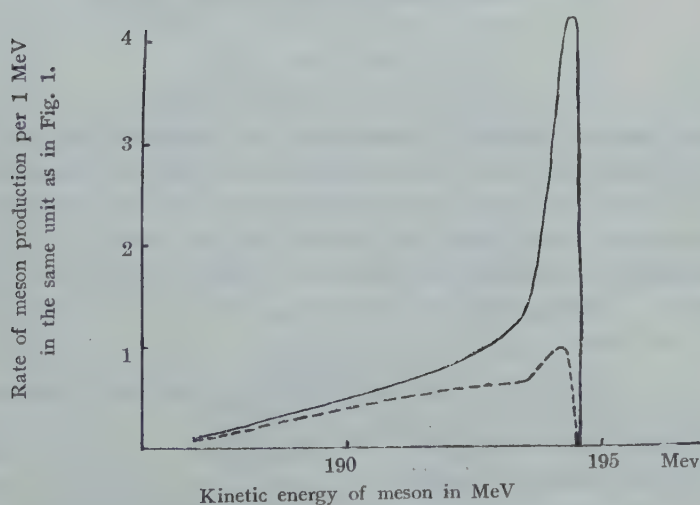


Fig. 2. Energy spectra of mesons produced at 0° by monochromatic 340 MeV γ -ray. — π^+ , - - - π^- .

as the one, which was pointed out by Gluckstern and Bethe,⁸⁾ in the analysis of n - d scattering.

In the case of the production of negative meson, the effect of Coulomb interaction between final two protons reduces the efficiency by an factor of $2\pi\eta/(\exp(2\pi\eta)-1)$ where $\eta=e^2M/n$. The dotted lines in Fig. 1 and 2 show the energy spectra of π^- produced by 340 MeV γ -ray.

(ii) Energy spectrum of mesons produced by 340 MeV X -ray

To compare with experiments it would be necessary to calculate the energy spectrum of meson produced by X -ray having $d\nu/\nu$ spectrum. The result at 90 degree is shown in Fig. 3. The shift of the peak to the low energy end is common to $d\nu/\nu$ spectrum, and does not so much depend on the detailed aspects of Fig. 1.

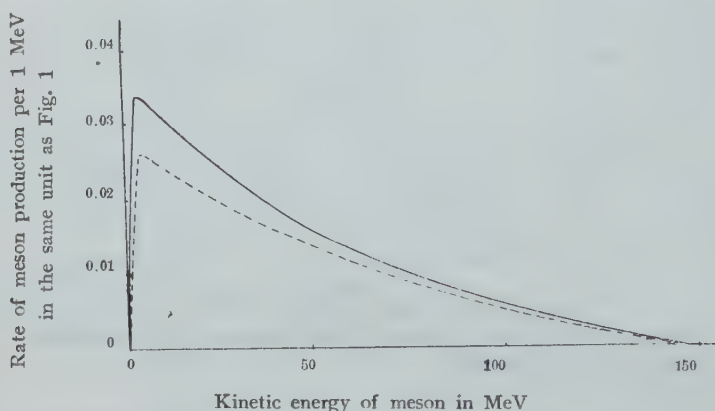


Fig. 3. Energy spectra of mesons produced at 90° by 340 MeV X -ray. — π^+ , - - - π^- .

(iii) Angular Distributions

In spite of the use of $|r_p|^2 = C/\mu_0\nu$, which is isotropic in angular distribution, the one in our case shows very sharp forward concentration, as is seen from the Fig. 1 and Fig. 2. This results mainly from the l^2 in the denominator of (20a), which becomes very small when the angle between photon and meson is very small. This feature is special for the case of photo-meson production from two nucleon system. This is also the case even if two nucleons are assumed to be free in their final state.

(iv) Yield ratio of mesons from free proton and deuteron

The cross section from free proton at rest is

$$d\sigma_p = d\mu(2\pi)^{-2} |r_p|^2 I_p, \quad (21)$$

$$I_p = \int d\mathbf{n} \delta(\mathbf{n} + \boldsymbol{\mu} - \boldsymbol{\nu}) \delta(n_0 + \mu_0 - \nu_0 - M). \quad (22)$$

Equation (21) can be rewritten as

$$d\sigma_p = \frac{\mu\mu_0 d\mu_0 d\Omega}{(2\pi)^2} |r_p|^2 \frac{M+\nu-\mu_0}{\mu\nu} \delta\left(\cos\theta - \frac{\mu_0}{\mu} + \frac{x^2}{2\nu\mu} + \frac{M(\nu-\mu_0)}{\nu\mu}\right), \quad (23)$$

where θ is the angle between μ and ν , and x is the mass of meson. This must be compared with (20), integrated with $d\nu/\nu$ spectrum. In $d\sigma_p$, the ν , which produces meson with definite energy and angle, is determined monochromatically by conservation law, i.e. the δ -function in (23). But, because of the deficiency of the resolving power in experiments of μ_0 , the effective ν has a width $\Delta\nu$. $\Delta\nu$ is determined by $\Delta\mu_0$ according to the conservation law as

$$\Delta\nu = \Delta\mu_0 (M + \nu + (\Delta\mu_0/\mu) \cos\theta) (M - \mu_0 - \mu \cos\theta)^{-1}. \quad (24)$$

The yield ratio $Y(\theta, \mu_0)$ now is written, if we put $|r_p|^2 = C/\mu_0\nu$ as above,

$$Y(\theta, \mu_0) = \frac{Y_a(\theta, \mu_0)}{Y_p(\theta, \mu_0)} = \frac{A^2}{2} \int_{\Delta\Omega} \int_{\Delta\mu_0} \int_{\nu^2}^{\mu} d\nu d\mu_0 d\Omega \bigg/ \int_{\Delta\Omega} \int_{\Delta\mu_0} \int_{\Delta\nu} \frac{M+\nu-\mu_0}{\nu^3} \delta\left(\cos\theta - \frac{\mu_0}{\mu} + \frac{x^2}{2\nu\mu} + \frac{M(\nu-\mu_0)}{\nu\mu}\right) d\nu d\mu_0 d\Omega,$$

where $\Delta\Omega$ is the angle interval, which is also determined by resolving power. This ratio gives at 135° , with $\mu_0 - x = 50$ MeV, when $\Delta\mu_0$ is assumed to be about 6 MeV,

$$Y(135^\circ, 50 \text{ MeV}) = 1/1.23. \quad (26)$$

Preliminary experimental ratio is⁹⁾

$$Y_{ex}(135^\circ, 50 \text{ MeV}) = (1.00 \pm 0.14)/(1.20 \pm 0.16). \quad (27)$$

The agreement between them is rather fair.

(v) Plus-minus ratio

There are strong evidences which show that the magnetic moment interaction is predominant in the photo-meson production, i.e. the angular distribution of mesons from hydrogen target is nearly flat, and the plus-minus ratio of mesons from carbon target is almost independent of its energy and angle, in spite of the obvious dependence of meson energy spectrum on angle.^{2), 12)} If we approve that this is the case, the plus-minus ratio of π^+ and π^- from free protons and neutrons at rest, respectively, is given, as was shown by Brueckner⁷⁾

$$\sigma(+)/\sigma(-) = [1 - 0.20(\mu_0/M)(1 - (v/c) \cos\theta)]^2. \quad (28)$$

If we put $\mu_0 - x = 50$ MeV ($\mu_0 = 193$ MeV) and $\theta = 135^\circ$, this ratio is 1/1.1.

As is clear from our formulation and from the fact that effective velocities of nucleons in deuteron is small compared to that of light, the plus-minus ratio from deuteron is exactly equal to that of free protons and neutrons at rest, i.e. equation (28), except for the fact that the production of π^- is reduced by the

impenetrability of Coulomb potential. This reduction, which is calculated as in (iii), amounts to about 0.81. Therefore the plus-minus ratio from deuteron becomes 0.94. The result of preliminary experiment is⁹⁾ $(1.00 \pm 0.14)/(1.00 \pm 0.12)$.

The agreement is not so good, but only 0.2% contribution from electric interaction is sufficient to get the experimental value.

§ 5. Concluding remarks

We carried out our calculation using the "impulse approximation". This approximation was used previously by Fermi¹⁰⁾ to the scattering of slow neutron by bound proton, and recently its usefulness to the analysis of n - d inelastic scattering was stressed by Chew.³⁾ Comparing the results of Fermi with those calculated exactly, Breit¹¹⁾ estimated the error brought into the results because of this approximation and show that it would amounts to the order of the ratio of the range of nuclear force to the extension of proton wave function. In our case this becomes $\sim a/\lambda$, i.e. about 10%.

So far there are only two experimental data concerning photo-meson production from deuteron, i.e. the yield ratio from proton and deuteron, and the plus-minus ratio. Agreement between theory and experiment are fairly well, though the experimental results are only preliminary ones.

We had also to consider the effect of exchange current, but take into account it is impossible if we use the impulse approximation, because it is essentially the two nucleon effect; and we did not include its effect in our calculation. However, its effect would be small at high energy reactions.

We wish to express our sincere thanks to Messrs. Y. Fujimoto and Y. Yamaguchi for their valuable discussions.

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On the Method of the Theory of Nuclear Forces

Mituo TAKETANI, *Nerima-Minamicho, Tokyo,*

Seitaro NAKAMURA, *Department of Physics, University of Tokyo,*
and

Munco SASAKI, *Tokyo Metropolitan University*

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From the standpoint of the meson theory of nuclear force, results of current analysis on phenomenological nuclear potentials are reexamined. A new method of the theory of nuclear forces is proposed. We employed meson potentials for the outside region, and phenomenological parameters for the inside region of the nuclear forces.

§ 1. Problem in the nuclear forces

The phenomenological treatments of the nuclear forces have long been investigated since last ten years, but no definite results were obtained yet. The main reason for it was the scarcity of experimental results to be fitted to and there remained diversity of possibilities. However, as the problem must be treated meson theoretically in its final stage, these possibilities, which do not suggest the connection to or contradict with it, must be put aside. Here we shall investigate some features which the present meson theory imposes to the problem of nuclear forces.

A. Nuclear force range

Recent developments have introduced two restrictions in the theory of nuclear forces. The one concerns with the force range and the other with the neutron-proton and the proton-proton scattering at high energy region.

The former restriction comes from the discovery of two mesons, of which π -meson interacts strongly with nucleons, while the μ -meson has nothing to do with the nuclear forces. The mass of π -meson is observed to be about $275 m_e$ (m_e being electron mass) and is heavier than that, assumed hitherto for nuclear forces. This fact indicates that the force range must be taken shorter, i.e., about 1.3×10^{-13} cm. Although this alteration introduces only small effects in the phenomena of neutron-proton system, the quadrupole moment of deuterons is an exception, because it behaves as the inverse square of meson mass and the heavier meson mass reduces its value considerably. To investigate the reduction, we made the calculations, with zero cut-off, assuming the general form of nuclear potential:

* Read in April and November, 1950.

$$V(x) \sim \left\{ a + b(\sigma_1 \cdot \sigma_2) + f S_{12} \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{3} \right) \right\} \frac{e^{-x}}{x},$$

where $x = \pi r$, $1/x$ being the Compton wave length for meson ; and

$$S_{12} = \left\{ \frac{3(\sigma_1 \cdot x)(\sigma_2 \cdot x)}{x^2} - \sigma_1 \cdot \sigma_2 \right\}.$$

The table shows that the quadrupole moment is still too small, unless we take unreasonably large cut-off radius. The best value of the quadrupole moment, 2.55×10^{-27} cm², is obtained for the coupling constants, $g = 1.111$ and $f = 0.982$; or to obtain the values in the usual form ($G^2/\hbar c$) we have only to multiply them by (μ/M) and they are 0.17 and 0.15 respectively. We observe, however, that the cut-off radius is somewhat too large. From this fact we can infer that single range force does not give the correct value of quadrupole moment.^{1),2)}

ordinary force $a+b=g$	tensor force f	quadrupole moment (in 10^{-27} cm ²)	cut-off radius (in 1.3×10^{-13} cm)
-0.307	0.491	1.086	0.32
0.307	0.645	2.097	0.37
1.111	0.982	2.55	0.69
1.228	1.474	2.92	0.82
1.842	1.965	3.34	1.10
2.456	2.456	3.95	1.28

Table I. The quadrupole moment of deuterons

While quadrupole moment depends strongly on the meson mass as mentioned above, it depends also on the shape of nuclear potential, though its amount is not so large. The shape of nuclear potential may be divided into two parts: (a) the "tail" and (b) the concentration of force near origin.³⁾ Here (b) contributes mainly to the interaction energy, while (a) serves to change the tail part of the wave function and consequently to increase the quadrupole moment. The $1/r^3$ term in the tensor force has the effect of (b) and reduces quadrupole moment. However, this term must not be omitted, although it has singularity at origin, for no meson theory gives the tensor force without $1/r^3$ term, and the omission of it should give considerable alteration.

Christian and Hart,⁴⁾ for instance, have employed the Yukawa type potential only both in central and tensor forces. This type of potential has long tail and considerably weak concentration, compared with the above case. As the result, they obtained fairly large value of quadrupole moment even for the range of 1.18×10^{-13} cm. But this tendency is special to Yukawa type potential and the situation is not so favourable for the square potential well, which has no tail. In this case, no sufficient amount of quadrupole moment can be obtained for the range shorter than 2.6×10^{-13} cm. Therefore, their results have nothing to suggest

the relation to the meson theory. In this substitution of Yukawa type potential the range had to be taken shorter.

B. High energy neutron-proton scattering

90-MeV neutron-proton scattering provides with another criticism, that is, the angular distribution. The scattering at this energy is symmetrical with respect to 90° and Serber proposed the potential which gives no contribution to the wave of odd parity, or meson theoretically the mixture of symmetrical (or charged) and neutral mesons in the same amounts. However, his proposal is too crude and it is possible only phenomenologically, for the mixture of mesons cannot give the exact cancellation of the potential for the odd parity wave. Of course, this cancellation may not be strictly needed. Indeed, the close examination shows that the angular distribution curve has the minimum at 80° or less and not at 90° .

However, the Serber potential imposes restriction to the meson theory of nuclear forces, in that the type of symmetrical (or charged) and neutral mesons must be different, because, otherwise, the cancellation will be too good and sometimes the potential for even parity wave will vanish at the same time. Therefore, the Serber potential can not be taken too seriously in the meson theory of nuclear forces.

§ 2. The proposition of the new method

1. After the π -meson was produced artificially, the experimental evidences concerning the meson theory have been accumulated by these artificial mesons. It has been revealed from these experiments that the interactions between π mesons and nucleons are strong enough and so there will be no doubt that most part of nuclear forces are due to π mesons. Therefore, the study on the nuclear forces must be based on the meson theory.

2. The conventional methods to obtain the nuclear forces from the meson theory are divided into two; i.e., the original Yukawa theory which assumes the single meson-interaction between mesons and nucleons, and pair theory. However, according to the experimental results on π and μ mesons, the meson pair theory has been excluded.^{2),5)}

3. The experimental results on nuclear forces have been investigated by several authors. They, however, employed there mainly the phenomenological potentials such as square well, exponential well, Yukawa well, and so on, and the meson theoretical potentials have scarcely been treated. Moreover, in the conventional studies the same simple phenomenological potentials have been applied to all the phenomena, regardless of the energies of the systems.

4. The analysis by Christian and Noyes⁶⁾ on the proton-proton scattering at high energies has revealed that the tensor type forces must be highly singular, which fact indicates that the potential due to the meson theory is superior to the

conventional, phenomenological potentials.

5. We have insisted previously⁷⁾ that in treating the nuclear forces one must separate the region near and further than the force range and that part in the neighbourhood of the nucleon. In the region near the nucleon, the potentials involve the effects due to the higher order perturbations, non-static forces, heavy mesons, relativistic parts, strong couplings and so on and thus the problem will become necessarily complicated.

6. On the contrary, in that region around and further than the force range, the potential is considered to be due to lower order terms of the perturbation expansions. This part of potential plays the important role in the nuclear scattering at low energies and the quadrupole moments of deuterons.

Here we could consider that the meson theory does not break down and the potentials due to the usual second and fourth order perturbation will give the correct results, i.e., the relativistic effect and thus non-static forces will not be important. Even if the excited states of the nucleons should exist, little influences will be given on the potentials for this region.

7. The fourth order perturbation will give the deviation from the second order calculations. Bethe has taken, in his "meson theory of nuclear forces",⁸⁾ for the cut-off radius the range where the magnitude of the second and the fourth perturbation will become comparable, i.e., the one-third of the nuclear force range.

Machida⁹⁾ has performed the calculations of the fourth order perturbation on the potential due to the pseudo-scalar meson field, and obtained the results that the fourth order perturbation gives strong enough contribution even in the region outside the range of the second order forces where only the static potential is important.

8. From these considerations we could conclude that in the investigation of the nuclear forces it will be meaningless to take the same simple potential from the neighbourhood of the nucleon to the outside region throughout. Therefore, we propose here for the analysis of nuclear forces that the problem is to be treated substantialistically in the outside region, while in the inside region phenomenologically, i.e., in the outside region we employ the second order and non-static fourth order potential, which is cut-off where it breaks down and substituted by the phenomenological potential, say square well potential, for the inside region. The depth and the width of the potential is so adjusted to fit the experimental results according as each process and according as each energy region. The spin- and charge-dependence of the potential is also chosen suitably according to the phenomena. This treatment corresponds to the statement in the previous section that more than two ranges are necessary for the nuclear forces.

9. For the outside region, the combinations of the four types of meson field are taken and the parameters for the potential in the inside region are adjusted. If the parameters cannot be chosen suitably for certain combination in the outside region, this type of the meson potential should be excluded.

10. If we could obtain the agreements with experiment by the suitable choice of parameters, the set of the parameters will give the data to lead to the future correct theory.

11. The static potential due to the fourth order calculation has not always meanings, since in the region where the fourth order calculation has sufficient enough magnitude, other non static effects will generally become strong too. However, in the outer region where the static potential due to the second order perturbation is influenced by the fourth order calculations, the spin charge and radial dependence of the latter will be taken into consideration.¹⁰⁾

12. On the type of meson we have attained to the conclusions as the results of analysis on the several phenomena that the pseudo-scalar meson will be best fitted.¹¹⁾

Onuma and Koide¹²⁾ have performed the calculations, employing the hard core for the inside potential as the simplification of our considerations, and the symmetrical pseudo-scalar meson potential for the substantialistical part of the potential, and obtained the results that if one takes 0.45 of the force range for the radius of hard core, the reasonable value for the quadrupole moments of deuterons will be obtained.

13. In the symmetrical vector meson theory, the quadrupole moment of deuterons becomes negative and it seems impossible to give the positive sign whatever characters one may give to the inside potential, because its range is too short. This fact is contrary to the expectation of Heisenberg.¹³⁾

14. Heisenberg¹³⁾ has emphasized to introduce vector π -meson to have the strong L - S coupling in the nuclear forces. However, it seems probable that the fourth order perturbations will give L - S coupling from the pseudo-scalar meson theory as its non-static part, and therefore we could not conclude that vector meson is indispensable.

15. Jastrow¹⁴⁾ has employed the exponential potential with hard core in its centre to explain the proton-proton scattering at high energies by the charge independent potential. The radius of the hard core has been taken as 0.6×10^{-13} cm. This is considered to be the simplified case of our method and from these investigations we could expect that our method will give good results so long as it concerned with the proton-proton scattering. However, Jastrow has employed phenomenological potential also for the outside region and thus, his method has no meanings for the investigation of the meson theory.

References

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Note on the Longitudinal and Scalar Photons

Ryôyû UTIYAMA, Tsutomu IMAMURA, Sigenobu SUNAKAWA and Tarô DODO

Department of Physics, Osaka University

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In this paper we have investigated a new kind of ambiguity and inconsistency pertaining to the longitudinal and scalar photons. We believe to have been able to give a clarification of this problem by introducing an artificial convergence factor g_n into the state vector.

Recently Belinfante¹⁾ pointed out that a new kind of ambiguities occurs in the calculation of the quantum electrodynamics owing to the Lorentz-condition. Speaking in more detail, any state Ψ which satisfies the Lorentz-condition is a superposition of all those states where the number of the longitudinal photons with a wave number vector \vec{k} is equal to that of the scalar photons with the same \vec{k} . From this fact the norm of Ψ diverges. As well there occur, occasionally, alternating series resulting from the very character of the state vector mentioned above, when we treat the components of the four-potential A_μ symmetrically in order to calculate the matrix elements corresponding to any physical process. These situations give rise to a new kind of ambiguity which has not been noticed before Belinfante's remark.

In fact Belinfante calculated the self-energy of an electron by making use of the ordinary perturbation method and showed that the self-energy of the order e^2 can be expressed as an alternating series which turns out to diverge as well known logarithmically if we make use of a particular method of summation, whereas it diverges quadratically if one makes use of another method. From his stand point it is not apparent why the former method is favourable compared with the latter.

According to Belinfante's remark, Dyson²⁾ has reinvestigated his theory of S -matrix and shown that the vacuum expectation value $\langle P(A_\mu(x)A_\nu(y)) \rangle_0$ is not equal to $\frac{1}{2}g_{\mu\nu}D_F(x-y)$ but equal to $\frac{1}{2}g_{\mu\nu}D_F(x-y) + \partial^2\phi(x-y)/\partial x^\mu\partial x^\nu$ on account of the Lorentz-condition.

But it has been shown by partial integration that this new additive term does not give any contribution to the S -matrix by virtue of its differential form. Now his argument seems to contain some inconsistencies pertaining to the commutation relations of A_μ if we follow his line of reasoning. The situation resembles that of contradiction between $\frac{\partial A_\mu}{\partial x_\mu}\Psi=0$ and $\langle [A_\nu, \frac{\partial A^\mu}{\partial x^\mu}] \rangle_0 \neq 0$. Further his short remark seems to be very unsatisfactory to us because his new additive term $\partial^2\phi/\partial x^\mu\partial x^\nu$

mentioned above really diverges, and hence it is very doubtful to integrate by parts when the integrand contains such an absolutely divergent factor. Ma²⁾ has also investigated these points, but his paper does not seem to bring any complete clarification about these problems.

Since all the difficulties mentioned above originate from the only fact that the norm of the state vector satisfying the Lorentz condition always diverges, we have tried to make the norm of Ψ converge absolutely by introducing an artificial convergence factor. By virtue of this factor we can calculate any matrix element pertaining to the longitudinal and scalar photons without any ambiguity. In the final step where all the calculations have been performed, we propose to make this convergence factor tend to unity. Then all the difficulties completely disappear. According to our method, it can be shown that the self-energy of an electron (of the order e^2) diverges logarithmically as usual, and Dyson's proposal reveals to be correct in spite of some doubtful points mentioned above.

We have, so far, discussed only about that case where the electromagnetic four-potential A_μ is symmetrically treated. On the contrary, if we eliminate, from the outset, the longitudinal and scalar parts of A_μ by making use of a unitary transformation as well known, then it might be conjectured that there appear no difficulties concerning these two parts stated above. But we can not justify the unitary character of the above transformation when the norm of the state vector diverges. The detailed proof for this justification by making use of our method will be given in the appendix.

§ 1. Preliminary discussion

In the present paper we shall, in general, follow Schwinger in notations³⁾ and formulations, but we prefer the real time $x^0=t$ to the imaginary one, and the hyper-plane σ to the curved one.

The Lorentz condition in the interaction representation runs, as well known,

$$\left\{ \frac{\partial A^\mu(x)}{\partial x^\mu} + \int_{\sigma'} D(x-x') j^\mu(x') n_\mu d\sigma' \right\} \Psi[\sigma'] = 0, \quad (1.1)$$

where n_μ is the unit normal to an arbitrary chosen space-like hyper-plane σ , and its components are chosen in such a way that the parameter τ , defined by

$$\tau = n_\mu x^\mu, \quad (1.2)$$

is a time-like variable, increasing in the direction of future. Then a σ -plane can be specified by a value of τ , and $\Psi[\sigma]$ turns into a function $\Psi(\tau)$ of the parameter τ , provided that a family of σ 's is arbitrarily fixed.

Now let us propose to assume that when we make the plane σ' in eq. (1.1) recede to the infinite past, i. e., $\tau' \rightarrow -\infty$, the integral part I of the left-hand side in eq. (1.1) vanishes and it holds

$$\frac{\partial A^\mu(x)}{\partial x^\mu} \Psi(-\infty) = 0. \quad (1.3)$$

Coester and Jauch¹⁾ have asserted that they have shown the validity of eq. (1.3) by utilizing the Fourier expansion of the expression I and at the same time they had given a mathematical justification for the often employed procedure of "switching on" the interaction at the infinite past, on the ground that the integral term drops out from the Lorentz condition.

Strictly speaking, however, their proof seems to be based on a vicious circle, because the current density $j^\mu(x)$ is implicitly assumed to vanish so rapidly at the infinite past as to be able to make j^μ transform into the Fourier integral with regard to the time. In fact eq. (1.3) can not be derived from eq. (1.1) without any additional assumption. Relying on the invariancy of the expression (1.1), let us, for the sake of simplicity, choose a special coordinate system, where the components of n_μ turn into (0, 0, 0, 1). In this special system of reference, the integral-term I of eq. (1.1) can be written in the following way, when $\tau' (=t')$ is put equal to $-\infty$,

$$I = \frac{1}{4\pi} \lim_{t' \rightarrow -\infty} (t-t') \int \rho(\vec{x} + |t-t'| \vec{e}, t') d\vec{e},$$

where \vec{e} is defined by

$$\vec{e} = \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|},$$

and ρ stands for the charge density j^0 .

Therefore eq. (1.3) holds if the charge density ρ is assumed to vanish so rapidly at the, temporally and spatially, infinitely remote world points, from the origin of the space-time coordinate as to make I vanish²⁾.

An alternative assumption is the following. Consider a function $f(t)$ of the time t , which has the properties

$$\begin{aligned} \text{i)} \quad & f(t) = 1 \quad \text{for finite } t, \\ \text{ii)} \quad & f(t) = 0 \quad \text{for sufficiently large } |t|. \end{aligned} \quad (1.4)$$

Let us assume the charge e not to be a constant but a function $ef(t)$. From this stronger assumption the vanishing of the expression I is easily seen. In this paper the latter assumption will be adopted, though the former less restrictive assumption should be examined in more detail in view of its importance in the various fundamental problems, such as of the self-stress and its covariance and of the S -matrix for finite time interval, etc.

In order to investigate the character of eq. (1.3) in greater detail, let us introduce the Fourier expansion of the electromagnetic four-potential $A_\mu(x)$. This is given by

$$A_{\mu}(x) = \frac{1}{\sqrt{2}(2\pi)^3} \sum_{\lambda=0}^3 \int e_{\mu}^{\lambda}(\vec{k}) \left\{ a_{\lambda}(\vec{k}) e^{ikx} + a_{\lambda}^{*}(\vec{k}) e^{-ikx} \right\} \frac{d\vec{k}}{\sqrt{k}}, \quad (1.5)$$

where

$$kx = k_{\mu} x^{\mu}, \quad k_{\mu} = (\vec{k}, -|\vec{k}|), \quad k = |\vec{k}|,$$

and

$$\left. \begin{aligned} e_{\mu}^{\lambda}(\vec{k}) &= \frac{k_{\mu} + n_{\mu}(nk)}{(nk)}, & e_{\mu}^0(\vec{k}) &= n_{\mu}, \\ e_{\mu}^{\lambda}(\vec{k}) e_{\nu}^{\lambda'}(\vec{k}) g^{\mu\nu} &= g^{\lambda\lambda'}, \\ e_{\mu}^{\lambda}(\vec{k}) e_{\nu}^{\lambda'}(\vec{k}) g_{\lambda\lambda'} &= g_{\mu\nu}. \end{aligned} \right\}^{(6)}$$

The commutation relation for a and a^{*} is

$$[a_{\lambda}(\vec{k}), a_{\rho}^{*}(\vec{k}')] = g_{\lambda\rho} \delta(\vec{k} - \vec{k}'), \quad (1.6)$$

from this relation the number operators are introduced by

$$\begin{aligned} n_{\lambda}(\vec{k}) &= a_{\lambda}^{*}(\vec{k}') a_{\lambda}(\vec{k}), & \lambda &= 1, 2, 3, \\ n_0(\vec{k}) &= b^{*}(\vec{k}) b(\vec{k}), \end{aligned}$$

with the definition

$$b^{*}(\vec{k}) = a_0(\vec{k}), \quad \text{and} \quad b(\vec{k}) = a_0^{*}(\vec{k}).$$

Substitute from (1.5) into (1.3), then it will result in

$$f(\vec{k}) \Psi(-\infty) = 0, \quad f^{*}(\vec{k}) \Psi(-\infty) = 0, \quad (1.7)$$

in place of eq. (1.3), where

$$f(\vec{k}) = a_3(\vec{k}) + a_0(\vec{k}) = a_3(\vec{k}) + b^{*}(\vec{k}).$$

From eq. (1.7), we see that

$$\Psi(-\infty) = \Psi' \Psi''$$

with

$$\left. \begin{aligned} \Psi''(-\infty) &= \prod_{\vec{k}} C(\vec{k}) \sum_{n=0}^{\infty} (-1)^n \chi_{n,n}(\vec{k}). \end{aligned} \right\} \quad (1.8)$$

Ψ'' represents a factor of the state vector pertaining to scalar and longitudinal photons, whereas Ψ' stands for the remaining factor. $\chi_{n,m}(\vec{k})$ is a normalized function representing a state where n longitudinal photons with the wave number vector \vec{k} and m scalar photons with the same \vec{k} are present, and finally $C(\vec{k})$ is a normalization constant⁷⁾.

From the expression (1.8), it is obvious that the norm of $\Psi(-\infty)$ diverges, for individual \vec{k}' s, in so far as the normalization constant $C(\vec{k})$ is kept finite.

Now, making use of (1.8), and putting $|C(\vec{k})|$ equal to $(\sum 1)^{-\frac{1}{2}}$, we obtain the following various expectation values:

$$\langle a_3 a_3^* \rangle \equiv (\Psi^*(-\infty), a_3 a_3^* \Psi(-\infty)) = \frac{\sum (n+1)}{\sum 1},$$

$$\langle a_3 a_0^* \rangle = -\frac{\sum n}{\sum 1}, \quad \langle a_3^* a_0 \rangle = -\frac{\sum (n+1)}{\sum 1},$$

$$\langle a_3^* a_3 \rangle = \frac{\sum n}{\sum 1},$$

$$\langle a_0^* a_3 \rangle = -\frac{\sum n}{\sum 1}.$$

Combinations of these expressions lead to the following relations:

$$\langle a_3 f^* \rangle = \frac{\sum (n+1) - \sum n}{\sum 1}, \quad (1.9a)$$

$$\langle a_3 f \rangle = \frac{\sum n - \sum (n+1)}{\sum 1}, \quad (1.9b)$$

$$\langle f^* a_3 \rangle = \frac{\sum n - \sum n}{\sum 1}. \quad (1.9c)$$

The right-hand sides of these expressions are all semiconvergent series. If we make the summation of (1.9) in the following way,

$$\sum (n+1) - \sum n = (1-0) + (2-1) + (3-2) + \dots,$$

$$\sum n - \sum n = (0-0) + (1-1) + (2-2) + \dots,$$

then they read:

$$\langle a_3 f^* \rangle = 1, \quad \langle a_3^* f \rangle = -1, \quad \langle f^* a_3 \rangle = 0$$

and

$$\langle [a_3, f^*] \rangle = \langle [a_3, a_3^*] \rangle = 1.$$

This result does not interfere with the commutation relations, whereas we get also

$$\langle a_3^* f \rangle \neq \langle f^* a_3 \rangle^*,$$

so that the above mentioned rule of summation destroys the hermitian character.

On the other hand, if we directly substitute from the condition (1.7) into expressions (1.9), then they turn into

$$\langle a_3 f^* \rangle = \langle a_3^* f \rangle = \langle f^* a_3 \rangle = 0.$$

In this case the commutation relation $[a_3, a_3^*] = 1$ is destroyed although the hermitian character is conserved. This situation just corresponds to the contradiction between

$$\frac{\partial A^\nu}{\partial x^\nu} \Psi(-\infty) = 0, \quad \left\langle \left[A_\mu(x), \frac{\partial A^\nu(x')}{\partial x'^\nu} \right] \right\rangle \neq 0.$$

The arguments stated so far based entirely upon the assumption that the charge e vanishes at the infinite past, and, in consequence, the expression I is equal to zero.

Now, let us consider the following eigenvalue problem, in order to treat more general cases not restricted by (1.4),

$$f(\vec{k})\Psi = \lambda\Psi, \quad f^*(\vec{k})\Psi = \lambda^*\Psi$$

and

$$\{n_\sigma(\vec{k}) - n_0(\vec{k})\}\Psi = \mu\Psi.$$

If we wish to rely upon a representation, where the number operators n_σ and n_0 are diagonal, in solving these equations, we are able to get a solution (1.8) when and only when both λ and μ are zero, as shown elsewhere⁷⁾.

This fact apparently justifies the proposal of eq. (1.3). But this assertion is doubtful, because, if we make use of another representation where a_σ and b are diagonal as was used in the paper of Tomonaga and his collaborators⁸⁾, we can obtain another solution with a nonvanishing eigenvalue λ . This situation implies that these two representations are not equivalent with each other, that is to say, the transformation between these two representations is not unitary.

All these contradictions stated so far caused entirely by the singular character of the state vector that its norm diverges.

§ 2. Introduction of the convergence factor

In order to make the norm of the state vector converge, let us introduce a convergence factor $e^{-\lambda n}$ in the following way;

$$\Psi''_c(-\infty) = \prod_{\vec{k}} C(\vec{k}) \sum (-1)^n e^{-\lambda n} x_{n,n}(\vec{k}), \quad (2.1)$$

where λ is a sufficiently small invariant parameter and is, for the sake of simplicity, assumed to be independent on \vec{k} . After all the calculations have been performed, we propose to make this convergence factor tend to unity.

The constant $C(\vec{k})$ appearing in eq. (2.1) turns out to be

$$\sqrt{1 - e^{-2\lambda}}.$$

The Lorentz condition (1.3) is invalidated so far as the parameter λ is not zero; in fact we obtain

$$\begin{aligned} N(f\Psi''_c) &\equiv \sqrt{(f\Psi''_c, f\Psi''_c)} = \frac{\sqrt{2\lambda^3}}{(1 - e^{-2\lambda})} \\ &\approx \frac{1}{\sqrt{2}} \sqrt{\lambda}, \quad 0 \lesssim \lambda \ll 1. \end{aligned} \quad (2.2)$$

In spite of this situation, we may assert that the modified state vector Ψ''_c reserves the essential character of Ψ'' , because the former consists of the superposition of

only $\chi_{n,n}$'s and does not contain $\chi_{n,m}$'s ($n \neq m$) as is the case for the latter, and hence the total energy of the electromagnetic field in the state Ψ''_c is positive definite and any physical process which is allowable for the state vector Ψ'' is also admitted for the state vector Ψ''_c and *vice versa*.

Computed for the modified state vector (2.1), the expressions (1.9) are turned into the following;

$$\left. \begin{aligned} \langle a_3 f^* \rangle_c &= \frac{1 - e^{-\lambda}}{1 - e^{-2\lambda}} \approx \frac{1}{2}, \\ \langle a_3^* f \rangle_c &= -\frac{e^{-\lambda}(1 - e^{-\lambda})}{1 - e^{-2\lambda}} \approx -\frac{1}{2}, \\ \langle f^* a_3 \rangle_c &= -\frac{e^{-\lambda}(1 - e^{-\lambda})}{1 - e^{-2\lambda}} \approx -\frac{1}{2}. \end{aligned} \right\} \quad (2.3)$$

From (2.3) we can easily see that

$$\langle [a_3 f^*] \rangle_c = \langle [a_3 a_3^*] \rangle_c = \frac{(1 - e^{-\lambda})(1 + e^{-\lambda})}{1 - e^{-2\lambda}} = 1,$$

$$\langle a_3^* f \rangle_c = \langle f^* a_3 \rangle_c^*.$$

Thus the contradictions mentioned in the preceding section completely disappear. It may be stressed that the expressions (2.3) do not vanish when we make λ tend to zero, in spite of the fact that $f\Psi''_c$ vanishes in this limit, whereas it is easily seen that the expressions $\langle f f^* \rangle_c$, $\langle a_\lambda f \rangle_c$ ($\lambda = 1, 2$) and other allied quantities equally vanish when λ is put equal to zero. This fact gives us the possibility of the complete separation of the longitudinal and scalar parts from the transversal part.

So far, we have been dealing with the simple convergence factor $e^{-\lambda n}$. But a more general form of the convergence factor may be of service. Let us replace the definition (2.1) with

$$\Psi''_c(-\infty) = \prod_{\vec{k}} C(\vec{k}) \sum_n (-1)^n g_n \chi_{n,n}(\vec{k}), \quad (2.4)$$

where g_n may be an arbitrary convergence factor satisfying the following condition

$$\lim_{g_n \rightarrow 1} N(f\Psi''_c) = \lim_{g_n \rightarrow 1} \left\{ \frac{\sum_{n=0}^{\infty} (2n+1) g_n^2 - \sum_{n=0}^{\infty} (2n+2) g_n g_{n+1}}{\sum_{n=0}^{\infty} g_n^2} \right\}^{\frac{1}{2}} = 0. \quad (2.5)$$

Also in general case, all the inconsistencies mentioned in § 1. can completely be resolved. The conclusions of the present paper do not depend on the special form of the convergence factor g_n , as will be shown later, though the precise values of (2.3) in general depend on the form of g_n .

§ 3. Expectation values in vacuum

In calculating the S -matrix in the manner of Dyson, we have to evaluate the following expectation value:

$$\langle P(A_\mu(x)A_\nu(x')) \rangle_{vac} = (\Psi_{vac}(+\infty), P(A_\mu(x)A_\nu(x'))\Psi_{vac}(-\infty)). \quad (3.1)$$

To do this, let us propose to define the vacuum state (pertaining to the electromagnetic field) by the following (c. f. Eq. (1.8))

$$\begin{aligned} \Psi_{vac}(\pm\infty) &= \Psi'_{vac}(\pm\infty)\Psi''(\pm\infty), \\ a_\lambda(\vec{k})\Psi'_{vac}(\pm\infty) &= 0, \quad (\lambda=1, 2) \\ \Psi''_{vac}(\pm\infty) &= \Pi \left\{ \sum_{\vec{k}} (-1)^n g_n \chi_{n,n}(\vec{k}) \right\} / \sqrt{\sum_n g_n^2}. \end{aligned} \quad (3.2)$$

The last definition is a substitute for the Lorentz condition, because it bears all the physical characteristics given by the Lorentz condition.

By using the definition (3.2) and the expressions (1.5) (1.6), let us compute the quantity (3.1). This is given by

$$\begin{aligned} \langle P(A_\mu(x)A_\nu(x')) \rangle_{vac} &= \frac{1}{2} g_{\mu\nu} D_F(x-x') \\ &+ \left(\frac{\sum_n g_n^2}{\sum_n g_n^2} \right) \frac{2}{2(2\pi)^3} \int \frac{k_\mu k_\nu}{(n_p k^p)^2 k} (e^{ik(x-x')} + e^{-ik(x-x')}) d\vec{k} \\ &- \frac{1}{2(2\pi)^3} \int \frac{k_\mu n_\nu + k_\nu n_\mu}{(n_p k^p) k} (e^{ik(x-x')} + e^{-ik(x-x')}) d\vec{k} \\ &+ \frac{1}{\sum_n g_n^2} \left\{ \sum_n (2n+1) g_n^2 - \sum_n 2(n+1) g_n g_{n+1} \right\} \\ &\times \frac{1}{2(2\pi)^3} \int \frac{1}{(n_p k^p)^2 k} \left\{ (n_\sigma k^\sigma) (k_\mu n_\nu + k_\nu n_\mu) + 2(n_\sigma k^\sigma)^2 n_\mu n_\nu \right\} \\ &\times \{ e^{ik(x-x')} + e^{-ik(x-x')} \} d\vec{k}, \end{aligned} \quad (3.3)$$

where D_F is given by

$$D_F(x) = D^{(1)}(x) + i\epsilon(x)D(x).$$

Putting symbolically

$$D^{(1)}(x) = -\frac{1}{2(2\pi)^3} \int \frac{1}{(n_p k^p)^2 k} (e^{ikx} + e^{-ikx}) d\vec{k}, \quad (3.4)$$

we can rewrite (3.3) in the following way,

$$\langle P(A_\mu(x)A_\nu(x')) \rangle_{vac} = \frac{1}{2} g_{\mu\nu} D_F(x-x')$$

$$\begin{aligned}
 & + 2 \left(\frac{\sum n_g^2}{\sum g_n^2} \right) \frac{\partial^2}{\partial x^\mu \partial x^\nu} Q^{(1)}(x-x') - \left(n_\rho \frac{\partial}{\partial x_\rho} \right) \left(n_\nu \frac{\partial}{\partial x^\mu} + n_\mu \frac{\partial}{\partial x^\nu} \right) Q^{(1)}(x-x') \\
 & + \frac{1}{\sum g_n^2} \cdot \sum \left\{ (2n+1)g_n^2 - 2(n+1)g_n g_{n+1} \right\} \\
 & \times \left\{ \left(n_\sigma \frac{\partial}{\partial x_\mu} \right) \left(n_\nu \frac{\partial}{\partial x^\mu} + n_\mu \frac{\partial}{\partial x^\nu} \right) + 2n_\mu n_\nu \left(n_\sigma \frac{\partial}{\partial x_\sigma} \right)^2 \right\} Q^{(1)}(x-x'). \quad (3.3)'
 \end{aligned}$$

Let, in the expression (3.3)', all the g_n tend to unity, then it turns into

$$\begin{aligned}
 \langle P(A_\mu(x) A_\nu(x')) \rangle_{vac} &= \frac{1}{2} g_{\mu\nu} D_F(x-x') \\
 & - \left(n_\rho \frac{\partial}{\partial x_\rho} \right) \left(n_\nu \frac{\partial}{\partial x^\mu} + n_\mu \frac{\partial}{\partial x^\nu} \right) Q^{(1)}(x-x') + 2 \lim_{g \rightarrow 1} \left(\frac{\sum n_g^2}{\sum g_n^2} \right) \frac{\partial^2}{\partial x^\mu \partial x^\nu} Q^{(1)}(x-x')
 \end{aligned} \quad (3.3)''$$

by virtue of the condition (2.5).

Thus the expression (3.3)'' contains an absolutely divergent term, since $\sum n_g^2 / \sum g_n^2$ can in no way be made convergent. Dyson⁹⁾ has suggested in his short note that $\langle P(A_\mu(x), A_\nu(x')) \rangle$ should have the following expression:

$$\langle P(A_\mu(x), A_\nu(x)) \rangle_{vac} = \frac{1}{2} g_{\mu\nu} D_F(x-x') + \frac{\partial^2 \Phi(x-x')}{\partial x^\mu \partial x^\nu}. \quad (3.5)$$

Comparing the expression (3.3)'' with Dyson's one, we see that our expression contains an extra term as the second member of the right-hand side.

Furthermore the last term of the right-hand side of Dyson's (3.5) is revealed to be mathematically meaningless, since in fact it corresponds to our divergent term.

Now we can show that the second and third terms of the righthand side in eq. (3.3)' do not give rise to any contribution to the S -matrix, as in case of Dyson, if we explicitly take the relation

$$\frac{\partial \epsilon(x-x')}{\partial x^\mu} [j^\mu(x), j^\nu(x')] = 0 \quad (3.6)^{11)}$$

for granted.

Proof: For the sake of simplicity, let us use the following abbreviations;

$$K^{\mu_1 \dots \mu_n} = P(j^{\mu_1}(x_1) j^{\mu_2}(x_2) \dots j^{\mu_n}(x_n)) \rightarrow P(j(1) j(2) \dots j(n))$$

and $\sigma(t_1 - t_2) \rightarrow \sigma(1, 2)$, where the function $\sigma(t)$ is defined by

$$\sigma(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

In terms of this σ -function, $K^{\mu_1 \dots \mu_n}$ can be expressed by

$$K^{\mu_1 \dots \mu_n} = \sum_P \sigma(12) \sigma(23) \dots \sigma(n-1, n) \{ j(1) j(2) \dots j(n) \},$$

where the summation is carried over all the permutations of the points $(1, 2, \dots, n)$.

In the first place let us prove that the expression

$$\frac{\partial K^{\mu_1 \dots \mu_n}}{\partial x_1^{\mu_1}} \quad (3.7)$$

vanishes, under the assumption (3.6). The expression (3.7) can be written in the following way by virtue of the equation of continuity;

$$\begin{aligned} \frac{\partial K^{\mu_1 \dots \mu_n}}{\partial x_1^{\mu_1}} = & \sum_{P'} \sum_{k=1}^n \sigma(23) \sigma(34) \dots \frac{\partial(\sigma(k, 1) \sigma(1, k+1))}{\partial x_1^{\mu_1}} \sigma(k+1, k+2) \dots \\ & \dots \sigma(n-1, n) \times \{ j(2) \dots j(k) j^{\mu_1}(1) j(k+1) \dots j(n) \}, \end{aligned}$$

the first summation being over the $(n-1)!$ permutations of the points $(2, 3, \dots, n)$.

Combine pairwise the typical terms in the above equation which contain $\frac{\partial \sigma(1, k)}{\partial x_1^{\mu_1}}$ or $\frac{\partial \sigma(k, 1)}{\partial x_1^{\mu_1}}$, then the expression (3.7) turns out into

$$\begin{aligned} & \frac{1}{2} \sum_{P'} \sum_{k=1}^n \sigma(23) \dots \sigma(k-1, k+1) \dots \sigma(n-1, n) j^{\mu_2}(2) \dots j^{\mu_{k-1}}(k-1) \\ & \times \frac{\partial \varepsilon(x_1 - x_k)}{\partial x_1^{\mu_1}} [j^{\mu_1}(1), j^{\mu_k}(k)] j^{\mu_{k+1}}(k+1) \dots j^{\mu_n}(n). \end{aligned}$$

Thus (3.7) is proved to vanish on the assumption (3.6).

In the second place we substitute from the second and third terms in (3.3)' into $\langle P(A_\mu(x), A_\nu(x')) \rangle_{vac}$ contained in the S -matrix, and integrate it by parts with regard to x^μ or x'^ν , then it vanishes owing to eq. (3.7) and the assumption (1.4), whereby the latter assumption warrants us the vanishing surface integrals. Q. E. D.

From the formal stand point, our method of proof stated above is completely the same with that of Dyson, but in his method there occurs a mathematically meaningless expression, whereas in our case we need not deal with such a quantity by virtue of the convergence factor g_n .

In the final step, let all the g_n tend to unity, then the fourth term or (3.3)' disappears, and only the contribution from (3.3)' to the S -matrix is that from the term $\frac{1}{2} g_{\mu\nu} D_F(x-x')$.

Thus we have arrived at the same conclusion as that of Dyson by entirely using the mathematically well defined procedure.

We have, so far, talked only about that method where the electromagnetic four-potential A_μ is symmetrically treated. On the contrary, one can also eliminate, from the outset, the longitudinal and scalar parts of A_μ by making use of a unitary transformation as shown by Schwinger. In this case there occurs

neither ambiguity nor inconsistency pertaining to the eliminated parts. But it must be remarked that Schwinger's procedure is justified, only provided that the norm of the state vector is made to converge as has been done in the present paper and also it is ascertained that there occur no such terms as appearing in eq. (2.3) in the calculation of the S -matrix. Specifically, one can put the term $-\frac{1}{c}n_\mu j^\mu(x)n_\nu \frac{\partial}{\partial x^\nu}(A(x)-A(x'))$ in Schwinger's paper equal to zero, only when the last situation just mentioned is ascertained.

In the appendix we shall give a proof for the equivalence of the two S -matrices obtained by using the two different methods.

§ 4. Application to the electron self-energy

The self-energy of an electron in the order of e^2 is obtained in the theory of S -matrix by evaluating the expression

$$(\text{one electron } |S_2| \text{ one electron}), \quad (4.1)$$

where S_2 is a part of the S -matrix of the order e^2 and is defined by

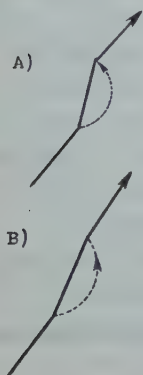
$$S_2 = \frac{(-i)^2}{2!} \int P(j^\mu(x), j^\nu(x')) P(A_\mu(x) A_\nu(x')) (dx)^4 (dx')^4. \quad (4.2)$$

Taking into account the conclusion obtained in the preceding section, the expression (4.1) can be written in the form

$$\begin{aligned} & -\frac{1}{4} \int (\text{one electron } |P(j^\mu(x), j_\mu(x'))| \text{ one electron}) \\ & \times D_F(x-x') (dx)^4 (dx')^4. \end{aligned}$$

This is just the well acquainted expression.

In this section, however, we shall give the self-energy of an electron at rest by using the ordinary perturbation theory based on our definition of vacuum (3.2), in order to clarify the origin of the ambiguity in Belinfante's paper. Contributions to the value of the self-energy from virtual processes are given as follows;



transversal photon

$$E_a \equiv E_t = \frac{e^2 m}{8\pi^2} \left[\log \frac{K + \sqrt{K^2 + m^2}}{m} - \frac{1}{2} \right], \quad (4.3)a$$

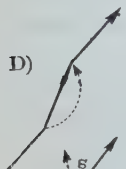
longitudinal photon

$$E_b = \sum (n+1) g_n^2 \cdot A \int_0^K F_+(k) k dk, \quad (4.3)b$$

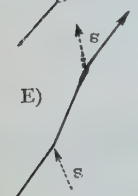


$$E_c = \sum n g_n^2 A \int_0^K F_-(k) \cdot k \cdot dk, \quad (4.3)c$$

scalar photon



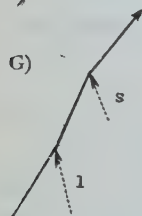
$$E_d = \sum (n+1) g_n^2 A \int_0^K F_-(k) \cdot \{2\sqrt{k^2+m^2} - k\} dk, \quad (4.3)d$$



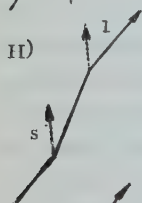
$$E_e = -\sum n g_n^2 A \int_0^K F_+(k) \{2\sqrt{k^2+m^2} + k\} dk, \quad (4.3)e$$



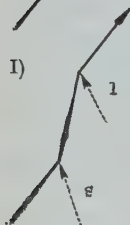
$$E_f = -\sum (n+1) g_n g_{n+1} A \int_0^K F_+(k) k dk, \quad (4.3)f$$



$$E_g = -\sum (n+1) g_n g_{n+1} A \int_0^K F_-(k) \cdot k dk, \quad (4.3)g$$



$$E_h = -\sum (n+1) g_n g_{n+1} A \int_0^K F_-(k) \cdot k dk, \quad (4.3)h$$



$$E_i = -\sum (n+1) g_n g_{n+1} A \int_0^K F_+(k) k dk, \quad (4.3)i$$

where $A = \frac{e^2 m}{8\pi^2} \left(\sum g_n^2 \right)$ and $F_{\pm}(k) = 1/\sqrt{k^2+m^2} (\sqrt{k^2+m^2} \pm k)$.

In every expression given above the contribution from the vacuum fluctuation has been subtracted.

Summing up the above expressions from E_b to E_t , we get the part of self-energy related to the longitudinal and scalar parts in the following closed form :

$$E_{t+s} = \frac{e^2}{4\pi^2 m} \frac{1}{\sum g_n^2} (\sum (2n+1) g_n^2 - \sum 2(n+1) g_n g_{n+1}) \int_0^\infty k dk + \frac{e^2 m}{4\pi^2} \int_0^\infty \frac{dk}{\sqrt{k^2 + m^2}} \quad (4.4)$$

In eq. (4.4), the first term, which quadratically diverges, drops out by virtue of the condition (2.5), and the remaining term, which is logarithmically divergent, is independent on the convergence factor g_n . Thus the self-energy in the order of e^2 of an electron at rest is given by

$$E = E_t + E_{t+s} = \frac{e^2 m}{8\pi^2} \left[3 \log \frac{K + \sqrt{K^2 + m^2}}{m} - \frac{1}{2} \right], \quad (4.5)$$

in complete agreement with the result of current theory.

§ 5. Conclusion

We have given a justification for the assertion of Dyson and others by using mathematically well defined procedures only. The mathematical manipulations employed in this paper may be summarized in the following way: in the first place one introduces an artificial convergence factor g_n in order to make the norm of the state vector absolutely converge; in the second place one calculates any desired matrix element, turning it into the form

$$A + \frac{1}{\sum g_n^2} \{ \sum (2n+1) g_n^2 - \sum 2(n+1) g_n g_{n+1} \} B.$$

Then A becomes independent on g_n , and agrees with the expression obtained by the treatment in which, from the beginning, the longitudinal and scalar parts of A_μ are separated. In the final step, where all the calculations have been performed, one lets g_n tend to unity, the second term of the above expression drops out. It must be quite emphasized that the result thus obtained is independent on the precise form of g_n provided that g_n satisfies the condition (2.5).

Recently a paper of S. N. Gupta¹¹⁾, where the same problem as ours has been treated in an elegant way, has come to our notice. Although he has analysed the nature of the difficulties in question very cleverly on the basis of the theory of indefinite metrics and come to the same conclusions as ours.

The merit of his theory may be said to be upset in introducing another transcendental element of indefinite metrics into the discussions.

Our theory, on the contrary, runs quite direct and elementary to bring into light the characters of the difficulties.

In conclusion the authors wish to thank Prof. K. Husimi, for much help in

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Appendix

We shall give a proof of equivalence of the two S -matrices obtained by the two different ways stated in the end of § 3.

If we separate the transversal part from A_μ , the Tomonaga-Schwinger equation is written

$$i \frac{\partial \Psi(\tau)}{\partial \tau} = \{ \bar{H}_1(\tau) + \bar{H}_2(\tau) \} \Psi(\tau), \quad (\text{a} \cdot 1)$$

where

$$\begin{aligned} \bar{H}_1(\tau) &= \int H_1(x) d\sigma = - \int_{\tau=\text{const.}} j^\mu(x) \mathfrak{A}_\mu(x) d\sigma, \\ \bar{H}_2(\tau) &= \int H_2(x) d\sigma \\ &= - \frac{1}{2} \iint n^\lambda \frac{\partial Q(x-x')}{\partial x^\lambda} n_\mu j^\mu(x) n_\nu j^\nu(x') d\sigma d\sigma' \end{aligned}$$

and \mathfrak{A}_μ is the transversal four-vector.

The S -matrix obtained from eq. (a.1) can be written in the following way

$$S = \sum_{n=0}^{\infty} S_n,$$

where S_n is given by

$$\begin{aligned} S &= \sum_{r=0}^{\leq \frac{n}{2}} \frac{(-i)^{n-r}}{(n-2r)! r!} \int P \{ H_1(x_1), \dots, H_1(x_{n-2r}), \\ &\quad H_2(y_1), \dots, H_2(y_r) \} dx_1 \dots dx_{n-2r} dy_1 \dots dy_r. \end{aligned} \quad (\text{a} \cdot 2)$$

By using Wick's⁽¹²⁾ method of decomposition of P -products, we can put eq. (a.2) in the following form:

$$S = \sum_{m=0}^{\leq \frac{n}{2}} \sum_{r=0}^m \frac{(-i)^{n-r} (-1)^{n-2r}}{(n-2m)! r! (m-r)! 2^{m-r}} F_n(r, m) \quad (\text{a} \cdot 3)$$

where $F_n(r, m)$ stands for

$$\begin{aligned} &\int P \{ j^\mu_1(x_1) \dots j^\nu_{n-2m}(x_{n-2m}) j^\mu_1(z_1) \dots j^\nu_{2(m-r)}(z_{2(m-r)}) H_2(y_1) \dots H_2(y_r) \} \\ &\quad \times : \mathfrak{A}_{\mu_1}(x_1) \dots \mathfrak{A}_{\mu_{n-2m}}(x_{n-2m}) : \\ &\quad \times (\mathfrak{A}_{\nu_1}(z_1) \mathfrak{A}_{\nu_2}(z_2)) \dots (\mathfrak{A}_{\nu_{2m-2r-1}}(z_{2m-2r-1}) \mathfrak{A}_{\nu_{2m-2r}}(z_{2m-2r})) \\ &\quad \times dx_1 \dots dx_{n-2m} \cdot dz_1 \dots dz_{2m-2r} \cdot dy_1 \dots dy_r. \end{aligned} \quad (\text{a} \cdot 3)'$$

Here we have permuted the tensor suffices, respectively integration variables, so as to lump together those terms which contain the same normal product.

Following the definition of contraction given by Wick, we get

$$\begin{aligned} (\mathfrak{A}_\mu^*(x), \mathfrak{A}_\nu^*(x')) &= (A_\mu^*(x) A_\nu^*(x')) \\ &- \frac{1}{2} \left\{ \frac{\partial^2}{\partial x^\mu \partial x^\nu} + \left(n \frac{\partial}{\partial x} \right) \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu} \right) \right\} Q^{(1)}(x-x') \\ &- \frac{i}{2} \varepsilon(x-x') \left\{ \frac{\partial^2}{\partial x^\mu \partial x^\nu} + \left(n \frac{\partial}{\partial x} \right) \left(n_\mu \frac{\partial}{\partial x^\nu} + n_\nu \frac{\partial}{\partial x^\mu} \right) \right\} Q(x-x'). \end{aligned} \quad (\text{a.4})$$

Now, if one, for instance, substitute from (a.4) into the factor $(\mathfrak{A}_{\nu_1}^*(z_1) \mathfrak{A}_{\nu_2}^*(z_2))$ in eq. (a.3)', then it is easily seen that the second term of eq. (a.4) does not give rise to any contribution to F if we integrate it by parts and at the same time take advantage of the assumption (1.4) and also the fact that the expression (3.7) is equal to zero.

On the contrary the contribution to F from the third term of (a.4) runs as follows;

$$\begin{aligned} \frac{i}{2} \int K^{\nu_1 \nu_2}(z_1 z_2) \varepsilon(z_1 - z_2) &\left\{ \frac{\partial^2}{\partial z_1^{\nu_1} \partial z_2^{\nu_2}} + \left(n \frac{\partial}{\partial z_1} \right) n_{\nu_1} \frac{\partial}{\partial z_2^{\nu_2}} + \left(n \frac{\partial}{\partial z_2} \right) n_{\nu_2} \frac{\partial}{\partial z_1^{\nu_1}} \right\} \\ &\times Q(z_1 - z_2) (dz_1)^4 (dz_2)^4, \end{aligned}$$

where the function K stands for the cofactor of $(\mathfrak{A}_{\nu_1}^* \mathfrak{A}_{\nu_2}^*)$ in the integrand of (a.3)'.

Remembering the expression (3.7) and the character of Q , i. e.

$$\left\{ \frac{\partial}{\partial x^\rho} + n_\rho \left(n \frac{\partial}{\partial x} \right) \right\} Q(x) = 0 \quad \text{for } (nx) = 0,$$

we can transform the above expression into

$$i \int_{-\infty}^{\infty} (dz)^4 \int d\sigma' K^{\mu\nu}(z\sigma') n_\mu n_\nu \left(n \frac{\partial}{\partial z} \right) Q(z - \sigma'),$$

where the inner integral is carried over a σ -plane containing the world point z . This expression can, further, be transformed into $-2iF_n(r+1, m)$ by making use of the definition (a.1).

If we similarly substitute all the contracted factors of (a.3) in the afore mentioned manner, then we get

$$S_n = \sum_{m=0}^{\leq \frac{n}{2}} \sum_{r=0}^m \sum_{k=0}^{m-r} \frac{(-1)^{n-2r} (-i)^{n-r}}{(n-2m)! 2^{m-r} r! (m-r)!} \cdot \frac{(-i)^{m-r-k} (m-r)!}{(m-r-k)! k!} F'_n(m-k, m), \quad (\text{a.5})$$

where F' has quite the same expression as F with the exception that every

$(\mathfrak{A}_\mu^* \mathfrak{A}_\nu^*)$ in the latter is replaced by $(A_\mu^* A_\nu^*)$ in the former. This exception results from the first term in (a.4). By interchanging the order of summations with regard to r and k , S_n becomes

$$S_n = \sum_{m=0}^{\leq \frac{n}{2}} \frac{(-i)^n}{n!} \frac{n! (-1)^n}{(n-2m)! m! 2^m} F'_n(o, m), \quad (\text{a.6})$$

where, specifically,

$$\begin{aligned} F'_n(o, m) = & \int P \{ j^\mu_1(x_1) \cdots j^\mu_{n-2m}(x_{n-2m}) j^\nu_1(y_1) \cdots j^\nu_{2m}(y) \} \\ & \times : (\mathfrak{A}_{\mu_1} \cdots \mathfrak{A}_{\mu_{n-2m}}) : (A_{\nu_1}^* A_{\nu_2}^*) \cdots (A_{\nu_{2m-1}}^* A_{\nu_{2m}}^*) \\ & dx_1 \cdots dx_{n-2m} \cdot dy_1 \cdots dy_{2m}. \end{aligned} \quad (\text{a.6})'$$

On the contrary, starting from the symmetrical treatment of A_μ we get the second S -matrix of the order e^n , say S'_n , in agreement with the expressions (a.6) (a.6)', but with the exception that in the former the normal product

$$: A_{\mu_1} \cdots A_{\mu_{n-2m}} : \quad (\text{a.7})$$

stands in place of the corresponding normal product of \mathfrak{A} in the latter.

Now consider the expectation value

$$(\Psi(+\infty) \int G^{\mu_1 \cdots \mu_r}(x_1 \cdots x_r) : A_{\mu_1}(x_1) \cdots A_{\mu_r}(x_r) : dx_1 \cdots dx_r \Psi(-\infty)), \quad (\text{a.8})$$

where $G^{\mu_1 \cdots \mu_r}$ stands for the cofactor of: $\mathfrak{A}_{\mu_1} \cdots \mathfrak{A}_{\mu_r}$ in (a.6)' with $r=n-2m$, and separate the A 's in (a.8) into the transversal and the remaining parts, then (a.8) is transformed into

$$\sum_{k=0}^{\leq \frac{r}{2}} L_r(k) \frac{r!}{2^k k! (r-2k)!},$$

where

$$\begin{aligned} L_r(k) = & (\Psi(+\infty), \int G^{\mu_1 \cdots \mu_{r-2k} \nu_1 \cdots \nu_{2k}} : \mathfrak{A}_{\mu_1} \cdots \mathfrak{A}_{\mu_{r-2k}} A_{\nu_1} \cdots A_{\nu_{2k}} : \\ & dx_1 \cdots dx_r \Psi(-\infty)), \end{aligned} \quad (\text{a.9})$$

and A_μ stands for

$$A_\mu - \mathfrak{A}_\mu = \frac{\partial}{\partial x^\mu} B + n_\mu C$$

with the abbreviation

$$\begin{aligned} B = & \frac{-i}{\sqrt{2}(2\pi)^3} \int \frac{1}{(nk)} \left\{ a_3 e^{ikx} - a_3^* e^{-ikx} \right\} \frac{d\vec{k}}{\sqrt{k}}, \\ C = & \frac{1}{\sqrt{2}(2\pi)^3} \int \left\{ f e^{ikx} + f^* e^{-ikx} \right\} \frac{d\vec{k}}{\sqrt{k}} \end{aligned} \quad (\text{a.10})$$

Introducing from these expressions into (a.9), we see that all the terms containing the function $B(x)$ drop off owing to the same reason as stated in §3. The only remaining term is

$$(\Psi(+\infty), \int G^{\mu_1 \dots \mu_{r-2k} \nu_1 \dots \nu_{2k}} : \mathcal{A}_{\mu_1} \dots \mathcal{A}_{\mu_{r-2k}} : \\ \times : C(x_{r-2k+1}) \dots C(x_r) : dx_1 \dots dx_r \Psi(-\infty)). \quad (\text{a.11})$$

If one remembers the very character of the state vector, (a.11) becomes a sum of only those terms which contain a factor of the form

$$(\Psi''(+\infty), \underbrace{f^* \dots f^*}_k \underbrace{f \dots f}_k \Psi''(-\infty)).$$

Now, it must be postulated that the convergence factor g_n should satisfy the following more stringent condition which is satisfied e.g. for $g_n = e$

$$\lim_{g_n \rightarrow 1} (\Psi''(+\infty), \underbrace{f^* \dots f^*}_k \underbrace{f \dots f}_k \Psi''(-\infty)) = 0. \quad (\text{a.12}) \\ (k=1, 2, \dots).$$

Thus the nonvanishing term in (a.8) is

$$(\Psi(+\infty), \int G^{\mu_1 \dots \mu_r} : \mathcal{A}_{\mu_1} \dots \mathcal{A}_{\mu_r} : dx_1 \dots dx_r \Psi(-\infty))$$

in complete agreement with the expression (a.6)'.

Q. E. D.

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β -Ray Spectrum and f t -Values in the Forbidden Transitions*Seitaro NAKAMURA, Hisao TAKEBE
and Minoru UMEZAWA*Department of Physics, University of Tokyo*

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It is shown that the β -spectra of Tc^{99} , Sb^{124} in the second-forbidden transition, and Tm^{170} in the first forbidden transition, can be explained by the tensor or vector interaction of the Fermi theory, if a suitable linear combination of matrix elements are chosen.

Using the ratio of the matrix elements to provide a good fitting with the experimental spectra, f t -values of each β -decay are evaluated on the corrected forbidden formula. Criticism is made with respect to the explanation of the beta spectrum of RaE by Konopinski and Uhlenbeck.

§ 1. Introduction

Forbidden β -ray spectra predicted by the Fermi theory have different shapes corresponding to various interaction types and selection rules of β -transitions. Discoveries of new type spectra verified the occurrence of forbidden transitions, i.e., 'a' type spectra imply the transition with $\Delta J = \pm 2$, parity change yes, and 'D₂' type spectra with $\Delta J = \pm 3$, parity change no. They can be explained by the tensor interaction, the former in the 1st forbidden, the latter in 2nd forbidden transition; both transitions select the unique nuclear matrix element, B_{ij} and S_{ijk} , respectively. Recently, another new type of forbidden spectrum of Cl^{36} was successfully explained by C. S. Wu and L. Feldman¹⁾ by means of a linear combination of A_{ij} and T_{ij} terms in the second forbidden tensor interaction. This provides a strong support for the forbidden theory, and stimulated us to develop further interpretation of forbidden spectra on the same principle. Correction factors given by Konopinski and Uhlenbeck can be written as follows, if the selection rule, $\Delta J = \pm 2$, parity change no, holds,

$$C_{2T}/\sum |T_{ij}|^2 \sim k_{2T}^2(3a) + \frac{1}{12}(3D_+ - c) - k_{2T}E,$$

$$C_{2V}/\sum |R_{ij}|^2 \sim k_{2V}^2(3a) + D_+ - 2k_{2V}E \quad (1)$$

for the second forbidden transition, and if the selection rule $\Delta J = \pm 1, 0$, parity change yes is valid,

$$C_{1T}/|\{\boldsymbol{\sigma} \times \mathbf{r}\}|^2 \sim k_{1T}^2 + A_+ - k_{1T}B,$$

$$C_{1V}/|\{\mathbf{r}\}|^2 \sim k_{1V}^2 + A_+ - k_{1V}B \quad (2)$$

for the first forbidden transition, where

$$\begin{aligned} k_{1T} &= \frac{|\mathbf{a}|}{|\mathbf{\sigma} \times \mathbf{r}|}, & k_{2T} &= \frac{\sum |A_{ij}|}{\sum |T_{ij}|}, \\ k_{1V} &= \frac{|\mathbf{a}|}{|\mathbf{r}|}, & k_{2V} &= \frac{\sum |A_{ij}|}{\sum |R_{ij}|} \end{aligned} \quad (3)$$

and another notations are indicated in Konopinski's paper.²⁾

If we plot $\sqrt{\frac{N}{C_{ij}F(\pm Z, W)\eta^2}}$ against W , the straight line Fermi plot should be obtained when the selected correction factor C_{ij} is really valid in the respective β -transition. C_{ij} represents C_{1V} , C_{1T} , C_{2V} , C_{2T} , etc., and $\eta = \sqrt{W^2 - 1}$, W_0 the maximum energy of the particle emitted. When the assumption $aZ \ll 1$ is not valid, the expressions (1) and (2) become bad approximations. For instance, C_{1T} should be replaced by the following formula: (See Osoba)³⁾

$$C_{1T}/|\mathbf{\sigma} \times \mathbf{r}|^2 = L_0 k_{1T}^2 + \dots \left(\frac{1}{6} K^2 L_0 + \frac{1}{2} L_1 + M_0 - \frac{2}{3} K N_0 \right) - k_{1T} \left(\frac{1}{3} K L_0 - N_0 \right)$$

$$\text{where } L_1 = \frac{F_1}{F} \frac{\eta^2}{9} \frac{2 + S_1}{4},$$

$$\begin{aligned} F_1/F &= \left[\frac{12\Gamma(1+2S)}{\Gamma'(1+2S_1)} \right]^2 (2\eta R)^{s(s_1-s-1)} \left| -\frac{1}{4} a^2 Z^2 + i a Z \frac{W}{\eta} \right|^2 \left(1 - \frac{a^2 Z^2}{4} C \right) \\ &+ \frac{a^2 Z^2}{4} \frac{a^2 Z^2 W^2}{\eta^2} \sum_{\nu=1}^{\infty} \frac{1}{\nu \left(\nu^2 + \frac{a^2 Z^2 W^2}{\eta} \right)} + i \left[a^2 Z^2 \frac{a Z W}{4\eta} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2 + \frac{a^2 Z^2 W^2}{\eta}} \right]^2 \end{aligned} \quad (4)$$

where C is Euler's constant, and the second column of (4) can be neglected. Another notations are all indicated in page 315 of Konopinski and Uhlenbeck's paper. (see ref. 14)

We have firstly tried to reinterpret the β -ray spectra of Tc^{99} and Sb^{124} by means of a correction factor C_{2T} or C_{2V} in (1), and of Tm^{170} by C_{1T} or C_{1V} in (2) with (4). The results are very satisfactory, and we obtained a straight line Fermi plot down to very low energy region from W_0 in each case. Next, we compare the required selection rules with the predictions⁴⁾ on the nuclear shell theory given by Mayer. Finally, using the k values determined by the requirement to provide a best matching of C_{2T} , or C_{1T} with the measurements in each β -decay, we evaluate their ft values on the forbidden probability functions. These ft values should be proportional to $\{\sum |T_{ij}|\}^{-2}$, or $|\mathbf{\sigma} \times \mathbf{r}|^{-2}$, respectively. The choice of interaction type will also be discussed.

§ 2. Tc^{99}

If we take $k_{2T}=7.7$, the forbidden Fermi plot becomes a straight line down to $W=1.35 \text{ } mc^2$ from the maximum energy $W_0=1.57 \text{ } mc^2$, as is seen in the Fig. 2.⁵⁾ The spin of the ground state of Tc^{99} was determined⁶⁾ to be $9/2$. On

the basis of the nuclear shell model, we can expect the ground state of Tc^{99} to arise from the combination of $g_{9/2}$ proton with a even state neutron; the resultant state should then have even parity. The ground state of Ru^{99} presumably may be constructed by the $d_{5/2}$ (or $g_{7/2}$) neutron with a even s state proton. If the ground state of Ru^{99} is $d_{5/2}$, the required selection rule, $\Delta J = \pm 2$, no parity change, will be satisfied.

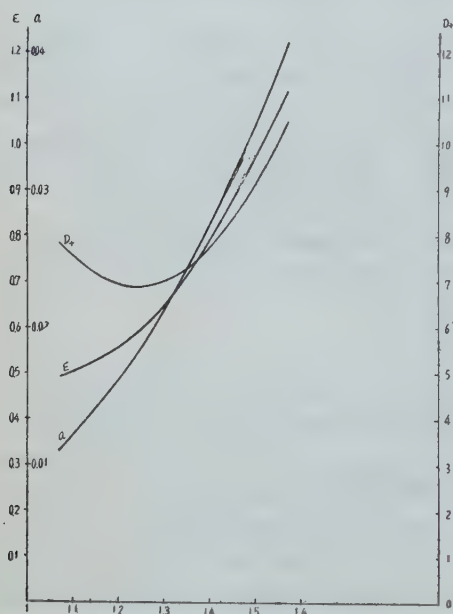


Fig. 1. Each of the second forbidden correction factors in the tensor interaction C_{2T} for Tc^{99} .

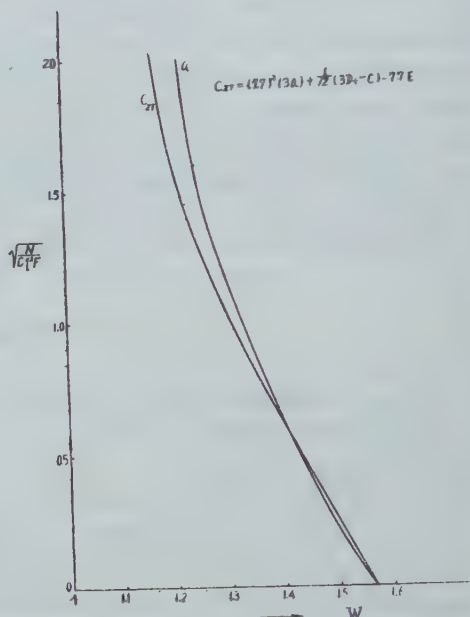


Fig. 2. ^{99}Tc . The forbidden Fermi plot in C_{2T} is straight from $W_0 = 1.57 mc^2$ down to $W = 1.3 mc^2$, if we take the ratio $k_{2T} = 7.7$.

§ 3. Sb^{124}

We found that $k_{2T} = 13$ results in a straight line forbidden Kurie plot from $W_0 = 5.49 mc^2$ down to $W = 4 mc^2$ for the highest energy β -ray group of Sb^{124} , (see Fig. 4). The data was presented by M. M. Langer, R. D. Moffat and H. D. Price,⁷⁾ Jr. who showed 'a' type interaction C_{1T} provides a good explanation for it. In the light of their analysis concerning $\text{Sb}^{124} \rightarrow \text{Te}^{124}$ decay schemes, we can take his alternative explanation: i.e., if C_{2T} transition ($\Delta J = \pm 2$, no) is valid in this transition, odd parity and spin 3 ground state ($g_{7/2}$, $h_{11/2}$) of Sb^{124} goes to the odd parity excited state (presumably spin 1) of Te^{124} , which is accompanied by the electric dipole radiation, 0.607 MeV γ -ray, reaching the even parity and spin 0 ground state of Te^{124} . On Mayer's nuclear shell model, however, it seems more reasonable to assume a ($g_{7/2}$ proton, $s_{1/2}$ neutron)-state, thus even parity, for the ground state of Sb^{124} , and consequently to assign even parity for

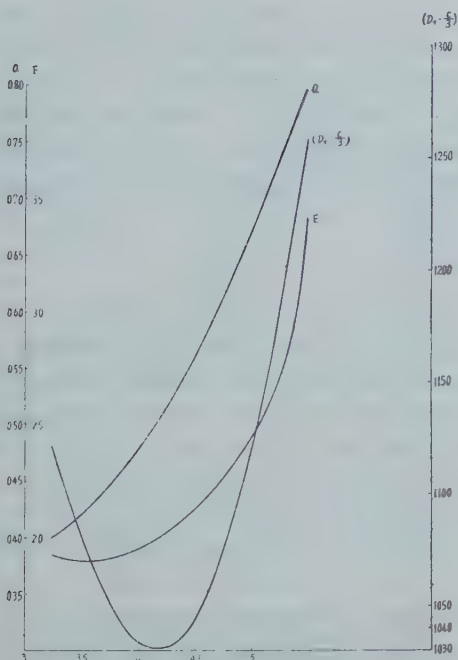


Fig. 3. Each of the second forbidden correction factors in the tensor interaction C_{2T} for Sb^{124} .

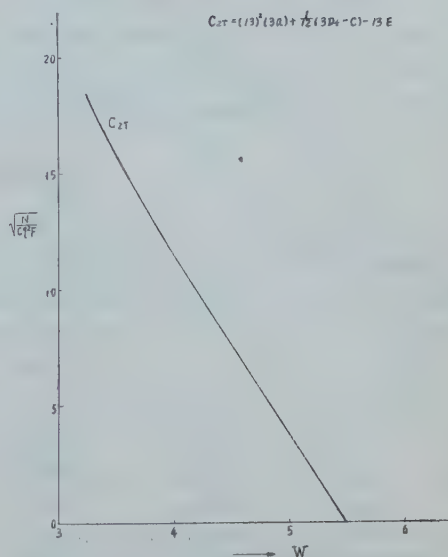


Fig. 4. Sb^{124} . The forbidden Fermi plot in C_{2T} is straight from $W_0=5.49 \text{ mc}^2$ down to $W=3.7 \text{ mc}^2$, if we take $k_{2T}=13$.

the excited state of Te^{124} , which is the final nucleus of 5.49 mc^2 β -ray. Anyhow, the portion of a straight line is greater for C_{2T} than for C_{1T} , as is seen in Fig. 4, which speaks in favor of our interpretation. In fact, C_{1T} ($4J=\pm 2$, yes) gives 10^{12} for ft values of Sb^{124} (See Table I), which seems too great to be classified to 'a' type groups, where ft values lies in the region $10^7 \sim 10^8$. (See M. Take-tani et al and J. Davidson.)

§ 4. ${}_{69}\text{Tm}^{170}$

The allowed Fermi plot presented by Agnew⁷⁾ shows a slight concave curvature to W axis at the low energy region. We took $k_{1T}=10.3$ and plot $\sqrt{\frac{N}{C_{1T} F(Z, W) \eta^2}}$ against W , and obtained a straight line forbidden plot (See Fig. 6).

In the case of $C_{1T}=10.5$ yields also a straight line Fermi plot. It is important to consider the accurate approximation on the correction factor, since the assumption $aZ \ll 1$ does not hold for ${}_{69}\text{Tm}^{170}$. If our interpretation is valid, this is a new type of first forbidden spectra following the selection rule $4J=\pm 1$, and parity change yes.

It is to be noted that the portion of a straight line is far greater for the forbidden plot C_{1T} or C_{1V} , than the allowed Fermi plot, although there was not

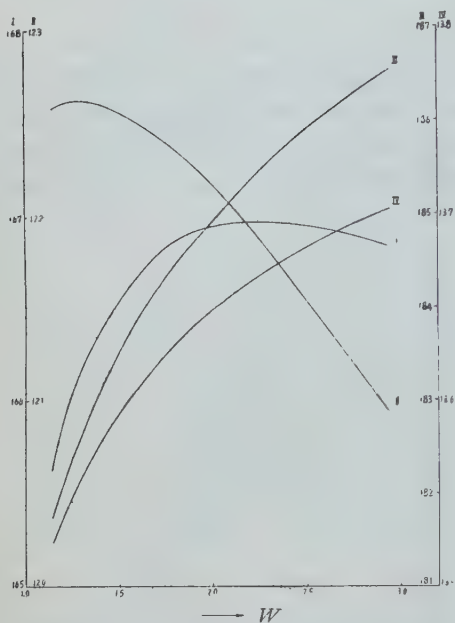


Fig. 5. Each of the first forbidden correction factors in the tensor interaction C_{1T} for Tm^{170} . I, II represent the 2nd and 3rd term of (2) with exact coulomb factors, (4), III, IV those with the assumption $aZ \ll 1$.

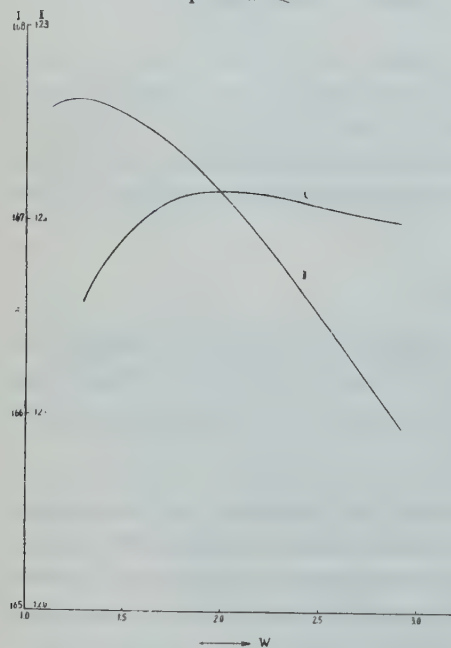


Fig. 7. Each of the first forbidden correction factors in the vector interaction C_{1V} for Tm^{170} , with exact coulomb factors (4).

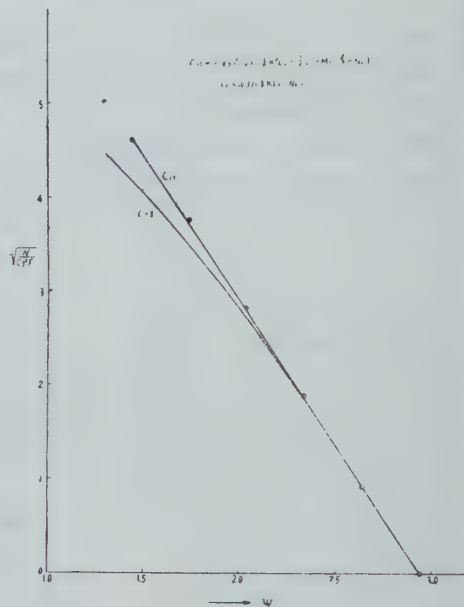


Fig. 6. Tm^{170} . The forbidden Fermi plot in C_{1T} is straight from $W_0 = 2.94 mc^2$ down to $W = 1.5 mc^2$, if one take $k_{1T} = 10.3$.

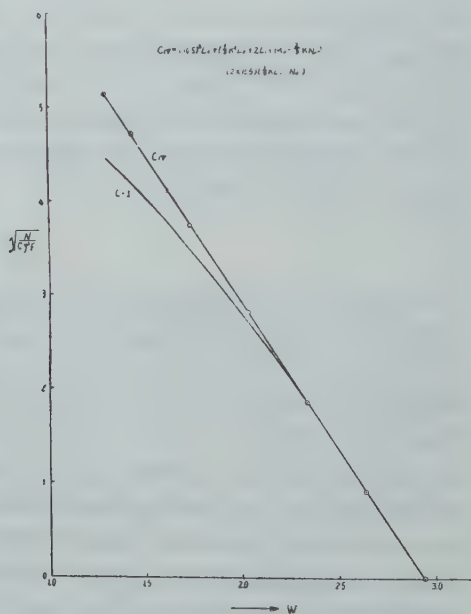


Fig. 8. The forbidden Fermi plot in C_{1V} is straight from $W_0 = 5.49 mc^2$ down to $W = 1.3 mc^2$, if we take $k_{1T} = 10.5$.

much to choose between C_{1T} and C_{1r} . According to the current shell model, one might expect the 69-th proton to $s_{1/2}$ or d . The 101-st neutron occurs in the $p_{1/2}$ state. The resultant state will be odd parity and presumably spin 3 state. The ground state of ${}_{70}\text{Yb}^{170}$ may be even, zero spin state and its excited state even and spin 2. Thus the required selection rule $\Delta J = \pm 1$, parity change yes, can be satisfied for the transition $\text{Tm}^{179} \rightarrow \text{Yb}^{170}$ accompanied by 85Kev γ ray emission.

§ 5. ft -values

Once the ratios k 's of the nuclear matrix elements are determined, we can immediately evaluate the ft values for each β -decay. Since the f -functions by Nakamura, Shima and Kobayasi⁽⁹⁾ was based on the assumption $aZ \ll 1$, we have multiplied them by the suitable factor perspected from the Figures of Feenberg and Trigg.⁽¹⁰⁾ The results are shown in Table I.

Elements	$t(\text{sec})$	$W_0(mc^2)$	yes, $\Delta J = \pm 2$ if $tf_{1T}(B_{1j})$	no, $\Delta J = \pm 2$ tf_{2T}	k
Cl ³⁶	1×10^{13}	2.4	6.1×10^{12}	9.8×10^{13}	$\sqrt{18}$
Tc ⁹⁹	7×10^{12}	1.57	1.1×10^{11}	8.4×10^{12}	7.7
Sb ¹²⁴	5×10^{16}	5.5	1.0×10^{10}	2.4×10^{12}	13

Table I. ft values of β -decay in the second forbidden transition

f_{1T} , f_{2T} are the first and the second forbidden f -functions, respectively; the coulomb corrections are also made. (see ref. 13. Appendix)

If we take into account the fact that all β emitters of 'a' type group ($\Delta J = \pm 2$, yes) have $ft = 10^8$ (see references 11, 12 and 13), it seems more natural to classify Cl³⁶, Tc⁹⁹, and Sb¹²⁴ into C_{2T} groups ($\Delta J = \pm 2$, no), rather than into 'a' type groups. Theoretically, tf_{2T} should be proportional to $\{\sum |T_{ij}|^2\}^{-1}$

The f -function for C_{1T} of (2) is⁽⁹⁾ (+ for electron, - for positron),

$$f_{1T}(|\sigma \times r|) = k_1 r^2 I_0 + \left\{ I_2 + \frac{a^2 Z^2}{4R^2} I_0 \pm \frac{az}{R} I_1 \right\} - k_{1T} \left\{ I_1 \pm \frac{aZ}{R} I_0 \right\},$$

$$I_0 = \left(\frac{W_0^4}{30} - \frac{3W_0^2}{20} - \frac{2}{15} \right) R + \frac{W_0}{4} L,$$

$$I_1 = \left(\frac{W_0^5}{90} - \frac{7W_0^3}{90} - \frac{9W_0}{40} \right) R + \left(\frac{W_0^2}{4} + \frac{1}{24} \right) L,$$

$$I_2 = \left(\frac{W_0^6}{210} - \frac{11W_0^4}{280} - \frac{901W_0^2}{5040} - \frac{1}{63} \right) R + \left(\frac{W_0^4}{6} + \frac{W_0}{16} \right) L,$$

$$R = (W_0^2 - 1)^{1/2}, \quad L = \ln(W_0 + (W_0^2 - 1)^{1/2}). \quad R: \text{nuclear radius.}$$

Elements	$t(\text{sec})$	$W_0(mc^2)$	yes, $\Delta J = \pm 1.0$ $tf_{1T}(\sigma \times r)$	k_{1T}
Tm ¹⁷⁰	1.09×10^7	2.94	6×10^{10}	10.3

Table II. ft values of β -decay in the first forbidden transition

Theoretically, $tf_{1T}(|\sigma \times r|)$ should be proportional to $|\sigma \times r|^{-2}$.

It is interesting to note that the empirical ft values of different transitions, i.e., the allowed ($\int \sigma$), the first forbidden ($B_{ij}, |\sigma \times r|$), and the second forbidden (T_{ij}, S_{ijk}), the third forbidden (S_{ijkl}) fall into distinct groups, i.e., 10^4 , $10^8 \sim 10^{10}$, and $10^{12} \sim 10^{13}$, 10^{16} , respectively. These ft values are very larger than those initially accepted for each forbidden transition. Since the mean kinetic energy of a nucleon in the nucleus is ~ 30 Mev, v/c , the ratio of the velocity of nucleons to light velocity, will be $\sim 1/50$, which may account for a part of the above figures.

§ 6. Discussion of results

Present analysis seems to show definitely that the tensor or vector interaction is indispensable for explanation of the forbidden transitions of Tc^{90} , Sb^{124} , Tm^{170} . Further selection between tensor and vector interactions may be indicated in developing other forbidden transitions. If the strict allowed shape should really appear in the transition $\Delta J=0$, yes, pseudoscalar interaction $\int \gamma_5$ would be necessary, since it is evident that tensor or vector interaction can not perfectly cancel out the contribution of other correction factors than that referred to $\int a$ in the 1st-forbidden formula.

The most likely explanation of the occurrence of different k values appears to be that the relation

$$\begin{aligned} \int a &= iW_0 \int r, \quad \sum |A_{ij}| = iW_0 \sum |R_{ij}|, \\ A_{ij} &= i \frac{W_0}{2} T_{ij} - \frac{i}{2} \int \frac{dx_j}{dt} [\sigma \times r]_i + \frac{dx_i}{dt} [\sigma \times r]_j \end{aligned} \quad (5)$$

hold in the Pauli approximations.¹⁴⁾ Actual values for k , indicated in the Table I, II, seem roughly to confirm these relations. The large deviations may be due to the difference between the maximum energy W_0 and the level spacing of the last nucleon which undergoes β -transition, in terms of Mayer's shell theory.

A simple calculation¹⁴⁾ shows that

$$\begin{aligned} \sum |T_{ij}|^2 &\approx \frac{4}{3} R^4, \\ \sum |B_{ij}|^2 &\approx \frac{20}{9} R^2 \end{aligned} \quad (6)$$

holds. If we take into consideration that, A being the mass number,

$$R = \frac{1.4 \times 10^{-13} \text{ cm. } A^{1/3}}{\hbar/mc}$$

and the characteristic time of β -decay T is 3.1×10^8 sec, we have

$$\sum |T_{ij}|^2 = \frac{3.1 \times 10^3 \ln 2 \text{ sec}}{t f_{2T} \text{ sec}} = \frac{4}{3} \left(\frac{1.4}{385} \right)^4 A^{4/3} = 2.3 \times 10^{-10} A^{4/3}, \quad (7a)$$

$$\sum |B_{ij}|^2 = \frac{3.1 \times 10^3 \ln 2 \text{ sec}}{t f_{1T} \text{ sec}} = \frac{20}{9} \left(\frac{1.4}{385} \right)^2 A^{2/3} = 29 \times 10^{-6} A^{2/3}. \quad (7b)$$

The figure of Table I, obtained from the present analysis seems to confirm these relations.

§ 7. Remarks on the beta spectrum of RaE

History of the explanation of the beta spectrum of RaE has many complications before the current theory of Konopinski and Uhlenbeck in Fermi theory of the second forbidden transition are accepted. The spectrum shape of RaE exhibited a marked deviation from that of the allowed transition in Fermi theory. The deviation could not be attributed to experimental errors, since many refined tools has repeatedly confirmed the same result. (The experimental materials up to 1950 are summarized in the excellent review articles by C. S. Wu).¹⁵⁾ In 1935, Konopinski and Uhlenbeck advanced a modification of the Fermi theory, which involves the time derivative of the neutrino wave function antisymmetrically. The success of the $K-U$ modification, however, has lost the light before the appearance of many beta spectra which have been very well investigated to show the exact Fermi type. Great advance has been made in the study of beta rays when Konopinski and Uhlenbeck¹⁴⁾ developed the calculations of the forbidden transitions in the Fermi theory. At that time, the most encouraging results of the forbidden theory appear in their explanation of the famous deviation of the spectrum of RaE from the allowed form, by means of a certain linear combination of several nuclear matrix elements of the Fermi theory, in the second forbidden transition with the vector or the tensor interactions. According to their formula, a correction factor to the allowed form will be approximately proportional to $(W_0 - W)^2$ for an element like RaE. This accounts for the agreements between the data and the so-called $K-U$ modification, without violating the success of the Fermi theory with respect to bulks of beta emitters.

Present analysis, indicated in the preceding chapters, will lead to a criticism of Konopinski and Uhlenbeck's view point on the explanation of the spectrum of RaE, by the following two reasons.

1) Following Konopinski and Uhlenbeck, the beta decay of RaE is classified to the second forbidden transition. Since the decay of RaE to the ground state of Po^{210} is, on the current version of the shell model, $(g_{9/2} \text{ or } i_{11/2}, h_{9/2})$ (even, even) and therefore involves a parity change, one would expect that the transition is first forbidden.

2) Conventional ft values of RaE is, according to the table of Feingold,¹⁷⁾ 1×10^8 . On the other hand, recent analysis of the second forbidden beta decay of Cl^{36} , Tc^{99} , Sb^{124} , and $\text{Cs}^{137, 160}$ by means of a linear combination of the several nuclear

matrix elements involved in the tensor or the vector interactions, indicate that these beta decays have $\sim 10^{13}$ for the conventional ft values. In the corrected ft values, shown in the Table I, they do not change very drastically. This speaks strongly in favor of the classification of RaE to the first forbidden, not to the second forbidden. In fact, with respect to the beta decays^{(11), (12), (13)} which have the 'a' type spectrum and obey the selection rule $\Delta J = \pm 2$, parity change yes, the ft values, are found to be $10^8 \sim 10^7$, Z dependence of which is little.

A reinvestigation of the beta spectrum of RaE is therefore undertaken on the first forbidden Fermi theory, using the detailed expansions of several gamma functions. (see the formula (4)). It is shown that in case of the transitions involving parity change,

1) $\Delta J = \pm 2$ is excluded. As is shown in Fig. 9, the correction factor of 'a' type is uniquely determined and the predicted shape is evidently at variance with the experiment.

2) $\Delta J = 0$ is also rejected. Since the ground state of Po^{210} is presumably spin 0 and even, the nuclear matrix element allowed in the transition is determined to be $\int \sigma \cdot r$. The correction factor corresponding to $\int \sigma \cdot r$ is almost energy independent, which can not be fitted to the experiment.

3) $\Delta J = \pm 1$ involves a linear combination of several matrix elements, i.e., $\int a$ and $\int \sigma \times r$ in the tensor interaction and $\int a$ and $\int r$ in the vector interaction. We have solved the algebraic equation in the three points. Unfortunately, any real value for the ratios of the nuclear matrix elements, k 's, can not afford the required shape of RaE (see Fig. 9).*

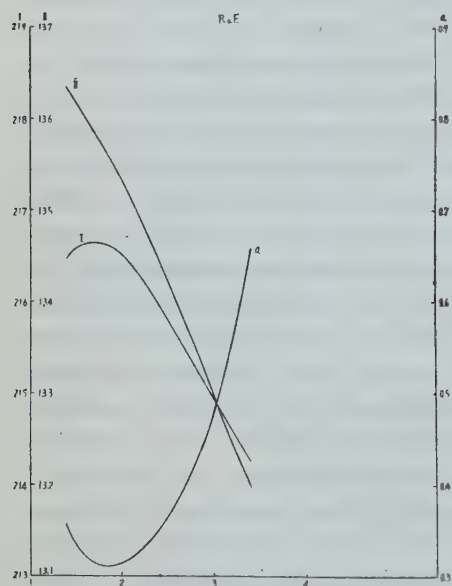


Fig. 9. The first forbidden correction factors in the tensor interaction, to be applied to the beta spectrum of RaE. a, I and II represent the correction factors corresponding to B_{ij} , $\int \sigma \times r$, and $(\int a) * \int \sigma \times r + cc$, respectively.

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Letters to the Editor

Generalized Furry's Theorem for Closed Loops

K. Nishijima

Department of Physics,
Osaka City University

June 15, 1951

In treating Feynman diagrams, we see that the "closed loops" reveal themselves as the most interesting objects. Furry's theorem¹⁾ and various selection rules for the meson decay²⁾ are well known, but we will try to prove them in a different way than usually did, since they are of formal interest.

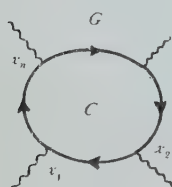


Fig. 1.

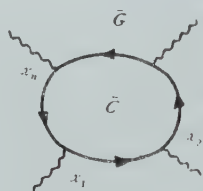


Fig. 2.

Now suppose a graph G containing a closed loop C . C is a closed directed Fermion polygon, at the vertices of which originate several Boson lines either free or virtual. (Fig. 1)

By reversing the direction of the closed loop C , we obtain another closed loop \bar{C} . (Fig. 2) Inserting \bar{C} into the graph G in place of C , we have a new graph \bar{G} . Apart from common factors, the contributions of G and \bar{G} to the S matrix are given respectively by

$$S_F(Q_1 S_F(x_1 - x_2) Q_2 \dots Q_n S_F(x_n - x_1)):$$

from C , (1)

$$S_F(S_F(x_1 - x_n) Q_n \dots Q_2 S_F(x_2 - x_1) Q_1):$$

from \bar{C} , (2)

and they must be added. The operator Q , expressing the type of coupling at each vertex point, can be decomposed into the direct product of a Dirac matrix O and an isotopic matrix T , i.e.

$$Q = O \cdot T. \quad (3)$$

Then the sum of contributions from G and \bar{G} is determined by the following quantity M :

$$M = a S_F(O_1 S_F(x_1 - x_2) O_2 \dots O_n S_F(x_n - x_1)) \\ + \bar{a} S_F(S_F(x_1 - x_n) O_n \dots O_2 S_F(x_2 - x_1) O_1),$$

where

$$a = S_F(T_1 T_2 \dots T_n), \quad \bar{a} = S_F(T_n \dots T_2 T_1)$$

Now making use of the familiar formula

$$S_F(\gamma_\alpha \gamma_\beta \dots \gamma_\lambda \gamma_\mu) \\ = S_F(\gamma_\mu \gamma_\lambda \dots \gamma_\beta \gamma_\alpha),$$

we have

$$M = a S_F(O_1 S_F(x_1 - x_2) O_2 \dots \\ O_n S_F(x_n - x_1)) + \bar{a} S_F(O_1^* S_F \\ (x_2 - x_1) O_2^* \dots O_n^* S_F(x_1 - x_n)), \quad (4)$$

where O^* is obtained from O by decomposing it into the product of elementary γ'_μ s and reversing the order. For instance,

$$O = \gamma_\alpha \gamma_\beta \dots \gamma_\lambda \gamma_\mu \longrightarrow O^* = \gamma_\mu \gamma_\lambda \dots \gamma_\beta \gamma_\alpha.$$

The essential relation in our discussion is

$$S_F(-x) = \gamma_5 S_F(x) \gamma_5. \quad (5)$$

Inserting (5) into (4), it follows

$$M = a S_F(O_1 S_F(x_1 - x_2) O_2 \dots O_n S_F(x_n - x_1)) \\ + \bar{a} S_F(O_1^* \gamma_5 S_F(x_1 - x_2) \gamma_5 O_2^* \gamma_5 \dots \\ \gamma_5 O_n^* \gamma_5 S_F(x_n - x_1) \gamma_5) \\ = a S_F(O_1 S_F(x_1 - x_2) O_2 \dots O_n S_F(x_n - x_1)) \\ + \bar{a} S_F(\bar{O}_1 S_F(x_1 - x_2) \bar{O}_2 \dots \bar{O}_n S_F(x_n - x_1)), \quad (6)$$

where $\bar{O} = \gamma_5 O^* \gamma_5 = \pm O$. We call the operator O an even (odd) matrix if $\bar{O} = O$ ($\bar{O} = -O$). (see Table). Now let the number of odd matrices in the Spur be N , then

$$M = (\alpha + (-1)^N \bar{\alpha}) S_P (O_1 S_{P'} (x_1 - x_2) O_2 \dots O_n S_{P'} (x_n - x_1)). \quad (7)$$

Calling our attention to the factor

$$\alpha + (-1)^N \bar{\alpha},$$

we have the following well known selection rules :

- (1) $\alpha = \bar{\alpha} = 0$, trivially forbidden.,
- (2) $\alpha = \bar{\alpha} \neq 0$, $N = \text{odd}$ is forbidden,
- (3) $\alpha = -\bar{\alpha} \neq 0$, $N = \text{even}$ is forbidden.

In quantum electrodynamics, $\alpha = \bar{\alpha} = 1$ and

Table of Parity

	S	P_S	P_V	V	T	$Pt(T)$
O	1	γ_5	$\gamma_5 \gamma_\mu$	γ_μ	$\gamma_{\mu\nu}$	$\gamma_5 \gamma_{\mu\nu}$
O^*	1	γ_5	$-\gamma_5 \gamma_\mu$	γ_μ	$-\gamma_{\mu\nu}$	$-\gamma_5 \gamma_{\mu\nu}$
\bar{O}	1	γ_5	$\gamma_5 \gamma_\mu$	$-\gamma_\mu$	$-\gamma_{\mu\nu}$	$-\gamma_5 \gamma_{\mu\nu}$

$O = \gamma_\mu$ (odd), thus we at once have Furry's theorem that the contributions from graphs containing odd order closed loops identically vanish.

What is interesting is the relation between the transformation $\bar{O} \rightarrow O$ and the charge conjugation, which we will investigate in the following. According to Schwinger²⁾, the charge conjugation matrix C has the following property :

$$C^{-1} \gamma_\mu C = -\gamma_\mu^T, \quad (8)$$

where O^T designates the transposed matrix of O .

By this definition of the matrix C , we immediately see

$$S_{P'}(-x)^T = C^{-1} S_{P'}(x) C, \quad O^T = C^{-1} \bar{O} C. \quad (9)$$

Inserting (9) into (2), follows the desired

relation ,

$$\begin{aligned} & S_P (S_{P'} (x_1 - x_n) O_n \dots O_2 S_{P'} (x_2 - x_1) O_1) \\ &= S_P (S_{P'} (x_1 - x_n) O_n \dots O_n S_{P'} (x_2 - x_1) O_1)^T \\ &= S_P (O_1^T S_{P'} (x_2 - x_1)^T O_2^T \dots O_n^T S_{P'} (x_n - x_1)^T) \\ &= S_P (C^{-1} \bar{O}_1 C \cdot C^{-1} S_{P'} (x_1 - x_2) C \cdot C^{-1} \bar{O}_2 C \dots C^{-1} \bar{O}_n C \dots) \\ &= S_P (\bar{O}_1 S_{P'} (x_1 - x_2) \bar{O}_2 \dots \bar{O}_n S_{P'} (x_n - x_1)). \end{aligned}$$

Thus the relation is clarified.

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On the Nature of V-Particles, I

Y. Nambu, K. Nishijima and
Y. Yamaguchi

Department of Physics,
Osaka City University

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Recently Butler et al.¹⁾ presented further evidences concerning the nature of the so-called V -particles which had been observed by Rochester and Butler²⁾ and by Anderson and collaborators³⁾. According to these authors: i) The V -particles are found among penetrating showers with a rate of the order of 1% ; ii) Their decay life is estimated to be about 10^{-10} sec ; iii) There are two kinds of them, charged and neutral, the latter being about 5 to 10 times more abundant than the former, presumably due to their difference in lives ; iv) According to Butler et al., moreover, they can be

classified in two groups of different masses. The heavier particles, with mass about 2200 to 2300 m_e , decay into a nucleon and a (π) meson, hence fermions. The lighter ones, with mass around 1000 m_e , decay into two (π) mesons, and hence bosons; v) One case is reported which shows a successive decay of a charged V through neutral V into three charged particles (three mesons or two mesons plus one proton), though no suggestion is given as to whether they belong to the heavier or the lighter group.

Then the arguments go on as follows. First of all, there is a possibility that the decay products of V and τ might include μ -mesons as well. This, however, seems rather improbable in view of the fairly large nuclear interaction of the decay mesons reported by Anderson et al.³⁾, and the relatively long life of the parents in contrast to their sizable production rate. A three particle decay, such as $V \rightarrow P + \mu + \nu$, is all the more unlikely according to Anderson et al.'s arguments about the coplanar character of the parent and daughter particles. It is also to be noted that soft showers are not associated with the V or τ events, which shows that decays giving rise to photon or electron must be rare compared to the main processes. The above mentioned contradiction between production and decay may be lifted if we postulate, as was just the case in the π - μ decay, that the observed V (though not necessarily including τ) are decay products of some unknown particles with sufficiently short life and strong nuclear interaction. But at the present stage we will try to solve this problem using only the observed particles, assuming the decay products to be π 's and nucleons (of course excluding the case V). This can be done as follows:

Now we face two alternative interpretations of the point v) above. One is to assume that the unknown particles involved in this reaction are V 's, decaying according to the scheme:

$$V_{\pm} \rightarrow V_0 + \pi_{\pm}, \quad V_0 \rightarrow N + \pi$$

$$(N = \text{nucleon}); \quad (1)$$

while the other is to assume them to be τ 's:

$$\tau_{\pm} \rightarrow \tau_0 + \pi_{\pm}, \quad \tau_0 \rightarrow 2\pi \quad (2)$$

According to (1), the V 's may be regarded as excited states of nucleons which make transitions to the lower states by emitting mesonic radiation. (τ -emission can be forbidden energetically.) Such a situation has long since been anticipated in the strong coupling theory of nucleon-meson interaction. This theory however, does not just seem to be very useful for our analysis. For it is too crude and incomplete to be relied upon quantitatively, especially in view of the extremely small width of the actual levels ($\sim 10^{-5}$ ev). In the present stage we had rather better treat them as different elementary particles, obeying Fermi statistics and having half-odd spins, possibly higher than 1/2, and introduce formal interactions which cause the observed transitions. Thus we assume the following scheme

Interaction	$\underbrace{V_{\pm} V_0 \pi_{\pm}, V_0 N \pi}_{G_1}$	$\underbrace{V_{\pm} N \tau, V_0 N \tau}_{G_2}$
Coupling const.	G_1	G_2
$V V \pi$	$N N \pi$	$V V \tau$
G_3	$g_3(\text{known})$	G_4
		g_4

$$(3)$$

The conditions to be considered are: a) decay life roughly all of order 10^{-10} sec.; b) competition among various possible decay modes (especially for the τ decay process); and c) production mechanism, to give a yield of $\sim 10^{-2}$ times that of ordinary mesons.

For the calculation of $\tau \rightarrow \pi + x$ ($x = \pi, \tau, \dots$) processes, the results of covariant calculation by many authors⁶⁾ can directly be applied with only a few alterations. Unfortunately, however, most of the calculation involved are not free from the diverging ambiguities which have to be disposed of with the aid of the Pauli regulator. Accordingly, in the present order of magnitude consideration, we check them with more rough and intuitive

estimation which only takes account of the coupling constants and the volume of phase space—in a manner more or less similar to Fermi's.⁷⁾ On the other hand, the various selection rules, such as described by Fukuda et al.,^{6e)} are more reliable and can be used to narrow down the possibilities.

In this way we get the following results:

$$G_1^2 \sim 10^{-11} - 10^{-13}, \quad G_2^2 \sim 10^{-2} - 10^{-3}, \\ g_4^2 \sim 10^{-7} - 10^{-9}$$

$$(G_3^2 \gtrsim g_3^2, \quad G_4^2 \lesssim g_4^2, \text{ not necessary}). \quad (4)$$

The transformation property of τ (or at least τ_0) must be either scalar or vector, since it is very likely that the π mesons are pseudo-scalar.⁸⁾ The range of values in (4) correspond to different assumptions as to Fermi's reaction volume (or alternatively the general trend of the covariant calculation), as well as the assignment of spin values for the V 's*. Thus discrepancies with experiment of the order, say, 100, should be tolerated.

Next let us examine the assumption (2). In this case, τ_{\pm} must be pseudoscalar, while τ_0 scalar or vector.* This choice, which may also be adopted in the first model (1), excludes the process $\tau_{\pm} \rightarrow 2\pi$ in favor of $\tau_{\pm} \rightarrow \tau_0 + \pi$ and $\tau_{\pm} \rightarrow 3\pi$ (Powell's case).^{5e)} There arise the following nine couplings:

$$\begin{array}{lll} VN\pi, & VN\tau_{\pm}, & VN\tau_0, \\ VV\pi, & VV\tau_{\pm}, & VV\tau_0, \\ NN\pi, & NN\tau_{\pm}, & NN\tau_0. \end{array} \quad (5)$$

Of these, $NN\pi$ is known, and $NN\pi_{\pm}$ and $NN\tau_0$ and $VN\tau_0$ turn out either unnecessary or harmful. Among many possible combinations of the remaining couplings few yield consistent results. Thus we take as a relatively reasonable choice (see Appendix):

$$\begin{array}{lll} VN\tau_0, & VV\tau_{\pm}, & VV\tau_0 \\ G_1^2 \sim 10^{-2}, & G_2^2 \sim 10^{-1.5}, & G_3^2 \sim 10^{-6}, \\ VV\pi, & VN\pi, & \text{others} = 0. \end{array} \quad (6)$$

$$G_4^2 \sim 10^{-1.5}, \quad G_5^2 \sim 10^{-12},$$

These estimations are of course susceptible

to fluctuations, by as large a factor as $\sim 10^2$, depending on different assumptions on the calculational procedure.

At present we cannot tell with confidence which of the above two alternatives (1) and (2) is the more preferable. The former is relatively free from ambiguities and confusions in determining the coupling constants. The idea of the existence of a series of excited levels of nucleons also attracts us. On the other hand, however, the latter cannot be excluded and even seems natural in view of the other evidences concerning τ -mesons, e.g. Powell's $\tau_{\pm} \rightarrow 3\pi$ decay, which can well compete with $\tau_{\pm} \rightarrow \tau_0 + \pi$ on this model. It is hoped that the forthcoming paper of Butler et al. will settle this question.

In conclusion, we should like to call attention to some effects which could be related to the V and τ mesons. First, the nuclear force would be modified. But the range being at most only $\sim 1/m_{\tau} \sim 2/m_N$, the effect would be too small to account for the existing N - N scattering data. Second, the conventional strong coupling theory, which allows the nucleon isobars of both charges to occur not as anti-particles, could be tested by the experiments, in which the stability of nuclei should also be taken account of. (According to our models, the anti-particles can only appear as pairs.)*** Third, the new particles would play some part in the anomalous magnetic moment of nucleons, though we do not know how and to what extent. Fourth, the production of these particles, though depending on the model, would occur mainly a V - τ pairs and V - V pairs, and to a lesser degree as single (or multiple) τ 's. The corresponding threshold energy would be about 1.1 Bev ($\gamma + N \rightarrow V + \tau$) and 0.8 Bev ($N + N \rightarrow V + V$) respectively in the laboratory system. So far these predictions are not definitely at variance with experiments.⁹⁾

A more detailed account of the present analysis will be given later.

Appendix

$$\cdot 10^{23} \text{sec}^{-1} \quad \text{for } \tau_0 \rightarrow 2\pi,$$

On the second model (Eq. (2)), the possible competing processes which need consideration are as follows**

$$\begin{aligned} \tau_0 &\rightarrow 2\pi, & \tau_0 &\rightarrow \pi_0 + \gamma, & \tau_0 &\rightarrow 2\gamma; \\ \tau_{\pm} &\rightarrow \tau_0 + \pi_{\pm} & \tau_{\pm} &\rightarrow 3\pi, & \tau_{\pm} &\rightarrow \pi_{\pm} + 2\gamma; \\ & & & & V &\rightarrow N + \pi. \end{aligned}$$

From these processes we can first determine the relative magnitude of the coupling constants so as to fit the experimental facts, and then normalize them by some process such as the production rate. A crude life-time formula is, for example, furnished by

$$1/t \sim (G_3^2)(G_4^2)^2 I_2 / m_{\pi} \sim (G_3^2)(G_4^2)^2$$

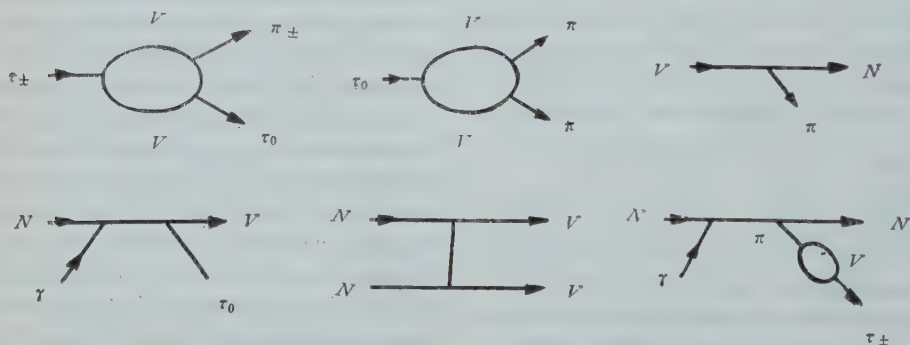
where I_2 means the available volume per unit energy of momentum space. An additional factor $\sim 10^{-3} (\sim (m_{\pi}/m_V)^3)$ may be introduced if we take account of the coordinate space volume corresponding to the third order process in which virtual V -pairs are created (see Figure). This estimation also agrees roughly with the results of covariant calculation using regulators. The numerical values given in (6) were obtained by the latter refined method assuming scalar and pseudoscalar coupling for the scalar τ_0 and pseudoscalar τ_{\pm} respectively.

The first model (1) may be treated analogously.

*) Spin 3/2 for V would make the decay $V \rightarrow N + \pi$ forbidden of first order if it were allowed for spin 1/2.

**) Actually pseudovector τ_{\pm} is to be discarded since they favor $\tau_{\pm} \rightarrow \pi_{\pm} + \gamma$ (assuming π to be pseudoscalar). For τ_0 , either pure neutral scalar or symmetrical neutral vector must be taken in order to allow the process $\tau_{\pm} \rightarrow \tau_0 + \pi_{\pm}$.

***) For example, the process: anti- $V_0' \rightarrow P_- + \pi_+$ would be much rarer than $V_0 \rightarrow P_- + \pi_+$ in our case. This is consistent with experiments.



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On the Nature of V-Particles, II

Y. Nambu, K. Nishijima
and Y. Yamaguchi

Osaka City University

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In a preceding letter¹⁾ we have proposed two possible interpretations of the phenomena associated with V -particles, mainly on the basis of Butler et al.'s recent observations.²⁾ Of course these interpretations have not been claimed to be the only and definite, but merely possible and more or less natural ones. Also the whole argumentation itself was not complete and exhaustive enough to get full insight into the nature of the V -particles. In view of these points, we will here supplement the previous letter with some more general considerations, while adding to our knowledge the experimental results of American groups³⁾ in so far as they are accessible to us.

Of the various properties of the V -particles which are known to us, the most remarkable seems to be their large yield and long life, two apparently contradicting properties on the basis of simple detailed balance consideration. They suggest that production and decay are not inverse processes and/or some kind of selection rules (in a very general sense) are at work in the decay reaction. On the other hand, among what are not yet clarified experimentally, there are such important things as the identification of the decay products of the V -particles and their decay modes. Thus at least some of the decay mesons could be μ -mesons (e.g. $\tau \rightarrow \pi + \mu$), and some of the decays could or should involve three or more product particles^{3a)} (e.g. $\tau + \pi_0 \rightarrow 2\pi_{\pm} V \rightarrow N + \mu + \nu$). Since no other crucial evidences are known (such as the mode of V production), we are led to a wide variety of possible interpretations if we will take all these points into account, and they can be recommended or disfavored only after a closer examination.

Following the above considerations, let us

first summarize as follows the various conceivable assumptions which shall serve to find out and characterize systematically the possible individual models: 1) Assume the V -particles as elementary entities obeying some basic field equations; or 1) Regard them as composite substance with substructure. 2) Regard V and τ as separate and independent events; or 2) Regard them as inherently related. 3) Assume the product mesons to be π 's only; or 3) They may involve μ 's as well. 4) Assume only two-particle decays; or 4) Allow of three or more particle decays. 5) In order to account for the large difference in production and decay rate: a) Use the relation in mass values (e.g. for production, $\tau + N \rightarrow V + \tau$, but for decay the inverse is forbidden energetically). b) Assume some unknown parents which, directly produced in nuclear events, decay instantly into the observed V 's (analogy of π and μ mesons). c) Assume an interaction mechanism such that V 's are generated in a nucleon-nucleon impact, but are hard to decay singly (e.g. $N + N \rightarrow V + V$). d) Take advantage of large internal angular momenta for V , which make the decay highly forbidden. e) Take advantage of parity and other selection rules related to transformation properties (including Furry's theorem). f) Other special devices. These various assumptions are not mutually exclusive, but may be appropriately combined. 6) Use the conventional perturbation theory (weak coupling); or 6) Exploit the strong interaction (like Fermi⁴⁾) or do not depend on the details of coupling at all (e.g. selection rules which can be enunciated by observing only the initial and final states).

By suitable combinations of these characteristic assumptions, we shall be able to arrive at a large number of models (or interpretations) which can explain more or less consistently the essential features of V events. We give below some of the representative models that follow in this way from our present general consideration, and make comments on

their merits and defects.

1) All observed processes are assumed to be direct ones. For example, we introduce the following couplings:

- a) $NNNV\tau$, $VN\pi$, $\tau\pi\pi$ ($\bar{1}2345c6$)*.
- b) $NV'\tau$, $V'V\pi$, $VN\pi$, $\tau\pi\pi$ ($m_{V'} > m_V + m_\pi$) ($12345b6$)⁵⁾,
- c) $NNVV$, $NN\tau\tau$, $VN\pi$, $\tau\pi\pi$ or $\tau\pi\mu$ ($12(3)45c6$).

This standpoint is equivalent to determining all the relevant terms of the S -matrix independently in so far as they do not lead to inconsistencies. Though formally possible, it is not a very attractive procedure since we little understand the nature of the events by such a highly phenomenological approach. The introduction of a short-lived parent V 's as in the above second model, seems to be an unnecessary complication unless some definite evidence on the existence of such a particle is presented.

2) Our two models proposed in I ($\bar{1}23(4)5ace6$).⁶⁾ They were specially designed to account for the successive decays of V or τ . But the pairwise production which they predict does not seem to be favored by experiments, if not yet rejected.

3) A three particle decay is assumed for V : $V \rightarrow N + \mu + \nu$, while the production occurs, for example, as $V + \pi$ pairs ($\bar{1}2345a6$). Since some evidences show that the decay does not necessarily follow a two-body scheme, such a possibility will not be excluded as responsible for at least part of the decays. (But a universal Fermi-type coupling ($g \sim 10^{-49}$) would lead to a rather long life ($> 10^{-6}$ sec).

4) Gamow-type model. Turning to the structural theories, the most naive one may be to regard V as a bound $N + \pi$ system ($\bar{1}2345cd6$). This, however, would require an unusually high potential barrier (or the order of several Bev) to assure the long life, and if this barrier were supplied by the centrifugal force, the angular momentum would have to be ~ 8 , which seems too high to be easily attained by nucleon-nucleon collisions

$(N+N \rightarrow V+V)$.

5) We may modify the above model so that V is a bound $N+\tau$ system. The assumed interaction is, for example,

$\tau\tau NN$ (for τ and V production), $\tau\pi\pi$ (for τ and V decay) (12345ac6),
or τNN , (for τ and V production), $\tau\pi\pi$ (for τ and V decay) („),

The decay of both V and τ are controlled by the same coupling $\tau\pi\pi$. The first model will predict the pairwise production, whereas on the second (τ must be a boson), the process $V \rightarrow N+\pi$ is threatened by $V \rightarrow N+\tau$.

6) Isomeric transition. Alternatively we amend the defects of model 4 by assuming that in the decay of V the π meson is radiated from a small volume of the dimensions of nucleon Compton wave length (12345cdf6). Using the well-known multiple radiation formula, the angular momentum L can be lowered to ~ 4 , which is not unreasonable to be realized by nuclear collisions. But in order to suppress the γ -emission, the bound meson field which is responsible for the excitation energy would have to be extremely rigid. If we calculate ad hoc the moment of inertia of the bound meson field around a nucleon and quantize this rigid body we obtain the observed excitation of ~ 200

Mev for $L=4\frac{1}{2}$ with the coupling constant $g^2 \sim 10$, and only the transitions to $L=1/2$ or $3/2$ are energetically allowed for π emission. Although this model is interesting, it is hard to be founded upon orthodox field theory. The conventional strong coupling theory, on the other hand, which is similar to above in physical ideas, does not seem to give the required long life of the excited states.

Some remarks may be added on the spin (or transformation property) of V and τ particles. We did not take it into account in the above classification since it did not seem essential in the characterization of the models. In general both fermion and boson property

may be allowed for these particles. But in any way the stability of protons and neutrons should be guaranteed. Thus, for instance, a coupling like $VN\mu$ (V =boson elementary particle), leading to $V \rightarrow \mu+N$, cannot be admitted.

In this way we have seen that at the present stage several different model can assert themselves just about as good. But, in general, we may say that the elementary particle theories (Alternative 1) are liable to predict pairwise production which remains to be confirmed by experiment, while the structural theories (Alternative 1) tend to suffer from high probabilities of radiative decay. It will be premature and useless to demand anything more definite from what we know at present about the V particles.

In concluding, we express deep gratitude to Prof. S. Hayakawa for informing us of the recent activities of American groups as his own opinions. The models mentioned here include those which have been proposed in America, but no published papers being available, we have refrained from explicit citation.⁷⁾ Thanks are also due to Messrs. G. Takeda, S. Takagi, H. Fukuda, K. Aizu, T. Kinoshita, H. Miyazawa and S. Ôneda for valuable discussions.

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- *) This notation signifies the nature of the model according to the above mentioned criteria.

The Effect of He³ Ingredient on the Thermal Rayleigh Disc Torque

S. Koide and T. Usui

Physics Department, Faculty of General Culture, University of Tokyo

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Recently¹⁾, we have studied the effect of He³ ingredient on the propagation velocity of sound in liquid helium II on the basis of the two fluid model. As a result of our analyses, it has been concluded that, in the sound fields, the He³ component moves in company with the centre of gravity. As one of the criteria of the above conclusion, it is suggested that thermal Rayleigh disc experiment on the second sound, as proposed by Pellam and Morse²⁾, may be considered valuable. The relevant formulae have been given in that paper. As its sequel, here will be given the results of some numerical computations.

With the same notations as in reference 1, the torque acting on a Rayleigh disc of radius a , set in the second sound field at the angle of attack $\pi/4$, is easily shown to be given by the following formula (compare reference 3) :

$$N = \frac{4}{3} \rho (1 - \xi_s) \frac{\bar{\xi}_n}{\xi_s} a^3 \left[\frac{\dot{H}}{T \rho \bar{\xi}_n (\partial S / \partial \bar{\xi}_n)_{p, T \xi_s}} \right]^2 \times \left(1 + \frac{1}{5} \left(\frac{\omega a}{c_2} \right)^2 + \dots \right),$$

where \dot{H} denotes the associate heat current density. Under the condition of constant \dot{H} , therefore, the ratio of torque acting on the same Rayleigh disc at the same temperature in the mixture to that in a pure helium II is given by

$$\frac{N}{N^0} = \frac{\xi_n^0 (1 - \xi_n^0)}{\xi_n (1 - \xi_n)} \left[\frac{(\partial S^0 / \partial \xi_n^0)_{p, T}}{(\partial S / \partial \xi_n)_{p, T \xi_s}} \right]^2 \times \left(1 + \frac{\omega^2 a^2}{5} \left(\frac{1}{c_2^2} - \frac{1}{c_2'^2} \right) + \dots \right),$$

where '0' denotes that the quantity is referred to the pure helium. Using the relation derived in reference 1 :

$$\left(\frac{\partial S}{\partial \xi_n} \right) = (1 - \xi_s) \left(\frac{\partial S^0}{\partial \xi_n^0} \right) \times \left[1 - \frac{R}{4 (\partial S^0 / \partial \xi_n^0)} \ln \frac{\bar{\xi}_n}{\xi_n + 4 \xi_s / 3 (1 - \xi_s)} \right],$$

and the necessary condition for H. London's condition⁴⁾

$$\left(\frac{\partial S^0}{\partial \xi_n^0} \right)_{p, T} = S_0(p, T) - S_{\lambda 0}(p),$$

we can calculate numerically the ratio N/N^0 , the results being listed in Table I. In this calculation, the values of $\xi_n(p, T, \xi_s)$ are taken from Table I of reference 1. This type of experiment will be highly desirable in order to check the above-mentioned picture.

$\xi_s(\%)$	$T(^{\circ}K)$	$N/N_0(\%)$
0.1	2.00	100
	1.90	99
	1.70	96
	1.50	92
0.5	2.00	103
	1.90	96
	1.70	86
	1.50	72
0.8	2.00	106
	1.90	95
	1.70	80
	1.50	64

Table I.

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Effects of Tensor Forces on the Elastic Scattering of Neutrons by Deuterons

H. Horie, T. Tamura and S. Yoshida

Department of Physics,
University of Tokyo

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Many authors calculated the cross sections of n - d scattering at high energies.¹⁾⁻³⁾ Recently, Gluckstern and Bethe⁴⁾ have carried out detailed calculation on the same subject, and compared the theoretical and experimental total cross section, limiting the interactions between two nucleons to central forces, and estimated the depth and exchange nature of n - n interaction potential. However, the importance of tensor forces to high energy nucleon-nucleon collision was pointed out by Christian *et al.*^{5), (6)}

We have calculated the elastic n - d scattering cross section at 90 MeV, using Born approximation, including tensor forces. Such calculation was also undertaken by Wu and Ashkin¹⁾, but they did not performed their calculation to its end.

As the procedure of our calculation is very similar to that of GB, we will show only the results. We obtained for the cross section

$$d\sigma(\theta) = \left(\frac{M}{3\pi\hbar^2}\right)^2 \left\{ \frac{2}{3} \left[\delta\xi(I_1 - I_2) + \frac{1}{2}(I_1 + I_2 - 2I_3) \right]^2 + \frac{8}{3} \sum_{m=-2}^2 \left[\gamma_{nm}(I_{1m} - I_{2m}) + \frac{1}{2}\gamma_{np}(I_{1m} + I_{2m} - 2I_{3m}) \right]^2 \right\}$$

$$+ \frac{1}{2}\gamma_{np}(I_{1m} + I_{2m} - 2I_{3m}) \Big]^2 + \frac{2}{3} \sum_{m=-2}^2 \left[\gamma_{nm}(I_{1m} - I_{2m}) + \frac{1}{2}\gamma_{np}(I_{1m} + I_{2m} + 4I_{3m}) \right]^2 \Big\} + \left(\frac{M}{3\pi\hbar^2}\right)^2 \left\{ \frac{1}{6} \left[\frac{3}{4}\xi(I_1 + I_2) + \frac{3}{8}\eta(I_1 + I_2) + \frac{1}{4}\xi\delta(I_1 - I_2) + \frac{1}{8}(I_1 + I_2 + 8I_3) \right]^2 + \frac{2}{3} \sum_{m=-2}^2 \left[\gamma_{nm}(I_{1m} - I_{2m}) + \frac{1}{2}\gamma_{np}(I_{1m} + I_{2m} - 2I_{3m}) \right]^2 \right\}$$

here ξ is the ratio of n - n singlet to n - p triplet potential depth.

δ is the ratio of n - n triplet to n - n singlet potential depth, ($\delta=1, 0$ and -1 for Wigner, Serber and Majorana forces, respectively).

η is the ratio of n - p singlet to n - p triplet potential depth.

γ_{np} and γ_{nm} are the ratios of n - p and n - n triplet tensor to n - p central potential depth, respectively. We employed for the n - p potential that used by Christian and Hart,⁵⁾ (Serber type Yukawa potential of range 1.35×10^{-13} cm). For n - n central potential, assuming charge independence, we used the same potential as Christian and Noyes,⁶⁾ and for n - n tensor potential the following three cases were used:

- (1) Christian and Noyes' ordinary Yukawa type attractive potential,
- (2) the same type repulsive potential,
- (3) no tensor force between n - n .

I_1 , I_2 and I_3 are the same as GB's formula (60), I_{1m} , I_{2m} and I_{3m} are the corresponding ones resulting from tensor potentials. I_1 , I_3 , I_{1m} and I_{3m} could be evaluated explicitly, but for the evaluation of I_2 and I_{2m} we had to integrate numerically. The elastic total cross sections obtained, together with the ones of GB and experimental data by Powell,⁷⁾ are tabulated in Table 1. We also calculated,

for comparison with GB, the cross section by pure central force, with the same range as above and the depth adjusted to low energy data. From this, we see, that the elastic total cross section without tensor forces is in good agreement with that of GB, in spite of the difference of force ranges. When tensor forces are included, we obtained results which agree nearly well with the experimental cross section, except for the case of Wigner forces.

In Fig. 1, we show the angular distributions of neutrons in c.m. system for the cases of

Table 1. Total elastic cross sections in mb.

n - n central forces exchange character	Wigner	Serber	Majorana
n - n attractive	96	50	49
n - n repulsive	101	55	52
n - n no tensor	91	45	42
no tensor force	119	50	26
GB	80	60	30
experimental value	48		

mb sterad

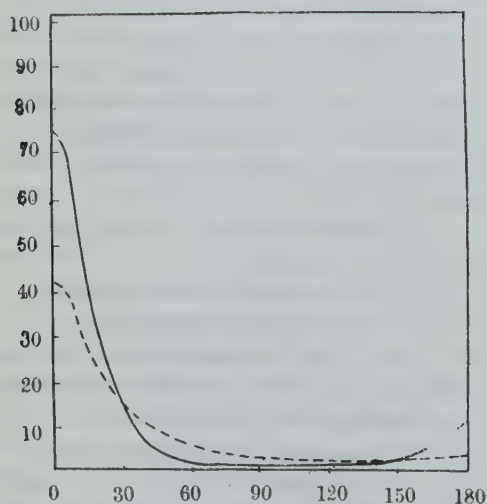


Fig. 1. Angular distribution of neutrons in c.m. system. — pure central force, central plus attractive tensor forces,

pure central and central plus attractive tensor force as a typical one (in both cases n - n central potentials are Serber type), and difference between them is marked. If the experiment on the angular distribution is performed, the nature of the tensor force between two neutrons may be revealed.

The authors wish to express our sincere thanks to Prof. T. Yamanouchi for his kind interest, and to Messrs. Y. Fujimoto and Y. Yamaguchi for their helpful discussions.

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Meson Current around the Nucleon

Y. Takahashi

*Institute of Theoretical Physics,
Nagoya University*

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In the previous papers, the magnetic moments of the nucleon was treated from the viewpoint of the *meson spectrum*.¹⁾ It has been shown by several authors that the contribution to the magnetic moment of the meson current around the nucleon is the same absolute value with opposite sign for proton and neutron either in the lowest order of the perturbation theory²⁾ or in the strong coupling³⁾ theory for a certain type of meson.

It is well known that the sum of the magnetic moments of proton and neutron deviates slightly from one nuclear magneton.

If the contribution from the meson current is the same magnitude with opposite sign for both nucleons, *this deviation is considered to be due to the nucleon current*. It is, therefore, interesting to examine the property of the meson current around the nucleon.

In the interaction representation, the scalar meson current⁽⁴⁾ of the first order in e is

$$j_\mu(x) = e \left\{ \frac{\partial \phi_1(x)}{\partial x_\mu} \phi_2(x) - \frac{\partial \phi_2(x)}{\partial x_\mu} \phi_1(x) \right\}. \quad (1)$$

This current becomes, in the mixed representation,

$$\begin{aligned} j_\mu(x) &= S(\infty) S^{-1}(\sigma) j_\mu(x) S(\sigma) \equiv \sum_{n=0}^{\infty} j_\mu^{(n)}(x) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_n P[j_\mu(x), H(x_1), \dots, \\ &\quad H(x_n)], \end{aligned} \quad (2)$$

with

$$H(x) = g_\lambda \bar{\psi}(x) \tau_\lambda \psi(x) \phi_\lambda(x). \quad (3)$$

The matrix element in which one nucleon and no meson are contained in the remote past and the infinite future becomes

$$\begin{aligned} \langle j_\mu(x) \rangle_{1,0} &= \sum_{n=0}^{\infty} \langle j_\mu^{(n)}(x) \rangle_{1,0}, \quad (4) \\ \langle j_\mu^{(n)}(x) \rangle_{1,0} &= (-i)^n e g^n \frac{1}{n!} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_n \langle P_N \\ &\quad \times [\bar{\psi}(x_1) \tau_{\lambda_1} \psi(x_1), \dots, \bar{\psi}(x_n) \tau_{\lambda_n} \psi(x_n)] \rangle_1 \\ &\quad \times \langle P_M[j_\mu(x), \phi_{\lambda_1}(x_1), \dots, \phi_{\lambda_n}(x_n)] \rangle_0, \end{aligned} \quad (5)$$

where obviously

$$\langle j_\mu^{(n)}(x) \rangle_{1,0} = 0 \quad (\text{for odd } n). \quad (6)$$

The meson part, for even n , $\langle P_M[j_\mu(x), \phi_{\lambda_1}(x_1), \dots, \phi_{\lambda_n}(x_n)] \rangle_0$ consists of terms of the form

$$\begin{aligned} &\left(\frac{1}{2}\right)^2 \frac{\partial \Delta_F(x-x_k)}{\partial x_\mu} \Delta_F(x-x_l) \\ &(\delta_{1\lambda k} \delta_{2\lambda l} - \delta_{2\lambda k} \delta_{1\lambda l}) \langle P[\phi_{\lambda_p}(x_p) \phi_{\lambda_q}(x_q)] \rangle_0 \dots \\ &\langle P[\phi_{\lambda_r}(x_r) \phi_{\lambda_s}(x_s)] \rangle_0 \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{2}\right)^{\frac{n}{2}+1} \frac{\partial \Delta_F(x-x_k)}{\partial x_\mu} \Delta_F(x-x_l) \\ &\times (\delta_{1\lambda k} \delta_{2\lambda l} - \delta_{2\lambda k} \delta_{1\lambda l}) \delta_{\lambda_p, \lambda_q} \dots \\ &\delta_{\lambda_r, \lambda_s} \Delta_F(x_p-x_q) \dots \Delta_F(x_r-x_s). \end{aligned} \quad (7)$$

Then, the nucleon part has one open polygon and some closed loops. Due to $\delta_{\lambda\lambda}, \dots$ in (7), the nucleon part of the matrix element corresponding to (7) has the product of odd τ_1 , odd τ_2 and even τ_3 . For the Dyson-diagram containing the closed loop, the product of the spur (of the product of selected τ 's from among the above set of τ) and remained τ 's appears in the matrix element. However, after taking spur, the non-vanishing product of τ is of the following two types:

- i) the product of odd τ_1 , odd τ_2 and odd τ_3 ,
- ii) the product of even τ_1 , even τ_2 and even τ_3 .

(zero is even!)

In the cases with one closed loop (above two cases) and with some closed loops (the combination of above two), we have, as the remained set of τ , two types of products of

- i') odd τ_1 , odd τ_2 and even τ_3 , or
- ii') even τ_1 , even τ_2 and odd τ_3 .

For both types i') and ii'), the product becomes $+\tau_3$ or $-\tau_3$. In all cases, $\langle j_\mu^{(n)}(x) \rangle_{1,0}$ is proportional to

$\bar{\psi}[\text{product of } S_F][\text{product of } S_F(S_F \dots S_F)] \times \tau_3 \psi$. Adding over all n , we find

$$\langle j_\mu(x) \rangle_{1,0} \propto \psi F[S_F, S_F(S_F \dots S_F), \Delta_F] \tau_3 \psi,$$

where F is the sum of $S_F, S_F[S_F, \dots, S_F]$ and Δ_F . The above procedure for scalar meson with scalar coupling is valid also for more complex types of meson, since the situation of τ in the interaction Hamiltonian is same in all cases. When the time derivative is contained in the interaction Hamiltonian, we may calculate according to the method⁽⁵⁾ proposed by Koba.

Therefore, we can conclude as follows:

The meson current around the nucleon has the same magnitude and opposite sign for neutron

and proton, independently of the coupling strength and the meson type.

The author is particularly indebted to Mr. H. Umezawa and Mr. S. Kamefuchi for their valuable discussions. He also wishes to thank Prof. S. Sakata for his constant encouragement.

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On the Interaction of Cohesive Field

K. Sawada

*Department of Physics,
Kyoto University*

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To avoid the divergence difficulties of self-energies, new field playing the role of cohesive force has been introduced by Pais¹⁾ and Sakata²⁾. These theories gave approximately correct answers for the mass defect of the proton compared with neutron. But the mass of the cohesive field thus introduced is to be assumed about 100 times electron mass in order to obtain the correct value for mass defect of proton³⁾, and this fact hinders us from understanding the mass difference of the mirror nuclei correctly as has been pointed out by Wightman⁴⁾. The mass difference between mirror nuclei is correctly understood by taking only pure Coulomb interactions between protons.

If we take this fact seriously, we must

assume that the nature of the cohesive field is to be such that it does not produce any appreciable change of mutual interaction between protons (i.e. Coulomb field) but play the role in virtual process as the cohesive agency. Such a interaction could not be considered as long as we confine ourselves to the usual interaction between proton and cohesive field. So that, the introduction of the new type of interaction seems to be necessary.

Now, a type of interaction, which hitherto being not taken into consideration, has recently proved experimentally its existence⁵⁾, and this interaction seems to satisfy the above demand concerning mirror nuclei. That is a sort of V -particle interaction. Confining our attention only to the interaction between proton and electromagnetic field together with cohesive field (which we take as scalar field), the interaction between proton and electromagnetic field is given by

$$ie\bar{\psi}\gamma^\mu\psi A_\mu \quad (1)$$

where ψ , A_μ are usual wave function of proton and electromagnetic field respectively. On the other hand, we assume the interaction between scalar meson and proton such as V -nucleon- π interaction:

$$f(\bar{\psi}\Psi + \Psi\psi)\phi \quad (2)$$

where Ψ , ϕ is the wave function of heavy proton (may be V -particle) and cohesive field (scalar meson) respectively.

It is to be noted that the interaction of type (2) cannot be obtained from the combination (i.e., higher order correction) of the interaction of type (1), it is essentially a new type of interaction.

The circumstances introduced by the system of proton, heavy proton, electromagnetic field and cohesive field with (1) and (2) are as follows: Firstly, interaction of type (2) plays the role of cohesive field, at least in second order approximation, under the condition

$$e^2 \cdot 3m - f^2 \left(M + \frac{m}{2} \right) = 0 \quad (3)$$

where m , M is the mass of proton and heavy-proton (may be V -particle). This condition makes the self-energy of proton to converge and giving correct mass defect by taking $M \approx 2000 m_{\text{electron}}$ and mass of cohesive field around $100 m_{\text{electron}}$. Secondly, mutual interaction between proton does not deviate from Coulomb law in e^2 , f^2 approximation, and deviation appears only in fourth order which is weak and its inverse range coming from cohesive meson is doubled and so plays no role in the problem of mirror nuclei. Thus the problem of mirror nuclei is solved.

Another point, which shows also advantages of these interaction scheme, is the problem concerning vacuum-polarization type of self-stress introduced by the cohesive field (scalar meson). This was proved to diverge by the usual interaction scheme, but in the scheme considered above there is no sort of v - p type of self-stress and so this difficulty is also solved at least in e^2 - f^2 approximation. In fact, simple calculation shows vanishing answer for the self-stress of proton.

Considering these circumstances, we may conclude that the transmutation interaction of type (2) seems to play some essential roles in connection with the so called difficulties in quantum field theories.

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On the Exchange Current

Y. Ataka

Department of Physics,
Kyoto University

July 4, 1951

The effects of the meson exchange current are represented as a sum of the electric and magnetic multipoles. The former can be expressed in terms of the nuclear potential, and the latter refers to the types of the meson theory.¹⁾ Therefore it seems very important to calculate the exchange magnetic moment. Here the calculation is done for the pseudo-scalar meson theory.

The meson current is given by

$$\mathbf{j}_{\text{meson}} = -\frac{ie}{\hbar} (\phi^* \text{grad } \phi - \phi \text{grad } \phi^*) \quad (1)$$

and the exchange moment is

$$\langle \mu_{\text{exch}} \rangle = \frac{1}{2c} \int \mathbf{r}' \times \langle \mathbf{j}_{\text{meson}} \rangle d\mathbf{r}'. \quad (2)$$

Using the perturbation theory, we get

$$\begin{aligned} \langle \mathbf{j}_{\text{meson}} \rangle &= -\frac{4ie}{\hbar} \int (\epsilon_K \epsilon_{K'})^{-1} \langle 0 | H_i | Y^+(\hbar \mathbf{K}) \rangle \\ &\quad \left(\begin{matrix} Y^+(\hbar \mathbf{K}) \\ 0 \end{matrix} \middle| \phi^* \text{grad } \phi - \phi \text{grad } \phi^* \middle| \begin{matrix} 0 \\ Y^+(\hbar \mathbf{K}) \end{matrix} \right) \\ &\quad \cdot \langle Y^+(\hbar \mathbf{K}) | H_i | 0 \rangle d\mathbf{K} d\mathbf{K}' (2\pi)^{-6} \\ &\quad + (\text{in the above } Y^+ \rightarrow Y^-) \end{aligned} \quad (3)$$

where H_i is the interaction energy of mesons and nucleons and ϵ_K is the energy of a free meson with momentum $\hbar \mathbf{K}$. The matrix elements are as follows²⁾:

$$\begin{aligned} \langle 0 | H_i | Y_+(\hbar \mathbf{K}) \rangle &= f(0 | \rho_2^{(1)} | -\hbar \mathbf{K}) \\ &\quad \times \frac{\tau_1 - i\tau_2}{\sqrt{2}} \frac{\hbar c}{\sqrt{2}\epsilon_K} e^{i\mathbf{K}\mathbf{r}} \\ &= \frac{if\hbar c}{\sqrt{2}\epsilon_K} \frac{\tau_1 - i\tau_2}{\sqrt{2}} \frac{(\sigma^{(1)} \mathbf{K})}{2\sqrt{\mathbf{K}^2 + \kappa_0^2}} e^{i\mathbf{K}\mathbf{r}}, \quad (4) \end{aligned}$$

$$\begin{aligned}
& (Y^+(\hbar\mathbf{K})|H_i|0) \\
&= -\frac{if\hbar c}{\sqrt{2}\epsilon_{K'}} \frac{\tau_1 + i\tau_2}{\sqrt{2}} \frac{(\sigma^{(2)}\mathbf{K}')}{2\sqrt{\mathbf{K}'^2 + \alpha_0^2}} e^{-i\mathbf{K}'\mathbf{r}}, \\
& \left(\begin{array}{c} Y^+(\hbar\mathbf{K}) \\ 0 \end{array} \middle| \phi^* \text{grad} \phi - \phi \text{grad} \phi^* \middle| \begin{array}{c} 0 \\ Y^+(\hbar\mathbf{K}') \end{array} \right) \\
&= \frac{\hbar c}{\sqrt{2}\epsilon_K} \frac{\hbar c}{\sqrt{2}\epsilon_{K'}} e^{i(\mathbf{K}' - \mathbf{K})\mathbf{r}'} i(\mathbf{K} + \mathbf{K}'). \quad (5)
\end{aligned}$$

Therefore it must be noted that the pseudoscalar coupling of the pseudoscalar theory is represented by

$$f\rho_2\phi \sim \frac{1}{2}f \frac{(\sigma\mathbf{r})}{\sqrt{-\mathbf{r}^2 + \alpha_0^2}} \phi. \quad (6)$$

If we substitute $\frac{2M}{\mu}g$ for f and take the limit of $M \rightarrow \infty$, the pseudovector coupling is obtained.

The calculation of (3) is very easy and the results are as follows:

$$\begin{aligned}
\langle \mu_{exch} \rangle &= \frac{ef^2}{8\hbar c} (\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)})_3 [(\boldsymbol{\sigma}^{(1)} \times \text{grad}) \\
& (\boldsymbol{\sigma}^{(2)} \text{grad}) - (\boldsymbol{\sigma}^{(1)} \text{grad}) (\boldsymbol{\sigma}^{(2)} \times \text{grad})] J(r) \\
& \quad (7)
\end{aligned}$$

$$\begin{aligned}
J(r) &= \int \frac{e^{i\mathbf{K}r} d\mathbf{K} (2\pi)^{-3}}{(\mathbf{K}^2 + \alpha^2)(\mathbf{K}^2 + \alpha_0^2)^2} \\
&= \frac{1}{\alpha_0^2 - \alpha^2} \left[\frac{e^{-\alpha r}}{8\pi r} - \frac{1}{\alpha_0^2 - \alpha^2} \left(\frac{e^{-\alpha r}}{4\pi r} - \frac{e^{-\alpha_0 r}}{4\pi r} \right) \right].
\end{aligned}$$

Using the above expression, the cross section for the radiative capture of slow neutrons by protons is calculated and the coupling constant $(f^2/4\pi\hbar c)(\mu^2/4(M^2 - \mu^2)) \simeq 0.05$, i.e. $(f^2/4\pi\hbar c) \simeq 8.7$ is in good agreement with the experimental value.

For the pseudovector coupling the exchange moment is the limiting value of (7) in $M \rightarrow \infty$. This is the same result as Ma and Yu's³⁾. In this case the coupling constant $(g^2/4\hbar c) \simeq 0.27$ is in good agreement with the experiment of the radiative capture.

The property (6) of the pseudoscalar coupling reduces the difficulties of the meson theory—for example, the r^{-3} singularity of

nuclear forces²⁾. Therefore the quantitative treatments have been possible for this case.

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On the Accuracy of the Molière Function, II

J. Nishimura

Department of Physics, Kobe University

K. Kamata

Scientific Research Institute

July 9, 1951

In our recent letter to the editor,¹⁾ it was shown that the Molière's²⁾ angular distribution function does not represent the structure of cascade showers accurately. In this paper we evaluate the integral lateral distribution function* $\Pi_1 2\pi r dr$ for a few shower ages s from the formula (5) of the reference (3), and compare them with Molière's results**.

As shown in Fig. 1, our distribution functions³⁾ become less steep with increasing shower age, as it must be. In spite of his crude approximation, his function agrees fairly well with ours of $s=1$ for $\frac{r}{r_1} < 2$. The discrepancy appearing for $\frac{r}{r_1} > 5$ seems to be surely due to the difference of the approximations used by each author, because the average number of particles lying in this region ($\frac{r}{r_1} > 5$) is much underestimated in the Arley's approximation used by him.

Nevertheless, Roberg-Nordheim⁴⁾ compared their calculated mean square deviation $\langle r^2 \rangle_{AV}^{***}$ with Molière's one, and found that his function is larger than exact one for large $\frac{r}{r_1}$. If we adopted this argument,

ours would not also be free from such adverse criticism, since ours is larger than Molière's results for $\frac{r}{r_1} > 5$. However, this conclusion must be revised because of the following reason. The $\langle r^2 \rangle_{AV}$ used by Roberg-Nordheim for the comparison with Molière's result corresponds to that of the particles having energies larger than $0.05 \epsilon^{***}$, while Molière's function and ours contain the contribution from all particles having energies larger than zero.

At first sight, the difference between $\langle r^2 \rangle_{AV} E=0.05\epsilon$ and $\langle r^2 \rangle_{AV} E=0$ seems to be

less significant. However, remembering the fact that the average number of the particles at shower maximum having energies less than 0.05ϵ is about 20% of the total number of shower particles and that the lateral deviations from shower axis of such low energy particles are much larger than that of high energy particles, the difference of the mean square deviations between two cases mentioned above, would become considerably important. In fact, the results calculated by the method of Roberg-Nordheim show that $\langle r^2 \rangle_{AV} E=0$ is about two times as large as $\langle r^2 \rangle_{AV} E=0.05\epsilon$.

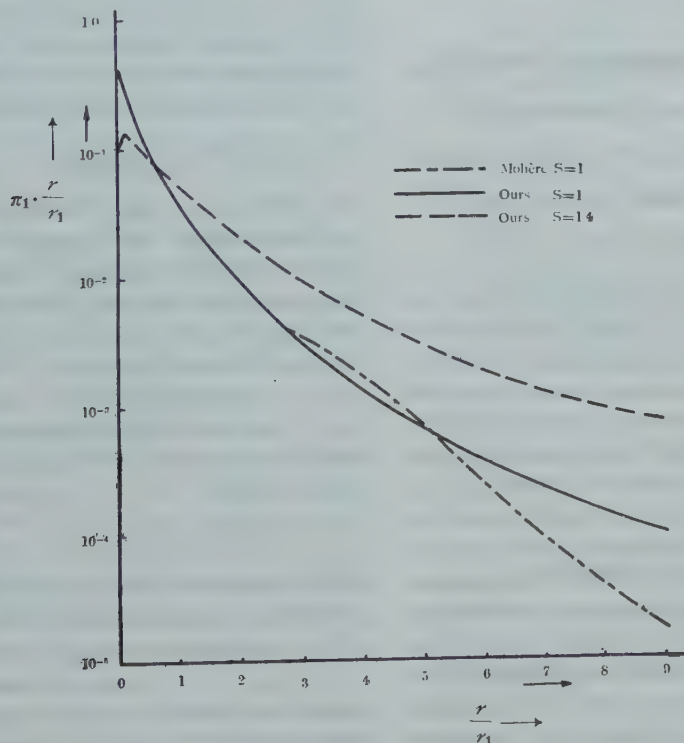


Fig. 1. Lateral distribution of shower electrons.

These functions are normalized as $\int_0^\infty \pi_1 2\pi \frac{r}{r_1} d\left(\frac{r}{r_1}\right) = 1$.

It should be noted that the contribution of single scattering are not taken into account in our case, while his function contains this contribution.

* In this paper we limit ourselves to the shower initiated by an electron of very high energy.

** Integral lateral distribution function $\pi_1(E, r)$
 $2\pi r dr$ represents the total number of electrons

with the energy larger than E and having the lateral distance from the axis between r and $r+dr$.

*** $\langle r^2 \rangle_{AV}$ is given by the formula

$$\langle r^2 \rangle_{AV} \epsilon = \epsilon' = \frac{\int_0^\infty \Pi_1(E', r, \epsilon) r^2 2\pi r dr}{\int_0^\infty \Pi_1(E', r, \epsilon) 2\pi r dr}.$$

**** ϵ represents the critical energy of the trans-
versing material.

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On a Possible Model of the V-Particle

K. Aizu* and T. Kinoshita**

*Department of Physics, St. Paul University**

*Department of Physics, Tokyo University***

July 15, 1951

Recent discovery by English and American observers¹⁾ of the V -shaped tracks in the cloud chamber photographs has given rise to much discussions among nuclear physicists because of their highly enigmatical characters. Especially the obvious inconsistency between the rather large frequency of production (a few per cents of pion production) and the exceedingly long life-time (10^{-10} sec.) of this event has made it difficult to understand in a sensible manner. Since the circumstance has some resemblance with the case of the two meson theory where mesons interact with matter very weakly though produced with large cross section, one may be led to suppose that at least two unknown particles participate in this phenomenon some of them being daughters of the others. Although such an approach might be a right one, it is so complicated and full of arbitrariness that it seems to be more reasonable for the moment to circumvent this possibility. The simplest starting point is then to assume that, only

one unknown elementary particle (which we call provisionally the " V -particle") plays a role in this event and try to find a consistent explanation assuming a suitable set of interactions with other known particles*. It seems also possible to prefer another view point that the observed phenomena are not caused by any new particle but by some known particles such as nucleons, pions, or possibly τ -mesons in a composite, excited, or isobaric state²⁾.

In this note we want to indicate that it is possible to explain, at least qualitatively, both the large cross section of production and the long life-time by the simple assumption that the V -particle is an elementary Fermi particle obeying the Dirac equation and that it interacts with the nucleon field through

$$H = G/M^2 \cdot \bar{\psi} Q \psi \cdot \bar{\psi} Q V + c.c. \quad (1)$$

where M is the mass of nucleons, ψ and V are the quantized field operators of nucleon and V -particle respectively, and Q specifies any one of the possible types of direct interaction analogous to the case of the Fermi's theory of beta-disintegration. This interaction gives directly the production cross section of the V -particle by nucleon-nucleon collision which is consistent with experiments if one assumes that

$$G^2/4\pi \sim 0.1-1. \quad (2)$$

The threshold kinetic energy of V -particle production in the laboratory system is ~ 400 Mev. for light nuclear target and ~ 200 Mev. for heavy one. It seems, however, very difficult to detect it in the experiments by the accelerators, since the production cross section is small near threshold and further the produced V -particle can hardly leave regions of large back ground because of its short range of flight.

Now, it is energetically possible that the produced V -particle (of assumed rest mass $\sim 2250m_e$) decays into a nucleon and a pion: Such a process can actually occur since a

V -particle becomes virtually three nucleons by the interaction (1) two of them subsequently annihilating each other to form a single pion. The latter process occurs through the ordinary interaction of a (pseudoscalar) pion with nucleons. In the computation of this process one encounters of course with the divergence difficulty of the present field theory which can be removed by making use of the ordinary regularization conditions³⁾

$$\begin{aligned}\sum e_i M_i^2 &= 0, \quad \sum e_i = 0, \quad \sum e_i M_i^2 \log M_i = 0, \\ \sum e_i \log M_i &= 0.\end{aligned}\quad (3)$$

In this manner one can obtain an unambiguous result only for the direct tensor interaction while all other types suffer from ambiguities which violate the validity of the equivalence theorem. Being able to drop these terms by suitable considerations, one obtains unambiguous results also in the latter cases. One can thus infer that the longest limit of the estimated life-time is of the order of 10^{-10} sec. for all five types, a value not inconsistent with the observed one. By the way, we remark that we can find several linear combinations of the direct interactions (1) that are free from such ambiguities.

One of the competing decay process is the disintegration of the V -particle into a nucleon and a gamma-ray quantum which, however, is completely forbidden to the first order. Other possibilities are the decay into a nucleon and an electron-positron pair. These are found to occur several thousands times slower than the above process.

In order to explain the difference between the life-times of V^0 and V^\pm , one may for instance assume that the latter is heavier than the former.

It is a matter of course that such a mechanism may be no more than a speculation and rejected by more precise measurements. It is well aware further that our result depends essentially on the rather dubious procedure of regularization and thus may

not be justified until a future correct theory is established. We shall only stress by discussing this model that it is not always inevitable to assume a complicated structure of the V -particle to understand their main features as far as we are concerned with what have been found.

The authors wish to thank Messrs. H. Fukuda and H. Miyazawa for their valuable discussions.

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* We do not intend here to treat all kinds of V -tracks but only those that seem to decay into a nucleon and a meson (assumed to be a pion). As to an attempt to include light V -particles, see the letter by Y. Nambu, K. Nishijima and Y. Yamaguchi, *Prog. Theor. Phys.* **6** (1951), 615, 619.

A Model for V-Particles*

H. Miyazawa

Department of Physics, University of Tokyo

July 15, 1951

Recent discoveries¹⁾ indicate the existence of two sorts of new unstable particles, one with rest mass heavier and the other lighter than proton, which are temporary called as V -particles. The heavier one decays probably into a proton and a meson, and the lighter into two mesons. We shall call the latter, for simplicity, as V_1 -particle, in distinction

from the former which shall be called as V_h -particle. A remarkable character of these particles is that they are frequently produced in penetrating showers and hence are rather strongly coupled to nuclear particles, yet have a considerably long life time of about 10^{-10} sec. A tempting interpretation to assume a V as an excited isobaric state of a nucleon, as appeared in the strong coupling meson theory, must be rejected, because an isobar decays too fast to be reconciled with experiments, by emitting a π -meson or a γ . π or γ emission can only be excluded if the excited state has an integral spin, which scheme will be discussed in the following.

The model proposed here is, that V_l 's are elementary fermions coupled to nucleons by the interaction

$$\eta \bar{\psi}_N O \psi_N \cdot \psi_{V_l} O \psi_{V_l}, \quad (1)$$

where O represents some Dirac matrices. As pointed out by Critchfield²⁾, for sufficiently strong coupling, a V_l can be trapped by a nucleon to form a bound state. We assume that the V_h is a composite particle formed in this way. The coupling constant η is determined so that the binding energy is equal to the mass difference $m_N + m_{V_l} - m_{V_h} \sim 500 m_e$. (Of course, to obtain a finite binding energy, we must smear the zero range interaction over a region of dimensions about $1/m_{V_l}$ ³⁾.)

The production of V -particles can be described as follows: A high energy nucleon in collision with another is deflected by nuclear force, and, by virtue of (1), emits an anti- V_l into free and a V_l into bound state, the latter being observed as a V_h . Therefore, our model predicts two sorts of heavy particles to be produced in pairs. However, since most of the V -particles escape without being observed, only one of the pair is likely to appear in a cloud chamber photograph. At present, there is only one photograph in which 3 V -tracks are observed⁴⁾.

The matrix element of (1) for the transition

$$N \rightarrow V_h + V_l$$

can be calculated by inserting for ψ_N the wave function of the nucleon that disappears, for ψ_{V_l} the wave function of the anti- V_l that appears and for $\bar{\psi}_N \bar{\psi}_{V_l}$ the bound state wave function of the produced V_h . The result is nearly equivalent to assume an interaction

$$F \bar{\psi}_{V_l} \psi_N \psi_{V_h}$$

with $F^2/4\pi$ of the order of 0.7, i.e., of the same order of the π -nucleon coupling constant. Nevertheless because of the considerably large energies required to produce these particles, V 's are rather hard to find except at extremely high energy collisions. The order of magnitude of the production cross section can most easily be estimated by comparing the statistical weights of various final states, as done by Fermi⁵⁾. It turns out that V 's are produced a few percent as frequently as π 's for incident energies of about 10Bev.; this seems a reasonable agreement with observation.

Finally, decay of these particles must be explained. Experimentally only a little is known about the nature of secondary particles. We assume that V_l decays into a π and a μ , in accordance with the previous assumption that V_l is fermion, through the interaction

$$f \bar{\psi}_\mu O' \psi_{V_l} \phi_\pi + c.c. \quad (2)$$

f should be adjusted so as to give the correct life time of V_l , i.e. $f^2/4\pi \sim 10^{-14}$. It is interesting to note that this value of coupling constant is almost the same order of magnitude as that of $\pi\mu\nu$ coupling. Then a V_h decays according to the following way: the bound V_l of the V_h dissolves virtually into a π and a μ through (2), and, having insufficient energy to set both free, the π is

subsequently absorbed by the nucleon, remaining the μ and the nucleon as decay products. That is, our decay scheme is

$$V_L \rightarrow \pi + \mu,$$

$$V_h = (N + V_L) \rightarrow N + \pi + \mu \rightarrow N + \mu.$$

The life time of V_h is estimated to be almost the same as that of V_L . Difference in the life time of neutral and charged V -particles, observed by Seriff et al¹⁾, may be accounted for by suitable asymmetrical choice of constants.

We have tentatively assumed that one of the secondaries is a μ -meson. There is yet no evidence that it is the case. If it is experimentally established that both secondary particles have strong nuclear interaction and cannot be μ -meson, the above model must be altered in that V_L 's are bosons, coupled to nucleons in pairs like (1), and decays into two π 's. Then a V_h decays into a nucleon and a π .

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Note on the Theory of V -Particles and τ -Mesons

S. Ôneda

Department of Physics, Kanazawa University

July 16, 1951

Following the discoveries of the *forked tracks* in cloud chamber by Rochester and Butler¹⁾, various evidences on τ -mesons have been reported by many authors²⁾. Assuming fairly strong interactions between τ -mesons and nucleons ($1/\hbar c g_\tau^2 \sim 10^{-3}$), and τ -mesons as Bosons (mass $\sim 900m$), the decay modes of τ -mesons into π -mesons via nucleon field were discussed thoroughly by Tôkyo group³⁾ and by us⁴⁾ independently. The modes $\tau^\pm \rightarrow \pi^\pm + \pi^0$ and $\tau^\pm \rightarrow \pi^\pm + \gamma$, if they are allowed among many selection rules, have generally very short life times between the range $10^{-17} - 10^{-14}$ sec. and $10^{-16} - 10^{-12}$ sec. respectively. However, when τ and π are both *ps*. mesons, the two particles decay are all forbidden, and $\tau \rightarrow 3\pi$ occurs with suitable life times of the order $10^{-12} - 10^{-9}$ sec. The latest informations⁵⁾ about the unstable heavy particles show the existence of nucleon-like particles (estimated mass $2100m \sim 2500m$) which seem to decay into a nucleon and a π -meson. The experimental evidences are not sure yet, but if they exist, they might play important roles in the future theory. Here we propose a theory which connects the new unstable particles with τ -mesons. These theories should overcome the contradiction between fairly long life times and notable abundance of their production. We treat this particle (V -particle) as a heavy nucleon. The following interactions are assumed.

V -nucleon- τ meson interaction

$$G_V (\bar{\psi} \tau_V O_V \psi, U^*), \quad (1)$$

nucleon- τ meson interaction

$$g_\tau (\bar{\psi} \tau_N O_\tau \psi, U^*), \quad (2)$$

nucleon- π meson interaction

$$g_{\pi}(\bar{\psi}\tau_N O_{\pi}\psi, \varphi^*) \quad (3)$$

where τ_V and τ_N are the isotopic spins which express the transitions between V and nucleons and between nucleons respectively. U and φ represent the wave function of τ and π meson, and O_V, O_{τ}, O_{π} the Dirac's matrices. Here we take the symmetrical theory for the charge of τ -meson. (A) V and τ -meson production According to (1) and (3), τ -mesons and V -particles may be produced in the following reactions at once.

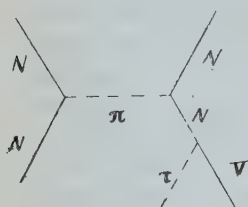


Fig. 1

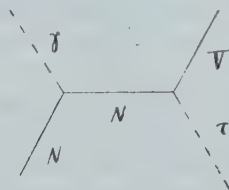


Fig. 2

Fig. 1 may be most effective. Compared with π -meson production, because the final V and τ -meson are respectively heavier than nucleon and π meson, the coupling constant G_V may be larger than $10^{-2}g_{\pi}$ or rather comparable with g_{π} in order to explain the production rates of V and τ -meson being a few percent of that of π -meson.

(B) $V \rightarrow N + \pi$ decay

Because $m_V < m_N + m_{\tau}$, $V \rightarrow N + \tau$ cannot occur by (1). This is the main point of our theory. Then the lowest order decay processes are the following (Fig. 3 and Fig. 4).

The other same order processes of the V -decay are the γ -decay (one of their diagrams are shown in Fig. 5). But as $l^2/\hbar c < g_{\pi}^2/\hbar c$, γ -decay life times will be a little longer than that of π -decay.

(C) τ -meson decay

$\tau^{\pm} \rightarrow \pi^{\pm} + \pi^0$, $\tau \rightarrow \pi^{\pm} + \gamma$ and $\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$ are thoroughly discussed in re-



Fig. 3



Fig. 4

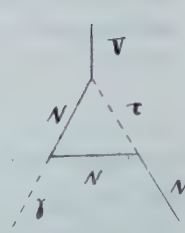


Fig. 5

ference (3) and (4). But here the situation changes so much. Because (1) is responsible for τ -meson production, we can take g_{τ} in (2) so small as to make τ -decay life times about 10^{-10} sec. In the rough estimation the reciprocal of the decay life times of the processes of (B) and (C) are $1/(\hbar c)^3 G_V^2 g_{\pi}^2 g_{\pi}^2 \cdot m_V c^2/\hbar$, $1/(\hbar c)^3 g_{\tau}^2 g_{\pi}^4 \cdot m_{\tau} c^2/\hbar$ ($\tau^{\pm} \rightarrow \pi^{\pm} \pi^0$) respectively. As discussed in (A) G_V should not be so smaller than g_{π} . Then choosing suitable value for g_{τ} , we can make both the life times of these transitions to be of the order 10^{-10} sec., though divergences makes their correct evaluations difficult.

(D) Calculations of the decay, $V \rightarrow N + \pi$ There remain the possibilities of taking the V -particle wave functions different in reflection properties from those of nucleons.⁽⁷⁾ Among many possibilities, we first checked the following cases.

(a) $\tau \dots ps(ps)$, $\pi \dots ps(ds)$, $V \dots pseudospinor$

$$O_V = 1, \quad O_{\tau} = \gamma_5, \quad O_{\pi} = \gamma_5$$

(b) $\tau \dots s(s)$, $\pi \dots ps(ps)$, $V \dots pseudospinor$

$$O_V = \gamma_5, \quad O_{\tau} = 1, \quad O_{\pi} = \gamma_5$$

These calculations are very similar to those of nucleon moments. The vertex (Fig. 3), the self energy (Fig. 4), and the vacuum polarization type processes will contain in general divergences which we omit by regulator. The approximate proper life times are given by

$$(a) \quad \frac{1}{\tau_0} = \frac{G_V^2 g_{\tau}^2 g_{\pi}^2}{(\hbar c)^3} \frac{1}{2} \frac{1}{\pi^2} \sqrt{(x_V - x)^2 - x_{\pi}^2} \\ \times C \times \frac{4}{10}$$

$$(\beta) \quad \frac{1}{\tau_0} = \frac{G_V^2 g_\tau^2 g_\pi^2}{(\hbar c)^3} \frac{1}{2} \frac{1}{\pi^2} \sqrt{(\mathbf{x}_V - \mathbf{x})^2 - \mathbf{x}_\pi^2} \\ \times C \times \frac{4}{100}$$

where \mathbf{x}_V , \mathbf{x} and \mathbf{x}_π denote the reciprocal Compton wave length of V , nucleon and π meson respectively. Taking $m_V = 2400m$, $m_N = 1800m$, $m_\tau = 900m$, $m_\pi = 300m$, and $g_\pi^2(\hbar c \sim 10^{-1})$, $G_V^2/\hbar c \sim 10^{-2}$ and $\tau_0 \sim 10^{-10}$ sec., we have (a) $g_\tau^2/\hbar c \sim 10^{-9}$ (b) $g_\tau^2/\hbar c \sim 10^{-8}$. In case (a), the lowest order τ -decay is $\tau \rightarrow 3\pi$, which has $\tau_0 \sim 10^{-7}$ sec. In case (b), if we take neutral theory for the coupling (3) we get $\tau_0 \sim 10^{-10}$ sec. ($\tau^\pm \rightarrow \pi^\pm + \pi^0$). Thus at least, the case (b) can explain the forked tracks of V and τ -decay consistently. The reason why we introduce pseudospinor is that when we take, for example, the case (b) scalar τ meson may behave as if it were pseudoscalar which is important in calculating nucleon moments.

(E) Another possibilities

It is possible to make only τ^0 meson interact with V -particle. That is, we assume the interactions $G^0 V N \tau^0$ ($G^0 \sim g_\pi^2$) and $g^0 N N \tau^0$. G^0 is responsible for the production of V and τ^0 meson, and g^0 for V -decay and $\tau^0 \rightarrow \pi^+ + \pi^-$, $\tau^0 \rightarrow 2\gamma$. (Assuming spin 0 for τ -meson, $\tau^0 \rightarrow \pi^0 + \gamma$, $\tau^0 \rightarrow e^+ + e^-$ are forbidden). For the charged τ -mesons, we treat them as reference (3) and (4). Then, adopting suitable models we can explain $\tau^\pm \rightarrow \pi^\pm + \pi^\mp + \pi^\pm$. The interactions (1) and (2) may have wider freedoms to use for the coming experimental results.⁽⁷⁾ On the other hand, taking G_V nearly equal to g_π , the interaction (1) may cause serious influences on nucleon phenomena such as nuclear forces or the anomalous magnetic moment of nucleon. More detailed investigations on these points are now in progress. The author wishes to express his sincere thanks to Prof. S. Ozaki for his encouragement and advices through this work. He is also greatly indebted to Messrs. S. Sasaki and S. Hori for their stimulating discussions.

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* Read at the Symposium on elementary particles at Tokyo (R.E.K.T.) July 7, 1951, where Nambu, Nisijima and Yamaguchi; Aizu and Kinoshita; and Miyazawa reported their theory on the same title. The author wishes to express his thanks to Tokyo group for their useful discussions.

On the Nuclear Forces and the Ground State of Deuteron

M. Taketani Tokyo

S. Ônuma and S. Koide

Physics Department, University of Tokyo

July 25, 1951

Recently, many authors have shown that, if one takes account of the relativistic effect

properly, the singularity of meson potential is not r^{-3} but only r^{-1} . With this non-singular potential, Araki and Mori¹⁾ and Sugawara²⁾ have discussed the ground state of deuteron. This simple replacement of r^{-3} singularity by r^{-1} is, however, not sufficient for the explanation of the difficulties, since besides this effect there should be many more questions to be considered at smaller distances of the nucleons, for instance, the fourth-order potential, the effect of strong coupling and the possible existence of heavier mesons.

In view of these situations we consider the nuclear forces as composed of two parts, namely the part due to " π -meson" and the part " X ". The latter represents inclusively the above mentioned various effects. As nothing is known about the properties of this " X " at present, we temporally adopt, for the sake of convenience, the square well (or wall) potential as the resultant one. The range and the depth (or height) of this well (wall) are to be determined so as to agree with the experimental results. The properties of the well (wall) represented by these quantities are not fixed but may depend on the sort of the phenomena and the relevant energy. This is the most fundamental difference between the Jastrow's hard-core-model³⁾ and ours.

The especially interesting points are :

1) Whether the pseudoscalar π -meson potential can interpret the low energy N-P interaction without exception.

2) Whether the main difficulty of symmetrical vector meson theory, that is, the negative value of the quadrupolemoment of deuteron, can be avoided by the modification of the inner potential. (Recently, Rosenfeld and Heisenberg⁴⁾ have stressed the importance of the vector meson theory.)

3) Whether the high energy nucleon-nucleon scattering cross sections are explainable by using the meson potential.

This letter is the preliminary report of these investigations. The results at present are as

follows :

a) Symmetrical pseudoscalar π -meson potential (outside), and infinite repulsive square wall (inside) :

$$\begin{aligned} {}^3V &= -\kappa g^2 \{ 1/3 + (1/x^2 + 1/x + 1/3) S_{12} \} \\ &\quad \times \exp (-x)/x, \quad x \geq x_0 \\ \kappa &= m_{\pi} c / \hbar = 1/1.40 \times 10^{-13} \text{ cm}, \quad x = \kappa r \\ {}^3V &= \infty, \quad x < x_0. \end{aligned}$$

κg^2 (Mev)	$g^2/\hbar c$	x_0	Q (in 10^{-27} cm ²)	p_D (%)
10.5	0.075	0.122	1.07	4.2
21.0	0.149	0.228	1.877	6.3
31.5	0.224	0.387	2.542	7.4
35.5	0.251	0.458	2.766	7.54

If we adopt the last constant, triplet effective range ${}^3r = 1.33 \times 10^{-13}$ cm and triplet scattering length ${}^3a = 5.12 \times 10^{-13}$ cm.

b) Symmetrical vector π -meson potential (outside), and infinite repulsive square wall (inside) :

$$\begin{aligned} {}^3V &= -V_0 \{ 1 + 2/3\gamma - \gamma S_{12} (1/3 + 1/x + 1/x^2) \} \\ &\quad \times \exp (-x)/x, \quad x \geq x_0 \\ {}^3V &= \infty, \quad x < x_0 \end{aligned}$$

V_0 (Mev)	γ	Q (in 10^{-27} cm ²)	x_0	p_D (%)
21.0	0.7	-2.43	0.25	8.1

As is readily seen from these results, pseudoscalar π -meson potential of the second-order is correct only outside the half of the range. In order to get the correct D -state probability, we have to consider the finite square wall, which probably has the independent range and height for central and tensor potential respectively. But in this respect quantitative estimation is not yet carried out. The small differences of the value of 3r and 3a with experimental value⁵⁾ are probably due to the approximate evaluations, that is, the deuteron wave functions are used instead of the zero-energy wave function.

It is, however, impossible to get the positive Q -value for the symmetrical vector potential, if one considers only the infinite

square wall inside. To see whether this situation is improved when the finite square wall with various exchange properties, are introduced, we have calculated the Q -value using the square well only, i.e., ${}^3V = -V_0 \{1 + \gamma S_{12}\}$. When the range 0.7×10^{-13} cm (half of the π -meson range) is adopted, rough estimation shows that the maximum Q -value is only about 0.3×10^{-27} cm², which is too small compared with the experimental value. As it has no significance to consider much longer range than used here, one may conclude that the symmetrical vector meson potential must be excluded.

Detailed discussions about the high energy scattering will be published in a later issue of this journal.

In conclusion, we would like to express our deep gratitude to Prof. S. Nakamura, Mr. M. Sasaki and Mr. Y. Fujimoto for their valuable discussions.

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Note on Some Type of Interaction

K. Sawada

Department of Physics, Kyoto University

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Relativistic invariant interaction between Boson and Fermion is usually taken as follows :

$$f\psi^*\gamma_4\gamma\psi + f^*\psi^*\gamma\gamma\psi \quad (1)$$

where ψ , Ψ , and ϕ .. refer to Fermions and Bosons wave function, and γ ..* means Hermite

conjugate of Dirac matrix γ ..

But, just as in the case of beta-decay theory in its Fermi's original form¹⁾, there is another relativistic invariant interaction, that is :*

$$g\phi\eta\gamma\psi + g^*\psi^*\gamma\gamma\phi^* \quad (2)$$

with

$$\eta = \gamma_1\gamma_3 \quad (3)$$

where η is γ_4 times the matrix δ defined by Fermi¹⁾.

This choice of interaction gives different reaction effects for the same type of Boson ϕ ..

It is easily be seen that if the ψ and Ψ fields are charged, then charge density which satisfies the conservation law, when ϕ ..-field is not charged, is :

$$e(\psi^*\phi - \Psi^*\Psi) \quad (4)$$

This shows that if ϕ -field has negatively charged negative energy sea, then χ -field has positively charged negative energy sea, and vice versa.

Then, we remark that the reaction effect arising from the interaction (2), when the Boson ϕ .. is normal (not pseudo-type) field, is just the same as the effect when one uses (1) and the Boson as pseudo-type, and vice versa.**

We have directly proved this situation by evaluating self-energy and the photon-meson production of nucleon. But the general feature shows that the situation holds in any other reaction processes also.

This will be related to the following expression derivable from (2) : if one expresses, in (2), ψ and ψ^* by their transformed wave-function by the matrix γ_2 :

$$\psi = (\psi')^*\gamma_2, \quad \psi^* = \gamma_2\psi' \quad (5)$$

where ψ' is related to the charge conjugated wave-function, since γ_2 generates charge conjugation. Then, we have for (2)

$$g(\psi')^*\gamma_2\eta\gamma\psi + g^*\psi^*\gamma\gamma\phi^* = g(\psi')^*\gamma_2\eta\gamma\psi + g^*\psi^*\gamma\gamma\phi^* \quad (6)$$

Now,

$$\gamma_2 \eta = \gamma_2 \gamma_1 \gamma_3 = -\gamma_1 \gamma_2 \gamma_3 = \gamma_4 \gamma_5, \\ \eta^* \gamma_2 = -\eta \gamma_2 = -\gamma_1 \gamma_3 \gamma_2 = \gamma_5^* \gamma_4^*. \quad (7)$$

Then, (6) gives

$$g(\psi')^* \gamma_4 \gamma_5 r \cdot \Psi \phi \cdot + g^* \Psi^* r \cdot \gamma_5^* \gamma_4^* \psi' \phi \cdot. \quad (8)$$

Comparing this expression with (1), one sees that extra matrix γ_5 appears.

Then, the use of the type (2) interaction as a cohesive interaction for the same field $\phi \cdot$, interacting on ordinary way through (1), becomes possible. Because it is the well known fact that in the ordinary theory the scalar-pseudoscalar and vector-pseudovector combination of fields have chance to make the diverging self-energy of Fermion to converge by the appropriate choice of parameters (charge f , mass etc.)².

The application of interaction (2) to the nucleon-meson system will play some role in accounting distinct behavior of neutral and charged meson on the single type of meson.

The details and the related problems will soon appear in this journal.

1) E. Fermi, *ZS. f. Phys.* **88** (1934), 161.

2) See in this connection, K. Sawada, *Prog. Theor. Phys.* **6** (1951), 626.

* This type of interaction was discussed by G. Wentzel in another connection: *ZS. f. Phys.* **104** (1937), 34.

** The situation is the same as in the paper by E. Fermi and C. N. Yang, *Phys. Rev.* **76** (1949), 1739.

Meson Theory of Nuclear Forces*

M. Taketani, *Tokyo*

S. Machida and S. Ônuma

Department of Physics, University of Tokyo

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Although it is generally accepted that, in the region where $r >$ about half the Compton

wave length of the meson, the fundamental defects of meson field theory reveals itself not so serious, and the relativistic and higher order effects are not large except the pseudoscalar meson theory with pseudoscalar coupling, we have found that the fourth order nuclear forces in the pseudoscalar meson theory with pseudovector coupling is unexpectedly large and, if we adopt the magnitude of the constant of pv coupling reasonably, their effects alter the nature of the second order nuclear forces completely. This result is due to the fact that many terms, appearing in the fourth order nuclear forces, cancel each other in general, but are all additive in the pv coupling case accidentally. (We have shown actually that the fourth order nuclear forces in the vector meson theory in vector coupling is small.)

We have obtained the fourth order static nuclear potentials in the *symmetrical pseudoscalar theory* by the method developed by Nambu¹⁾. The results are ($\hbar=c=1$)

$$V_{ps}^{(4)} = \left(\frac{f^2}{4\pi}\right)^2 \frac{3}{8\pi} \frac{\kappa}{M^2 r^2} K_1(2\kappa r), \quad (1)$$

$$V_{pr}^{(4)} = \left(\frac{g^2}{4\pi}\right)^2 \cdot \kappa \cdot [(\tau^{(1)} \tau^{(2)}) U_\tau(\kappa r) \\ + (\sigma^{(1)} \sigma^{(2)}) U_\sigma(\kappa r) + S_{12} U_T(\kappa r)], \quad (2)$$

where

$$U_\tau(x) = \frac{8}{\pi} (-A_1 + 2A_2 - 4A_3 + 2A_4),$$

$$U_\sigma(x) = -\frac{8}{\pi} (2A_2 - 2A_3 + A_4),$$

$$U_T(x) = \frac{4}{\pi} (2A_2 - 5A_3 + 4A_4),$$

$$A_1 = \frac{K_0''''(2x)}{x}, \quad A_2 = \frac{K_0'''(2x)}{x^2},$$

$$A_3 = \frac{K_0''(2x)}{x^3}, \quad A_4 = \frac{K_0'(2x)}{x^4},$$

M and κ are the masses of a nucleon and a meson respectively, and K_0 and K_1 are the modified Bessel functions.* Cross terms between ps and pv couplings do not survive in

the static approximation. Since $V^{(4)}_{ps}$ is much smaller than $V^{(4)}_{pv}$, if $f^2/4\pi$ is not much greater than $g^2/4\pi$, we will drop $V^{(4)}_{ps}$ hereafter, and assume $g^2/4\pi$ to be $0.05 \sim 0.1$. Then one can obtain the net result of second and fourth order nuclear potential. The results are summarized in Table I.

Table I. Qualitative aspects of nuclear potentials derived from the symmetrical pseudoscalar meson theory including fourth order effects.

state	central force	tensor force
triplet even	strong, repulsive	attractive
singlet even	strong, attractive	—
triplet odd	weak, attractive	weak, repulsive
singlet odd	repulsive	—

Since the triplet even potential is qualitatively similar to that of the theories of Bethe²⁾ and Ferretti³⁾, so the deuteron problem might be solved well by our potential using a reasonable cut-off procedure. Actually, if we adopt the coupling constant $g^2/4\pi \simeq 0.07 \sim 0.08$, low energy phenomena are all explainable.

Low and high energy proton-proton scattering data would be reproduced fairly well by our potential assuming a short range hard sphere, since our singlet even and triplet odd potentials are rather similar numerically to that assumed by Jastrow⁴⁾

The fact that the odd state interactions are much weaker than the even state interaction, which is required to explain the neutron-proton scattering data,⁵⁾ is reproduced in our potentials. Especially it can easily be shown that the S - P interference term for n - p scattering is very small by virtue of cancelling between the effects of central forces of triplet and singlet P states. But, to account for the n - p scattering data at various energies, more detailed calculations are necessary, since the triplet even potential is rather complex.

Full accounts will soon appear in this journal.

* About the method we have adopted to attack the problems of nuclear forces, we refer to the paper in this issue of this journal written by one of us (M.T.).

** The same result has been obtained by Nishijima using the method of canonical transformation in the \not{p} coupling case. (To be published in this journal.)

- 1) Y. Nambu, Prog. Theor. Phys. **5** (1950), 614.
- 2) H. A. Bethe, Phys. Rev. **57** (1940), 260, 390.
- 3) B. Ferretti, Ric. Sci. **12** (1941), 993.
- 4) R. Jastrow, Phys. Rev. **81** (1951), 165.
- 5) R. Christian and E. W. Hart, Phys. Rev. **77** (1950), 441.

β -Matrix Formalism as $\kappa \rightarrow 0^*$

D. C. Peaslee**

Washington University,
St. Louis, Missouri, U.S.A.

August 7, 1951

The following question is examined: to what extent can the properties of the photon field be regarded as those of a vector meson field, which passes continuously to the limit $\kappa \rightarrow 0$? The method used is to postulate for the meson field the β -matrix equation¹⁾ (ten-rowed for spin 1, five-rowed for spin 0)

$$(\beta_\mu \partial_\mu + \kappa)\psi = 0, \\ \beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \beta_\lambda \delta_{\mu\nu} + \beta_\nu \delta_{\mu\lambda} \quad (1)$$

and to consider this equation in the limit of zero rest mass. It appears that for $\kappa \neq 0$ both the Klein-Gordon equation $(\square^2 - \kappa^2)A_\mu = 0$ and the subsidiary condition $\partial_\mu A_\mu = 0$ are inherent in (1); whereas for $\kappa = 0$ these two equations are not independent, but one is true only if the other is postulated. Since it is, in fact, generally assumed that $\square^2 A_\mu = 0$ in this case, the photon field can be regarded as the continuous limit of the vector field. In particular this implies that the self-energy of a photon is $\delta\kappa \sim \kappa \rightarrow 0$. The same treatment is applied to the β -formalism for a particle of spin zero, and it is found that no subsidiary condition exists; therefore the

limit $\kappa \rightarrow 0$ is approached continuously without further postulates.

The following characteristics of the β_μ are of importance: i) β_μ and $(\beta_\mu \partial_\mu)$ have no inverses; ii) $(\beta_\mu \partial_\mu)^2 \neq \square^2$ identically, but $(\beta_\mu \partial_\mu)^3 = \square^2 (\beta_\mu \partial_\mu)$; iii) the matrix $M = \sum \beta_\mu^2$ has the commutation property that $M\beta_\nu = \beta_\nu(5-M)$ for all β_ν ; therefore $M(5-M)$ commutes with all β and is a constant times the identity matrix. This constant is 6 for the 10-rowed, 4 for the 5-rowed representation, so that the corresponding eigenvalues of M are (2,3) and (1,4). This is easily verified by direct construction with the representations of reference (1); the eigenvalues (2, 3) and (1,4) have respective multiplicities (6, 4) and (4,1).

Case of spin 1

Now consider specifically the spin 1 representation. Applying the operators $(3-M)$ and $(M-2)$ from the left to the equation of motion, one has

$$\beta_\mu \partial_\mu A + F = 0, \quad \beta_\mu \partial_\mu F + \kappa^2 A = 0 \quad (2)$$

where $(M-2)\psi = \sqrt{\kappa} A$, $(3-M)\psi = 1/\sqrt{\kappa} F$. Here A and F are independent 10-element column matrices with 4 and 6 non-zero components. In the representation of reference (1) they are

$$A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}, \quad F = \begin{pmatrix} F_{14} \\ F_{24} \\ F_{34} \\ -F_{23} \\ -F_{31} \\ -F_{12} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A + F = \psi \quad (3)$$

where A_μ and $F_{\mu\nu}$ correspond to the usual definitions in the electromagnetic case. The individual components may be extracted by the operations

$$A_\mu = (1 - \beta_\mu^2) A; \quad F_{\mu\nu} = \beta_\mu^2 \beta_\nu^2 F = \beta_\nu^2 \beta_\mu^2 F. \quad (4)$$

Note that the $F_{\mu\nu}$ obtained by (4) is defined

with the sign convention as given by (3); the only way obtaining $F_{\nu\mu}$ in this formalism is by $F_{\nu\mu} = -F_{\mu\nu}$.

The Klein-Gordon equations of motion are obtained from (2) as follows:

$$\begin{aligned} (\beta_\mu \partial_\mu)^2 A &= -(\beta_\mu \partial_\mu) F = \kappa^2 A, \\ (\beta_\mu \partial_\mu)^3 A &= \square^2 (\beta_\mu \partial_\mu) A = \kappa^2 (\beta_\mu \partial_\mu A), \\ \square^2 F &= \kappa^2 F. \end{aligned} \quad (5)$$

Likewise $(\beta_\mu \partial_\mu)^2 F = \kappa^2 A$ leads to

$$\square^2 A = \kappa^2 A. \quad (6)$$

To exhibit the subsidiary condition explicitly, write out the first of equations (5)

$$\begin{aligned} \kappa^2 A &= (\beta_\mu \partial_\mu)^2 A = \square^2 A + \\ &\quad \{ (\beta_\mu^2 - 1) \partial_\mu^2 + \sum_{\mu \neq \nu} \beta_\mu \beta_\nu \partial_\mu \partial_\nu \} A. \end{aligned} \quad (7)$$

The λ th row of this matrix equation becomes

$$\kappa^2 A_\lambda = \square^2 A_\lambda - \{ \partial_\lambda (\partial_\mu A_\mu) \}. \quad (8)$$

Hence by (6)

$$\partial_\lambda (\partial_\mu A_\mu) = 0, \quad \partial_\mu A_\mu = C, \quad (9)$$

where C may as well be zero. Thus the subsidiary condition is inherent in the equations of motion (1) and not independent.

Suppose now that $\kappa \rightarrow 0$; in this case one cannot start with (1) but must proceed from the derived equations of motion (2) which become

$$\beta_\mu \partial_\mu A + F = 0, \quad \beta_\mu \partial_\mu F = 0. \quad (2')$$

As before

$$\begin{aligned} (\beta_\mu \partial_\mu)^2 A &= -(\beta_\mu \partial_\mu) F = 0, \\ (\beta_\mu \partial_\mu)^3 A &= \square^2 \beta_\mu \partial_\mu A = 0, \\ \square^2 F &= 0. \end{aligned} \quad (5')$$

Now, however, the derivation of the corresponding equation $\square^2 A = 0 \dots (6')$ fails completely, and hence only $\square^2 F = 0$ is implied by the equations of motion. Again writing out equation (5') for the λ th component, one has

$$0 = \square^2 A_\lambda - \{ \partial_\lambda (\partial_\mu A_\mu) \}. \quad (8')$$

Therefore when $\kappa \rightarrow 0$ the equations of motion do not in themselves imply (6') and (9)

separately, but only the combined equation (8'). Of course (6') and (9) are compatible with the equations of motion; and if we require either one, the other automatically follows. This is in fact usually the case, since one supposes $\square^2 A = 0$.

The electromagnetic field can then be regarded as the continuous limit of a vector field as $\kappa \rightarrow 0$ in the following sense: If the equation of motion $(\square^2 - \kappa^2)A = 0$ is taken to pass continuously to the limit as $\kappa \rightarrow 0$, the subsidiary condition $\partial_\mu A_\mu = 0$ is automatically imposed in all cases by the original equations of motion. Since the self-energy of a vector meson is $\delta\kappa \sim \kappa$, the photon self-energy is implicitly zero. This conclusion is altered only if one makes the theoretically permissible but otherwise pathologic choice of a functional discontinuity at $\kappa = 0$ in the right-hand side of $(\square^2 - \kappa^2)A = 0$.

Case of spin 0

Finally, this procedure may be repeated for the spin 0 representation to see what happens as $\kappa \rightarrow 0$. One obtains

$$\beta_\mu \partial_\mu S + D = 0, \quad \beta_\mu \partial_\mu D + \kappa^2 S = 0, \quad (10)$$

where

$$S = \frac{1}{\sqrt{\kappa}} \left(\frac{M-1}{3} \right) \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ U \end{pmatrix},$$

$$D = \sqrt{\kappa} \left(\frac{4-M}{3} \right) \phi = \phi \begin{pmatrix} -\partial_1 U \\ -\partial_2 U \\ -\partial_3 U \\ -\partial_4 U \\ 0 \end{pmatrix}, \quad (11)$$

$$S + D = \phi.$$

For $\kappa \neq 0$, the same procedure as (5) and (6) leads from (10) to

$$\square^2 S = \kappa^2 S, \quad \square^2 D = \kappa^2 D. \quad (12)$$

To find the subsidiary condition, write out the intermediate form

$$\kappa^2 S = (\beta_\mu \partial_\mu)^2 S = \square^2 S + \{(\beta_\mu^2 - 1) \partial_\mu^2 + \sum_{\mu \neq \nu} \beta_\mu \beta_\nu \partial_\mu \partial_\nu\} S. \quad (13)$$

With the five-rowed β_μ it turns out that the bracketed expression in (13) is identically zero; therefore no subsidiary condition exists.

When $\kappa \rightarrow 0$, the only relations obtainable directly from (10) are $\square^2 D = 0$ and the analogue of (13) with $\kappa^2 = 0$. Now the identical vanishing of the subsidiary condition means that also $\square^2 S = 0$. Thus the original equations of motion imply that the Klein-Gordon equations for both S and D pass continuously to the limit as $\kappa \rightarrow 0$.

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** Now at Columbia University, New York.

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On the Self-Energies of Nucleons

Hiroshi ENATSU

Department of Physics, Kyoto University

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The second order self-energies of nucleons are evaluated by making use of the recent covariant formalism. If the symmetrical pseudoscalar type with pseudovector coupling is assumed for the π meson, it will be shown that, in order to cancel divergences of self-energies of nucleons due to π mesons, it is necessary to introduce heavy mesons which obey the symmetrical scalar field with vector coupling. Their coupling constants are larger than those of π mesons, and their masses are $1474 m_\pi$.

§ 1. Introduction

As is well known, the most fundamental difficulties appeared in the theory of the wave fields are the divergences of self-energies of the elementary particles and the problem of vacuum polarization. In order to overcome these problems many attempts have been made by several authors.

With respect to the first case Pais¹⁾ and Sakata²⁾ have independently proposed the theory of the cohesive meson as a possible method which makes the self-energy of an electron finite. This theory was then extended to the cases of nucleons, though the attempt proved to be by no means successful because the right mass difference of proton and neutron could not be obtained for the non-vanishing mass of the cohesive meson.

As to the second case, it is confronted with two types of difficulties. They are the appearance of non-gauge invariant terms and the infinite self-charge. In spite of various trials, at the present stage, the method of mixed fields is of no use for the solution of divergences mentioned above. This may be due to our insufficient knowledge on the properties of the elementary particles as well as to difficulties of the present field theory which prevent us from deriving reasonable conclusions out of physical principles.

Generally speaking, there are two points of view about the subduction of the present difficulties. In relation to this point, Dirac stressed,³⁾ "The interlocking of the difficulties impress certain people very much and they think that the way out of these difficulties will be ultimately attained by some one making a master stroke which will simultaneously remove all of the difficulties." Then he stated another view that one might try to separate the difficulties as far as possible, thus one could dispose of the difficulties one at a time and one would then have an easier problem to face.

In our opinions, the self-energies of the elementary particles (mass-types) are

easier to attack than the problem of vacuum polarization, because the former have the correspondence in their relations to classical theory, but on the other hand the latter is only a quantum mechanical effect.

Therefore, in the present and next papers we will pick up at first the problems of the self-energies of nucleons due to π mesons and photon, and investigate how far we are able to overcome these difficulties by means of the prevailing covariant formalism and the method of mixed fields which is based essentially on the realistic view point. As to this point Sakata and Umezawa⁴⁾ have expressed their view that part of difficulties appeared is due to the overlooking of intimate relations between the elementary particles, and one must investigate the behavior of each elementary particle by taking account of the effects of all the other elementary particles which interact with it at the same time.

Furthermore, according to the recent experiments⁵⁾ about cosmic rays, it seems very likely that there are a few kinds of mesons, including π mesons, which interact with nucleons. On the basis of these considerations we adopt the realistic point of view in the following.

To such problems which include the vacuum polarization we shall come back elsewhere.

As regards the principle of the mixed theory (M. T. for short) itself, which was useful in the history of development of the theory of elementary particles, we have given some remarks in the previous papers.⁶⁾ We shall summarize briefly our proposal.

In his work on the classification of possible meson wave fields, Kemmer⁷⁾ has treated four different fields of mesons, namely, scalar, vector, pseudovector and pseudoscalar ones. It is well known that the various M. T. which have been made up to now were based upon the Kemmer's classification. However, if the interactions with source particles were omitted, the differences between dual cases would be merely a matter of notation. In this connection, it may be noticed that we can write the Maxwell's equations, which are expressed in the vector form, in two alternative ways. That is, the Maxwell's equations

$$\begin{cases} \text{curl } \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = 0, & \text{div } \mathbf{E} = 0, \\ \text{curl } \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, & \text{div } \mathbf{H} = 0, \end{cases} \quad (c=1) \quad (1)$$

are converted into the next forms. Following Dirac,⁸⁾ one possibility of writing them in tensor form is

$$\begin{cases} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0, \\ \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0. \end{cases} \quad (2)$$

Alternatively, however, in his dual notation Eqs. (1) can be transcribed as

$$\begin{cases} \frac{\partial (F^\dagger)_{\mu\nu}}{\partial x^\lambda} + \frac{\partial (F^\dagger)_{\nu\lambda}}{\partial x^\mu} + \frac{\partial (F^\dagger)_{\lambda\mu}}{\partial x^\nu} = 0, \\ \frac{\partial (F^\dagger)_{\mu\nu}}{\partial x_\nu} = 0, \end{cases} \quad (3)$$

where $(F^\dagger)_{01} = F_{23}$, etc. and \dagger denotes the dual quantity. When the electromagnetic field is coupled with the source charge, it is seen as usual that source density may be added to the right sides of above equations. In Eqs. (2) it is the charge-current density j_μ , and in Eqs. (3) it is probably of the form $\epsilon_{\mu\nu\lambda\rho} j^\rho$, where $\epsilon_{\mu\nu\lambda\rho}$ is the antisymmetric tensor of fourth rank in the Minkowski space.

In this way we see clearly that two equations (2) and (3) describe the same substance (electromagnetic field).

As has been shown by Schoenberg,⁹⁾ this procedure can be extended to the meson theory by making use of the tensor $\epsilon_{\mu\nu\lambda\rho}$. Actually, we can define the dual tensors¹⁰⁾ as follows:

$$\begin{cases} (\phi^\dagger)^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta\gamma\delta} \phi, \\ (\phi^\dagger)^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta} \phi_\delta, \\ (\phi^\dagger)^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \phi_{\gamma\delta}, \end{cases} \quad (4)$$

It should be, however, noticed that the dual quantities of source functions could not be defined uniquely by $\epsilon_{\mu\nu\lambda\rho}$ owing to their bilinear forms of nucleonic wave functions. In other words, the source vector $i\bar{\psi}\gamma^\mu\psi$ cannot be connected with the source pseudovector $i\bar{\psi}\gamma^\mu\psi$ by ϵ . In spite of these ambiguities, we may suppose that the scalar meson is the same one with the pseudoscalar field, and the vector field is identified with the pseudovector one. In the following we shall call this treatment the mixed interaction theory (briefly M. I. T.).

In such a situation it is interesting to point out the resemblance between M. I. T. and other theories. At first, it is to be noted that the M. I. T. for spin 1 meson field may correspond to the magnetic pole theory introduced by Dirac.⁸⁾⁶⁾ On physical ground, however, this analogy is not so complete because of the ambiguities stated above. That is to say, in the meson theory such an attempt as M. I. T. is merely a speculation for which there is no physical basis, on the other hand, Dirac's assumption for the magnetic pole is related to the well established standpoint of the Maxwell theory.

Next, let us now consider the case of spin 0. Contrary to the spin 1 field, the concept of M. I. T. for spin 0 is assumed *ad hoc*. Indeed, we have no counterpart for it in the classical theory. In connection with the mixing prescription of M. I. T., it is, however, interesting to refer to the results by Yang and Tiomno.¹¹⁾ Considering pseudospinors, they found that for the scalar meson there is a possibility of having a pseudoscalar type interaction. In addition, Moseley and

Rosen¹²⁾ have shown that, when one regards a π meson as the composite particle of nucleons, in the case of spin 0, one may attain to nuclear potentials which are interpreted as a mixture of the scalar and pseudoscalar types in the relativistic region.

The discussions will be performed in several steps: In the next section we shall review briefly Pais' work concerning the self-energies of nucleons which were calculated by making use of the old method. Then we shall explain in § 3 the method of evaluation and state the criterion for the applicability of it. In § 4 the derivation of the second order self-energies of nucleons are carried out. The examination of the results obtained and the discussion of mass spectrum will be done in the last section.

§ 2. Reviw of previous work by Pais¹⁾

In attacking the problem of self-energies of nucleons our first move, following the next reasoning, is to find the relations between four independent coupling constants which appear in the spin 0 meson field of M. I. T..

In the previous note⁶⁾ we have obtained the nuclear potential for it, and, actually, in order to reduce the numbers of constants, we set up the postulate,

$$|f_1| = |f_1'| = |f_2'| \tag{5}$$

where they are the coupling constants of scalar, pseudoscalar and pseudovector interactions respectively (vector one dropped by Dyson's Theorem). Excepting that they have the dimension of an electric charge, there is no other reason for assuming the relation (5). To carry the analysis further requires us to find a

Table I

Type	C_{qu}	C_{log}
f_s	—	$-\frac{1}{2}f^2$
g_s	$-g^2\eta^2$	$\frac{4}{3}g^2\eta^2$
f_sg_s	—	—
g_s : direct	$-ng^2\eta^2$	$\frac{4}{3}\eta g^2\eta^2$
f_{ps}	—	$\frac{1}{6}f^2$
g_{ps}	$\frac{1}{2}g^2\eta^2$	$-\frac{2}{3}g^2\eta^2(1-\eta^{-2})$
$f_{ps}g_{ps}$	$\frac{1}{2}fg\eta$	$-\frac{2}{3}fg\eta(1+\eta^{-2})$
g_{ps} : direct	$ng^2\eta^2$	$-\frac{4}{3}ng^2\eta^2$

$$W^{(1)} = \frac{C_{qu}}{\pi^2 M \hbar} \int_0^\infty p dp + \frac{3MC_{log}}{8\pi^2 \hbar} \int_0^\infty \frac{dp}{p} + finite \ term.$$
$$\eta = \frac{M}{\mu}$$

new idea. We shall turn to the suggestion made by Pais.¹⁾ It is a characteristic feature of the Pais' theory of compensation as well as of that of Sakata, that, supposing the elementary particles to be the source of sets of fields in such a way that their infinite contributions to the self-energies cancel one another, one may find relations between the constants of the fields responsible for the self-energies. This procedure is convenient to apply to our problems.

According to Pais,¹⁾ by employing the non-relativistic method of evaluation, the coefficients of divergences of self-energies of nucleons due to spin 0 mesons on hole theory are given in Table I after his notation. It may be remarked that for the M. I. T. there arise cross terms besides them shown above, for example, $f_s f_{ps}$, $f_s f_{pv}$, $g_s g_{ps}$, ... etc.. However, they contain γ_5 , so that, finally they vanish by taking their traces. Then, if the mass of scalar meson is equal to that of pseudoscalar one, the difference between M. I. T. and M. T. can not be seen in the present approximation. Now, we shall return to our own objectives of M. I. T.. In order to compensate the divergences one another, one must solve a cross word puzzle of Table I. It will be easily seen that the convergence conditions are

$$\begin{aligned} |f_{ps}| &= \eta |g_{ps}|, \\ |g_s| &= |g_{ps}|, \end{aligned} \quad (6)$$

and
$$|f_s| = \frac{\eta}{\sqrt{3}} |g_{ps}|.$$

In a word, though the spin 0 field has apparently four constants of interaction, only one of them is the independent variable. Thus, instead of assuming the relation (5) *ad hoc*, one may take the conditions (6) on the physical ground. It is, further, interesting to point out that the first condition of (6) may illuminate another question. Namely, by Dyson's theorem concerning the equivalence between the pseudoscalar and pseudovector couplings, one meets frequently a factor

$$f_{ps} + \frac{2M}{\mu} g_{ps}. \quad (7)$$

One may ask why the factor $\frac{2M}{\mu}$ does not occur in the first term of (7). By use of the first relation of (6), one finds that the expression (7) leads to

$$\frac{M}{\mu} g_{ps} + \frac{2M}{\mu} g_{ps}, \quad (8)$$

which is just the answer to the above question.

In this way, if we adopt the spin 0 field of M. I. T. (or M. T.) for the π mesons, we know that the nucleon self-energy due to it is finite in virtue of (6) without introducing another meson. Considering these circumstances, in the following sections we shall inquire whether this is true or not for the covariant computation.

§ 3. Method of calculation

We shall now proceed to the evaluations of self-energies of nucleons to the lowest order by the Feynman-Dyson's formalism. Of course we can perform them in the usual way, but in the meson theories Dyson's formula for the S -matrix suffers from the complication owing to so-called normal dependent parts of interactions, therefore, we shall make use of the method suggested by Koba¹³⁾ to attain the results without being bothered by the presence of those terms.

Following his method, the S -matrix is given by

$$S = U(+\infty, -\infty) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar c} \right)^n \int_{-\infty}^{\infty} dx_1 \cdots dx_n P^*(L(x_1), \dots, L(x_n)), \quad (9)$$

where $P^*(\dots)$ is the modified P -symbol and $L(x_i)$ is an interaction Lagrangian density. In the following the whole theory can be formulated in a neutral way for brevity of calculation. In a symmetrical theory the first approximation to the self-energy is twice that of the neutral one. The next table II gives the notations for coupling constants and wave functions. Here, the classification by Kemmer is employed, for it is trivial to transform to cases of M. I. T. and M. T..

The systems of the nucleon field and the meson fields in interaction are described by the Lagrangian densities,

Table II

Type of fields	Coupling constants	Wave functions
$S(s)$ $S(v)$	f_1 f_2	ϕ
$Ps(ps)$ $Ps(pv)$	F_1 F_2	
$V(v)$ $V(\epsilon)$	g_1 g_2	
$Pv(pv)$ $Pv(pt)$	G_1 G_2	

$$L_s = f_1 \bar{\psi} \psi \phi + i \left(\frac{f_2}{x} \right) \bar{\psi} \gamma_\mu \psi \frac{\partial \phi}{\partial x_\mu}, \quad (10)$$

$$L_{ps} = i F_1 \bar{\psi} \gamma_5 \psi \phi + i \left(\frac{F_2}{x} \right) \bar{\psi} \gamma_5 \gamma_\mu \psi \frac{\partial \phi}{\partial x_\mu}, \quad (11)$$

$$L_v = i g_1 \bar{\psi} \gamma_\mu \psi A_\mu + i \left(\frac{g_2}{x} \right) \bar{\psi} \gamma_\mu \gamma_\nu \psi \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right), \quad (12)$$

$$L_{pv} = i G_1 \bar{\psi} \gamma_5 \gamma_\mu \psi A_\mu + i \left(\frac{G_2}{x} \right) \bar{\psi} \gamma_5 \gamma_\mu \gamma_\nu \psi \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right), \quad (13)$$

where $\bar{\psi}$, ψ , ϕ and A_μ satisfy the following equations and commutation relations,

$$\bar{\psi} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M \right) = 0, \quad M = \frac{mc}{\hbar} \quad (14)$$

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M \right) \psi = 0, \quad (15)$$

$$(\square - x^2)\phi = 0, \quad (16)$$

$$(\square - x^2)A_\mu = 0, \quad (17)$$

$$\{\phi_\alpha(x), \bar{\psi}_\beta(x')\} = \frac{1}{i} S_{\alpha\beta}(x-x'), \quad (18)$$

$$[\phi(x), \phi(x')] = i\hbar c \Delta(x-x'), \quad (19)$$

$$[A_\mu(x), A_\nu(x')] = i\hbar c \delta_{\mu\nu} \Delta(x-x'), \text{ etc.} \quad (20)$$

Before entering into the evaluation of the second order S -matrix, we shall briefly summarize the method of P^* -symbol. It is defined as follows:

$$P^*(\phi(x_1), \phi(x_2)) \equiv \frac{1 + \epsilon^*(x_1, x_2)}{2} \phi(x_1)\phi(x_2) + \frac{1 - \epsilon^*(x_1, x_2)}{2} \phi(x_2)\phi(x_1), \quad (21)$$

where the ϵ^* -symbol denotes

$$\epsilon^*(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 > x_2 & \text{time-likely,} \\ 0 & \text{for } x_1 \text{ and } x_2 \text{ are space-like,} \\ -1 & \text{for } x_1 < x_2 & \text{time-likely.} \end{cases} \quad (22)$$

The vacuum expectation value of $P^*(\phi(x_1), \phi(x_2))$ is given by the modified Δ_F function,

$$\Delta_F^*(x_1 - x_2) = \Delta^{(1)}(x_1 - x_2) + i\epsilon^*(x_1, x_2) \Delta(x_1 - x_2), \quad (23)$$

as

$$\langle P^*(\phi(x_1), \phi(x_2)) \rangle_0 = \frac{\hbar c}{2} \Delta_F^*(x_1 - x_2), \quad (24)$$

$$\langle P^*\left(\frac{\partial\phi(x_1)}{\partial x_{1\mu}}, \phi(x_2)\right) \rangle_0 = \frac{\hbar c}{2} \frac{\partial \Delta_F^*(x_1 - x_2)}{\partial x_{1\mu}}, \quad (25)$$

$$\langle P^*\left(\frac{\partial\phi(x_1)}{\partial x_{1\mu}}, \frac{\partial\phi(x_2)}{\partial x_{2\nu}}\right) \rangle_0 = \frac{\hbar c}{2} \frac{\partial^2 \Delta_F^*(x_1 - x_2)}{\partial x_{1\mu} \partial x_{2\nu}}, \quad (26)$$

and

$$\langle P^*(\bar{\psi}(x_1), \phi(x_2)) \rangle_0 = \frac{1}{2} S_F^*(x_2 - x_1). \quad (27)$$

That is to say, neglecting the normal dependent part from the beginning, one may carry through the straightforward differentiation of the Δ_F^* function in the P^* symbol. The Fourier representations of them and S_F^* function are

$$S_F^*(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M \right) \Delta_F^*(x), \quad (28)$$

$$\Delta_F^*(x) = \frac{-2i}{(2\pi)^4} \int \frac{e^{ik_\mu x_\mu} (dk)}{k^2 + M^2}, \quad (29)$$

$$S_F^*(x) = \frac{-2i}{(2\pi)^4} \int \frac{(i\gamma_\mu k_\mu - M) e^{ik_\mu x_\mu}}{k^2 + M^2} (dk), \quad (30)$$

$$A_F^*(x) = \frac{-2i}{(2\pi)^4} \int \frac{e^{ik_\mu x_\mu}}{k^2 + M^2} (dk), \quad (31)$$

$$\bar{\psi}(x) = \bar{\psi}(p') e^{-ip'x_\mu}, \quad \psi(x) = \psi(p) e^{ipx_\mu}. \quad (32)$$

Then, substituting the above representations for the second order S -matrices which correspond to the Fig. 1, and uniting the

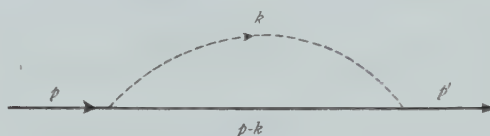


Fig. 1

denominators of them into a single term by means of the obvious formula

$$\frac{1}{ab} = \int_0^1 \frac{du}{[au + b(1-u)]^2}, \quad (33)$$

we obtain the following expressions,

$$S^{(2)} = \frac{1}{2!} \left(\frac{i}{\hbar c} \right)^2 \int dp \int dp' \delta(p - p') \bar{\psi}(p') J \psi(p), \quad (34)$$

$$J = N \hbar c \int_0^1 du \int (dk) \frac{I'}{[(k_\mu - u p_\mu)^2 + M^2 u^2 + (p_\mu^2 + M^2) u (1-u) + x^2 (1-u)]^2}, \quad (35)$$

where

	N	I'
$S(s)$	$-2F_1^2$	$[i\gamma(k-p) + M]$
$S(v)$	$2\left(\frac{f_2}{x}\right)^2$	$(\gamma k)[i\gamma(k-p) + M](\gamma k)$
$S(s, v)$		0
$Ps(p_s)$	$2F_1^2$	$-[i\gamma(k-p) - M]$
$Ps(pv)$	$2\left(\frac{F_2}{x}\right)^2$	$(\gamma k)[i\gamma(k-p) - M](\gamma k)$
$Ps(p_s, pv)$	$i\left(\frac{F_1 F_2}{x}\right)$	$(\gamma k)[i\gamma(k-p) - M] + [i\gamma(k-p) - M](\gamma k)$
$V(v)$	$-4g_1^2$	$[i\gamma(k-p) - 2M]$
$V(t)$	$-2\left(\frac{f_2}{x}\right)^2$	$8i(\gamma k)[\gamma(p-k)](\gamma k)$ $+ 4i[\gamma(p-k)](\gamma k)(\gamma k)$ $+ 12M(\gamma k)(\gamma k)$

$$\begin{aligned}
 V(v, t) & -i\left(\frac{g_1 g_2}{x}\right) & 6i[\gamma(k-p)](\gamma k) + 6i(\gamma k)[\gamma(k-p)] \\
 & & -12M(\gamma k) \\
 Pv(pv) & -4G_1^2 & i\gamma(k-p) + 2M \\
 Pv(pt) & -2\left(\frac{G_2}{x}\right)^2 & 8i(\gamma k)[\gamma(k-p)](\gamma k) \\
 & & + 4i[\gamma(k-p)](\gamma k)(\gamma k) \\
 & & + 12M(\gamma k)(\gamma k) \\
 Pv(pv, pt) & & 0 \quad . \quad (36)
 \end{aligned}$$

(γk) and $\gamma(k-p)$ mean the products $\gamma_\mu k_\mu$ and $\gamma_\mu(k_\mu - p_\mu)$ respectively, and $P_s(p_s, p_v)$ denotes the contribution which arises from the cross-term proportional to $f_1 f_2$.

Now, it is to be noted that we here meet with an ambiguity for the evaluation of divergent integrals. No actual convention to avoid such a situation has been found at the present stage. According to Peaslee,¹⁴⁾ it seems appropriate to separate the problems into two aspects, though they are closely related to the divergence difficulty. At first, we shall consider the case of variable changes of the momentum integrals. Namely, the denominator of the integral

$$\int f(k_\mu, p_\mu)(dk) \quad (37)$$

is made symmetrical in k by a transformation $k_\mu \rightarrow k_\mu + u p_\mu$, and it becomes¹⁵⁾

$$\int (1+D)g(k_\mu, p_\mu)(dk), \quad (38)$$

where

$$D = -u p_\mu \frac{\partial}{\partial k_\mu} + \frac{1}{2!} u^2 p_\mu p_\lambda \frac{\partial^2}{\partial k_\mu \partial k_\lambda} - \dots \quad (39)$$

Peaslee asserted that the terms of type D (translation effect) may be neglected. It is, however, necessary for our case to take account of such an effect. For it is our main aim to cancel divergences by mixing various fields. Actually, as will be shown in the later section, the first integral of (38) is always divergent, then the operator D will induce finite or infinite values according to the degree of its divergence. Therefore, for the sake of completeness whole divergent terms which come from the translation effect must be added to the original ones in applying the method of compensation. On another observation, however, we note that the inclusion of D may be preferable to our theory, for it will be shown in the next paper that the right mass difference of proton and neutron may be obtained only when such an effect is taken up.

Secondly, let us consider another obscure point. In the recent work concerning the ambiguities of the present field theory, Fukuda and Kinoshita¹⁶⁾ pointed out

that in order to remove them one need two sets of conditions,

$$(I) \quad \int k^2 \{ (k^2 + B) \delta^n(k^2 + B) + n \delta^{n-1}(k^2 + B) \} (dk) = 0 \quad (40)$$

and

$$(II) \quad \int \frac{\partial}{\partial k_\mu} \{ k^\mu \delta^{n-1}(k^2 + B) \} (dk) = 0, \quad (41)$$

or after Peaslee's notation (for a special case)

$$(I) \quad \int k^2 G_n(dk) = -n \int G_{n-1}(dk) - B \int G_n(dk) \quad (42)$$

and

$$(II) \quad \int k^2 G_n(dk) = \frac{1}{2} \int k_\mu \frac{\partial G_{n-1}}{\partial k_\mu} (dk), \quad (43)$$

where

$$G_n = \delta^n(k^2 + B) = \frac{(-1)^n n!}{(k^2 + B)^{n+1}}. \quad (44)$$

They are mutually independent and contradictory to each other. From our point of view that divergences must be eliminated by compensation, it seems, however to be superfluous to impose the condition (II) on some integrals. Apparently, the expression (II) means that one must always drop such surface integrals as those which are introduced by considering the operator D , so that nothing is gained by taking that effect. This is explained in detail for the cases of $PS(p_s)$ and $S(s)$ (logarithmic divergences) in Appendix.

As a result of these investigations we shall summarize our prescription in the following way:

- (1) The operator D must be taken into account when making transformation of type $k_\mu \rightarrow k_\mu + u p_\mu$.
- (2) Then the condition (I) may be consistently used throughout the calculation.
- (3) The condition (II) be always neglected.

The above method was adopted by Karplus and Kroll.¹⁵⁾

§ 4. Second order self-energies of nucleons

$$I = \int_0^1 du \int \frac{\Gamma(dk)}{[(k_\mu - u p_\mu)^2 + M^2 u^2 + (p_\mu^2 + M^2)u(1-u) + x^2(1-u)]^2}. \quad (45)$$

After the transformation of variable,

$$k_\mu \rightarrow k_\mu + u p_\mu, \quad (46)$$

the translation effects have to be taken into account in the integrand of (45). Considering the degree of divergence with respect to the k -integration, it turns

out to be necessary to pick up the following terms from the series of ascending power of u for each of interactions,

$$\left\{ \begin{array}{l} D = -u p_\mu \frac{\partial}{\partial k_\mu}, \quad S(s), Ps(ps), V(v) \text{ and } Pv(pv), \\ D = -u p_\mu \frac{\partial}{\partial k_\mu} + \frac{1}{2!} u^2 p_\mu p_\nu \frac{\partial^2}{\partial k_\mu \partial k_\nu}, \quad Ps(ps, pv) \text{ and } V(v, t), \\ D = -u p_\mu \frac{\partial}{\partial k_\mu} + \frac{1}{2!} u^2 p_\mu p_\nu \frac{\partial^2}{\partial k_\mu \partial k_\nu} - \frac{1}{3!} u^3 p_\mu p_\nu p_\lambda \frac{\partial^3}{\partial k_\mu \partial k_\nu \partial k_\lambda}, \\ S(v), Ps(pv), V(t) \text{ and } Pv(pt). \end{array} \right. \quad (47)$$

Then, (45) becomes

$$I = \int_0^1 du \int (dk) (1 + D) \frac{1}{A^2} \Gamma|_{k_\mu \rightarrow k_\mu + u p_\mu}, \quad (48)$$

where

$$\begin{aligned} A &= k^2 + M^2 u^2 + x^2 (1 - u) + (p_\mu^2 + M^2) u (1 - u) \\ &= k^2 + A^2. \end{aligned} \quad (49)$$

In the evaluation of the right-hand side of (48), we shall make use of the next substitutions from the symmetric property of the k -integration,

$$k_\alpha \rightarrow 0, \quad (50)$$

$$k_\alpha k_\beta \rightarrow \frac{1}{4} k^2 \delta_{\alpha\beta}, \quad (51)$$

$$k_\alpha k_\beta k_\gamma \rightarrow 0, \quad (52)$$

$$k_\alpha k_\beta k_\gamma k_\delta \rightarrow \frac{1}{24} (k^2)^2 (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}), \quad (53)$$

$$k_\alpha k_\beta k_\gamma k_\delta k_\epsilon \rightarrow 0, \quad (54)$$

$$\begin{aligned} k_\alpha k_\beta k_\gamma k_\delta k_\epsilon k_\eta \rightarrow & \frac{1}{192} (k^2)^3 (\delta_{\alpha\beta} \delta_{\gamma\delta} \delta_{\epsilon\eta} + \delta_{\alpha\beta} \delta_{\gamma\epsilon} \delta_{\delta\eta} + \delta_{\alpha\beta} \delta_{\gamma\eta} \delta_{\delta\epsilon} \\ & + \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{\epsilon\eta} + \delta_{\alpha\gamma} \delta_{\beta\epsilon} \delta_{\delta\eta} + \delta_{\alpha\gamma} \delta_{\beta\eta} \delta_{\delta\epsilon} \\ & + \delta_{\alpha\delta} \delta_{\beta\gamma} \delta_{\epsilon\eta} + \delta_{\alpha\delta} \delta_{\beta\epsilon} \delta_{\gamma\eta} + \delta_{\alpha\delta} \delta_{\beta\eta} \delta_{\gamma\epsilon} \\ & + \delta_{\alpha\epsilon} \delta_{\beta\gamma} \delta_{\delta\eta} + \delta_{\alpha\epsilon} \delta_{\beta\delta} \delta_{\gamma\eta} + \delta_{\alpha\epsilon} \delta_{\beta\eta} \delta_{\gamma\delta} \\ & + \delta_{\alpha\eta} \delta_{\beta\gamma} \delta_{\delta\epsilon} + \delta_{\alpha\eta} \delta_{\beta\delta} \delta_{\gamma\epsilon} + \delta_{\alpha\eta} \delta_{\beta\epsilon} \delta_{\gamma\delta}). \end{aligned} \quad (55)$$

After performing the cumbersome differentiations with respect to k_μ , one obtains the following results in which all terms are rearranged into two groups. That is, one of them has a factor $(i\gamma p + M)$ and the other has not, but the

former vanishes in the final results on account of the assumption that a free nucleon satisfies the Dirac equation.

$$I = \int_0^1 du \int (dk) \mathcal{Q}, \quad (56)$$

$$\mathcal{Q}_{S(s)} = (i\gamma p + M) \left[-\frac{1}{A^2} + \frac{uk^2}{A^3} \right] + \frac{2M}{A^2} - \frac{uMk^2}{A^3}, \quad (57)$$

$$\begin{aligned} \mathcal{Q}_{S(v)} = & \left(\frac{1}{A^2} \right) \left[(i\gamma p + M) \left\{ -\frac{1}{2} (3u^2 + u - 1) k^2 - 3u^5 M^2 \right\} \right. \\ & \left. + \frac{1}{2} (3u^2 - u + 1) M k^2 + (3u^5 - 3u^4 - u^3) M^3 \right] \\ & + \left(\frac{1}{A^3} \right) \left[(i\gamma p + M) \{ uk^4 + (6u^4 + 2u^3 + u^2) M^2 k^2 - 2(u^5 - u^4) M^4 \} \right. \\ & \left. - uMk^4 - (6u^4 - 2u^3 - u^2) M^3 k^2 + 2(u^5 - u^4) M^5 \right] \\ & + \left(\frac{1}{A^4} \right) \left[(i\gamma p + M) \{ -6u^3 M^2 k^4 + 3(u^5 - u^4) M^4 k^2 \} \right. \\ & \left. + 3(2u^3 - u^2) M^3 k^4 - 3(u^5 - 2u^4) M^5 k^2 \right], \end{aligned} \quad (58)$$

$$\mathcal{Q}_{S(s,v)} = 0, \quad (59)$$

$$\mathcal{Q}_{P^s(p^s)} = (i\gamma p + M) \left[\frac{1}{A^2} - \frac{uk^2}{A^3} \right] + \frac{uMk^2}{A^3}, \quad (60)$$

$$\begin{aligned} \mathcal{Q}_{P^s(pv)} = & \left(\frac{1}{A^2} \right) \left[(i\gamma p + M) \left\{ -\frac{1}{2} (3u^2 + u - 1) k^2 - 3u^5 M^2 \right\} \right. \\ & \left. + \frac{3}{2} (u^2 + u - 1) M k^2 + (3u^5 + 3u^4 + u^3) M^3 \right] \\ & + \left(\frac{1}{A^3} \right) \left[(i\gamma p + M) \{ uk^4 + (6u^4 + 2u^3 + u^2) M^2 k^2 - 2(u^5 - u^4) M^4 \} \right. \\ & \left. - 3(2u^4 + 2u^3 + u^2) M^3 k^2 - uMk^4 + 2u^5 M^5 \right] \\ & + \left(\frac{1}{A^4} \right) \left[(i\gamma p + M) \{ -6u^3 M^2 k^4 + 3(u^5 - u^4) M^4 k^2 \} \right. \\ & \left. + 3(2u^3 + u^2) M^3 k^4 - 3u^5 M^5 k^2 \right], \end{aligned} \quad (61)$$

$$\begin{aligned} \mathcal{Q}_{P^s(p^s, pv)} = & \left(\frac{1}{A^2} \right) [k^2] \\ & + \left(\frac{1}{A^3} \right) \left[(i\gamma p + M) \{ uMk^2 + 2u^3 M^3 \} + 2u^2 M^2 k^2 - 2u^4 M^4 \right] \\ & + \left(\frac{1}{A^4} \right) \left[(i\gamma p + M) \{ -3u^3 M^3 k^2 \} - 3u^2 M^2 k^4 + 3u^4 M^4 k^2 \right], \end{aligned} \quad (62)$$

$$\mathcal{Q}_{V(u)} = (i\gamma p + M) \left[-\frac{1}{A^2} + \frac{uk^2}{A^3} \right] - \frac{M}{A^2} - \frac{uMk^2}{A^3}, \quad (63)$$

$$\begin{aligned} \mathcal{Q}_{V(u)} = & \left(\frac{1}{A^2} \right) [3Mk^2] \\ & + \left(\frac{1}{A^3} \right) [(i\gamma p + M) \{ -3uk^4 - 6(2u^5 - u^4)M^4 \} \\ & \quad + 3uMk^4 + 6u^2M^3k^2 + 12u^5M^5] \\ & + \left(\frac{1}{A^4} \right) [(i\gamma p + M) \{ -3(3u^3 + u^2)M^2k^4 + 9(5u^5 - 3u^4)M^4k^2 \} \\ & \quad + 3(3u^3 - 2u^2)M^3k^4 - 45u^5M^5k^2] \\ & + \left(\frac{1}{A^5} \right) [(i\gamma p + M) \{ 12u^3M^2k^6 - 12(3u^5 - 2u^4)M^4k^4 \} \\ & \quad - 12u^3M^3k^6 + 36u^5M^5k^4], \end{aligned} \quad (64)$$

$$\begin{aligned} \mathcal{Q}_{V(v,t)} = & \left(\frac{1}{A^2} \right) [-2k^2] \\ & + \left(\frac{1}{A^3} \right) [(i\gamma p + M) \{ -2uMk^2 - 4u^3M^3 \} - 4u^2M^2k^2 - 4u^4M^4] \\ & + \left(\frac{1}{A^4} \right) [(i\gamma p + M) \{ 6u^3M^3k^2 \} + 6u^2M^2k^4 - 6u^4M^4k^2], \end{aligned} \quad (65)$$

$$\mathcal{Q}_{P^{vv}(pv)} = (i\gamma p + M) \left[-\frac{1}{A^2} + \frac{uk^2}{A^3} \right] + \frac{3M}{A^2} - \frac{uMk^2}{A^3}, \quad (66)$$

$$\begin{aligned} \mathcal{Q}_{P^{vv}(vt)} = & \left(\frac{1}{A^2} \right) [3Mk^2] \\ & + \left(\frac{1}{A^3} \right) [(i\gamma p + M) \{ 3uk^4 + 6(2u^5 - u^4)M^4 \} \\ & \quad - 3uMk^4 + 6u^2M^3k^2 - 12(u^5 - u^4)M^5] \\ & + \left(\frac{1}{A^4} \right) [(i\gamma p + M) \{ -3(3u^3 - u^2)M^2k^4 - 9(5u^5 - 3u^4)M^4k^2 \} \\ & \quad - 3(3u^3 + 4u^2)M^2k^4 + 9(5u^5 - 6u^4)M^4k^2] \\ & + \left(\frac{1}{A^5} \right) [(i\gamma p + M) \{ -12u^3M^2k^6 + 12(3u^5 - 2u^4)M^4k^4 \} \\ & \quad + 12u^3M^3k^6 - 12(3u^5 - 4u^4)M^5k^4], \end{aligned} \quad (67)$$

$$\mathcal{Q}_{P^{vv}(pv,pt)} = 0, \quad (68)$$

The divergent and convergent elements of these integrals are separated by using

the following formulas: Before performing the integrals on k , the k^{2n} terms in the numerators are replaced by the equivalent substitutions,

$$\begin{aligned} k^2 &\rightarrow (k^2 + A^2) - A^2, \\ k^4 &\rightarrow (k^2 + A^2)^2 - 2A^2(k^2 + A^2) + A^4, \\ &\dots\dots\dots, \end{aligned} \quad (69)$$

which are derived by the condition (I). As an example of them, we shall show that

$$\int \frac{k^4 (dk)}{(k^2 + A^2)^3} = \int \frac{(dk)}{(k^2 + A^2)} - 2A^2 \int \frac{(dk)}{(k^2 + A^2)^2} + A^4 \int \frac{(dk)}{(k^2 + A^2)^3}. \quad (70)$$

The condition (I) can be obtained from the formal identity

$$\int (dk) (k^2 + A^2)^a \delta(k^2 + A^2) = 0, \quad a=1 \text{ or } 2, \quad (71)$$

by the differentiation with respect to A^2 . For $a=1$, it leads to

$$\int (dk) \{ (k^2 + A^2) \delta'(k^2 + A^2) + \delta(k^2 + A^2) \} = 0, \quad (72)$$

$$\int (dk) \{ (k^2 + A^2) \delta''(k^2 + A^2) + 2\delta'(k^2 + A^2) \} = 0. \quad (73)$$

For $a=2$, by differentiating (71) two times

$$\begin{aligned} &\int (dk) \{ (k^4 - A^4) \delta''(k^2 + A^2) \} \\ &= \int (dk) \{ -2A^2(k^2 + A^2) \delta''(k^2 + A^2) - 4(k^2 + A^2) \delta'(k^2 + A^2) \\ &\quad - 2\delta(k^2 + A^2) \}. \end{aligned} \quad (74)$$

But from application of (73) on the right-hand side of (74), it becomes

$$\int (dk) \{ -4k^2 \delta'(k^2 + A^2) - 2\delta(k^2 + A^2) \}. \quad (75)$$

Finally, considering the identity (72), one obtain

$$\begin{aligned} &\int (dk) \{ (k^4 - A^4) \delta''(k^2 + A^2) \} \\ &= \int (dk) \{ 4A^2 \delta'(k^2 + A^2) + 2\delta(k^2 + A^2) \}. \end{aligned} \quad (76)$$

More conveniently, we have at last

$$\begin{aligned} \frac{1}{2} \int k^4 \delta''(k^2 + A^2) (dk) &= \int \delta(k^2 + A^2) (dk) + 2A^2 \int \delta'(k^2 + A^2) (dk) \\ &\quad + \frac{A^4}{2} \int \delta''(k^2 + A^2) (dk). \end{aligned} \quad (77)$$

This is the result (70) which we have intended to prove. The other ones are justified in a similar way.

Futhermore, in the following treatment, the denominators are reduced by the relation

$$\frac{1}{[k^2 + A^2]^n} = \frac{1}{[k^2 + M^2 A_0^2]^n} - n(p_\mu^2 + M^2)u(1-u) \times \int_0^1 \frac{dt}{[k^2 + M^2 A_0^2 + (p_\mu^2 + M^2)u(1-u)t]^{n+1}}, \quad (78)$$

where

$$A_0^2 = u^2 + \lambda^2(1-u), \quad (79)$$

and the terms which contain the factor $(p_\mu^2 + M^2)$ are omitted.

Thus, we have the expressions which are non-vanishing contributions to the self-energies.

$$I_{S(s)} = \frac{3}{2}M \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M \int_0^1 \left[u - (u^2 - 4u) \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du \quad (80)$$

$$I_{S(v)} = \frac{M}{4} \int \frac{(dk)}{A_0} - \frac{M^2}{3} \left(\frac{1}{5} + \frac{\lambda^2}{4} \right) \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{6} M^3 \int_0^1 [-3A_0^2 - 9u^5 + 33u^4 - 35u^3 + 12u^2] du + \frac{i\pi^2}{12} M^3 \int_0^1 \left[10u^6 - \frac{134}{5}u^5 + (22 + 3\lambda^2)u^4 - (6 + 13\lambda^2)u^3 + 15\lambda^2u^2 - 6\lambda^2u \right] \left(\frac{2u - \lambda^2}{A_0^2} \right) du, \quad (81)$$

$$I_{S(s,v)} = 0, \quad (82)$$

$$I_{S(s,ps)} = \frac{M}{2} \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M \int_0^1 \left[-u + u^2 \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (83)$$

$$I_{S(s,pv)} = -\frac{3}{4}M \int \frac{(dk)}{A_0} + M^2 \left(\frac{1}{3} + \frac{5}{4}\lambda^2 \right) \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M^3 \int_0^1 \left[-uA_0^2 - 3u^5 + 7u^4 - \frac{5}{3}u^3 - 2u^2 \right] du + \frac{i\pi^2}{4}M^3 \int_0^1 \left[\frac{10}{3}u^6 - 2u^5 + (2 + \lambda^2)u^4 - (2 + 3\lambda^2)u^3 + \lambda^2u^2 + 6\lambda^2u \right] \left(\frac{2u - \lambda^2}{A_0^2} \right) du, \quad (84)$$

$$I_{S(s,ps,pv)} = \int \frac{(dk)}{A_0} - x^2 \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M^2 \int_0^1 \left[3u^2 - 2\lambda^2u \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (85)$$

$$I_{V(v)} = -\frac{3}{2}M \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M \int_0^1 \left[u - (u^2 + 2u) \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (86)$$

$$I_{V(v)} = \frac{9}{2}M \int \frac{(dk)}{A_0} - \frac{M^3}{2}(1 + 9\lambda^2) \int \frac{(dk)}{A_0^2} \\ + \frac{i\pi^2}{2}M^3 \int_0^1 \left[3uA_0^2 + 11u^3 + 4u^2 + \{ -3u^4 + (2 + 3\lambda^2)u^3 - 6\lambda^2u^2 - 6\lambda^2u \} \right. \\ \left. \times \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (87)$$

$$I_{V(v, v)} = -2 \int \frac{(dk)}{A_0} + 2\lambda^2 \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M^2 \int_0^1 \left[-6u^2 + 4\lambda^2u \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (88)$$

$$I_{Pv(pv)} = \frac{5}{2}M \int \frac{(dk)}{A_0^2} + \frac{i\pi^2}{2}M \int_0^1 \left[u - (u^2 - 6u) \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (89)$$

$$I_{Pv(pv)} = \frac{3}{2}M \int \frac{(dk)}{A_0} + \frac{M^3}{2}(1 - 3\lambda^2) \int \frac{(dk)}{A_0^2} \\ + \frac{i\pi^2}{2}M^3 \int_0^1 \left[-3uA_0^2 - 11u^3 + 14u^2 \right. \\ \left. + \{ 3u^4 - (2 + 3\lambda^2)u^3 + 6\lambda^2u^2 - 6\lambda^2u \} \left(\frac{2u - \lambda^2}{A_0^2} \right) \right] du, \quad (90)$$

$$I_{Pv(pv, pv)} = 0, \quad (91)$$

where $A_0 = k^2 + M^2$, and $\lambda = \frac{x}{M}$, and in deriving them

$$\int \frac{(dk)}{(k^2 + L^2)^3} = \frac{i\pi^2}{2L^2}, \quad (92)$$

$$\int \frac{(dk)}{(k^2 + L^2)^4} = \frac{i\pi^2}{6L^4}, \quad (93)$$

and

$$\int \frac{(dk)}{(k^2 + L^2)^5} = \frac{i\pi^2}{12L^6}, \quad (94)$$

have been used. In addition to them we shall make use of the formulas,

$$\int \frac{(dk)}{A_0} = 2i\pi^2 \left(KK_0 - M^2 \log \frac{K + K_0}{M} \right), \quad (95)$$

and

$$\int \frac{(dk)}{A_0^2} = 2i\pi^2 \left(\log \frac{K + K_0}{M} - 1 \right), \quad K_0 = \sqrt{K^2 + M^2}, \quad (96)$$

which will be called hereafter as the quadratic and logarithmic parts respectively, nevertheless they contain logarithmic and constant terms.

Now we shall give here the final forms for the mesonic masses of nucleons ($\kappa < M$).

$$\begin{aligned} \delta m_{S(s)} = m \left(\frac{f_1^2}{4\pi\hbar c} \right) & \left[-\frac{3}{4\pi} \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & - \frac{1}{8\pi} \left\{ -\lambda^2 + \frac{15}{2} + (\lambda^4 - 6\lambda^2) \log \lambda \right. \\ & \left. \left. - (\lambda^5 - 8\lambda^3 + 16\lambda) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (97) \end{aligned}$$

$$\begin{aligned} \delta m_{S(v)} = m \left(\frac{f_2^2}{4\pi\hbar c} \right) & \left(\frac{1}{\lambda^2} \right) \left[\frac{1}{8\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) - \frac{1}{120\pi} (4 + 5\lambda^2) \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & + \frac{1}{120\pi} \left\{ 25\lambda^{10} - \frac{329}{2}\lambda^8 + \frac{931}{3}\lambda^6 - \frac{1375}{12}\lambda^4 - \frac{55}{6}\lambda^2 + \frac{158}{15} \right\} \\ & + \frac{1}{120\pi} \left\{ -25\lambda^{12} + 202\lambda^{10} - 530\lambda^8 + 455\lambda^6 - 65\lambda^4 \right\} \log \lambda \\ & \left. + \frac{1}{120\pi} \left\{ 25\lambda^{13} - 252\lambda^{11} + 884\lambda^9 - 1161\lambda^7 + 473\lambda^5 \right\} \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right], \quad (98) \end{aligned}$$

$$\delta m_{S(s,v)} = 0, \quad (99)$$

$$\begin{aligned} \delta m_{P(s)} = m \left(\frac{F_1^2}{4\pi\hbar c} \right) & \left[\frac{1}{4\pi} \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & \left. + \frac{1}{8\pi} \left\{ \lambda^2 + \frac{1}{2} - (\lambda^4 - 2\lambda^2) \log \lambda + (\lambda^5 - 4\lambda^3) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (100) \end{aligned}$$

$$\begin{aligned} \delta m_{P(s,pv)} = m \left(\frac{F_2^2}{4\pi\hbar c} \right) & \left(\frac{1}{\lambda^2} \right) \left[-\frac{3}{8\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) + \frac{1}{120\pi} (20 + 75\lambda^2) \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & + \frac{1}{120\pi} \left\{ 25\lambda^{10} - 120\lambda^8 + \frac{515}{12}\lambda^6 - \frac{45}{2}\lambda^4 + 95\lambda^2 - \frac{20}{3} \right\} \\ & + \frac{1}{48\pi} \left\{ -10\lambda^{12} + 63\lambda^{10} - 105\lambda^8 + 44\lambda^6 - 42\lambda^4 \right\} \log \lambda \\ & \left. + \frac{1}{48\pi} \left\{ 10\lambda^{13} - 83\lambda^{11} + 211\lambda^9 - 168\lambda^7 + 72\lambda^5 - 96\lambda^3 \right\} \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right], \quad (101) \end{aligned}$$

$$\begin{aligned} \delta m_{P(s,ps,pv)} = m \left(\frac{F_1 F_2}{4\pi\hbar c} \right) & \left(\frac{1}{\lambda} \right) \left[-\frac{1}{2\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) + \frac{\lambda^2}{2\pi} \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & \left. + \frac{1}{8\pi} \left\{ 4\lambda^2 - 1 - 2\lambda^4 \log \lambda + (2\lambda^5 - 8\lambda^3) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (102) \end{aligned}$$

$$\delta m_{V(v)} = m \left(\frac{g_1^2}{4\pi\hbar c} \right) \left[\frac{3}{2\pi} \left(\log \frac{K+K_0}{M} - 1 \right) + \frac{1}{4\pi} \left\{ \lambda^2 + \frac{9}{2} - \lambda^4 \log \lambda + (\lambda^5 - 2\lambda^3 - 8\lambda) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (103)$$

$$\begin{aligned} \delta m_{V(v)} = m \left(\frac{g_2^2}{4\pi\hbar c} \right) \left(\frac{1}{\lambda^2} \right) & \left[-\frac{9}{\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) + \frac{1}{\pi} (1 + 9\lambda^2) \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & + \frac{1}{24\pi} (12\lambda^4 + 198\lambda^2 - 42) \\ & - \frac{1}{2\pi} (\lambda^6 + 6\lambda^4) \log \lambda \\ & \left. + \frac{1}{2\pi} (\lambda^7 + 4\lambda^5 - 32\lambda^3) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right], \quad (104) \end{aligned}$$

$$\begin{aligned} \delta m_{V(v,t)} = m \left(\frac{g_1 g_2}{4\pi\hbar c} \right) \left(\frac{1}{\lambda} \right) & \left[\frac{3}{\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) - \frac{3}{\pi} \lambda^2 \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & \left. - \frac{3}{4\pi} \left\{ 4\lambda^2 - 1 + (2\lambda^4 - 4\lambda^2) \log \lambda + (2\lambda^5 - 8\lambda^3) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (105) \end{aligned}$$

$$\begin{aligned} \delta m_{Fv(pv)} = m \left(\frac{G_1^2}{4\pi\hbar c} \right) & \left[-\frac{5}{2\pi} \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & \left. + \frac{1}{4\pi} \left\{ \lambda^2 - \frac{23}{2} - (\lambda^4 - 8\lambda^2) \log \lambda + (\lambda^5 - 10\lambda^3 + 24\lambda) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right\} \right], \quad (106) \end{aligned}$$

$$\begin{aligned} \delta m_{Fv(pv)} = m \left(\frac{G_2^2}{4\pi\hbar c} \right) \left(\frac{1}{\lambda^2} \right) & \left[-\frac{3}{\pi} \left(\frac{KK_0}{M^2} - \log \frac{K+K_0}{M} \right) - \frac{1}{\pi} (1 - 3\lambda^2) \left(\log \frac{K+K_0}{M} - 1 \right) \right. \\ & - \frac{1}{24\pi} (12\lambda^4 - 90\lambda^2 + 16) \\ & + \frac{1}{2\pi} (\lambda^6 - 6\lambda^4) \log \lambda \\ & \left. - \frac{1}{2\pi} (\lambda^7 - 8\lambda^5 + 16\lambda^3) \frac{1}{\sqrt{4-\lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4-\lambda^2}}{\lambda} \right) \right], \quad (107) \end{aligned}$$

$$\delta m_{Fv(pv,pv)} = 0. \quad (108)$$

§ 5. The convergence relations and mass spectra

In the preceding two sections we have carried out in detail the self-energies of nucleons to the second order. It is the purpose of this section to try to obtain finite ones, or to find out the convergence relations and mass spectra.

As we know that it was suggested by Heisenberg, the concept of nucleon

means that the proton and neutron are different states of the same particle with respect to the electric charge, so that the problems of nucleon self-energies must be divided into two parts:

1) Is it possible to make the electromagnetic self-energy of a proton finite and to explain at the same time the mass difference of proton and neutron?

2) Can the divergences of the self-energies of nucleons due to mesons be compensated by assuming any other meson?

As to the first question, it will be shown in the next article that at least a neutral scalar meson with scalar coupling satisfies the requirement mentioned above as Pais¹⁾ and Sakata²⁾ anticipated. Therefore it will not be examined in detail here.

We now turn to the second question. Before entering into various discussions, we shall reconsider our previous inquiry which was stated in the end of § 2. Namely, is it true for the results of covariant evaluation of self-energies that they are made finite by themselves for the spin 0 field of M. I. T. (or M. T.; $\chi_s = \chi_{ps}$)? We can easily see that this may not hold in the present treatment. Hence, we must look for an alternative possibility.

The recent trend of experimental evidences about π mesons seems to one to show that there exist three kinds of them, i.e. positive, negative and neutral mesons, and they are all pseudoscalar types. Consequently, from the point of view of meson theory, these imply the necessity of assuming the symmetrical pseudoscalar theory. As regards couplings between nucleons and π mesons, we are not yet sure. However, the pseudovector one is the most probable case, in that it is by itself not at variance with the experimental information,¹⁷⁾ at least qualitatively. Thus we shall take up provisionally the symmetrical pseudoscalar type with pseudovector coupling for π mesons. Moreover, there are a few conditions, not yet taken into account, which are useful for the selection of fields to be mixed. According to Pais,¹⁾ one must take the following ones;

- i) The theory of nuclear forces shall be charge independent.
- ii) The theory shall give the right relative position of the 3S and 1S level of the deuteron.

Nevertheless, it should be noticed that, whatever the restrictions may be, the type to be mixed is uniquely determined for our case. That is to say, on the assumption that the counter meson interacts with nucleons through single coupling, one will find that for π meson it is fulfilled by introducing the symmetrical scalar meson with vector coupling. The convergence relations for the quadratic and logarithmic parts (in our definitions, cf. § 4) are respectively

$$f_2^2 \chi^2 = 3F_2^2 \mu^2 \quad (109)$$

and

$$(20m^2 + 75\chi^2) - 3(4m^2 + 5\mu^2) = 0, \quad (110)$$

where μ is the mass of cohesive meson (C' meson, for brevity).

It is more or less interesting to point out that the relations obtained above have the same form as those which were given by Pauli and Villars.¹⁸⁾ Applying the method of regularization to the divergence problems, they have used the two conditions,

$$\sum_i C_i = 0 \quad (111)$$

and

$$\sum_i C_i M_i^2 = 0 \quad (112)$$

in order to cancel the singularities. Generally the mass spectrum can be derived from the compensation of logarithmic singularities.

Taking the masses of nucleons and π mesons to be $m=1837m_e$ and $\kappa=276m_e$ ($\kappa^+=\kappa^-=\kappa^0$), one gets

$$\mu = 1474m_e. \quad (113)$$

In this way we are led to the view that the introduction of heavy mesons is necessary for no self-energy divergence. In connection with them we shall summarize briefly some remarks as follows:

(I) On account of the Dyson's theorem, the C' mesons give no effect on the nuclear forces, anomalous magnetic moments of nucleons and $\gamma-C'$ productions to the second power in the nucleon-meson coupling constants.

(II) The force range of the C' meson is much smaller than that of the π meson, so that almost all results attained up to the present time in the theory of nuclear forces due to π mesons would not be affected by mixing C' mesons with them.

(III) The higher order processes may be predominant for the phenomena in which, because of largeness of their coupling constants, C' mesons play important roles.

(IV) The production of C' mesons may be probably due to the nucleon-nucleon collisions.

Finally, it is to be noted that there have been some examples for the existence of mesons other than π and μ mesons, that is, τ mesons and V -particles.⁵⁾ It turned out by the recent experiments that the estimated mass values of the one of the latter are $800m_e \sim 1400m_e$, but the scarcity of observations prevents one from drawing decisive conclusions about them. Therefore, we shall be here content only with indicating the theoretical requirement of introducing heavy mesons.

Anyway, the considerations of this paper show that it would seem possible to attain finite self-energy of nucleon in the lowest order approximation by the method of mixed fields.

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Appendix

For the cases of $S(s)$ and $P_s(ps)$, we shall now consider the integral (45), i.e.

$$I = \int_0^1 du \int (dk) \frac{\Gamma}{[(k_\mu - u p_\mu)^2 + M^2 u^2 + x^2(1-u) + (p_\mu^2 + M^2)u(1-u)]^2}. \quad (\text{A.1})$$

After the transformation $k_\mu \rightarrow k_\mu + u p_\mu$, one gets the following results,

$$I = \int_0^1 du \int (dk) Q, \quad (\text{A.2})$$

i) if the surface term D is omitted,

$$Q_{S(s)} = (i\gamma p + M) \left(\frac{u-1}{A^2} \right) + \frac{(2-u)M}{A^2}, \quad (\text{A.3})$$

$$Q_{P_s(ps)} = (i\gamma p + M) \left(\frac{1-u}{A^2} \right) + \frac{uM}{A^2}, \quad (\text{A.4})$$

or ii) if the surface term D is included,

$$Q_{S(s)} = (i\gamma p + M) \left(-\frac{1}{A^2} + \frac{uk^2}{A^3} \right) + \frac{2M}{A^2} - \frac{uMk^2}{A^3}, \quad (\text{A.5})$$

$$Q_{P_s(ps)} = (i\gamma p + M) \left(\frac{1}{A^2} - \frac{uk^2}{A^3} \right) + \frac{uMk^2}{A^3}. \quad (\text{A.6})$$

From the condition (II) ((44)) for $n=2$, we obtain the formula,

$$\int \frac{k^2(dk)}{(k^2 + A^2)^3} = \int \frac{(dk)}{(k^2 + A^2)^2}. \quad (\text{A.7})$$

Applying it to the right-hand sides of (A.5) and (A.6), we write them as

$$(\text{A.5}) \rightarrow (i\gamma p + M) \left(-\frac{1}{A^2} + \frac{u}{A^2} \right) + \frac{2M}{A^2} - \frac{uM}{A^2}, \quad (\text{A.8})$$

$$(\text{A.6}) \rightarrow (i\gamma p + M) \left(\frac{1}{A^2} - \frac{u}{A^2} \right) + \frac{uM}{A^2}. \quad (\text{A.9})$$

Thus one finds that these results are the same expressions as (A.3) and (A.4).

Considering that the conditions (40) and (41) which were found by Fukuda and Kinoshita¹⁶⁾ by omitting the surface effect D are very strong, one may find weaker conditions by including such an effect from the beginning. However, we have not confirmed it in a general way. Recently, Koba et al.¹⁹⁾ have shown that, by considering the translation effects, one can get consistent results for the problems of gauge invariance and equivalence.

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On the Mass Difference of Nucleons and the Cohesive Mesons

Hiroshi ENATSU and Pong Yul PAC

Department of Physics, Kyoto University

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§ 1. Introduction

Pais¹⁾ and Sakata²⁾ have independently introduced the cohesive meson field in order to remove the divergence difficulties of electromagnetic self-energies of electron and proton. As was well known, it seemed that the theory of C meson would give a reasonable explanation for the mass difference between the proton and the neutron. So far one had expected that the mass of C meson could be determined by means of it. But these attempts were not successful, because the best value which could be obtained, amounted only to 75% of the observed mass difference, even if the mass of C meson would be zero.^{3) 4)}

Concerning the self-energies of nucleons due to π mesons, one of the present authors has shown in the previous paper⁵⁾ that, being assumed the symmetrical pseudoscalar type with the pseudo-vector coupling for π mesons, one could remove the divergences by introducing heavy mesons. In connection with above problems, we have considered the mass difference of nucleons and the C meson for the electromagnetic case.

The present paper⁶⁾ aims to show that, when the methods of calculations of preceding authors^{1) 2)} are modified along the line of the previous notes⁵⁾, the right value of mass difference and the finite masses of C mesons which may explain the observed one can be given. Furthermore, we shall try to investigate whether the existence of scalar C meson is compatible with the recent various experiments^{7) 8)} on neutral meson or not. Here the emphasis will be on the consistent application of the method of evaluation in which the translation effects are involved at all times, and another important problem with respect to the mirror nuclei¹⁷⁾ will not be considered in detail.

In relation to the mass difference of nucleons, we shall derive in § 2 the finite masses of C mesons for which the scalar type with scalar coupling and the pseudovector type with pseudovector coupling are supposed. Then the comparison between the results of the scalar C meson theory derived above and the recent experiments of π^- -capture will be made and the neutral meson production by γ -rays is considered in § 3. In the final section the discussion will be given.

§ 2. The mass difference of nucleons⁶⁾

As to the mass difference of proton and neutron, it was found that the theoretical prediction has the right sign and order of magnitude, but Kawabe and Umezawa⁴⁾ have shown that one can not obtain an adequate mass of C meson to explain the observed mass difference, $-2.47 m_e$, although the calculation was carried out in a relativistically invariant way. That is, the best value of the mass difference, even if one would make the mass of C meson tend to zero, amounted only to $-2.13 m_e$. Hence, the theory of C meson seems to be quantitatively unsatisfactory, though it is the unique theory to explain qualitatively the mass difference of nucleons at the present stage.

It seems to be difficult for us to remove such a stumbling block, so far as one keeps the method of calculation of the Dyson's formalism⁹⁾, because, as is well known, Kawabe-Umezawa's " w "-method gives the same results which are equivalent to ones by the former.

According to Kinoshita¹⁰⁾, by making use of the modified method of computation, i. e. Karplus and Kroll's¹¹⁾ one which takes account of the so-called surface terms (or translation effects), one would observe that some considerable contributions would arise in the cases of the mass-type divergent integrals, but this would not be the fact for the Lamb-shift type. Actually, one can add small contributions to the finite terms in the cases of C mesonic and electromagnetic self-energies of nucleons in which the logarithmic divergences would appear. It was our starting point to reconsider the problem, and we have carried out the evaluation of the mass difference of nucleons by considering such an effect.

In other words, it is the same method as that adopted in the previous paper. In that way, we obtain the electromagnetic and cohesive mesonic masses of a proton, with the results:^{5) 6)}

$$\delta m = \delta m_1 + \delta m_2 + \delta m_3 \quad (2.1)$$

For a scalar C meson,

$$\delta m_1 = m \left[\frac{3}{2\pi} \left(\frac{e^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} - \frac{3}{4\pi} \left(\frac{f_1^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} \right], \quad (2.2)$$

$$\delta m_2 = -m \left[\frac{1}{4\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{8\pi} \left(\frac{f_1^2}{4\pi\hbar c} \right) \left\{ 1 - \lambda^2 + (\lambda^4 - 6\lambda^2) \log \lambda \right. \right. \\ \left. \left. + \frac{(-\lambda^5 + 8\lambda^3 - 16\lambda)}{\sqrt{4 - \lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4 - \lambda^2}}{\lambda} \right) \right\} \right], \quad (2.3)$$

$$\delta m_3 = -m \left[\frac{1}{8\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{16\pi} \left(\frac{f_1^2}{4\pi\hbar c} \right) \right], \quad (2.4)$$

and for a pseudovector C meson*

* We should like to thank Professor A. Pais for kindly pointing out this case.

$$\delta m_1 = m \left[\frac{3}{2\pi} \left(\frac{e^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} - \frac{5}{2\pi} \left(\frac{g_1^2}{4\pi\hbar c} \right) \log \frac{K+K_0}{M} \right], \quad (2.5)$$

$$\delta m_2 = -m \left[\frac{1}{4\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{4\pi} \left(\frac{g_1^2}{4\pi\hbar c} \right) \left\{ 1 - \lambda^2 + (\lambda^4 - 8\lambda^2) \log \lambda \right. \right. \\ \left. \left. + \frac{(-\lambda^3 + 10\lambda^3 - 24\lambda)}{\sqrt{4 - \lambda^2}} \cdot \tan^{-1} \left(\frac{\sqrt{4 - \lambda^2}}{\lambda} \right) \right\} \right], \quad (2.6)$$

$$\delta m_3 = -m \left[\frac{1}{8\pi} \left(\frac{e^2}{4\pi\hbar c} \right) + \frac{1}{8\pi} \left(\frac{g_1^2}{4\pi\hbar c} \right) \right], \quad (2.7)$$

where m : Proton mass,

x : C meson mass,

m_e : Electron mass,

$$\left(\frac{f_1^2}{4\pi\hbar c} \right), \left(\frac{g_1^2}{4\pi\hbar c} \right): \text{Coupling constants of scalar and pseudovector C mesons,} \quad (2.8)$$

$$\lambda = \frac{x}{m}, \quad M = \frac{mc}{\hbar}, \quad K_0 = \sqrt{M^2 + K^2}, \quad K \rightarrow \infty.$$

Accordingly, the divergent parts are eliminated by the conditions

$$f_1^2 = 2e^2 \quad (2.9)$$

and

$$g_1^2 = \frac{3}{5} e^2. \quad (2.10)$$

The last term, δm_3 , comes from the surface integral part, $-u p_\mu (\partial/\partial k_\mu)$, and is not negligible compared to δm_2 . The terms which contain the higher derivatives than the first vanish identically for both cases. The results of numerical evaluation are given in Figs. 1 and 2. For the case of scalar C meson, the curve obtained by Kawabe and Umezawa is also shown, which is entirely in agreement with our finite term δm_2 .

Thus one will find that the masses of C mesons are, in order that the right value of the mass difference of proton and neutron shall be given, the following ones,

$$x_s = 110 m_e, \quad (2.11)$$

and

$$x_{pv} = 13 m_e, \quad (2.12)$$

in deriving them we have put, $m = 1837 m_e$ and $e^2/4\pi\hbar c = 1/137.3$. The former is of the expected order of magnitude by Pais¹⁾ and Sakata²⁾.

In this way, on the basis of the present and previous investigations we see that there is a possibility of obtaining simultaneously a finite self-energy of

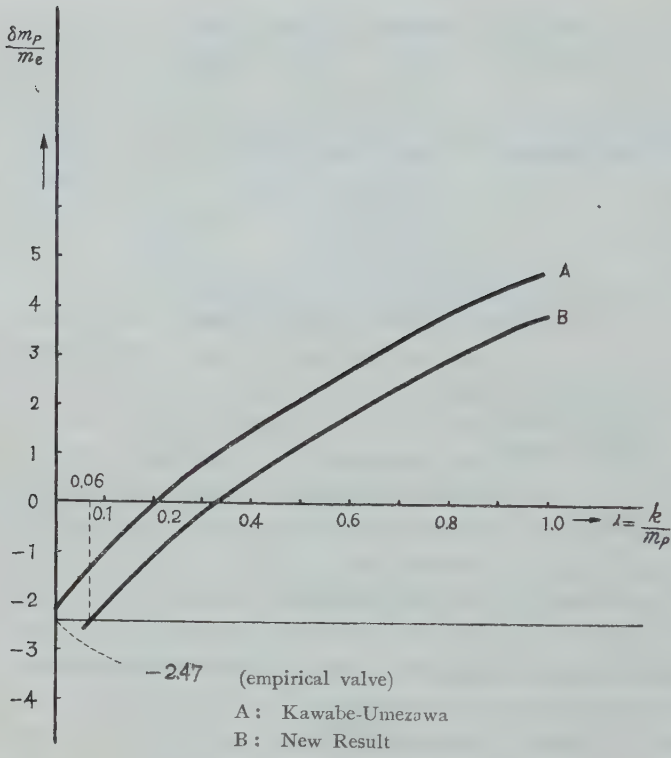


Fig. 1

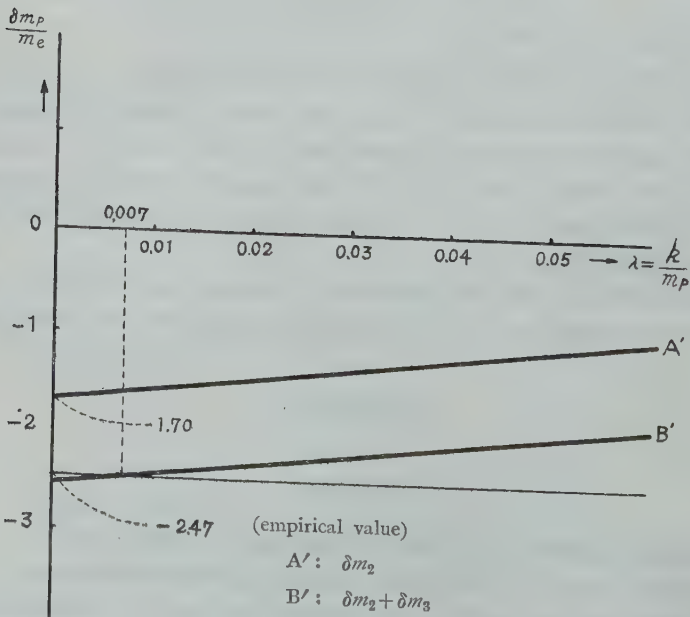


Fig. 2

proton and neutron. Of course, one has been inclined to make the tentative assumptions which could not be verified in the present situation, one should like to state that the method of mixing meson fields would seem fruitful. There are, however, some other matters into which we must look. One of them is the existence of C mesons. In the next section we shall discuss more closely the observability of the scalar C meson as an example.

§ 3. Observability of the scalar C meson

Evidently, the direct effects of C meson may appear in the recent experiments concerning artificial mesons (especially neutral mesons).^{7) 8)} Therefore we shall try to examine whether the existence of such a C meson is compatible with the results mentioned above or not. For this purpose it seems to be preferable to take up the processes of the π^- -capture by a proton and the neutral meson production by γ -rays.

As for the first case, Panofsky et al.⁸⁾ have observed that the following two processes take place to the same extent;

$$\pi^- + P \rightarrow N + \gamma \quad (\gamma) \quad (3.1)$$

$$\pi^- + P \rightarrow N + \pi^0 \rightarrow N + 2\gamma \quad (\pi^0) \quad (3.2)$$

Furthermore, it was inferred from the width ΔE of the energy spectra of γ -rays around 70 MeV that the mass difference of charged and neutral mesons would be about $10 m_\pi$. If the C meson is assumed for the proton, the process

$$\pi^- + P \rightarrow N + C \rightarrow N + 2\gamma \quad (C) \quad (3.3)$$

has to be added to them. However, by virtue of a rough theoretical estimation¹²⁾ one finds the ratio of the probabilities $W_{(e)}$ and $W_{(r)}$ as

$$\frac{W_{(e)}}{W_{(r)}} < \frac{1}{42}, \quad (3.4)$$

on the other hand, the processes (γ) and (π^0) occur with comparable frequencies. Therefore, we may conclude that the process (C) will not be incompatible with the data of recent experiments.

As for the second case, i. e. the neutral meson production by γ -rays, various theoretical discussions have been made by many authors.^{13) 14) 15)} It has been, however, shown that, so long as one confines oneself to the lowest order computation, one cannot illustrate the observed facts. In order to avoid this difficulty, it seems somewhat useful that another effect is taken into account, for example, the higher processes or the simultaneous productions of different kinds of mesons. Here, we shall consider the latter assuming that a proton emits a scalar C meson as well as a π^0 meson which is of the pseudoscalar type with pseudovector coupling.

By employing the Dyson's formalism,⁹⁾ our evaluations are carried out in the lowest order approximation, and the cross-section for the production of C meson

will be compared with that of π^0 meson. The differential cross-section for them are¹⁶⁾

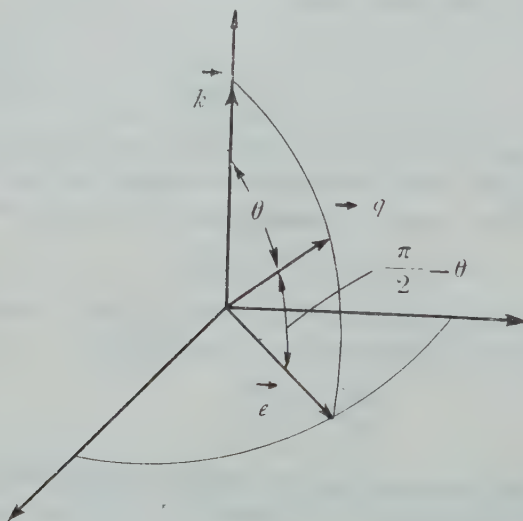
$$S(s) : d\sigma = \frac{1}{4} \cdot \frac{e^2}{4\pi\hbar c} \cdot \frac{f^2}{4\pi\hbar c} \cdot \frac{q'(m - q'_0 + q' \cos \theta)}{x(mq'_0 - x^2/2)^2} |f(q', \theta)|^2 dq'_0 d\Omega', \quad (3.5)$$

$$|f(q', \theta)|^2 = \left\{ \frac{(mq'_0 - \frac{1}{2}x^2)}{m(m - q'_0 + q' \cos \theta)} + \left(2m^2 - \frac{1}{2}x^2 \right) \cdot \frac{q'^2 \sin^2 \theta}{(mq'_0 - \frac{1}{2}x^2)^2} \right\},$$

$$P_s(pv) : d\sigma = \frac{1}{4} \cdot \left(\frac{2m}{x} \right)^2 \cdot \frac{e^2}{4\pi\hbar c} \cdot \frac{g^2}{4\pi\hbar c} \times \frac{q'(m - q'_0 + q' \cos \theta)}{x(mq'_0 - x^2/2)^2} |f(q', \theta)|^2 dq'_0 d\Omega', \quad (3.6)$$

$$|f(q', \theta)|^2 = \left\{ \frac{(mq'_0 - \frac{1}{2}x^2)}{m(m - q'_0 + q' \cos \theta)} - \frac{1}{2}x^2 \cdot \frac{q'^2 \sin^2 \theta}{(mq'_0 - \frac{1}{2}x^2)^2} \right\},$$

in which the notations are as follows;



	photon	meson
incident	$k_\mu (K = \sqrt{k_1^2 + k_2^2 + k_3^2}, k_0)$	$q_\mu (q = \sqrt{q_1^2 + q_2^2 + q_3^2}, q_0)$
virtual	—	—
final	—	q'_μ

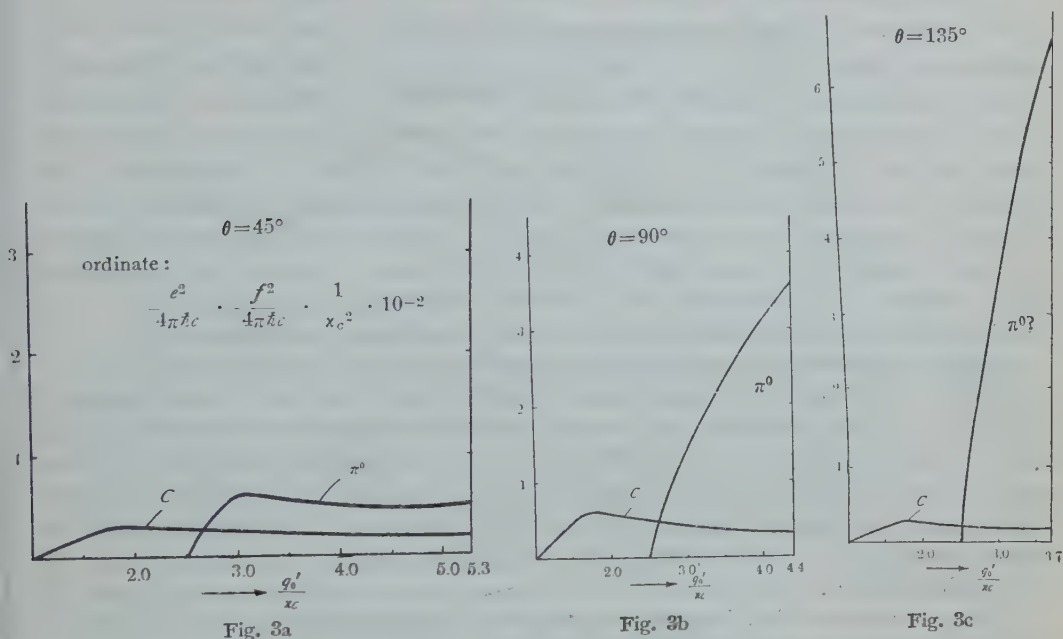
$$K = \frac{mq'_0 - \frac{1}{2}x^2}{m - q'_0 + q' \cos \theta} \leq 330 \text{ MeV},$$

where m : Proton mass ($=1837 m_e$),

x : Meson mass ($x_c = 110 m_e$, $x_\pi = 276 m_e$),

$$\frac{f^2}{4\pi\hbar c} = \frac{2e^2}{4\pi\hbar c} = \frac{2}{137.3}, \quad \frac{g^2}{4\pi\hbar c} = 0.1.$$

The energy spectra for various angles are given in Figs. 3, and also the θ -dependence of the cross-section is shown in Fig. 4.

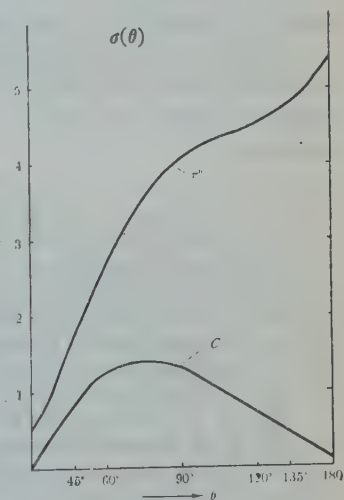


Finally, one may obtain the average total cross-sections for C meson and π meson productions per one photon by integrating them. They are

$$\sigma_{av} = \begin{cases} 2.7 \times 10^{-31} \text{ cm}^2 & (C) \\ 9.6 \times 10^{-31} \text{ cm}^2 & (\pi^0), \end{cases} \quad (3.7)$$

where $\frac{e^2}{4\pi\hbar c} \cdot \frac{f_c^2}{4\pi\hbar c} \cdot \frac{1}{x_c^2} \times 10^{-2} = 1.37 \times 10^{-31} \text{ cm}^2$.

These results are smaller than the empirical value (order of 10^{-28} cm^2). One may change the coupling constant for π meson field by considering nuclear forces and anomalous magnetic moments of nucleons, and make it large to a certain extent. On the contrary, one cannot do with that for the C meson field because of the relation $f_c^2 = 2e^2$. Thus, under our tentative assumptions it will be found that the cross-section of the C meson production is not large as compared with that of the π^0 meson. Therefore, even if the effect of the scalar C meson is taken into account, it seems to be hardly possible to improve the present discrepancy between the theoretical prediction and empirical data.



§ 4. Discussions

Having shown that the result obtained above is rather favourable for the mass difference of proton and neutron, unlike the previous investigations, some problems for heavy nuclei still remain unattacked so far. In fact, as Wightman¹⁷⁾ showed, the mass of C meson should be of the order of $1373 m_e$ or larger so as to be in conforming with the experimental informations of the mass differences of heavy nuclei. Evidently, for the case mentioned above the condition is not satisfied. We shall, however, need to work out quantitatively the binding energies of heavy nuclei for the pseudovector C meson, because the coupling constant of it is not so small in comparison with that of the scalar C meson that one cannot know whether the modification of Coulomb potential has a drastic effect on the binding effect or not, without performing the quantitative computation.

In our opinion, it seems that the pseudovector C meson is difficult to be reconciled with the recent experiments which indicate the two γ -rays of neutral mesons. In this way, one may state that the present form of cohesive meson theory is not adequate for the description of electromagnetic phenomena for heavy nuclei. Another difficulty of C meson theory concerns the convergence of the fourth-order self-energy. As was pointed out by Kinoshita,¹⁸⁾ in fact, the divergent terms which appear in the fourth order calculation cannot be removed only by the condition (2.9).

However it may be, it seems worthwhile at the present stage that the theory of C meson is the only one which will give a reasonable explanation for the mass difference of proton and neutron.

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Recoil Effect on Electron-Proton Forces and Inapplicability of Energy Law

Gentaro ARAKI and Sigeru HUZINAGA

Faculty of Engineering, Kyoto University

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Bethe and Fermi showed that Breit's formula for the two-electron interaction transmitted by photons can be obtained according to the quantum electrodynamics if the recoil effect is neglected. It was shown by Breit himself that his formula was not in complete agreement with experiment. The discrepancy was shown to come from a term which was proportional to e^4 in the Pauli approximation, but its true origin has not yet fully been known. In the present paper it is shown that this surplus term is due to the neglect of the recoil effect and the discrepancy disappears when the interaction is derived correctly taking into account the effect of the electron and proton recoils according to the method of the unitary transformation. Further it has generally been known that the electron-proton interaction transmitted by photons takes the form which was previously derived by Møller for the two-electron interaction if the conservation law is applied to the total energy of the two particles in the free state. In the present paper it is shown that the application of the energy law gives rise to an incorrect result in case of hydrogen atom. According to Møller's formula the interval between $2^2S_{1/2}$ and $2^2P_{1/2}$ levels of the hydrogen atom becomes too large. Its value amounts to 7273 Mc/s in contradiction to the Lamb-Retherford experiment. Therefore we arrive at the conclusion that the energy law can not be made use of in general when forces between Fermions in the bound states are derived.

Introduction and Summary

Recently one of the present authors studied the effect of the nucleon recoil on nuclear forces⁽¹⁾. Although it may bring one into compliance that the effect is important in its theoretical aspect as a general problem on two-Fermion forces transmitted by Bosons, any definite conclusion can not be drawn from the result because the related phenomena have not yet been studied experimentally as well as theoretically in a sufficient exactness and even the meson theory has not yet arrived at its satisfactory form. There is no ambiguity of such a kind in case of two-electron or electron-proton forces. The fine structure of hydrogen levels was precisely analysed by the recent experiment of Lamb and Retherford⁽²⁾. We can therefore examine the effect in question by comparing theory with experiment in this case as a test of the general problem on forces between two Fermions. This is the purpose of the present paper.

We shall seek for the answers to two questions: (i) What is the difference between the new and conventional formulas when we correctly take into account the effect of electron and proton recoils on their interaction? (ii) What effect

arises when we apply the conservation law to the total energy of two Fermions in their free state in deriving their interaction?

A formula for the electromagnetic interaction between two electrons in the configuration space was derived by Breit⁽³⁾ by considering a quantum-mechanical correspondence of the classical formula. Another formula for the interaction in the form of matrix elements was deduced by Møller⁽⁴⁾ according to Klein's method. Bethe and Fermi⁽⁵⁾ showed on the basis of quantum electrodynamics that Breit's formula can be obtained if the effect of electron recoil is neglected and further that Møller's formula is equivalent to Breit's when the conservation law of energy is valid.

It seems to have been believed by many authors that the neglect of the recoil effect^{(6) (7) (8)} and the application of the energy law^{(7) (9) (10)} are adequate, or the former effect was considered in the wrong way^{(5) (7)}. However the neglect of the former effect can not be justified for the reason which was discussed in A*. In fact it was shown by Breit⁽¹¹⁾ himself that his formula can not be in complete agreement with the experiment on the He triplet. The discrepancy comes from a term which is proportional to ϵ^4 in the Pauli approximation. Recently Ishidzu⁽¹²⁾ has shown in case of the electron-proton interaction that a similar term can not be reconcilable with the experiment on the hyperfine structure of hydrogen.

Explanations^{(13) (14) (15)} of the justification for omitting this surplus term of ϵ^4 were tried, but they could not convince us in a satisfactory way. On the standpoint of quantum electrodynamics, Breit's formula can be considered as a term of ϵ^2 in the two-electron interaction transmitted by photons. One may think on this basis, as was noted by Rosenfeld⁽¹⁵⁾, that the ϵ^4 -term must be omitted in order to consistently take into account only ϵ^2 -terms. The calculation in A was carried out on this standpoint. As will be shown in the third section below, however, the above mentioned ϵ^4 -term can not distinctly be separated from the remaining terms because we find another ϵ^4 -term in Breit's formula.

Nambu⁽⁷⁾ pointed out that the fourth order term in the ϵ -perturbation cancels the second order term in the ϵ^2 -perturbation. However, this was not yet clear-cut, though it might be thought most convincing, because the Pauli approximation is the second degree term in a power series in $1/\epsilon$ whereas Nambu's perturbation is a series in ϵ .

In the present paper the problem is considered from another viewpoint. The electron-proton interaction transmitted by photons is derived taking into account the effect of electron and proton recoils considering both the positive and negative energy states of them. The calculation will be carried out in the second section according to the method mentioned in A. The apparent form of the new formula is entirely different from Breit's, but we find in the Pauli approximation that it

* The paper cited in Ref. (1) will be referred to as A.

coincides with the latter except for the above mentioned surplus term. This will be shown in the third section. Consequently we see that Breit's surplus term of ϵ^4 is due to the neglect of the recoil effect and the correct method is capable of avoiding the defect of Breit's formula and that two-electron or electron-proton forces predicted by quantum electrodynamics is in quite agreement with experiment.

As is mentioned above, Møller's formula can not be obtained without assuming the energy law, according to quantum electrodynamics. The application of the law implies the neglect of the term in the form $[H_0 S']$ where H_0 is the Hamiltonian without interaction, S' is a skew-Hermitian operator, and the bracket denotes a commutator. The neglect of such a term has no validity unless its influence is proved to be negligibly small, except for the case of a scattering problem where the influence exactly vanishes.

If the interaction between two Fermions is very weak the influence of the above mentioned term is a second order correction and it may consequently be neglected compared with a first order correction. In the present case, however, the influence of the Coulomb interaction is so large that the energy eigenstates of an electron-proton system are wholly different from those in case of no interaction. Therefore the effect of $[H_0 S']$ is by no means negligible compared with Breit's term.

It has often been considered⁽⁷⁾⁽¹⁶⁾ that $[H_0 S']$ can be unitarily transformed into the higher order term in ϵ and that the ϵ^2 -forces involve an ambiguity in this sense, but, in our opinion, such a transformation is superfluous in principle. In the present paper the two-Fermion forces transmitted by Bosons are derived by a well-defined unitary transformation which makes the Hamiltonian, of a system consisting of Fermions and Bosons, diagonal with respect to the number of Bosons. This method uniquely defines the forces and there is no ambiguity. This is more concretely accounted for in the first section.

The effect of neglecting $[H_0 S']$ will be examined in the fourth section by comparing Møller's formula with the experiment of Lamb and Retherford⁽⁹⁾. It will be shown that the calculated result can not be reconcilable with their experiment. According to Møller's formula both the $2^2P_{1/2, 3/2}$ levels of the hydrogen atom are deeper than the position predicted by Dirac's theory while the $2^2S_{1/2}$ level is not influenced. The shift amounts to 7273 Mc/s which is wholly inconsistent with the observed value of 1062 Mc/s. Generally speaking, this proves the inadequacy of applying the conservation law of energy to the derivation of two-Fermion forces transmitted by Bosons. Thus we finally obtain the definite answers to the two questions mentioned in the beginning.

§ 1. Problem and Method

We shall first define the method of deriving the electron-proton forces. Consider a system which consists of electrons, protons and photons. The Hamiltonian

of the system is written in the form $H=H_0+U+H'$ where H_0 is the Hamiltonian without interaction, U denotes the Coulomb interaction between the Fermions, and H' is the interaction of the Fermions with photons.

The first two parts of the H are diagonal and the last is non-diagonal with respect to the number of photons. We unitarily transform H into a diagonal form with respect to the number of photons where the possibility of the transformation is of course an assumption. The new Hamiltonian then takes the form H_0+U+H'' and its matrix becomes of a staircase form as is shown in Fig. 1.

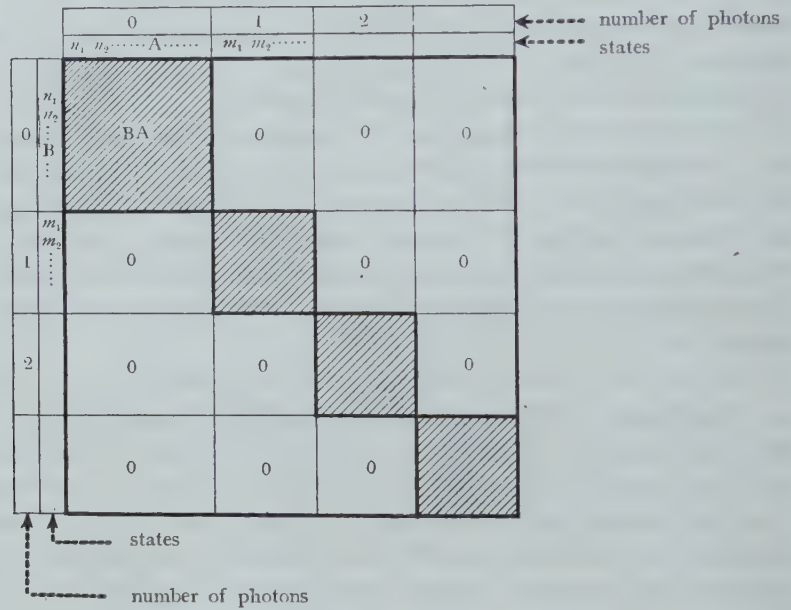


Fig. 1.

Let W be the part, of H'' , which has its non-vanishing matrix-elements in the BA -part of Fig. 1 and depends on the relative position between an electron and a proton. Then W represents their interaction transmitted by photons, and the total interaction between an electron and a proton is given by $u+W$ where u denotes the Coulomb interaction between them. (u is not equal to U).

We denote the unitary transformation by $\exp S$ where S is skew-Hermitian and *non-diagonal with respect to the number of photons*. We again assume that S and the transformed Hamiltonian can be expanded into powers in ϵ :

$$S=S^{(1)}+S^{(2)}+S^{(3)}+\dots \quad (1.1)$$

$$(\exp S)H \exp(-S)=H_0+U+H^{(2)}+H^{(4)}+H^{(6)}+\dots \quad (1.2)$$

where $S^{(n)}$ and $H^{(n)}$ are proportional to ϵ^n . In order that (1.2) is diagonal with respect to the number of photons $S^{(n)}$'s must satisfy the following system of simultaneous equations:

$$[H_0 S^{(1)}] = H' \quad (1.3)(a)$$

$$H^{(2)} + [H_0 S^{(2)}] = (1/2)[S^{(1)} H'] \quad (1.3)(b)$$

$$[H_0 S^{(3)}] = -(1/2)[H_0 [S^{(1)} S^{(2)}]] + (1/3)[S^{(1)} [S^{(1)} H']] + [S^{(1)} U] \quad (1.3)(c)$$

$$\begin{aligned} H^{(4)} + [H_0 S^{(4)}] = & -(1/2)[H_0 [S^{(1)} S^{(3)}]] - (1/6)[S^{(2)} [S^{(2)} H_0]] \\ & - (1/12)[S^{(1)} [S^{(1)} [S^{(2)} H_0]]] + (1/3)[S^{(1)} [S^{(2)} H']] \\ & + [S^{(2)} U] + (1/2)[S^{(1)} [S^{(1)} U]] \end{aligned} \quad (1.3)(d)$$

etc.

where the bracket denotes the commutator. The *non-diagonal* property of S with respect to the number of photons is capable of avoiding the ambiguity in $S^{(2)}$. $H^{(n)}$ represents the diagonal part of the e^n -term in (1.2) and $S^{(n)}$ is so determined that $[S^{(n)} H_0]$ cancels the non-diagonal part of the rest. In this method $S^{(2)}$ is *unique* and there is no ambiguity in it.

The simultaneous equations can be actually solved by introducing any orthonormal set. The correct solution can be obtained provided that the set is *complete*. We can adopt the eigenfunctions of H_0 as the orthonormal set. In order to fulfil the requirement of *completeness* we have to consider all eigenfunctions corresponding to both the positive and negative energy. None of them can be omitted in the present case of interacting two Fermions, as was stressed in A.

We shall consider only the e^2 -term in W . This comes from $H^{(2)}$. Its matrix elements are, by (1.3)(b), given by

$$W_{BA} = (1/2)[S^{(1)} H']_{BA} \quad (1.4)$$

where A and B are two arbitrary states in the BA -part of Fig. 1. The conventional formula⁽¹⁷⁾⁽¹⁸⁾⁽¹⁹⁾ is given by $-(S^{(1)} H')_{BA}$ in contrast with (1.4), but this does not represent a physical quantity because it is not Hermitian. Therefore the conventional formula is incorrect while (1.4) is Hermitian.

If we insert the explicit formula for H' in (1.4) we have

$$W_{BA} = -\frac{2\pi}{V} e_1 e_2 \left\{ \sigma^{(1)} \sigma^{(2)} - \frac{(\sigma^{(1)} \mathbf{k})(\sigma^{(2)} \mathbf{k})}{\mathbf{k}^2} \right\} \frac{\rho_1^{(1)} \rho_1^{(2)} \delta(\Delta \mathbf{p})}{\mathbf{k}^2 - (\epsilon - \epsilon_0)^2} + (1 \leftrightarrow 2) \quad (1.5)$$

where $\mathbf{k} = \mathbf{p} - \mathbf{p}_0$, $\mathbf{p} = \mathbf{p}_B^{(1)}$, $\mathbf{p}_0 = \mathbf{p}_A^{(1)}$, $\epsilon = \epsilon_B^{(1)}$, $\epsilon_0 = \epsilon_A^{(1)}$, and $\Delta \mathbf{p} = \mathbf{p}_B^{(1)} + \mathbf{p}_B^{(2)} - (\mathbf{p}_A^{(1)} + \mathbf{p}_A^{(2)})$. Further e_1 and e_2 are respectively the charges of the first and second Fermions, V is the volume of the normalization space, \mathbf{p} and ϵ are respectively the momentum and energy of a Fermion, and all quantities are measured in $\hbar=1$ units. Quantities in the state A (or B) are indicated by the subscript A (or B), the quantities of the first (or second) Fermion except for the charge and mass are indicated by the superscript (1) (or (2)), and $(1 \leftrightarrow 2)$ denotes the expression obtained from the preceding one by interchanging the first and second particles. ρ_1 , ρ_2 , ρ_3 , and σ are Dirac's matrices, and their matrix subscripts for appropriate parts of Dirac's wave functions are omitted for the sake of simplicity. The formula

(1.5) is valid for the two-electron interaction too.

The denominators of (1.5) are different from those of the conventional formula⁽¹⁷⁾ owing to the above mentioned difference between their general forms. When the energy is conserved the difference disappears. In the present case we are considering the case in which the energy is not conserved. We have therefore to sharply distinguish these two forms. The term $(\epsilon - \epsilon_0)^2$ represents the retardation effect according to Klein-Møller's method,^{(4) (5)} or it represents the effect of the energy change due to the recoil. We shall refer to this term as a recoil term. The first problem in the present paper is to examine the rôle of this term.

Making use of Dirac's equation which is satisfied by the wave function of electron or proton we can transform (1.5) as follows*:

$$W_{RA} = -\frac{2\pi}{V} e_1 e_2 \left\{ \frac{\rho_1^{(1)} \rho_1^{(2)} \boldsymbol{\sigma}^{(1)} \boldsymbol{\sigma}^{(2)} - 1}{\mathbf{k}^2 - (\epsilon - \epsilon_0)^2} + \frac{1}{\mathbf{k}^2} \right\} \delta(\Delta \mathbf{p}) + (1 \rightleftharpoons 2) + [H_0 S']_{RA} \quad (1.6) (a)$$

where

$$S'_{RA} = -\frac{2\pi}{V} \frac{e_1 e_2}{\mathbf{k}^2} \frac{(\epsilon - \epsilon_0) \delta(\Delta \mathbf{p})}{\mathbf{k}^2 - (\epsilon - \epsilon_0)^2} + (1 \rightleftharpoons 2) \quad (1.6) (b)$$

In order to obtain Møller's formula we have to omit $[H_0 S']$ in (1.6)(a). The second problem in the present paper is to study the effect of $[H_0 S']$, in another word, to study the difference between Møller's and exact formulas.

§ 2. Electron-Proton Forces

The method of correctly taking into account the recoil effect for deriving two-Fermion forces was accounted for in A. The denominator of (1.5) can be written in the following form:

$$\frac{1}{\mathbf{k}^2 - (\epsilon - \epsilon_0)^2} = \frac{\frac{1}{2} \left(1 + \frac{\epsilon \epsilon_0 + \mathbf{p} \mathbf{p}_0}{m_1^2 c^2} \right)}{\mathbf{k}^2 + \frac{\mathbf{k}^2 \mathbf{p}_0^2 - (\mathbf{k} \mathbf{p}_0)^2}{m_1^2 c^2}} \quad (2.1)$$

where m_1 is the mass of the first particle. (The mass of the second particle will be denoted by m_2). The right side is inserted in (1.5). ϵ and ϵ_0 are then replaced by $\rho_1^{(1)} \boldsymbol{\sigma}^{(1)} \mathbf{p} + m_1 c \rho_3^{(1)}$ and $\rho_1^{(1)} \boldsymbol{\sigma}^{(1)} \mathbf{p}_0 + m_1 c \rho_3^{(1)}$ respectively and the former is placed on the left and the latter on the right of $\rho_1^{(1)}$ and $\boldsymbol{\sigma}^{(1)}$ in (1.5). The denominator of the right side in (2.1) is expanded in a power series in $1/c^2$. This series converges independently of the magnitude of \mathbf{p} provided that $|\mathbf{p}_0| < m_1 c$. We calculate the right side of (1.5) up to the terms of $1/c^2$ in this way.

* The result obtained on the basis of the conventional formula is different, as was given by Hamilton⁽¹⁸⁾

Making use of the relations

$$\frac{4\pi}{V} \frac{\delta(\Delta \mathbf{p})}{\mathbf{k}^2} = (\varphi_B, r^{-1} \varphi_A) \tag{2.2} (a)$$

$$\frac{4\pi}{V} \frac{(\mathbf{v} \mathbf{k})(\mathbf{w} \mathbf{k})}{\mathbf{k}^2} \frac{\delta(\Delta \mathbf{p})}{\mathbf{k}^2} = 2^{-1} (\varphi_B, r^{-1} \{ \mathbf{v} \mathbf{w} - r^{-2} (\mathbf{v} \mathbf{x})(\mathbf{w} \mathbf{x}) \} \varphi_A) \tag{2.2} (b)$$

we write (1.5) in the form which contains the expressions in the right side of these formulas where $\mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$, $r = |\mathbf{x}|$, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the position vectors of the first and second Fermions respectively, φ_A and φ_B are the space parts of the eigenfunctions of two free Fermions in the states A and B respectively, and \mathbf{v} and \mathbf{w} are vectors. We next replace remaining \mathbf{p} and \mathbf{p}_0 by $-i\nabla^{(1)}$ and place the former on the left and the latter on the right of r^{-1} , r^{-2} , and \mathbf{x} . We have then the interaction between two Fermions transmitted by photons in the configuration space as follows:

$$W^{(A)} = \frac{1}{4m_1c} \rho_3^{(1)} \rho_1^{(2)} \frac{e_1 e_2}{r} \left\{ i \sigma^{(2)} \nabla^{(1)} + \frac{i}{r^2} (\sigma^{(2)} \mathbf{x})(\mathbf{x} \nabla^{(1)}) - \frac{\mathbf{x}}{r^2} [\sigma^{(1)} \sigma^{(2)}] \right\} + (1 \leftrightarrow 2) \tag{2.3}$$

where the thick bracket denotes a vector product of two vectors embraced in it. This formula is exact up to $1/c^2$ in the correct sense.

If we omit the recoil term in (1.5) we have Breit's formula as follows:

$$W^{(B)} = -\rho_1^{(1)} \rho_1^{(2)} \Gamma \tag{2.4} (a)$$

where

$$\Gamma = \frac{e_1 e_2}{2r} \left\{ \sigma^{(1)} \sigma^{(2)} + \frac{(\sigma^{(1)} \mathbf{x})(\sigma^{(2)} \mathbf{x})}{r^2} \right\} \tag{2.4} (b)$$

The apparent forms of (2.3) and (2.4) are entirely different. The difference between them is the contribution of the recoil term. This will be studied in the next section. The neglect of the recoil term is to take the zero degree term in the incorrect expansion as follows:

$$\frac{1}{\mathbf{k}^2 - (\epsilon - \epsilon_0)^2} = \frac{1}{k^2} \sum_{n=0}^{\infty} \left(\frac{\epsilon - \epsilon_0}{k} \right)^{2n} \tag{2.5}$$

The incorrectness of such an expansion was fully discussed in A. The arguments of Bethe and Fermi⁽⁶⁾ and Nambu⁽⁷⁾ were based on this incorrect expansion.

§ 3. Recoil Effect

We shall examine the Pauli approximation of $W^{(A)}$ and $W^{(B)}$ in order to study the influence of the recoil term. The method was already explained in the previous papers^{(1)(20)*}. When there is no photon the Hamiltonian of the two-Fermion system is, according to (1.2), given by

* In the present case photons have the vanishing rest mass whereas mesons have the finite mass. Therefore the Pauli approximation was considered in a somewhat different way in A.

$$H = -ic\rho_1^{(1)}\boldsymbol{\sigma}^{(1)}\boldsymbol{\nabla}^{(1)} - ic\rho_1^{(2)}\boldsymbol{\sigma}^{(2)}\boldsymbol{\nabla}^{(2)} + m_1c^2(\rho_3^{(1)} - 1) + m_2c^2(\rho_3^{(2)} - 1) + u + W \quad (3.1)$$

where $u = e_1e_2/r$ is the Coulomb interaction, and the rest energy of the system (including the self-energy) is subtracted for the sake of later convenience. We write the eigenfunction of the Hamiltonian in the form

$$\phi = \phi^{++}\chi^+(1)\chi^+(2) + \phi^{+-}\chi^+(1)\chi^-(2) + \phi^{-+}\chi^-(1)\chi^+(2) + \phi^{--}\chi^-(1)\chi^-(2) \quad (3.2)$$

where $\chi^+(1)$ and $\chi^-(1)$ are the eigenfunctions of $\rho_3^{(1)}$ belonging to its eigenvalues $+1$ and -1 respectively. The coefficient ϕ^{++} etc. are functions of space and spin coordinates. The eigenvalue equation $H\phi = E\phi$ then splits into simultaneous equations for ϕ^{++} etc.. If we eliminate ϕ^{+-} , ϕ^{-+} , and ϕ^{--} from them, we have an equation in the form $H'\phi^{++} = E\phi^{++}$ where H' involves E .

We can expand H' in a power series in $1/c$. E in H' can be eliminated by replacing $E\phi^{++}$ with $H'\phi^{++}$. Thus we have

$$H' = -\frac{1}{2m_1}A^{(1)} - \frac{1}{2m_2}A^{(2)} - \frac{1}{8m_1^3c^2}(A^{(1)})^2 - \frac{1}{8m_2^3c^2}(A^{(2)})^2 + \dots + u + W' \quad (3.3)$$

where u is the Coulomb interaction, $u + W'$ stands for a part depending on the relative position of two particles. Now H' does not contain E and the equation $H'\phi^{++} = E\phi^{++}$ is still valid. We can consider H' as the effective Hamiltonian because its eigenvalue is E . $u + W'$ represents an effective interaction. If we reserve only the terms up to the second degree and omit higher degree terms from (3.3), H' is the Pauli approximation of the Hamiltonian and W' is the Pauli approximation of W .

In this way we have the Pauli approximation of $W^{(A)}$ and $W^{(B)}$ as follows:

$$\begin{aligned} W'^{(A)} = & -\frac{1}{4m_1^2c^2} \frac{e_1e_2}{r^3} (\mathbf{x}\boldsymbol{\nabla}^{(1)} + \boldsymbol{\sigma}^{(1)}\mathbf{L}^{(1)}) \cdot \frac{1}{4m_2^2c^2} \frac{e_1e_2}{r^3} (-\mathbf{x}\boldsymbol{\nabla}^{(2)} + \boldsymbol{\sigma}^{(2)}\mathbf{L}^{(2)}) \\ & + \frac{1}{2m_1m_2c^2} \frac{e_1e_2}{r} \left\{ \boldsymbol{\nabla}^{(1)}\boldsymbol{\nabla}^{(2)} + \frac{1}{r^2}\mathbf{x}(\mathbf{x}\boldsymbol{\nabla}^{(1)})\boldsymbol{\nabla}^{(2)} - \frac{1}{r^2}(\boldsymbol{\sigma}^{(1)}\mathbf{L}^{(2)} + \boldsymbol{\sigma}^{(2)}\mathbf{L}^{(1)}) \right\} \\ & + \frac{1}{4m_1m_2c^2} \frac{e_1e_2}{r^3} \left\{ \boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)} - \frac{3(\boldsymbol{\sigma}^{(1)}\mathbf{x})(\boldsymbol{\sigma}^{(2)}\mathbf{x})}{r^2} \right\} \end{aligned} \quad (3.4)$$

$$W'^{(B)} = W'^{(A)} + \frac{\Gamma^2}{2(m_1 + m_2)c^2} \quad (3.5)$$

where Γ is given by (2.4) (b), $\mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$, $\mathbf{L}^{(1)}$ is the orbital angular momentum of the first particle with respect to the second particle, and $\mathbf{L}^{(2)}$ is that of the second particle with respect to the first:

$$\mathbf{L}^{(1)} = -i[\mathbf{x}\boldsymbol{\nabla}^{(1)}] \quad \mathbf{L}^{(2)} = i[\mathbf{x}\boldsymbol{\nabla}^{(2)}] \quad (3.6)$$

The difference between the new formula and Breit's is the last term of (3.5). This is just the surplus term mentioned in the introduction, and the defect of Breit's formula was to contain this term. Breit⁽¹¹⁾ showed in case of the two-

electron interaction that if the I^2 -term in (3.5) was reserved the calculated interval of He 2^3P was not reconcilable with experiment. On the contrary, according to the formula given by (3.4) the calculated interval was in good agreement with experiment.⁽¹¹⁾⁽²¹⁾ Further Ishidzu⁽¹²⁾ has recently shown in case of the electron-proton interaction that the I^2 -term is in contradiction to the experiment on the hyperfine structure of hydrogen. Thus we see that the new formula is in good agreement with experiment.

In order to see the nature of each terms and the degree of approximation we consider the case of hydrogen. For the sake of simplicity the electron mass is adopted as a mass unit and the proton mass is assumed to be infinite. Inserting $W^{(A)}$ into (3.3) we have

$$H' = -\frac{1}{2}A - \frac{e^2}{r} - \frac{1}{8c^2}A^2 + \frac{1}{4c^2} \frac{e^2}{r^3} (\mathbf{x}\nabla + \sigma\mathbf{L}) \quad (3.7)$$

This is the well-known Pauli formula. When its eigenvalue is evaluated considering e^{-2} -terms as perturbations it can be written in the form

$$H' = -\frac{1}{2}A - \frac{e^2}{r} - \frac{1}{2c^2} \left(E_0 + \frac{e^2}{r} \right)^2 - \frac{e^2}{4c^2 r^3} (\mathbf{x}\nabla - \sigma\mathbf{L}) \quad (3.8)$$

where E_0 is the unperturbed eigenvalue. Its eigenvalue is given by

$$E = -\frac{e^4}{2n^2} - \frac{e^8}{2n^3 c^2} \left(\frac{1}{J+1/2} - \frac{3}{4n} \right) \quad (3.9)$$

where n and J are the principal and inner quantum number. This is exactly equal to the zeroth and first degree terms in the power series in e^4/c^2 of Dirac's eigenvalue without the rest energy.⁽²²⁾ Therefore our method is mathematically correct. Since the second degree terms in e^4/c^2 is very small owing to the smallness of $e^4/c^2 = 137^{-2}$, the error of the Pauli approximation is very small.

The I^2 -term in Breit's formula (3.5) is proportional to $(e^2/r)^2$, as is seen from (2.4)(b). Its factor is independent of e and r . The formula (3.8) contains $(e^2/r)^2$ also, but this can not be omitted in order to obtain the correct result, as we have just seen. The contributions from the various e^{-2} -terms in (3.8) are all proportional to e^8 as well as those from the terms in (3.4) and (3.5) including the I^2 -term, where each $1/r$ or $\partial/\partial r$ gives a factor e^2 . Therefore it can not be a correct reason for omitting the I^2 -term that is proportional to e^4 .⁽¹⁵⁾

§ 4. Comparison of Møller's Formula with Experiment

We shall compare the theoretical result deduced from Møller's formula with the experiment on the fine structure of hydrogen in order to examine the effect of omitting $[H_0S']$ from (1.6). The calculation is carried out as before. If we omit $[H_0S']$ from (1.6) we have

$$\begin{aligned}
 W^{(M)} = & \frac{1}{4m_1^2c^2} \frac{e_1e_2}{r^3} \{ (\mathbf{L}^{(1)})^2 - \sigma^{(1)} \mathbf{L}^{(1)} \} + \frac{1}{4m_1c} \frac{e_1e_2}{r^3} \rho_2^{(1)} \sigma^{(1)} \mathbf{x} \\
 & + \frac{1}{4m_1c} \rho_3^{(1)} \rho_1^{(2)} \frac{e_1e_2}{r} \left\{ i\sigma^{(2)} \left(2\nabla^{(1)} - \frac{\mathbf{x}}{r^2} \right) - \frac{\mathbf{x}}{r^2} [\sigma^{(1)} \sigma^{(2)}] \right\} + (1 \leftrightarrow 2) \quad (4.1)
 \end{aligned}$$

where the following relation is made use of:

$$-\Delta + \frac{2}{r^2} \mathbf{x} \nabla + \frac{1}{r^2} \mathbf{x} (\mathbf{x} \nabla) \nabla = \frac{1}{r^2} \mathbf{L}^2 \quad (4.2)$$

The Pauli approximation of this interaction is given by

$$\begin{aligned}
 W'^{(M)} = & W'^{(A)} + \frac{e_1e_2}{4c^2r^3} \left\{ \frac{1}{m_1^2} (\mathbf{L}^{(1)})^2 + \frac{1}{m_2^2} (\mathbf{L}^{(2)})^2 \right\} \\
 & + \frac{1}{2m_1m_2c^2} \frac{e_1e_2}{r} \left\{ \nabla^{(1)} \nabla^{(2)} + \frac{\mathbf{x}}{r^2} (\nabla^{(1)} - \nabla^{(2)}) - \frac{1}{r^2} \mathbf{x} (\mathbf{x} \nabla^{(1)}) \nabla^{(2)} \right\} \quad (4.3)
 \end{aligned}$$

where $W'^{(A)}$ is given by (3.4). The difference between $W'^{(M)}$ and $W'^{(A)}$ is the influence of omitting $[H_0S']$. This difference is by no means negligibly small. It gives rise to a serious contradiction with experiment as will be shown below.

We shall compare (4.3) with the level structure of hydrogen atom. We shall calculate the shift of fine structure levels caused by $W'^{(M)} - W'^{(A)}$ in atomic units. Let the first particle be the electron and the second be the proton. In this case $e_1e_2 = -1$, $m_1 = 1$ and $m_2 = M = 1840$. In the centre-of-mass system we have

$$W'^{(M)} - W'^{(A)} = -\frac{1}{4c^2} \left(1 + \frac{1}{M} \right)^2 \frac{\mathbf{L}^2}{r^3} \quad (4.4)$$

Its expectation value in the nL -eigenstate of the zero-degree Hamiltonian in $1/c$ is given by

$$\begin{aligned}
 \langle (W'^{(M)} - W'^{(A)}) \rangle &= 0 & \text{for } L=0 \\
 &= -\frac{1}{4c^2 \left(1 + \frac{1}{M} \right) n^3 \left(L + \frac{1}{2} \right)} & \text{for } L \neq 0 \quad (4.5)
 \end{aligned}$$

where n and L are respectively the principal and azimuthal quantum numbers of the state.

The shift of levels from the position predicted by Dirac's theory is given by (4.5) because the energy eigenvalue calculated on the interaction $u + W'^{(A)}$ is equal to Dirac's eigenvalue up to $1/c^2$, as was shown in the preceding section. Therefore S -levels are unchanged and P , D , F , ... levels are lowered. For 2^2P the shift amounts to 7273 Mc/s, where $1/2c^2 \text{ au} = 5.822 \text{ cm}^{-1}$ is used.⁽²³⁾ The interval between $2^3P_{3/2}$ and $2^3P_{1/2}$ is 1.5 times of this value.⁽¹²⁾⁽²⁴⁾ Thus, according

to Møller's formula, we have

$$E(2^2P_{3/2}) - E(2^2S_{1/2}) = 3637 \text{ Mc/s} \quad (4.6) (a)$$

$$E(2^2S_{1/2}) - E(2^2P_{1/2}) = 7273 \text{ Mc/s} \quad (4.6) (b)$$

The level structure is shown in Fig. 2. The value given by (4.6) (b) is inconsistent with the observed value $1062 \pm 5 \text{ Mc/s}$ of Lamb and Retherford.⁽²⁾ Therefore we see that $[H_0S']$ in (1.6) can not be omitted.

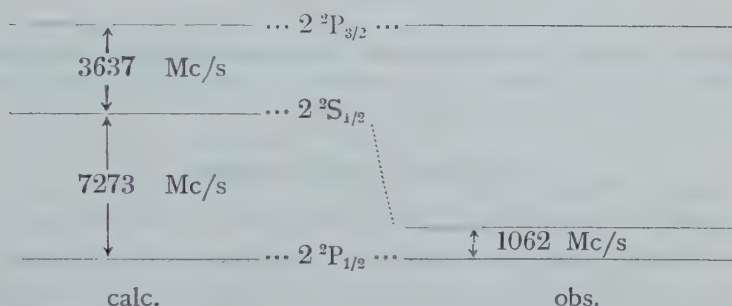


Fig. 2. Level structure of hydrogen atom according to Møller's formula

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Relativity of Representation Coordinates and its Consequences

Giiti IWATA

Department of Physics, Tokyo University

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Relativity of space-time coordinates has been fully appreciated in theoretical physics. It seems, however, to me that relativity of representation coordinates is not clearly acknowledged. Here are pursued briefly consequences of this relativity.

§ 1. Unitary transformations

Conventional field theories are cast in the frame of the x -diagonal representation. For example, gauge transformation, Hamilton formalism and the second quantization are formulated only in the x -representation. It is useful in many cases to employ the x -representation, but there are many representations other than the x -representation. It seems there is no apriori reason for a special choice of representation. So I propose that field theories should be formulated so as to be independent from representation coordinates. As a first step I assume that representation coordinates connected with each other through unitary transformations are equivalent. The x -representation coordinate is clearly equivalent to the p -representation coordinate.

Under the transformation from the ξ -representation to the η -representation, wave functions $\psi(\xi)$, $\bar{\psi}(\xi)$ and matrix element $(\xi|A|\xi')$ of an operator A transform as

$$\begin{aligned}\psi(\xi) &\rightarrow \psi(\eta) = (\eta|\xi)\psi(\xi), & \bar{\psi}(\xi) &\rightarrow \bar{\psi}(\eta) = \bar{\psi}(\xi)(\xi|\eta), \\ (\xi|A|\xi') &\rightarrow (\eta|A|\eta') = (\eta|\xi)(\xi|A|\xi')(\xi'|\eta')\end{aligned}$$

where $(\xi|\eta)$ and $(\eta|\xi)$ are transformation functions complex-conjugate to each other, satisfying the unitary conditions

$$\begin{aligned}(\xi|\eta)(\eta|\xi') &= (\xi|\xi') = \delta(\xi - \xi'), \\ (\eta|\xi)(\xi|\eta') &= (\eta|\eta') = \delta(\eta - \eta'),\end{aligned}$$

while the convention

$$(*|\xi)(\xi|**) = \int_{-\infty}^{\infty} (*|\xi)d\xi(\xi|**)$$

is used, $\psi(\xi)$, $\bar{\psi}(\xi)$ being counted for $(\xi|)$, $(|\xi)$.

We extend next the usual gauge transformation

$$\begin{aligned}\psi(x) &\rightarrow e^{i\lambda}\psi(x), & \lambda: \text{real function of } x \\ \bar{\psi}(x) &\rightarrow e^{-i\lambda}\bar{\psi}(x)\end{aligned}$$

that leaves unaltered $\bar{\psi}(x)\psi(x)$, to the unitary transformation

$$\begin{aligned}\psi(x) &\rightarrow' \psi(x) = (x|U|x')\psi(x'), \\ \bar{\psi}(x) &\rightarrow' \bar{\psi}(x) = \bar{\psi}(x')(x'|U^{-1}|x), \\ (x'|U^{-1}|x) &= (x'|U^+|x),\end{aligned}$$

or, in matrix forms,

$$\begin{aligned}\psi &\rightarrow' \psi = U\psi, \quad \bar{\psi} \rightarrow' \bar{\psi} = \bar{\psi}U^{-1} \\ U^{-1} &= U^+.\end{aligned}$$

The unitary transformation makes invariant not $\bar{\psi}(x)\psi(x)$ (not integrated with respect to x) but $\bar{\psi}\psi = \bar{\psi}(x)\psi(x)$ integrated with respect to x , since

$$\bar{\psi}\psi \rightarrow' \bar{\psi}'\psi = \bar{\psi}U^{-1}U\psi = \bar{\psi}\psi.$$

It is to be remarked that commutation relations imposed usually on $\psi(x)$, $\bar{\psi}(x)$

$$\begin{aligned}[\psi(x), \bar{\psi}(x')] &= \psi(x)\bar{\psi}(x') - \bar{\psi}(x')\psi(x) = c\delta(x-x'), \\ \{\psi(x), \bar{\psi}(x')\} &= \psi(x)\bar{\psi}(x') + \bar{\psi}(x')\psi(x) = c\delta(x-x') \quad (c: \text{const.})\end{aligned}$$

are invariant under both representation coordinate transformations and extended gauge transformations. In fact, we have

$$\begin{aligned}[\psi(\xi), \bar{\psi}(\xi')] &\rightarrow [\psi(\eta), \bar{\psi}(\eta')] = (\eta|\xi)(\xi'|\eta')[\psi(\xi), \bar{\psi}(\xi')] \\ &= (\eta|\xi)(\xi'|\eta')c\delta(\xi-\xi') = c(\eta|\xi)(\xi|\eta') \\ &= c\delta(\eta-\eta') \quad \text{etc.}\end{aligned}$$

Inversely if we assume the invariance of $\psi(\xi)\bar{\psi}(\xi') \pm \bar{\psi}(\xi')\psi(\xi) = (\xi|F|\xi')$ against unitary transformations, we have

$$F' = UFU^{-1} = F \quad \text{i.e.} \quad UF = FU.$$

Since U is arbitrary, F must be a multiple of the unit I . Hence $(\xi|F|\xi') = c\delta(\xi-\xi')$.

§ 2. Spinors in pseudo-euclidian spaces

To get spinor wave equations invariant under all reflections both space-like and time-like, we consider spinors in the n -dimensional space where the squared length of a vector $x(x^1, x^2, \dots, x^n)$ is defined by

$$(x^1)^2 + \dots + (x^r)^2 - (x^{r+1})^2 - \dots - (x^n)^2 = g_{ij}x^i x^j.$$

We introduce here α -algebra $\alpha_1, \dots, \alpha_n, \beta$ such that

$$\left. \begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2g_{ij} \\ \alpha_i \beta + \beta \alpha_i &= 0 \\ \beta^2 &= 1 \end{aligned} \right\} \quad (1)$$

If an element X is associated to the vector x through the relation

$$X = a_i x^i$$

the squared length of x is given by

$$(xx) = g_{ij} x^i x^j = X^2.$$

The vector x transforms under the reflection with respect to a vector a as

$$x \rightarrow x - 2(ax)a(aa)^{-1}.$$

The transformation of x may be expressed in terms of algebras associated to vectors as

$$X \rightarrow X - (AX + XA)AA^{-1} = -AXA^{-1},$$

where A is conditioned only so as to have A^{-1} .

When an element $\Xi = i\beta X$ is employed in place of X , the above expression may be changed into

$$\Xi \rightarrow -i\beta AXA^{-1} = A(i\beta X)A^{-1} = A\Xi A^{-1},$$

since β anticommutes with A .

In the representation where a_i and β are represented by faithful matrices, two spinors ψ and ϕ are introduced which should transform a priori under the reflection A as

$$\psi \rightarrow A\psi, \quad \phi \rightarrow \phi A^{-1}.$$

Under all reflections there are then invariant expressions

$$\phi\psi, \quad \phi\Xi\psi, \quad \phi\Xi_{(2)}\psi, \dots$$

or

$$\phi\Xi_{(p)}\psi, \quad \Xi_{(p)}: \text{p-vector}, \quad p=0, 1, \dots, n.$$

It is convenient to introduce γ -algebras defined by

$$\gamma_0 = 1, \quad \gamma_j = i\beta a_j, \quad \gamma_{ij} = \gamma_{[i}\gamma_{j]} = (\gamma_i\gamma_j - \gamma_j\gamma_i)/2,$$

$$\dots \quad \gamma_{ij\dots i} = \gamma_{[i}\gamma_{j}\dots\gamma_{i]},$$

γ -algebras also satisfy the relations (1) in place of a -algebras, i.e.

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}, \quad \gamma_i\beta + \beta\gamma_i = 0. \quad (2)$$

The coefficients of contravariant components of p -vector in the expression $\phi\Xi_{(p)}\psi$ constitute covariant components of antisymmetric tensor of p -th rank. They are typically represented by $\phi\gamma_{ij\dots k}\psi$ and total 2^n in number with all p -values ranging from 0 to n .

Since complex conjugation, hermite-conjugation and transposition leave unaltered the relations (1) or (2) to be satisfied by representation matrices of a_i or γ_i , there must be certain matrices c, h, t , or C, H, T such that

$$\bar{A} = cAc^{-1} \quad \bar{\Xi} = C\Xi C^{-1} \quad - : \text{complex-conjugation}$$

$$\begin{aligned} A^+ &= h A h^{-1} & \Xi^+ &= H \Xi H^{-1} & + : \text{hermite-conjugation} \\ A^* &= i A i^{-1}, & \Xi^* &= T \Xi T^{-1} & * : \text{transposition.} \end{aligned}$$

Such matrices are easily found in the representation where $\alpha_i, \beta_i, \gamma_i$ are represented, except for numerical coefficients, by direct products of hermitean matrices P, Q, I, I'

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for each of the preceding operations replaces γ_i by γ_i or $-\gamma_i$ so that the matrix corresponding to that operation commutes with γ_i 's that remain unaltered and anticommutes with γ_i 's that change their sign.

If A is so restricted that $A^2 = 1$, ϕ and ϕ^+ transform as

$$\phi \rightarrow \phi A^{-1} = \phi A, \quad \phi^+ \rightarrow \phi^+ A^+ = \phi^+ h A h^{-1}.$$

Hence

$$\phi^+ h \rightarrow \phi^+ h A.$$

So ϕ may be identified with $\phi^+ h$ in the rotation group that contains no single reflection with respect to time-like unit vector. There is otherwise no relation between ϕ and ψ except that they transform contragradient to each other. Therefore it is impossible to reserve one and discard the other if no restriction is imposed on A .

§ 3. Gauge transformations

The expression $-g_{ij} (p^i - e A^i) (p^j - e A^j) - m^2$ can be factorized by virtue of γ_i into $(i\gamma_j (p^j - e A^j) + m) (i\gamma_i (p^i - e A^i) - m)$. Regarding p^j and A^j as hermitean operators we set up the Lagrangian

$$L = \phi (i\gamma_j (p^j - e A^j) + m) \psi \quad (3)$$

which is written in the ξ -representation

$$L = \phi(\xi) (i\gamma_j (\xi | p^j | \xi') - i\gamma_j e (\xi | A^j | \xi') + m (\xi | \xi')) \psi(\xi').$$

If ϕ represents the wave function charge-conjugate to ψ , there is a certain matrix S such that

$$p^{j*} = S p^j S^{-1}, \quad A^{j*} = -S A^j S^{-1}$$

since we see that the expression

$$\begin{aligned} L &= \phi^* (i\gamma_j^* p^{j*} - i\gamma_j^* e A^{j*} + m) \phi^* \\ &= \phi^* T (i\gamma_j p^{j*} - i\gamma_j e A^{j*} + m) T^{-1} \phi^* \end{aligned}$$

should correspond to the expression in which p^{j*} and A^{j*} are replaced by p^j and $-A^j$ respectively.

In the p -representation we see $p^{j*} = p^j$, so S may be supposed to be the unit transformation. A^{j*} must be then equal to $-A^j$.

On the other hand the expression (3) should be invariant under the gauge transformation

$$\phi \rightarrow U\phi, \quad \phi \rightarrow \phi U^{-1}, \quad p^j \rightarrow p^j, \quad A^j \rightarrow {}'A^j.$$

Hence we have

$$p^j - e' A^j = U(p^j - e A^j)U^{-1}$$

or

$${}'A^j = U A^j U^{-1} - e^{-1}(U p^j U^{-1} - p^j).$$

It is difficult to harmonize the gauge transformation with the consideration for charge-conjugation because the gauge transformation does not conserve the relation $A^{j*} = -A^j$.

Even if U is confined within such that

$$U = \bar{U} = U^{*-1},$$

we have

$$\begin{aligned} {}'A^* &= U A^* U^{-1} - e^{-1}(U p^* U^{-1} - p^*) \\ &= -U A U^{-1} - e^{-1}(U p U^{-1} - p) \neq -{}'A. \end{aligned}$$

To get through the difficulty, we represent the charge not by the scalar quantity e but by the skew-symmetric hermitean matrix ϵ

$$\epsilon = \begin{pmatrix} 0 & -ie \\ ie & 0 \end{pmatrix}, \quad \epsilon^* = -\epsilon.$$

Further we restrict U within such that

$$U = \bar{U} = U^{*-1}, \quad U\epsilon = \epsilon U$$

and we regard p and A as two dimensional matrices

$$p \rightarrow p = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad A \rightarrow A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then A is transformed

$${}'A = U A U^{-1} - \epsilon^{-1}(U p U^{-1} - p)$$

by the gauge transformation which conserves the relation $A^* = -A$.

Thus we have the Lagrangian

$$L_M = \phi(i\gamma_j P^j - i\gamma_j \epsilon A^j + m)\phi \quad (4)$$

where each of ϕ and ϕ is composed of two 4-component spinors

$$\phi = (\phi_I, \phi_{II}), \quad \phi = \begin{pmatrix} \phi_I \\ \phi_{II} \end{pmatrix}.$$

The Lagrangian L_M expressed with these components

$$L_M = \phi_I(i\gamma_3 p^3 + m)\psi_I + \phi_{II}(i\gamma_3 p^3 + m)\psi_{II} + e\phi_{II}\gamma_3 A^3\psi_{II} - e\phi_{II}\gamma_3 A^3\psi_I \quad (5)$$

remains unaltered by the interchange of suffices *I* and *II* together with reversal of the sign of *e*.

Next we shall set up the Lagrangian for radiation field. The electromagnetic field \mathbf{F}_{ij} defined by

$$i\epsilon\mathbf{F}_{ij} = [\mathbf{p}_i - \epsilon\mathbf{A}_i, \mathbf{p}_j - \epsilon\mathbf{A}_j]$$

or

$$\mathbf{F}_{ij} = i[\mathbf{p}_i, \mathbf{A}_j] - i[\mathbf{p}_j, \mathbf{A}_i] - i\epsilon[\mathbf{A}_i, \mathbf{A}_j]$$

transform

$$\mathbf{F}'_{ij} = U\mathbf{F}_{ij}U^{-1}$$

under the gauge transformation, so there exists the invariant Lagrangian

$$\left. \begin{aligned} L_R &= 1/8 \cdot \text{Tr} \text{tr}(\mathbf{F}_{ij}\mathbf{F}^{ij}) \\ &= 1/4 \cdot \text{Tr}(F_{ij}F^{ij}) - e^2/4 \cdot \text{Tr}([A_i, A_j][A^i, A^j]), \\ F_{ij} &= i[\mathbf{p}_i, \mathbf{A}_j] - i[\mathbf{p}_j, \mathbf{A}_i] \end{aligned} \right\} \quad (6)$$

where *tr* denotes the trace of 2-dimensional matrix, *Tr* the trace of operator.

§ 4. Field equations and the second quantization

Field equations can be derived from the unified Lagrangian

$$L = L_M + L_R = \phi(i\gamma_3 p^3 - i\gamma_3 \epsilon \mathbf{A}^3 + m)\psi + 1/8 \cdot \text{Tr} \text{tr}(\mathbf{F}_{ij}\mathbf{F}^{ij}) \quad (7)$$

with the aid of variational principle as follows,

$$\left. \begin{aligned} (i\gamma_3 p^3 - i\gamma_3 \epsilon \mathbf{A}^3 + m)\psi &= 0, \\ \phi(i\gamma_3 p^3 - i\gamma_3 \epsilon \mathbf{A}^3 + m) &= 0, \\ i[F_{ij}, p^i] - e^2[[A_i, A_j]A^i] - s_j &= 0, \\ (\phi|s_j|\phi') &= \frac{1}{2}\{\phi(\phi')i\gamma_3 \epsilon \psi(\phi) - \phi(\phi)i\gamma_3 \epsilon \psi(\phi')\} \\ &= e/2\{\phi_I(\phi')\gamma_3\psi_{II}(\phi) - \phi_I(\phi)\gamma_3\psi_{II}(\phi')\} - e/2 \cdot \{\phi_{II}(\phi')\gamma_3\psi_I(\phi) - \phi_{II}(\phi)\gamma_3\psi_I(\phi')\}. \end{aligned} \right\} \quad (8)$$

The equation to ψ may be expanded

$$(i\gamma_3 p^3 + m)\psi_I + e\gamma_3 A^3\psi_{II} = 0,$$

$$(i\gamma_3 p^3 + m)\psi_{II} - e\gamma_3 A^3\psi_I = 0.$$

From the above equations it can be inferred that

- 1, neither I-field alone nor II-field alone create radiation field,
- 1', I-field and II-field have no interaction between them when radiation field is absent,
- 2, diagonal elements of the charge current vector s_j are absent,
- 2', total charge and current vanish,

$2''$, a particle produces no action on itself.

Invariance under unitary transformations and statistics to be satisfied by fields give the commutation relations

$$\left. \begin{aligned} \{\phi_a(p), \phi_\beta(p')\} &= \{\phi_a(p), \phi_\beta(p')\} = 0, \\ \{\phi_a(p), \phi_\beta(p')\} &= \delta_{a\beta} \delta(p-p'), \quad a, \beta = \dots 1, 2, \dots, 8, \\ [(p|A_j|p'), (q|A_k|q')] &= i g_{jk} \cdot 1/2 \cdot (\delta(p-q') \delta(p'-q) - \delta(p-q) \delta(p'-q')), \end{aligned} \right\} \quad (9)$$

other commutators vanish. (p, p', q, q' being eigenvalues of the operator p).

It is to be noted that commutators $[\psi, L]$, $[\phi, L]$, $[A_j, L]$ calculated with these commutation relations yield the left-side members of the field equations (8) except for numerical coefficients.

Hence, in place of simultaneous field equations (8), we may treat

$$[\psi, L] = 0,$$

$$[\phi, L] = 0,$$

$$[A_j, L] = 0.$$

These equations however have no meanings in themselves. They represent the conditions that should be satisfied by the state function Ψ and its adjoint Φ . So the faithful relations should be

$$\left. \begin{aligned} \Phi[\psi, L]\Psi &= 0, \\ \Phi[\phi, L]\Psi &= 0, \\ \Phi[A_j, L]\Psi &= 0. \end{aligned} \right\} \quad (10)$$

We make use of a short cut to solve these simultaneous equations.

If l denotes an eigenvalue of the operator L , there exist eigenstate Ψ_l and its adjoint Φ_l such that

$$\left. \begin{aligned} L\Psi_l &= l\Psi_l, \\ \Phi_l L &= \Phi_l l. \end{aligned} \right\} \quad (11)$$

Hence we have

$$\Phi_l[\psi, L]\Psi_l = \Phi_l[\psi, l]\Psi_l = 0, \quad \text{etc.}$$

So the solution of simultaneous equations (10) may be reduced to the solution of the eigenvalue problem (11) of the operator L .

On the Antiferromagnetism of Single Crystals

Kei YOSIDA

Department of Physics, Osaka University

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The antiferromagnetism of a single crystal is considered, basing on the Van Vleck model, modified by introducing an anisotropy energy of one spin, arising mainly from the crystalline field of the surrounding anions. On this model are derived the anisotropy of the susceptibility above the Curie point, the temperature variation of the anisotropy constant below the Curie point and the field dependency of the susceptibility under a comparatively strong magnetic field.

§ 1. Introduction

Recently T. Nagamiya¹⁾ has developed a general theory of antiferromagnetism, on the basis of the idea that the anisotropy energy is essential for the antiferromagnetism, and derived the relation which represents the field dependency of the susceptibility when an applied field is comparatively weak. This theory has an advantage that it is applicable not only to powder specimens, but to single crystals. As far as a weak field is concerned, however, he has shown that it brings essentially different results from those obtained by the Van Vleck theory²⁾ only when it is applied to the phenomenon of the magnetic resonance absorption.

On the other hand, according to the new experiments for a single crystal made by C. J. Gorter and his coworkers³⁾, the spins in an antiferromagnetic material take a parallel orientation to the preferential direction determined by the crystal structure, analogous to the easy direction of magnetization in a ferromagnetic material, and when an applied field becomes stronger than a certain critical value, these spins turn towards a perpendicular direction to the applied field. Concerning these experimental facts, it will be of some interest to extend Nagamiya's theory for a comparatively strong field.

In this paper, to simplify the calculation, we shall confine our considerations mainly to the case of MnF_2 , the magnetic properties of which have experimentally been studied by J. W. Stout and M. Griffel⁴⁾, since its crystal structure is tetragonal and the magnitude of the spin of a Mn^{2+} ion is larger than $1/2$. Also for the other crystal structures, especially for MnO , however, the present theory can easily be extended, but another treatment will be needed for the case of a spin of $1/2$.

§ 2. Hamiltonian

Let us consider a single crystal of MnF_2 as an antiferromagnetic material.

Its crystal structure is of the rutile type. each Mn ion occupying a body-centered lattice point and being surrounded by six F⁻ions. Therefore a Mn²⁺ion is immersed in the crystalline electric field of orthorhombic symmetry, arising mainly from the surrounding six F⁻ions.

Since a Mn²⁺ion has no orbital angular momentum in its free state of ⁶S, its six-fold spin degeneracy is lifted only by the higher order perturbation, through its excited states, of this crystalline electric field combined with the spin-orbit interaction^{b)}. This combined action of the crystalline field and the spin-orbit interaction gives rise to the following Hamiltonian of one Mn²⁺ion :

$$H = -D_z S_z^2 - D_\xi S_\xi^2 - D_\eta S_\eta^2 + A(S_1^4 + S_2^4 + S_3^4), \quad (1)$$

where D and A are the constants determined by the crystalline field and the electronic wave function of an ion, and S_z , S_ξ , S_η , are the component of the spin angular momentum in the unit of $\hbar/2\pi$, referred to the orthorhombic axes and S_1 , S_2 and S_3 are its components referred to the cubic axes. The first term in (1) arises from the combined action of the orthorhombic part of the crystalline field and the spinorbit interaction, and the last from the cubic part. The former may be a dominant part in the tetragonal crystal such as MnF₂, while in the cubic crystal such as MnO, the latter may be a main part.

The Hamiltonian of the total magnetic ions is given by the sum of the energy of each ion given by (1), the Zeeman energy and the exchange energy between nearest neighbouring magnetic ions, namely

$$H = -D \sum_i S_{zi}^2 + g\beta H \cdot \sum_i S_i - 2/\sum_{i,j} J_{ij} S_i \cdot S_j, \quad (2)$$

where g , β and H denote the Lande factor, the Bohr magneton and the external magnetic field, respectively, and J is the exchange integral having a negative sign, and moreover the cubic term in (1) is omitted in order to confine the calculation to the tetragonal crystal of MnF₂. Actually, the principal axes of the crystalline field at the Mn²⁺ ion take the different directions for two Mn²⁺ ions in a unit cell. In this expression of the Hamiltonian, however, only $D_z S_z^2$ in the orthorhombic parts is, for simplicity, adopted, since that of the three principal axes, whose direction coincides with the c-axis, is common for the two Mn²⁺ ions and the whole crystal has the tetragonal symmetry.

In this Hamiltonian it is supposed that the first term causes the anisotropy of the whole crystal. Strictly speaking, besides this we have to take into account other causes of the anisotropy like the magnetic interaction and the anisotropic exchange coupling between spins. In this paper, however, we shall suppose that these effects are implicitly contained in the constant D and shall not discuss them any further. When the spin quantum number S is equal to 1/2, this anisotropy energy given by (1) becomes only a constant value, and gives no anisotropy. In this case, we may have explicitly to introduce the anisotropic coupling between two spins, such as the anisotropic exchange energy or the coupling of the dipole-

dipole interaction type⁶⁾. For an S larger than $1/2$, however, the anisotropy energy of the type (1) represented by one spin variable, will probably play a predominant rôle in the anisotropy of the whole crystal at least in an antiferromagnetic material.

§ 3. Anisotropy of the susceptibility above the Curie point

To begin with, we shall calculate the susceptibility above the Curie point, making use of the Hamiltonian of the total spin system just established by (2).

The partition function Z and two principal susceptibilities $\chi_{||}$, χ_{\perp} are, as usual, given by the following relations:

$$Z = \text{Trace} [\exp(-H/kT)], \quad (3)$$

$$\chi_{||} = \frac{kT}{H_x} \frac{1}{Z} \frac{\partial Z}{\partial H_x}, \quad \chi_{\perp} = \frac{kT}{H_x} \frac{1}{Z} \frac{\partial Z}{\partial H_x}. \quad (4)$$

The evaluation of Z above the Curie point can easily be performed by the diagonal sum method developed by Van Vleck⁷⁾. Expanding Z in a power series of $(kT)^{-1}$ and the magnetic field H , and neglecting the higher terms than $(kT)^{-3}$ and H^2 , we obtain the following expressions for $\chi_{||}$ and χ_{\perp} .

$$\chi_{||} = \frac{Ng^2\beta^2 S(S+1)}{3kT \left(1 + \frac{2|J|z S(S+1)}{3kT}\right)} \left[1 + \frac{D}{kT} \left\{\frac{4S(S+1)}{15} - \frac{1}{5}\right\}\right], \quad (5)$$

$$\chi_{\perp} = \frac{Ng^2\beta^2 S(S+1)}{3kT \left(1 + \frac{2|J|z S(S+1)}{2kT}\right)} \left[1 - \frac{D}{kT} \left\{\frac{2S(S+1)}{15} - \frac{1}{10}\right\}\right]. \quad (6)$$

Here N denotes the number of magnetic ions, and z the number of nearest neighbouring magnetic ions. These two equations show that the Curie temperature θ is given by

$$\theta = \frac{2|J|z S(S+1)}{3k}. \quad (7)$$

The susceptibility χ_p for a powder specimen is represented by

$$\chi_p = \frac{1}{3}\chi_{||} + \frac{2}{3}\chi_{\perp} = \frac{Ng^2\beta^2 S(S+1)}{3k(T+\theta)}, \quad (8)$$

which contains the constant D no longer.

Subtracting (6) from (5), we obtain

$$\chi_{||} - \chi_{\perp} = \chi_p \frac{D}{kT} \left\{\frac{2}{5}S(S+1) - \frac{3}{10}\right\}. \quad (9)$$

The difference of the two principal susceptibilities in a single crystal of MnF_2 has been measured by J. W. Stout and M. Griffel⁸⁾, whose results are shown by a solid curve in Fig. 1. A dashed curve is a plot of Eq. (9), where S is replaced by $5/2$, the experimental values are used for χ_p , which has been measured by H.

Bizette and B. Tsai⁸⁾, and de Haas, Schultz and Koolhaas⁹⁾, and the value of D is selected for two curves so as to accord with each other at the higher temperature side.

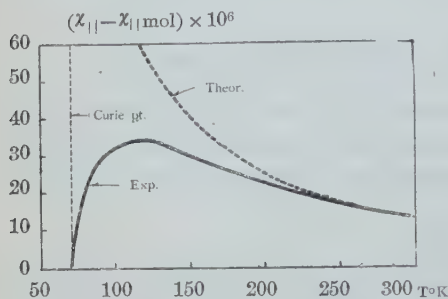


Fig. 1. The anisotropy of the susceptibility above the Curie point. The solid curve is a plot of the experimental values obtained by J. W. Stout and M. Griffel. The dashed curve is a theoretical one.

The departure of the theoretical curve from the experimental one near the Curie point is, to be supposed, probably due to the fluctuation of the molecular field and to explain this disaccordance it will be necessary to take into account the higher terms in the series expansion of the partition function.

§ 4. Susceptibility below the Curie point, I

The value of D estimated from the anisotropy of the susceptibility above the Curie point in the preceding paragraph, is very small, compared with the exchange energy. Therefore, in the absence of an external magnetic field, the circumstances below the Curie point are predominantly determined by the exchange energy and are not different from those derived by Van Vleck²⁾, as pointed out by T. Nagamiya¹⁾.

The crystal is, therefore, divided into two sublattices, which are each occupied by the spin equal in magnitude but opposite in direction. In the case of MnF_2 , it may be supposed that the crystal is resolved into the successive layers normal to the tetragonal axis, each having an alternating \pm spin orientation.*

The average magnitude S_0 of \pm spin on each sublattice in a zero field is given by the following relations:

$$S_0 = SB_0 \left[\frac{2|J| z SS_0}{kT} \right]. \quad (10)$$

The spin having this magnitude S_0 takes a parallel orientation to the easy axis on account of the anisotropy energy.

* Note added in proof: According to the neutron diffraction studies on MnF_2 , recently made by R. A. Erickson and C. G. Shull (Phys. Rev. **83** (1951), 208), the magnetic unit cells and chemical unit cells are of the same size in MnF_2 .

As to the effect of the magnetic field, it is convenient to consider this in two steps. In this paragraph, as the first step, we shall consider the relative change of the spin vectors $\pm \mathbf{S}_0$ on two sublattices, produced by the external field under the assumption that the spin vectors $\pm \mathbf{S}_0$ are fixed not to rotate as a whole without changing their mutual relation. The second step is to determine the direction of the vector \mathbf{S}_0 with respect to the easy axis or the external field.

The relative change of $\pm \mathbf{S}_0$ will predominantly be determined by the exchange energy and so we shall be able to forget the small anisotropy energy. Then we are led to the same results as Van Vleck. According to his theory²⁾ the changes of two vectors $\pm \mathbf{S}_0$ produced by the external field are equal, and the components of this change $\delta \mathbf{S}$ parallel and perpendicular to the vector \mathbf{S}_0 are, respectively, given by the following equations:

$$\delta S_{||} = \frac{\delta S_{\perp}}{H_{\perp}} \frac{2H_{||}}{1 + \frac{kT}{2|J|zS^2} B_s^{-1} \left[\frac{2|J|zSS_0}{kT} \right]}, \quad (11)$$

$$\delta S_{\perp} = \frac{g\beta}{4|J|z} H_{\perp}, \quad (12)$$

where $H_{||}$ and H_{\perp} are, respectively, the components of the magnetic field H parallel and perpendicular to the vector \mathbf{S}_0 . From these equations, the following two quantities $\chi_{||}$ and χ_{\perp} , which will turn out to be the susceptibilities parallel and perpendicular to the easy direction for a comparatively weak field, will be defined:

$$\chi_{||} = Ng\beta \frac{\delta S_{||}}{H_{||}}, \quad \chi_{\perp} = Ng\beta \frac{\delta S_{\perp}}{H_{\perp}}. \quad (13)$$

Here it is to be noted that the field components $H_{||}$ and H_{\perp} in Eqs. (11) and (12), and therefore $\delta \mathbf{S}$ remain still undetermined until the direction of \mathbf{S}_0 with respect to the external field H is established. Because, although it is true that the vector \mathbf{S}_0 takes the easy direction in the absence of an external field, when an external field is applied, the spins as a whole may possibly rotate without changing their mutual orientations. The direction of \mathbf{S}_0 will be determined by the anisotropy energy. When it has been established, we can calculate the real susceptibility from (11) and (12). This second step will be discussed in the following paragraph.

§ 5. Temperature dependency of the anisotropy constant

Before calculating the susceptibility below the Curie point, we shall have to consider the anisotropy energy of the whole crystal. This energy is given by the statistical average of the microscopic anisotropy energy represented by Eq. (1).

Now let us calculate the statistical average of $-DS_z^2$, when the direction of the spins as a whole is deviating on an average from the easy direction along the

z -axis by an angle θ . Let the average direction of the spins be the ζ -axis and the plane including the z - and ζ -axis be η - ζ plane, then $-DS_z^2$ can be written as

$$\begin{aligned} -DS_z^2 &= -D(\cos \theta S_\zeta + \sin \theta S_\eta)^2 \\ &= -D[\cos^2 \theta S_\zeta^2 + \sin^2 \theta S_\eta^2 + \sin \theta \cos \theta (S_\eta S_\zeta + S_\zeta S_\eta)]. \end{aligned} \quad (14)$$

If each spin is quantized in the direction of the ζ -axis, the average value of DS_z^2 in the state that S_ζ has its eigenvalue M , is given by the corresponding diagonal element of the matrix of DS_z^2 in the representation making the matrix of S_ζ diagonal. Using (14) and the following relations:

$$\begin{aligned} (M | S_\zeta S_\eta | M) &= 0, \\ (M | S_\eta^2 | M) &= \frac{1}{2} S(S+1) - \frac{1}{2} M^2, \end{aligned}$$

this diagonal element corresponding to the eigenvalue M of S_ζ becomes

$$D(M | S_z^2 | M) = D[M^2(\cos^2 \theta - \frac{1}{2} \sin^2 \theta) + \frac{1}{2} S(S+1) \sin^2 \theta].$$

Therefore, the statistical average of the anisotropy energy is represented by

$$-D\overline{S_z^2} = -D\left\{(\cos^2 \theta - \frac{1}{2} \sin^2 \theta) \overline{M^2} + \frac{1}{2} S(S+1) \sin^2 \theta\right\}. \quad (15)$$

Here $\overline{M^2}$ means the statistical average of M^2 , and can be calculated from the following relation:

$$\overline{M^2} = \frac{\sum_{M=-S}^S M^2 \exp\left(\frac{M}{S} x\right)}{\sum_{M=-S}^S \exp\left(\frac{M}{S} x\right)}, \quad x = \frac{2|J| z S S_0}{kT}. \quad (16)$$

The summation in the right hand side of this equation can easily be performed and gives the following expression for $\overline{M^2}$:

$$\overline{M^2} = S^2 \left(\frac{S+1}{S} - 2 \operatorname{B}_s[x] \frac{1}{2S} \coth \frac{1}{2S} x \right). \quad (17)$$

From (17) and (15), we arrive at the final expression for the average value of the microscopic anisotropy energy when the average direction of each spin is deviating by an angle of θ from the easy direction, namely the macroscopic anisotropy energy, apart from the constant term independent of θ :

$$D' \sin^2 \theta = DS^2 \left\{ \frac{S+1}{S} - 3 \operatorname{B}_s[x] \frac{1}{2S} \coth \frac{1}{2S} x \right\} \sin^2 \theta, \quad (18)$$

where D' is considered as the anisotropy constant and it identically vanishes when $S=1/2$. For the case of $S>1/2$, D' becomes $DS(S-1/2)$ at the absolute zero of temperature and becomes zero at the Curie point.

Fig. 2 illustrates the relation between D' and the temperature, calculated from

Eq. (18) in the case of $S=5/2$. If we adopt the cubic form of (1) as the microscopic anisotropy energy, we shall find a more rapid decrease of D' near the Curie point.

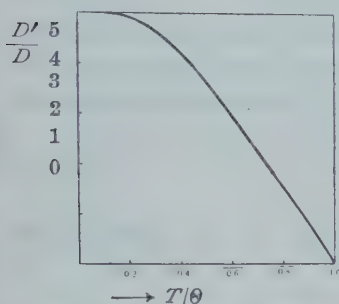


Fig. 2. The temperature dependence of the anisotropy constant for $g=5/2$.

In an antiferromagnetic material with a comparatively high Curie point, differing from the case of the ferromagnetism, this anisotropy constant can not, in general, be measured experimentally, but it will be possible for the case of a crystal having a very low Curie point.

§ 6. Susceptibility below the Curie point, II

In the preceding paragraph, we have given the expression of the macroscopic anisotropy energy.

Now we shall consider the real susceptibility of an antiferromagnetic crystal.

From Eqs. (11) and (12) representing the change of the vector \mathbf{S}_0 under an applied field when the original direction of \mathbf{S}_0 is fixed, we can immediately find that the magnetic energy is larger in its absolute value in the case that \mathbf{S}_0 is perpendicular to H than in the case parallel to it, because δS_{\perp} is larger than δS_{\parallel} . Therefore, the vector \mathbf{S}_0 , which takes an orientation parallel to the easy direction in the absence of an external field, has a tendency to turn to a perpendicular direction to the external field. This tendency will, however, be prevented by a restoring force arising from the anisotropy energy. The vector \mathbf{S}_0 will, therefore, take such an orientation that both these tendencies balance with each other.

Let an angle between the easy direction and the applied magnetic field H be β and that between the vector \mathbf{S}_0 and H be ϕ . Then the change $\delta \mathbf{S}$ of \mathbf{S}_0 , when the angle ϕ is kept constant, is given by Eqs. (11) and (12). Since $H_{\perp} = H \sin \phi$ and $H_{\parallel} = H \cos \phi$ in this orientation, each component of $\delta \mathbf{S}$ can be written as

$$\delta S_{\perp} = \frac{\chi_{\perp}}{Ng\beta} H \sin \phi, \quad \delta S_{\parallel} = \frac{\chi_{\parallel}}{Ng\beta} H \cos \phi. \quad (19)$$

Since the change of the magnetic moment of one spin is given by $g\beta\delta \mathbf{S}$, the magnetic energy produced by an external field becomes

$$-\frac{1}{2} g\beta \{ \delta S_{\perp} H \sin \phi + \delta S_{\parallel} H \cos \phi \}. \quad (20)$$

Using Eq. (19) for δS_{\perp} and δS_{\parallel} , this can be written as

$$-\frac{1}{2N} \{ \chi_{\perp} \sin^2 \phi + \chi_{\parallel} \cos^2 \phi \} H^2. \quad (21)$$

On the other hand, the anisotropy energy in this orientation is given by

$$D' \sin^2(\phi - \beta), \quad (22)$$

and the susceptibility χ is

$$\chi = \chi_{\perp} \sin^2 \phi + \chi_{\parallel} \cos^2 \phi = \chi_{\parallel} + (\chi_{\perp} - \chi_{\parallel}) \sin^2 \phi. \quad (23)$$

Neglecting the change of the exchange energy which is an order of δS^2 , ϕ is, then, determined by the following condition that the sum of the magnetic energy and the anisotropy energy, given by (21) + (22), takes its minimum value:

$$D' \sin 2(\phi - \beta) - \frac{1}{2N} (\chi_{\perp} - \chi_{\parallel}) H^2 \sin 2\phi = 0, \quad (24)$$

which we have by differentiating (21) + (22) with respect to ϕ . Solving (24), we obtain

$$\tan 2\phi = \frac{\sin 2\beta}{\cos 2\beta - \lambda}, \quad (25)$$

where

$$\lambda = \frac{H^2}{H_c^2} \quad (26)$$

and

$$H_c = \left[\frac{2ND'}{\chi_{\perp} - \chi_{\parallel}} \right]^{\frac{1}{2}}. \quad (27)$$

The calculation performed here is entirely the same with that given by L. Néel⁽¹⁰⁾ in discussing the susceptibility of an antiferromagnetic substance at the absolute zero of temperature on the basis of a classical spin model, and therefore, our relation between ϕ and β coincides with his result.

If we eliminate ϕ from (23) using (25), we obtain

$$\chi = \chi_{\parallel} + (\chi_{\perp} - \chi_{\parallel}) \frac{1}{2} \left\{ 1 - \frac{\cos 2\beta - \lambda}{[1 - 2\lambda \cos 2\beta + \lambda^2]^{\frac{1}{2}}} \right\}. \quad (28)$$

According to this expression for the real susceptibility, we obtain, in the case $\beta = 0$ that an external field is parallel to the easy direction,

$$\chi = \chi_{\parallel}, \quad \text{when } \lambda < 1 \text{ or } H < H_c, \quad (29)$$

and

$$\chi = \chi_{\perp}, \quad \text{when } \lambda > 1 \text{ or } H > H_c.$$

For the case $\beta = \frac{\pi}{2}$ that an external field is perpendicular to the easy direction, we obtain a constant susceptibility χ_{\perp} independent of H . When $0 < \beta < \frac{\pi}{2}$ the susceptibility χ increases continuously from $\chi_{\parallel} + (\chi_{\perp} - \chi_{\parallel})(1/2)(1 - \cos 2\beta)$ for $H=0$ to χ_{\perp} for $H=\infty$.

When the external field applied to the preferential direction becomes stronger than a critical field H_c , the vector S_0 averts abruptly its orientation from the easy axis to the direction perpendicular to it. This critical field is given by (27). The relation between this field and the temperature is illustrated in Fig. 3, where (18) for D' and (11), (12) and (13) for $\chi_{||}$ and χ_{\perp} are used respectively. In this case, H_c increases slowly with a rising temperature, on account of a more rapid decrease of $\chi_{\perp} - \chi_{||}$ than that of D' . But, there may possibly be the cases that H_c decreases with an increasing temperature, D' decreasing more rapidly than $\chi_{\perp} - \chi_{||}$.

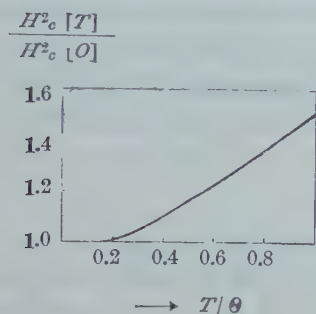


Fig. 3. The relation between the square of the critical field and the temperature when $S=5/2$.

Since $\chi_{||}=0$, $\chi_{\perp} = \frac{Ng^2\beta^2}{4|J|z}$, and $D' = DS(S - 1/2)$, at the absolute zero of temperature, the critical field becomes from (27) and (7)

$$(H_c)_{T=0} = \left[\frac{12k\theta Dg^{-2}\beta^{-2} S(S - \frac{1}{2})}{S(S + 1)} \right]^{\frac{1}{2}}. \quad (30)$$

This critical value for MnF_2 is estimated as 0.7×10^5 oersteds, using the value of D determined from the anisotropy of the susceptibility above the Curie point. This is so large that it is impossible to find the transition of the susceptibility from $\chi_{||}$ to χ_{\perp} in the usual experiments. H_c represented by (30) is, however, proportional to the square root of the Curie temperature and so it is expected that this transition will possibly be found in an antiferromagnetic substance having a Curie point at the temperature of an order of $1^\circ K$, where H_c amounts to several thousands oersteds, assuming that D' is the same order with that for MnF_2 . The experiments on $CuCl_2 \cdot 2H_2O$ carried out by C. J. Gorter and his collaborators³⁾ may surely correspond to this case, but our results can not be strictly applicable to this material on account of the spin of $1/2$ for Cu^{2+} ion. In general, if the critical field is measured experimentally, it will be possible to find the temperature dependency of the anisotropy constant from (27) in an antiferromagnetic material.

In the case that an external field is small compared with H_c , we can expand (28) in a power series with respect to λ . Then we obtain as χ :

$$\chi = \chi_{||} + (\chi_{\perp} - \chi_{||}) \frac{1}{2} \left\{ 1 - (\cos 2\beta - \lambda) \sum_{l=0}^{\infty} \lambda^l P_l(\cos 2\beta) \right\}. \quad (31)$$

Here, neglecting higher terms than λ , χ is approximated by the following expression:

$$\chi = \chi_{||} + (\chi_{\perp} - \chi_{||}) \left\{ \sin^2 \beta + \frac{\chi_{\perp} - \chi_{||}}{2ND'} H^2 2 \sin^2 \beta \cos^2 \beta \right\}. \quad (32)$$

As to the susceptibility for a powder specimen we obtain, averaging this expression over all the solid angles

$$\chi_p = \chi_{||} + (\chi_{\perp} - \chi_{||}) \left\{ \frac{2}{3} + \frac{4}{15} \frac{H^2}{H_c^2} \right\}. \quad (33)$$

The expressions of (32) and (33) are in accord with those obtained by T. Nagamiya¹⁾ when an S tends to infinity.

The field dependency of the susceptibility for a powder specimen of MnF_2 has been measured by de Haas, Schultz and Koolhaas¹⁰⁾. Their results do not, however, show a parabola-like dependency of χ_p on H . This fact may probably be due to the impurities contained in their sample. If we assume, however, that $\chi_{||}$ obeys a parabola-law, and estimate H_c roughly from (33), we obtain $H_c \sim 0.8 \times 10^5$ oers-eds which is in accord in its order with that estimated from the susceptibility above the Curie point.

§ 7. Conclusion

Introducing the microscopic anisotropy energy represented by one spin variable into the model used by Van Vleck, the susceptibility both above and below the Curie point was calculated for an antiferromagnetic single crystal in the case that the spin quantum number S of the magnetic ions is larger than $1/2$. For $S = 1/2$, it would be necessary to introduce the anisotropic coupling between two spins.

The calculation of the susceptibility below the Curie point was performed in two steps. Firstly, we calculate the change δS of the vector S_0 , arising from an external field, which is equal for the spins on both sublattices, when the orientation of the S_0 is fixed. In this calculation, we can neglect the anisotropy energy which is very much small compared with the exchange coupling, and, therefore, the results are in complete accord with those obtained by Van Vleck. The susceptibility obtained in such a way, however, does not give a true one. Secondly, the direction of the S_0 is determined, taking account of the anisotropy energy after the way shown by L. Néel. As the results obtained in this manner, it has been found that the two susceptibilities $\chi_{||}$ and χ_{\perp} given by Van Vleck, which imply the susceptibilities parallel and perpendicular to the vector S_0 , are equal to those parallel and perpendicular to the easy direction, as far as the external field strength is smaller than the critical field, as pointed out by T. Nagamiya.

When the external field parallel to the easy direction arrives at the critical value, the susceptibility jumps abruptly from $\chi_{||}$ to χ_{\perp} , while in the case of the external field perpendicular to the easy direction, the susceptibility keeps the constant value χ_{\perp} independent of the field strength. When the external field is applied to the intermediate direction, the susceptibility approaches the value χ_{\perp} continuously with an increasing field strength. Thus for the field strength com-

parable to the critical value, we have to take into consideration the rotation of the spin, as L. Néel did.

In this paper, we confined the discussions of the susceptibility to the crystal of tetragonal symmetry. The considerations for the other crystal structures will be developed in another paper.

In conclusion, the writer is very much indebted to Prof. T. Nagamiya for his continual interest and illuminating discussions on this problem.

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On the Zero Point Entropy of Methane Crystal*

Takeo NAGAMIYA

Department of Physics, Osaka University, Osaka

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Possible crystalline structures are suggested for the low temperature modifications of CH_4 , CD_4 , and CH_3D crystals from the consideration of interatomic distances, assuming that the molecules are vibrating rather than rotating in the crystals. The degrees of degeneracy of several lowest eigenstates of the molecular motion are investigated by group theoretical considerations on the simplifying assumption that each molecule is subjected to a field of a certain symmetry, namely, of octahedral, tetrahedral, trigonal, or of no symmetry. It results that the lowest rotational state of the para- CH_4 molecule, $j=2$, splits into several components in the octahedral and tetrahedral field, of which the lowest state has a degree of spatial degeneracy 4 and 2, respectively. An explanation of Clusius's measurement of the zero-point entropy of the CH_4 crystal can be given by assuming a tetrahedral field and assuming that the conversion between different symmetry types of the molecule does not occur. The alternative and more probable explanation can be given by the assumption that 12 lowest levels resulting from the symmetry operations of the molecule are quasi-degenerate and the symmetry of the field is tetrahedral or lower. The occurrence of the conversion does not matter in the latter case. The zero-point entropy of CH_3D can also be accounted for by the second type of assumption.

§ 1. Introduction

The thermal properties of methane crystal at low temperatures (above 10°K) have been investigated mainly by Clusius and his coworkers.¹⁾ CH_4 crystal undergoes a phase transition at 20.42°K accompanied by a sharp peak of the specific heat curve. CH_3D and CD_4 crystals have two transition points, with two corresponding sharp peaks in their specific heat curves, and the crystals are optically anisotropic in their lowest modifications. A similar behaviour is found for CH_2D_2 , but not in detail. The transition in CH_4 corresponds to the upper transition in CH_3D and CD_4 , as shown clearly by Eucken's²⁾ study of the mixed crystals of CH_4 and CD_4 . With increasing concentration of CD_4 the transition point increases monotonically, and a straight line is obtained which connects the transition point of the pure CH_4 with the upper transition point of the pure CD_4 . The lower transition appears for the concentrations of CD_4 higher than 20%, and the corresponding curve runs nearly parallel to the former. The transition points of the crystal of CH_3D coincide nearly with the points on these curves with 25% CD_4 .

Eucken²⁾ derived the rotational part of the specific heat for CH_4 crystal and found that it is about 3.0 cal/Mol above the transition point, showing that the molecules are in the state of free rotation. He also found the corresponding value

* The main part of this work was read at the Meeting of the Phys. Soc. of Japan in Osaka, November, 1947 (Symposium on Phase Transition).

of $3.8 \sim 3.3$ cal/Mol for CD_4 above its upper transition point, and inferred that there is some hindrance for the molecular rotation. However, these results cannot be considered as conclusive, for the theoretical rotational specific heat obtained by Maue³⁾ shows a maximum value of 3.7 cal/Mol at about 25°K for CH_4 and that the classical value of 3 cal/Mol can only be attained above 50°K . Alpert's⁴⁾ proton magnetic resonance experiments seemed at first to show that the transition in CH_4 and the upper transition in CH_3D are the transition from the state of oscillation to the state of nearly free rotation, but a more recent research by himself⁵⁾ indicates the result which favours rather the oscillational state nearly throughout the solid state.

The most recent X-ray structure analysis of CH_4 is that carried out by Schallamach.⁶⁾ Although some anomalous patterns were observed in the transition range, no definite conclusion was drawn from them, and the structure (f.c.c.) seems to remain unchanged.

It is the purpose of the present paper to make some inference to the crystal structures of light and heavy methanes and to compute their zero-point entropies on the basis of the model that each molecule in the crystal is subjected to a field with definite symmetry. We assume that the interaction between neighbouring molecules can be replaced by a mean potential with a definite symmetry. Clusius and Frank¹⁾ took a point of view that the molecular interaction is so strong that it is meaningless to speak of any motion of a molecule independent of the others, and they assumed therefore that the proton or deuteron spins are distributed at random in the crystal. They showed that their measurements of the zero-point entropy can be satisfactorily explained in this way. The point of view here presented may be considered as the opposite approximation to the problem, but it can be shown that the results of the measurements are equally well explained.

Let us inquire somewhat closely into the comparison between the picture presented by Clusius and Frank and that presented here. Consider an eigenstate of the system of molecules in a CH_4 crystal and suppose that the molecules are moving like gears coupled tightly together, or rather better that each molecule is tightly bound to a definite direction with respect to the crystal axes, oscillating with a small amplitude. We may take another eigenstate with the same energy eigenvalue which differs from the former by a symmetry operation of any one molecule. If the coupling between molecules, or the coupling between each molecule and the crystalline field, is not infinitely strong, then there will be a resonance between these two states. Clusius and Frank's hypothesis is equivalent to assuming that the energy associated with this resonance is not large compared with the nuclear spin-spin coupling energy between neighbouring protons. The latter is of the order of 10^{-22} erg per proton pair, so that the resonance frequency must not be greater than $10^{-22}/h \sim 10^4 \text{ sec}^{-1}$, in order that the assumption of Clusius and Frank is correct. The nuclear magnetic resonance data show, however,

that this frequency ($1/\tau_0$) is⁷⁾ of the order of $10^6 \sim 10^7 \text{ sec}^{-1}$, which is surely greater. We are thus lead to conclude that the eigenstates of the molecular motions are independent of the spin configuration. Every eigenfunction of the space coordinates alone, describing the molecular motions, must have a certain symmetry character of the tetrahedral group with respect to the symmetry operations of any one molecule in the crystal, and, by the Pauli principle for the protons, the spin function of any one molecule must have the corresponding symmetry character. This is the point of view we are adopting here.

§ 2. On the crystal structure of methane crystal at low temperatures

The X-ray analysis of the CH_4 crystal shows that it has a f. c. c. lattice both above and below the transition point. In the lower modification, it may be supposed that the molecules are arranged in a most regular way. There are two possible space groups for it. One is $F\bar{4}3m$ which corresponds to the parallel arrangement of all the molecules with all their C-H arms directed towards the body diagonals. The other is $P2_13$ where each of the four molecules in the unit cell has one of its C-H arms directed to one of the four diagonals $[111]$, $[\bar{1}\bar{1}\bar{1}]$, $[\bar{1}11]$, $[1\bar{1}\bar{1}]$ (different for the four molecules), and its remaining three C-H arms directed as nearly as possible to the three cubic axes. The distances between atoms in the neighbouring molecules and the numbers of the corresponding pairs of atoms in the unit cell are listed in Table 1.

Table 1. The atomic distances between neighbouring molecules. The figures in parentheses give the numbers of the corresponding pairs in the unit cell. The lattice constant is taken to be 5.84 Å. for the cubic lattices. For the hexagonal lattices the nearest carbon-carbon distance in the basal plane is taken to be equal to that in the cubic lattices, and the axial ratio c/a is taken to be $2\sqrt{2/3} (1-u)$.

Structure	$F\bar{4}3m$	$P2_13$	hexagonal ($u=0$)	hexagonal ($u=0.05$)
H, H distance	2.35 (24)	2.56 (12) 2.69 (12) 3.06 (12) 3.28 (12) 3.58 (12) 3.63 (12)	2.35 (6) 2.82 (12)	2.35 (6) 2.71 (12)
	3.58 (96)			
C, H distance	3.30 (48)	3.30 (36) 3.65 (24) 3.98 (12) 4.27 (24) 4.55 (12)	3.30 (12) 3.65 (6)	3.24 (12) 3.41 (6)
	4.27 (96)			
C, C distance		4.13	4.13 4.13	4.09 4.13

If we take 1.2 Å. for the van der Waals radius of H, the nearest H,H dis-

tance in $F\bar{4}3m$, 2.35 (24), is nearly equal to twice this radius. The nearest H,H distance in $P2_13$, 2.56 (12), is somewhat larger, but there are abundant next-nearest distances in this structure. As to the C,H distances, if we take for the carbon radius 2.0A. (the van der Waals radius of the CH_3 group), we find that the nearest C,H distance in $F\bar{4}3m$, 3.30 (48), is nearly equal to $2.0 + 1.2 = 3.2$. The same distance is found in $P2_13$ with 36 pairs. We are here assuming that the molecules are at rest, but actually they are oscillating, and the atomic distances are distributed continuously. It is not possible to draw a definite conclusion from the consideration above of the atomic distances alone which of the two structures is more stable, but it appears to the writer that the structure $F\bar{4}3m$ is more stable.

A possible structure of the lowest modification of heavy methanes, CH_3D and CD_4 , is a hexagonal close packing. The best packing condition is obtained by arranging the molecules as shown in Fig. 1, where the molecules in each of the basal planes are parallel to one another, but the orientations in the successive two layers are different through a rotation of 180° about the hexagonal axis. A small contraction along the hexagonal axis may be possible compared with the closest-packing structure.

Each molecule in the structure $F\bar{4}3m$ moves in a field of tetrahedral symmetry, and in other structures here considered, it moves in a field of trigonal symmetry. In the highest modification, the field must be of octahedral symmetry in the statistical average.

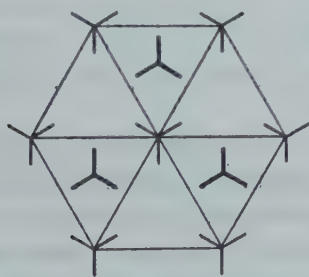


Fig. 1.

Projection of two successive net planes upon one of them.

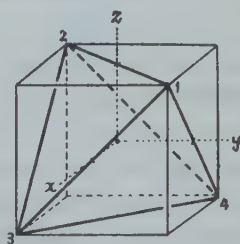


Fig. 2.

§ 3. The lowest eigenstates of CH_4 and CD_4 in a field of various symmetries

We assume in the following that each molecule in the crystal is subjected to a field with a definite symmetry and moves each independent of others. This approximation may be valid near absolute zero where the amplitude of the molecular oscillation is small.

The eigenfunction describing the motion of any one molecule can be classified according to the irreducible representations of the tetrahedral group for CH_4 and

CD₄. As the starting approximation, let us take an eigenfunction which describes the lowest oscillational state of a molecule about one of its equilibrium positions and 11 other eigenfunctions which are obtained from the first by the symmetry operations of the molecule. We shall denote the first by $\psi(E)$, and others by

$$\begin{aligned} \psi(C_x), \psi(C_y), \psi(C_z), \psi(C_1), \psi(C_2), \psi(C_3), \psi(C_4), \\ \psi(C_1^2), \psi(C_2^2), \psi(C_3^2), \psi(C_4^2), \end{aligned} \quad (1)$$

where C_x means the rotation through π about the x -axis (see Fig. 2), C_1 the rotation through $2\pi/3$ about the line joining the carbon atom and the first hydrogen atom, C_1^2 the double this operation, and so on. By linear combinations of these functions we have the following irreducible eigenfunctions:

$$\text{Total symmetric: } \psi_1 = \sum_C \psi(C), \quad (2)$$

where C takes E and those appearing in (1).

Doubly degenerate:

$$\begin{aligned} \psi_2 = \psi(E) + \sum_{x,y,z} \psi(C) + \epsilon^2 \sum_{1,2,3,4} \psi(C_1) + \epsilon \sum_{1,2,3,4} \psi(C_1^2), \\ (\epsilon = e^{2\pi i/3}) \end{aligned} \quad (3)$$

ψ_2^* = the conjugate complex of ψ_2 ,

$$\text{Three sets of triply degenerate functions: } \psi_{ij} = \sum_C u_{ij}(C^{-1}) \psi(C), \quad (4)$$

where $(u_{ij}(C))$ is the three-dimensional unitary irreducible representation of the tetrahedral group and will be given in Appendix. It can be shown that the functions with the same i but with different j 's mix with each other by the symmetry operations of the group (Appendix).

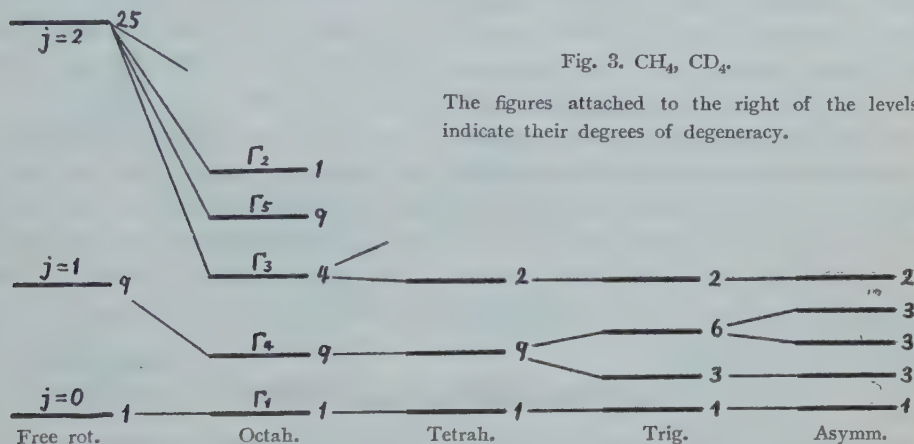
If the symmetry of the field is tetrahedral and the minimum of the potential energy corresponds to the coincidence of the molecular symmetry axes with the symmetry axes of the field, then it can be shown that the functions with the same i but with different j 's mix with each other by the symmetry operations of the field, so that these functions form a nine-fold degenerate state. In a trigonal field, this level splits into a three-fold level and a six-fold level, if the minimum position corresponds to the coincidence of the trigonal axis of the field with one of the trigonal axes of the tetrahedron, and in an asymmetric field it splits into three three-fold levels.

In a field of octahedral symmetry, again assuming that the minimum position corresponds to the coincidence of the symmetry axes of the tetrahedron with those of the field, we have two sets of singly and nine-fold degenerate levels and one four-fold degenerated level. All the mathematical details will be postponed to Appendix. In Fig. 3 is shown an example of the level scheme for various symmetries of the field.

§ 4. The zero-point entropy of CH₄ and CD₄

We can now compute the zero-point entropies of CH_4 and CD_4 . We distinguish between two cases: 1. All the energy levels computed above are very close to each other compared with the lowest kT of the measurements, and 2. their separations are very large compared with it. As cited before, the nuclear resonance data show a flipping frequency ($1/\tau_0$) of the order of $10^6 \sim 10^7 \text{ sec}^{-1}$ which corresponds to level splittings of the order of 10^{-20} erg , or the temperature of the order of $10^{-4} \text{ deg. abs.}$ If this is truly the case, the case 1 is valid, and the degeneracy of the levels is almost perfect.

There are three types of CH_4 molecule, namely, those with resultant spin 2, 1, 0. Their spin functions belong to the three irreducible representations of the tetrahedral group of dimensions 1, 3, 2. The spin weights are thus $5 \times 1 = 5$, $3 \times 3 = 9$, $1 \times 2 = 2$, respectively. In normal methane they are contained with the



ratio 5:9:2, the most abundant being called ortho, the next meta, and the least abundant para. As Maue³⁾ has shown, each type couples only with the molecular motion of the same irreducible representation, and the total weight of a given level is equal to its spatial degeneracy for ortho and para, and multiplied by 5 for meta.

We shall at first consider the first case cited above.

a) Octahedral field. Weights for meta, ortho, para are $2 \times 5 = 10$, 18, 4, respectively. The abundance ratio is thus unchanged whether the conversion between different types occurs or it does not occur, but the zero-point entropy is greater by $R \ln 2$ compared with that measured.

b) Tetrahedral or lower field. Weights for meta, ortho, para are $1 \times 5 = 5$, 9, 2 respectively. Therefore the zero-point entropy is just equal to the measured value whether or not the conversion occurs.

We next go to the second case, and assume that conversion does not occur.

For, if it occurs, the molecule will all drop to the non-degenerate lowest level of total symmetry and becomes meta at sufficiently low temperatures.

a) Octahedral field. The spatial degeneracies of the lowest states for meta, ortho, and para molecules are 1, 9, and 4, respectively, so that the total weights are 5, 9, and 4. The zero-point entropy is again greater than the measured value by $(2/16) \cdot R \ln 2$.

b) Tetrahedral field. The spatial degeneracies are 1, 9, 2, so that just the required value is obtained.

c) Trigonal field. The spatial degeneracies are 1, 3 or 6, and 2, so that the zero-point entropy is smaller than the required value by $(9/16) \cdot R \ln (9/3)$ or $(9/16) \cdot R \ln (9/6)$. The former value applies also to the case of an asymmetric field.

We are thus lead to conclude that the symmetry of the field is tetrahedral or lower and the ground levels are spaced very closely, or, alternatively, that they are largely spaced, the field is of tetrahedral symmetry, and the conversion between different types of the spin symmetry does not occur. The former is more probable in view of nuclear magnetic resonance experiments.

Similar considerations can be carried out for CD_4 , and the same conclusions can be drawn, with minute differences that the excess in entropy in the case of second a) is $(12/81) \cdot R \ln 2$, and the deficiency of it in the case of second c) is $(54/81) \cdot R \ln 3$ or $(54/81) \cdot R \ln (3/2)$. The quasi-degeneracy of the lowest levels for CD_4 may be more valid because of its larger moment of inertia. However Clusius does not seem to have given explicitly the value of the zero-point entropy for this methane.

§ 5. The zero-point entropy of CH_3D

The lowest levels for this molecule in a field of definite symmetry can be treated in a similar way as for CH_4 , the mathematical detail of which will be given in Appendix.

Clusius obtained for this methane an excess zero-point entropy of $R \ln 4$ over that one would expect from the random distribution of spins. He explained this by pointing out that D occupies every corner of the molecular tetrahedron in random way. Similar explanation can be given from our point of view.

There are two types of spin configuration for this molecule; one is invariant against the rotation about the trigonal axis of the molecule, and the other has the degenerate character 1, ϵ , ϵ^2 or 1 ϵ^2 , ϵ . Normal gas contains them with a ratio of 1:1, because the proton spin weight of the former (quartet) is 4 and that of the latter (doublet) is also $2 \times 2 = 4$. The former couples with the total symmetric modes of molecular motion, and the latter with the degenerate modes. The total weight of a level is equal to its spatial degeneracy multiplied by 4 for the former type, and equal to that multiplied by 2 for the latter type, as was shown by Maue.

In a field of tetrahedral symmetry or of lower symmetry, we have 12 lowest levels, and if these are considered as quasi-degenerate, we have 4 levels for the first type of methane and 8 levels for the second. The zero-point entropy is therefore

$$R \left[\frac{1}{2} \ln(4 \times 4) + \frac{1}{2} \ln(8 \times 2) - \frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln \frac{1}{2} \right] = R \ln 32$$

(the last two terms in the bracket is the mixing entropy), namely, greater by $R \ln 4$ than that required from the random distribution of proton spins. Here again is immaterial whether the conversion between different types occurs or it does not.

If the level distances are large and the non-occurrence of the conversion is assumed, we have too small a value of the zero-point entropy even for the case of free rotation, as one would see from the level scheme shown in Fig. 4.

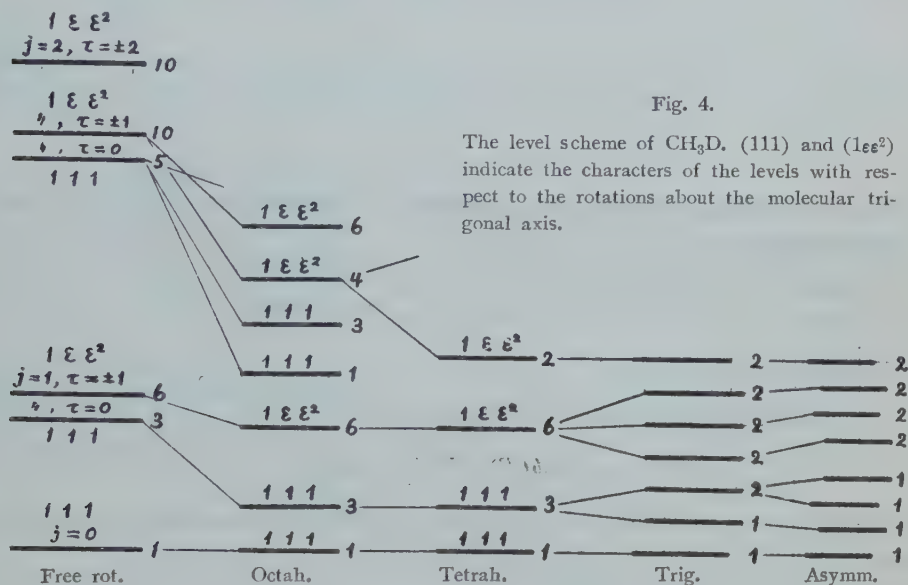


Fig. 4.

The level scheme of CH₃D. (111) and (1εε²) indicate the characters of the levels with respect to the rotations about the molecular trigonal axis.

§ 6. Discussion

The present consideration would appear as not very much different from that given by Clusius and Frank. But the following point must be pointed out. In their point of view the spins are distributed at random in the crystal, and as its consequence the vapours of methanes in equilibrium with their corresponding crystals at low temperatures would contain the greatest amount of the most stable type of the molecule (meta type in the case of CH₄), because the conversion from one type of molecule to another can take place through the crystals. In our point of view of the quasi-degenerate levels the occurrence of the conversion is not necessary, so that the vapour can contain different types in the normal ratio. The problem of the conversion in the crystals of methanes requires another analysis.

Appendix

1. The irreducible representations of the tetrahedral group

The Table 2 gives the simple characters and the following is the three-dimensional unitary irreducible representation :

$$\begin{array}{cccc}
 E & C_x & C_y & C_z \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon^2 - 2\epsilon \\ -2\epsilon & -1 & 2\epsilon^2 \\ -2\epsilon^2 & 2\epsilon & -1 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon - 2\epsilon^2 \\ -2\epsilon^2 & -1 & 2\epsilon \\ -2\epsilon & 2\epsilon^2 - 1 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2 & -2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix} \\
 C_1 & C_2 & C_3 & C_4 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon^2 - 2\epsilon \\ -2\epsilon^2 & -\epsilon & 2 \\ -2\epsilon & 2 & -\epsilon^2 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon - 2\epsilon^2 \\ -2 & -\epsilon & 2\epsilon^2 \\ -2 & 2\epsilon & -\epsilon^2 \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2 & -2 \\ -2\epsilon & -\epsilon & 2\epsilon \\ -2\epsilon^2 & 2\epsilon^2 & -\epsilon^2 \end{pmatrix} \\
 C_1^2 & C_2^2 & C_3^2 & C_4^2 \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon - 2\epsilon^2 \\ -2\epsilon & -\epsilon^2 & 2 \\ -2\epsilon^2 & 2 & -\epsilon \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2 & -2 \\ -2\epsilon^2 & -\epsilon^2 & 2\epsilon^2 \\ -2\epsilon & 2\epsilon & -\epsilon \end{pmatrix} & \frac{1}{3} \begin{pmatrix} -1 & -2\epsilon^2 - 2\epsilon \\ -2 & -\epsilon^2 & 2\epsilon \\ -2 & 2\epsilon^2 & -\epsilon \end{pmatrix}
 \end{array}$$

2. The transformation properties of the eigenfunctions

Operation of a group element A on (4), § 3 gives

$$\begin{aligned}
 A\psi_{ij} &= \sum_C u_{ij}(C^{-1}) \psi(AC) \\
 &= \sum_B u_{ij}(B^{-1}A) \psi(B) \quad (B=AC) \\
 &= \sum_B \sum_k u_{ik}(B^{-1}) u_{kj}(A) \psi(B) = \sum_k \psi_{ik} u_{kj}(A),
 \end{aligned}$$

and successive operations of two elements give the corresponding product of the matrices.

3. The energy eigenvalues (CH_4 , CD_4)

We first construct the energy matrix elements from 12 functions, $\psi(E)$ and 11 functions given by (1), § 3:

$$\begin{aligned}
 \int \psi(E) H \psi(E) d\tau &= \alpha, \\
 \int \psi(C_x) H \psi(E) d\tau &= \beta_x, \quad \int \psi(C_y) H \psi(E) d\tau = \beta, \quad \int \psi(C_z) H \psi(E) d\tau = \beta_z, \\
 \int \psi(C_i) H \psi(E) d\tau &= \gamma_i = \int \psi(C_i^2) H \psi(E) d\tau, \quad i=1,2,3,4.
 \end{aligned}$$

Here we assume that all ψ 's are real. The symmetry of the field can be described by certain relations among these parameters. For a tetrahedral field, one has

Table 2.

Class :	E	(C_x)	(C_1)	(C_1^2)
Simple character :	1	1	1	1
	1	1	ϵ	ϵ^2
	1	1	ϵ^2	ϵ
	3	-1	0	0

$$\beta_x = \beta_y = \beta_z \equiv \beta, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma,$$

and for a trigonal field, one has, taking the trigonal axis to coincide with the C_4 -axis when the molecule takes the original position E ,

$$\beta_x = \beta_y = \beta_z \equiv \beta, \quad \gamma_2 = \gamma_3 = \gamma_4 \equiv \gamma.$$

The case of an octahedral field will be treated afterwards.

The Hamiltonian matrix elements with respect to the eigenfunctions given by (4), § 3 can be calculated as follows:

$$\begin{aligned} \int \phi_{ij}^* H \phi_{kl} d\tau &= \sum_A \sum_B u_{ij}^*(A^{-1}) u_{kl}(B^{-1}) \int \phi(A) H \phi(B) d\tau \\ &= \sum_C \sum_B u_{ij}^*(C^{-1} B^{-1}) u_{kl}(B^{-1}) \int \phi(BC) H \phi(B) d\tau \\ &= \sum_C \sum_B u_{ij}^*(C^{-1} B^{-1}) u_{kl}(B^{-1}) H_C, \end{aligned}$$

where

$$H_C = \int \phi(C) H \phi(E) d\tau.$$

Since we are taking a unitary representation, we have

$$\begin{aligned} &\sum_B u_{ij}^*(C^{-1} B^{-1}) u_{kl}(B^{-1}) \\ &= \sum_B u_{ji}^*(BC) u_{kl}(B^{-1}) = \sum_B \sum_m n_m^j(B) u_{mi}(C) u_{kl}(B^{-1}) \\ &= \sum_m \left(\sum_B u_{jm}(B) u_{kl}(B^{-1}) \right) u_{mi}(C). \end{aligned}$$

The sum in the parenthesis vanishes unless $m=k, j=l$, and when these relations hold it is equal to h/f (h =order of the group=12, f =dimension of the irreducible representation=1 or 3).⁸⁾ We thus have

$$\begin{aligned} \int \phi_{ij}^* H \phi_{kl} d\tau &= 0, \quad (j \neq l) \\ \int \phi_{ij}^* H \phi_{kj} d\tau &= \frac{h}{f} \sum_C u_{ki}(C) H_C. \end{aligned}$$

Thus the Hamiltonian matrix elements exist only between those ϕ_{ij} 's which have a common j . For the three-dimensional irreducible representation we have thus to solve a three-dimensional secular equation.

The results for the energy eigenvalues are given in Table if we neglect the overlap integral between any two different $\phi(C)$'s.

Table 3.

ϕ_1 :	ϕ_2, ϕ_3^* :	$\phi_{ij} (i, j=1,2,3)$:
$\alpha + 3\beta + 8\gamma$	$\alpha + 3\beta - 4\gamma$	$\alpha - \beta$ (9) ...tetrahedral
$\alpha + 3\beta + 2\gamma_1 + 6\gamma_2$	$\alpha + 3\beta - \gamma_1 - 3\gamma_2$	$\begin{cases} \alpha - \beta + 2(\gamma_1 - \gamma_2) & (3) \\ \alpha - \beta - (\gamma_1 - \gamma_2) & (6) \end{cases}$...trigonal

The second row refers to a tetrahedral field and the third and the fourth rows refer to a trigonal field. The parenthesized figures in the third column give the degrees of degeneracies.

4. Tetrahedron in an octahedral field

We may here rather change the rôles of the symmetry operations of the molecule and those of the field. Thus C now implies a symmetry operation of the octahedral field. The reduction of the regular representation of the octahedral group yields the following irreducible representations:

$$\Gamma_1 + \Gamma_2 + 2\Gamma_3 + 3\Gamma_4 + 3\Gamma_5,$$

where Bethe's notation is used.⁹⁾ Γ_1 and Γ_2 are one-dimensional, and should be both total-symmetric with respect to the symmetry operations of the molecule. Γ_3 is two-dimensional, and each of the two should contain both the characters $(1, 1, \epsilon, \epsilon^2)$ and $(1, 1, \epsilon^2, \epsilon)$ with respect to the symmetry operations of the tetrahedral subgroup of the field. But these two Γ_3 are degenerate, as one can verify directly, or as follows. The function ϕ_2 , constructed previously, is an eigenfunction with respect to both molecular symmetry and field-symmetry, and we can construct another function ϕ_2^1 by a rotation through $\pi/2$ about one of the cubic axis of the field. Then ϕ_2 and ϕ_2^1 are bases of Γ_3 . This set and its conjugate complex evidently belong to the same level. Γ_4 and Γ_5 are three-dimensional, and both have the same character with respect to the tetrahedral subgroup, so that both must contain the nine-fold degenerate functions constructed previously. Thus we have two non-degenerate levels, one four-fold degenerate level and two nine-fold degenerate levels.

5. CH_3D in a field of various symmetries

Tetrahedral field We have only to take the case of putting a tetrahedron in a trigonal field and change the rôles of the symmetries of the molecule and those of the field. We have the following scheme (Table 4):

Table 4.

Degree of degeneracy	Character with respect to the rotation about a molecular trigonal axis through			Energy eigenvalue
	0°	120°	240°	
1	1	1	1	$a + 3\beta + 2\gamma_1 + 6\gamma_2$
6	1	ϵ, ϵ^2	ϵ^2, ϵ	$a - \beta - \gamma_1 + \gamma_2$
3	1	1	1	$a - \beta + 2(\gamma_1 - \gamma_2)$
2	1	ϵ, ϵ^2	ϵ^2, ϵ	$a + 3\beta - \gamma_1 - 3\gamma_2$

Octohedral field As before, all levels with the degrees of degeneracy 1, 6, 3 are doubled, and the level with degeneracy 2 becomes a level with degeneracy 4.

Trigonal field We distinguish two cases. a) The C—D arm oscillates about the trigonal axis of the field. The level scheme is then as follows (Table 5):

Table 5.

Degree of degeneracy	Character with respect to the rotation about the trigonal axis			Energy eigenvalue
1	1	1	1	$a+2\gamma$
2	1	ϵ, ϵ^2	ϵ^2, ϵ	$a-\gamma_1$

b) One of the C—H arms oscillates near the trigonal axis. Denoting by γ'_1 the H_C for the rotation about the C—D arm, by γ'_2 the H_C for the rotation about the trigonal axis of the field, by γ'_3 the H_C for the combination of these two rotations in the same sense, by γ'_4 that in the opposite sense, and putting $H_E = a'$, we have the following scheme (Table 6):

Table 6.

Degree of degeneracy	Character about the molecular axis			Character about the field axis			Energy eigenvalue
1	1	1	1	1	1	1	$a'+2\gamma'_1+6\gamma'_2'$
2	1	1	1	1	ϵ, ϵ^2	ϵ^2, ϵ	$a'+2\gamma'_1'-\gamma'_2'-\gamma'_3'-\gamma'_4'$
2	1	ϵ, ϵ^2	ϵ^2, ϵ	1	1	1	$a'-\gamma'_1'+2\gamma'_2'-\gamma'_3'-\gamma'_4'$
2	1	ϵ, ϵ^2	ϵ^2, ϵ	1	ϵ, ϵ^2	ϵ^2, ϵ	$a'-\gamma'_1'-\gamma'_2'-\gamma'_3'+2\gamma'_4'$
2	1	ϵ, ϵ^2	ϵ^2, ϵ	1	ϵ^2, ϵ	ϵ, ϵ^2	$a'-\gamma'_1'-\gamma'_2'+2\gamma'_3'-\gamma'_4'$

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Quantum-Statistical Theory of Liquid Helium

Takeo MATSUBARA

Department of Physics, Osaka University

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The lambda transition in liquid helium is investigated on the basis of quantum statistical mechanics. In the first place, the partition function for an assembly of Bose particles is formulated in such a way that the effect of Bose statistics appears as a strong attractive potential between particles and gives rise to molecular clustering of various sizes, so that the assembly can be considered as a system of molecules and atoms obeying Boltzmann statistics. This formulation is then applied to the ideal Bose gas and to the actual helium liquid, and the following results are obtained: (a) An ideal Bose gas undergoes a phase transition similar in its formal property to the condensation of a classical imperfect gas. (b) By taking into account the interatomic potential, but neglecting all the quantum effects but the statistics, we obtain a jump in the specific heat curve at the transition temperature and a value of the transition temperature coming nearer to the actuality. (c) Beside the molecular clustering, or the "Bose-excitation," the quantum effects bring in the "Debye phonons" and the "Landau rotons," the former corresponding to the translational motions of the molecules and the atoms, and the latter to the rotational motions of the molecules. Although the quantum effects are not dealt with in detail, the agreement between theory and experiments is shown to be improved by taking them into account. It is also shown that these effects probably are not of very great importance even near the lambda point.

§ 1. Introduction

The peculiar properties of liquid helium II have hitherto been well explained by phenomenological theories based on two-fluid model, although the molecular-theoretical interpretation of the model itself still remains almost unsolved. Two well-known theories, one due to London and Tisza¹⁾ and the other due to Landau²⁾, have so far appeared in competition, and some of the behaviours of helium II stand in favour of the former but others are more favourable to the latter. Experiments on mixtures of He³ and He⁴ or on pure He³ show that He³ does not take part in superfluidity³⁾ and this gives a support to the London-Tisza theory which starts from the concept of Bose-Einstein condensation, while Landau's theory is incapable of explaining this fact. On the other hand, experiments on the temperature and pressure dependencies of second sound velocity rather support the predictions from Landau's theory⁴⁾, while some theories developed recently along the line of London-Tisza's idea do not seem successful in explaining its behaviours near absolute zero⁵⁾. These circumstances indicate the imperfectness of both theories, and one will be lead to scrutinize their foundations.

Confining ourselves to the properties of liquid helium in thermal equilibrium,

as we shall do in this paper, the following points must thus be expected to result from a correct theory:

a) Helium remains to be liquid at absolute zero because of the light mass of the atom and a weak interatomic force.

b) A phase change occurs at a certain temperature on account of the Bose-Einstein statistics for He^4 ; the lower phase possesses a peculiar character of allowing for the existence of two velocity fields.

c) There exists in the lower phase a quantity which characterizes its internal structure, for instance a parameter x that defines the mixing ratio of the two velocity fields.

We attempt in the present paper to develop a theory of liquid helium on the basis of orthodox quantum-statistical mechanics, with a hope to obtain some clues to the real solution, and we believe that an answer to some of the points cited above is really given by our treatment. We first develop a general method suitable for analyzing the effect of Bose-Einstein statistics for a system composed of a large number of particles (§ 2). This method is then applied to an ideal gas, where we show that we can interpret the Bose-Einstein condensation in terms of space coordinates alone (§ 3). This application provides us a ground to go a step further and treat a system of interacting particles. We first neglect all the quantum-mechanical effects except for the statistics in treating a system of interacting particles, and show that the behaviours of the phase transition comes nearer to actuality by inclusion of the interatomic forces (§ 4). Finally, the quantum effects are taken into consideration, though not perfectly (§ 5). We show there that these effects do not influence essentially the behaviours of the phase transition, but that they rather provide us some possibilities of accounting for the anomalous behaviours found at lower temperatures.

§ 2. General quantum-statistical mechanics of an assembly of Bose particles

Let the Hamiltonian of a system of N Bose particles be H . Then we have the density matrix of the system in coordinate representation in the form

$$R_N(\mathbf{r}_1 \dots \mathbf{r}_N; \mathbf{r}'_1 \dots \mathbf{r}'_N) = \sum_k \Psi_k^*(\mathbf{r}'_1 \dots \mathbf{r}'_N) e^{-\beta H} \Psi_k(\mathbf{r}_1 \dots \mathbf{r}_N), \quad (2.1)$$

where the wave functions $\{\Psi_k(\mathbf{r}_1 \dots \mathbf{r}_N)\}$ are an arbitrary set of orthogonal normalized functions symmetric with respect to the coordinates. For instance, we may construct them from a set of orthogonal normalized functions of one coordinate, $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$, $\phi_3(\mathbf{r})$,, in the following way:

$$\Psi_{n_1 n_2 \dots}(\mathbf{x}) = \frac{1}{\sqrt{N! n_1! n_2! \dots}} \sum_P P \Psi_{n_1 n_2 \dots}(\mathbf{x}), \quad (2.2)$$

where we mean by $\Psi_{n_1 n_2 \dots}(\mathbf{x})$ a product of $n_1 \phi_1, n_2 \phi_2, \dots (n_1 + n_2 + \dots = N)$ with $(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \mathbf{x}$ arranged in their order, that is,

$$\Psi_{n_1 n_2 \dots}(\mathbf{x}) = \underbrace{\phi_1(\mathbf{r}_1) \phi_1(\mathbf{r}_2) \dots \phi_1(\mathbf{r}_{n_1})}_{n_1} \underbrace{\phi_2(\mathbf{r}_{n_1+1}) \dots}_{n_2} \dots,$$

and by P a permutation operated upon the coordinates, the summation being extended over all the permutations. Substituting (2.2) in (2.1) and taking the trace of the density matrix, we have the sum over states:

$$\begin{aligned} Z_N &= \sum_{\{n_k\}} \sum_P \sum_Q \frac{1}{N! n_1! n_2! \dots} \int \dots \int P \Psi_{n_1 n_2 \dots}^*(\mathbf{x}) e^{-\beta H} Q \Psi_{n_1 n_2 \dots}(\mathbf{x}) d\mathbf{x} \\ &= \frac{1}{N!} \sum_{\{n_k\}} \sum_P \frac{N!}{n_1! n_2! \dots} \int \dots \int P \Psi_{n_1 n_2 \dots}^*(\mathbf{x}) e^{-\beta H} \Psi_{n_1 n_2 \dots}(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (2.3)$$

This expression can be written in another way. We may namely start from another set of functions which we obtain from $\Psi_{n_1 n_2 \dots}(\mathbf{x})$ by changing the order of arrangement of the ϕ 's in the product through a definite permutation. Evidently we then have the same result. Since there are just $N!/n_1! n_2! \dots$ essentially different such permutations for a given set n_1, n_2, \dots , (2.3) can be written as

$$Z_N = \frac{1}{N!} \sum_P \sum_k \int \dots \int P \Psi_k^*(\mathbf{x}) e^{-\beta H} \Psi_k(\mathbf{x}) d\mathbf{x}, \quad (2.4)$$

if we imply by $\{\Psi_k(\mathbf{x})\}$ the complete set of functions $\{\phi_{k_1}(\mathbf{r}_1) \phi_{k_2}(\mathbf{r}_2) \dots \phi_{k_N}(\mathbf{r}_N)\}$ with $k_i = 1, 2, 3, \dots (i = 1, 2, \dots, N)$. Furthermore, we have the same value of the sum behind \sum in the above expression when P runs over every element of the same class of the symmetric group S_N . To see it, we take a set k'_1, k'_2, \dots, k'_N which we obtain from k_1, k_2, \dots, k_N by changing the order of arrangement of the latter through a permutation T . It merely means a change of order of the summation over k . But this permutation is equivalent to replacing the integrand by

$$PT^{-1} \Psi_k^*(\mathbf{x}) e^{-\beta H} T^{-1} \Psi_k(\mathbf{x})$$

whose integral is equal to

$$\int \dots \int TPT^{-1} \Psi_k^*(\mathbf{x}) e^{-\beta H} \Psi_k(\mathbf{x}) d\mathbf{x}$$

as one sees by the change of variables $T\mathbf{x} = \mathbf{x}'$.

Therefore, the summation over P can be substituted by a sum over the classes of the symmetric group S_N , namely,

$$Z_N = \sum_{\substack{\{m_i\} \\ \sum_i l m_i = N}} h_{m_1 m_2 \dots m_N} Z(m_1 m_2 \dots m_N) / N!, \quad (2.5)$$

$$h_{m_1 m_2 \dots m_N} = \frac{N!}{\prod_{i=1}^N m_i! \, l^{m_i}}, \quad (2.6)$$

$$Z(m_1 m_2 \dots m_N) = \sum_k \int \dots \int P \Psi_k^*(x) e^{-\beta H} \Psi_k(x) dx, \quad (2.7)$$

P = any element of the class $(m_1 m_2 \dots m_N)$.

Here (2.6) is the number of elements in the class characterized by m_1, m_2, \dots, m_N , where m_1 is the number of cycles of length 1 in the permutation, m_2 that of length 2, and so on, so that $\sum_i l m_i = N$.

The expression here obtained is especially suited for investigating the effect of the statistics upon the thermal properties of the assembly because $\Psi_k(x)$'s are not the symmetrized functions. Indeed, the term which corresponds to the identical permutation,

$$\frac{1}{N!} Z(N, 0, \dots, 0) = \frac{1}{N!} \sum_k \int \dots \int \Psi_k(x) e^{-\beta H} \Psi_k(x) dx, \quad (2.8)$$

is nothing but the sum over states (divided by $N!$) for an assembly of particles which obey Boltzmann statistics, and should be expected to be dominant at high temperatures, while other terms represent corrections due to Bose statistics. In the next section, we shall see by applying (2.5) to the case of ideal gas that our division into the classes of S_N bears a close relationship to the Bose condensation. Moreover, since the present formalism is quite general, it is possible to extend our computations to the case of interacting particles, and at the same time to criticize how far reaching is the application of the ideal gas model to liquid helium.

§ 3. The condensation of ideal Bose gas

In the case of an ideal gas, the Hamiltonian is a sum of commuting operators:

$$H = \frac{1}{2m} \sum_{s=1}^N p_s^2. \quad (3.1)$$

In such a case, we can obtain the sum over states in a compact way, and indeed exactly in the case of an ideal gas. Writing the density matrix of one particle as

$$\rho(r; r' | \beta) = \sum \phi_k^*(r') e^{-\beta p^2/2m} \phi_k(r), \quad (3.2)$$

we have straightforwardly

$$Z(m_1 m_2 \dots m_N) = \prod_{i=1}^N (I_i)^{m_i}, \quad (3.3a)$$

$$I_i = \int \dots \int \rho(\mathbf{r}_1 \mathbf{r}_2 | \beta) \rho(\mathbf{r}_2 \mathbf{r}_3 | \beta) \dots \rho(\mathbf{r}_i \mathbf{r}_1 | \beta) d\mathbf{r}_1 \dots d\mathbf{r}_i \\ = \int \rho(\mathbf{r} \mathbf{r} | l\beta) d\mathbf{r} \quad (3.3b)$$

($l\beta = l$ times β). Now we define "cluster integrals b_i " by

$$I_i = V l b_i. \quad (V = \text{volume of the assembly}) \quad (3.4)$$

Then we have

$$Z_N = \sum_{\substack{m_i \\ \sum l m_i = N}} \prod_{l=1}^N \frac{(V l b_l)^{m_l}}{m_l!} \quad (3.5)$$

which is the familiar form obtained in the case of a classical imperfect gas.⁶⁾

Taking plane waves for $\psi_k(\mathbf{r})$'s, we obtain after a simple calculation the following expression of b_l for sufficiently large volume V :

$$\left. \begin{aligned} b_l &= \frac{1}{l^{5/2} \lambda^3} + \frac{1}{lV} \equiv b_{0l} + \frac{1}{lV}, \\ \lambda^2 &= \frac{\hbar^2}{2\pi m k T}. \end{aligned} \right\} \quad (3.6)$$

The term $1/lV$ is contributed from the state of zero momentum; it is negligible at high temperatures or for large volume, but becomes important when condensation begins. Inserting (3.6) into (3.5), we have

$$Z_N = \sum_{\substack{N_0=0 \\ \sum l m_l = N - N_0}}^N \left[\sum_{\{m_l\}} \sum_{\{\mu_l\}} \left\{ \prod_l \frac{(V l b_{0l})^{m_l}}{m_l!} \prod_l \frac{1}{\mu_l! l^{\mu_l}} \right\} \right] \quad (3.7)$$

$$= \sum_{N_0=0}^N \left\{ \sum_{\{m_l\}} \prod_{l=1}^{N-N_0} \frac{(V l b_{0l})^{m_l}}{m_l!} \right\} \left(\because \sum_{\{\mu_l\}} \prod_l \frac{1}{\mu_l! l^{\mu_l}} = 1 \right). \quad (3.7')$$

Obviously N_0 is the number of particles in the lowest state of zero momentum. We now adopt the approximation of replacing $\log Z_N$ by the logarithm of the maximum term of the right-hand side of (3.7'). The logarithm of the maximum term for a given N_0 is

$$\log Z(N_0) = (N - N_0)a + V \sum_l b_{0l} e^{-l\alpha}, \quad (3.8)$$

where a is determined from

$$V \sum_l l b_{0l} e^{-l\alpha} = N - N_0. \quad (3.9)$$

Next, differentiating $\log Z(N_0)$ with respect to N_0 , we have $a=0$. Putting $a=0$ in (3.9), we have the equation which determines N_0 :

$$V \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \sum_{l=1}^{\infty} \frac{1}{l^{3/2}} = N - N_0. \quad (3.10)$$

Since, however, N_0 cannot be negative, $a=0$ is not allowed for temperatures higher than T_0 determined by

$$V \left(\frac{2\pi mkT_0}{h^2} \right)^{3/2} \sum_{l=0}^{\infty} \frac{1}{l^{3/2}} = N. \quad (3.11)$$

For such temperatures $N_0=0$, $a>0$, and a is determined by

$$V \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \sum_{l=0}^{\infty} \frac{e^{-la}}{l^{3/2}} = N. \quad (3.12)$$

Therefore, we have two phases below and above T_0 :

$$\begin{aligned} T < T_0: \quad N_0 \neq 0, \quad a=0, \\ T > T_0: \quad N_0=0, \quad a>0. \end{aligned} \quad (3.13)$$

This is the well-known Bose condensation. We shall show in the next section that the method of derivation here presented can be followed almost without modification in the case of interacting particles.

§ 4. Theory of the Bose liquid, taking into account only the effect of the statistics

As the first step to the study of the influence of interatomic forces on the condensation, we shall neglect all the quantum-mechanical effects except for the statistics. Such a treatment is interesting because it provides us an insight to see whether the Bose statistics is essential for the phase transition in liquid helium or not. Indeed, we shall see below that the knick in the specific heat curve in the case of an ideal gas changes to a discontinuity by inclusion of the interatomic forces.

The neglect of the quantum effect is equivalent to treating the momentum operators to commute with the potential energy. We take the Hamiltonian as

$$H = \frac{1}{2m} \sum_{s=1}^N \mathbf{p}_s^2 + \sum_{s>t} \varphi(r_{st}) = \frac{1}{2m} \sum_s \mathbf{p}_s^2 + \phi(\mathbf{r}_1, \dots, \mathbf{r}_N), \quad (4.1)$$

where $\varphi(r_{st})$ is the potential energy between the particles s and t . Taking for $\{\Psi_k(\mathbf{x})\}$ again the set of plane waves

$$\Psi_{\mathbf{p}_1 \dots \mathbf{p}_N}(\mathbf{x}) = \frac{1}{V^{N/2}} \exp \left[\frac{i}{\hbar} (\mathbf{p}_1 \cdot \mathbf{r}_1 + \dots + \mathbf{p}_N \cdot \mathbf{r}_N) \right], \quad (4.2)$$

the sum over k is replaced approximately by an integral over $\mathbf{p}_1 \dots \mathbf{p}_N$:

$$\sum_k \rightarrow \frac{V^N}{h^{3N}} \int_{-\infty}^{\infty} \dots \int d\mathbf{p}_1 \dots d\mathbf{p}_N,$$

and (2.5) becomes

$$Z_N = \sum_{\substack{\{m_l\} \\ \sum l m_l = N}} \left(\prod_{l=1}^N \frac{1}{m_l! l^{m_l}} \right) Z(m_1 m_2 \dots m_N), \quad (4.3)$$

with

$$Z(m_1 m_2 \dots m_N) = \frac{1}{\lambda^{3N}} \int \dots \int_{(V)} e^{-\beta \Phi(\mathbf{r}_1 \dots \mathbf{r}_N) - \frac{\pi}{\lambda^2} \sum_{s=1}^N (\mathbf{r}_s - \mathbf{r}_{s'})^2} d\mathbf{r}_1 \dots d\mathbf{r}_N, \quad (4.4)$$

where we understand by s' the index of coordinate which we have from s through a permutation P in the class $(m_1 \dots m_N)$. Thus, m_1 terms in $\sum_{s=1}^N (\mathbf{r}_s - \mathbf{r}_{s'})^2$ vanish, m_2 pairs of terms appear in closed form like $r_{12}^2 + r_{21}^2$, m_3 sets of three terms appear in closed form like $r_{34}^2 + r_{45}^2 + r_{53}^2$, and so on. The permutation P introduces, therefore, an apparent potential energy of the form

$$\frac{\pi}{\lambda^2 \beta} [r_{12}^2 + r_{23}^2 + \dots + r_{l-1l}^2 + r_{l1}^2], \quad (l=2, 3, \dots, N) \quad (4.5)$$

and this gives a tendency for m_l sets of l atoms to form m_l "molecules," with $l=2, 3, \dots, N$.

We now introduce an interatomic potential of the form

$$\varphi(r) = \epsilon \left\{ \left(\frac{a}{r} \right)^{12} - 2 \left(\frac{a}{r} \right)^6 \right\}. \quad (4.6)$$

Then the apparent potential for a pair of atoms which are contained in a molecule is

$$\phi(r) = \epsilon \left\{ \left(\frac{a}{r} \right)^{12} - 2 \left(\frac{a}{r} \right)^6 \right\} + \frac{\pi}{\lambda^2 \beta} r^2. \quad (4.7)$$

$\phi(r)$ depends on temperature. The parameters ϵ and a will be chosen as

$$\epsilon = 0.827 \times 10^{-15} \text{ erg}, \quad a = 2.95 \times 10^{-8} \text{ cm}. \quad (4.8)$$

($m = 6.64 \times 10^{-24}$ gr.) The corresponding curve of $\phi(r)$ for $T = 2.19^\circ \text{K} = T_\lambda$ is shown in Fig. 1.

It will be seen that the minimum of $\phi(r)$ is very steep, and is even steeper for higher temperatures. We may expand $\phi(r)$ near its minimum:

$$\beta\phi(r) = \beta\Delta + \frac{\gamma}{2} \left(\frac{r-r_0}{a} \right)^2 + \dots, \quad (4.9)$$

with

$$\frac{1}{\sqrt{\gamma}} \approx \left(\frac{\lambda^2}{28\pi a^2} \right)^{3/2} = 0.21 \left(\frac{T_\lambda}{T} \right)^{1/2}. \quad (4.10)$$

The breadth of the minimum, $a/\sqrt{\gamma}$, is very small compared with a even for T_λ ; it increases with decreasing temperature.

The analysis given above suggests us the following model of the internal structure of liquid helium: each term $Z(m_1 \dots m_N)$ in Z_N corresponds to an assembly of strongly bonded m_2 diatomic molecules, m_3 triatomic molecules, and so on, interacting with each other by a weak van der Waals type attraction. Taking the molecular binding energy equal to $l\Delta$, as given by (4.9), for an l -atomic molecule, we may therefore expect for $Z(m_1 \dots m_N)$ the following form:

$$Z(m_1 \dots m_N) \sim \Pi \left\{ \left(\frac{2\pi M_l kT}{h^2} \right)^{3/2} e^{-\beta l \Delta} \right\}^{m_l} \Omega(m_1 \dots m_N), \quad (4.11)$$

where M_l is the effective mass of an l -atomic molecule, $\Omega(m_1 \dots m_N)$ is the configurational partition function of a liquid consisting of atoms and molecules indicated by $(m_1 \dots m_N)$. The evaluation of the latter is difficult, although it may be handled with the aid of the theories of multi-component solutions, and we shall not enter into its analysis.

Actually, $Z(m_1 \dots m_N)$ can be calculated a little more exactly. When carrying out the integration with respect to l coordinates $r_1 \dots r_l$ forming a molecule, we notice that the integrand depends on $r_1 \dots r_l$ very slowly except through the factor

$$e^{-\beta[\phi(r_{12}) + \phi(r_{23}) + \dots + \phi(r_{l1})]}, \quad (4.12)$$

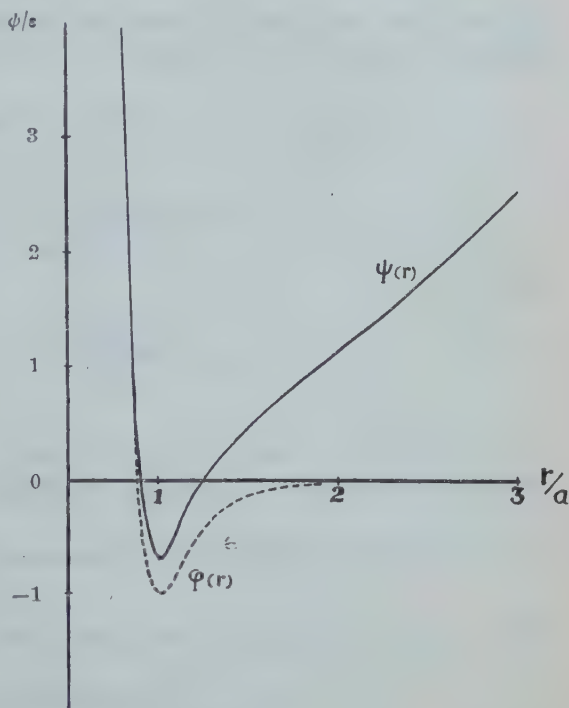


Fig. 1. The interatomic potential $\phi(r)$, and the apparent interatomic potential $\psi(r)$ for $T_\lambda = 2.19^\circ \text{K}$.

so that other part than this factor can be put outside the integral. Using the Fourier integral theorem, we have

$$\begin{aligned} b_i &= \left(\frac{1}{\lambda}\right)^{3i} \frac{1}{l} \int \dots \int e^{-\beta[\psi(r_{12}) + \psi(r_{23}) + \dots + \psi(r_{i1})]} dr_2 \dots dr_i \\ &= \frac{1}{l} \frac{1}{2\pi^2} \int_0^\infty t^2 [G(t)]^i dt, \end{aligned} \quad (4.13)$$

where

$$G(t) = \frac{4\pi}{\lambda^3} \int_0^\infty r^2 e^{-\beta\psi(r)} \frac{\sin tr}{tr} dr. \quad (4.14)$$

Thus b_i depends no more on r_1 . We now use the expansion (4.9) and carry out the integration by the method of steepest descent. Then

$$G(t) = 2 \left(\frac{2\pi a^2}{\gamma \lambda^2} \right)^{3/2} e^{-\beta\Delta} \left\{ \cos r_0 t + \gamma \left(\frac{r_0}{a} \right) \frac{\sin r_0 t}{at} \right\} e^{-(\alpha^2 t^2 / 2\gamma)},$$

and b_i can be written in the following form:

$$b_i = \frac{e^{-i\beta\Delta}}{\lambda^3 l^{5/2}} A_i = \left(\frac{2\pi M_i kT}{h^2} \right)^{3/2} \frac{e^{-i\beta\Delta}}{l^{5/2}}, \quad (4.15)$$

where

$$\left(\frac{M_i}{m} \right)^{3/2} = A_i = 2 \left(\frac{2\pi a^2}{\gamma \lambda^2} \right)^{3/2} (l^{-1}) \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^2 \left[\cos \sigma \xi + l \sigma \frac{\sin \sigma \xi}{\xi} \right]^i e^{-(1/2)\xi^2} d\xi, \quad (4.16)$$

$$l \sigma^2 = \gamma \left(\frac{r_0}{a} \right)^2. \quad (4.17)$$

Carrying out such an approximate integration successively for every molecule, we finally arrive at an integral which contain $m_1 + m_2 + \dots + m_N$ coordinates, one per each molecule, as integration variables. This integral will be denoted as $\Omega(m_1 \dots m_N)$ and will not be analyzed further. In any way, we have

$$Z_N = \sum_{\sum l m_l = N} \frac{b_1^{m_1} b_2^{m_2} \dots}{m_1! m_2! \dots} \Omega(m_1 \dots m_N). \quad (4.18)$$

The most important result here obtained is that, as the consequence of the Bose statistics, the atoms are excited to form molecules whose energies are higher by Δ per atom consisting the molecule than the energy of atoms not forming molecules. It has already been shown by Bijl, de Boer, Michels,⁷⁾ and Toda⁸⁾ that in such a case we have a strong dependence of entropy on temperature and a discontinuity in the specific heat curve.

To simplify our calculations, we now neglect the interactions between molecules and between molecules and atoms, and we assume a constant mean potential energy ϵ_0 for atoms.

Corresponding with (4.18) the sum over states Z_{N-N_0} for a given number of zero atoms N_0 can be written as

$$Z_{N-N_0} = \sum_{\substack{m_l \\ \sum l m_l = N - N_0}} \left\{ \prod_{l=1}^{N-N_0} \frac{(V b_l)^{m_l}}{m_l!} \right\} e^{-N_0 \beta \epsilon_0}, \quad (4.19)$$

$$b_1 = \left(\frac{2\pi m k T}{h^2} \right)^{3/2} e^{-\beta \epsilon_0}, \quad b_l = \left(\frac{2\pi M_l k T}{h^2} \right)^{3/2} \frac{e^{-l^3 \Delta}}{l^{5/2}} \quad (l \geq 2), \quad (4.20)$$

and the total sum over states is given by

$$Z_N = \sum_{N_0=0}^N Z_{N-N_0}. \quad (4.21)$$

As in § 3 we shall adopt the approximation of replacing $\log Z_N$ by the maximum term of (4.19). For a fixed N_0 we have

$$\begin{aligned} \log Z_{N-N_0} &= N a - N_0 (a + \beta \epsilon_0) + V \sum_l b_l e^{-l a}, \\ V \sum_l l b_l e^{-l a} &= N - N_0, \end{aligned}$$

and taking the maximum with respect to N_0 ,

$$a' = a + \beta \epsilon_0 = 0,$$

so that, putting $\Delta - \epsilon_0 = \Delta'$, we obtain the following result:

For $T < T_c$:

$$V \left(\frac{2\pi m k T}{h^2} \right)^{3/2} \left\{ 1 + \sum_{l=2}^{\infty} \frac{e^{-l^3 \Delta'} A_l}{l^{2/2}} \right\} = N - N_0. \quad (4.22)$$

For $T > T_c$: $N_0 = 0$, and

$$V \left(\frac{2\pi m k T}{h^2} \right)^{3/2} \left\{ e^{-a'} + \sum_{l=2}^{\infty} \frac{e^{-l a' - l^3 \Delta'} A_l}{l^{3/2}} \right\} = N. \quad (4.23)$$

The transition point is determined from Eq. (4.23) by putting $a' = 0$. Formulas for the free energy, the entropy,⁶ the specific heat, etc. can be given easily, and they are of the similar forms as those given by Toda,⁸⁾ although the calculations in the present case are more complicated owing to the dependencies of A_l and Δ' on temperature. In Fig. 2 and 3 are shown the curves of $x = (N - N_0)/N$ and of the molar heat as functions of temperature, where we assumed (4.6) and (4.8) for the interatomic potential, the actual volume of liquid helium for $V/N = v$, and $-\epsilon$ for ϵ_0 . We obtained the value of the transition temperature as 2.9°K,

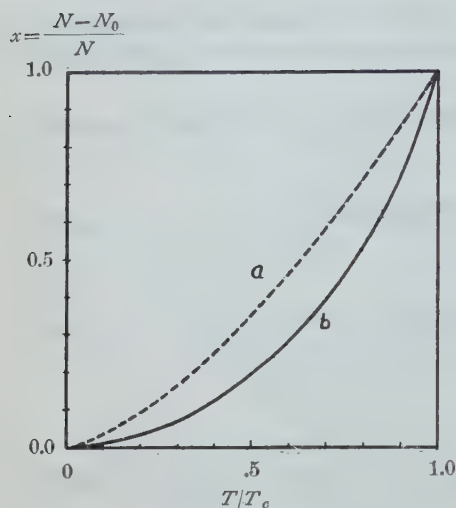


Fig. 2. Number of moving molecules as a function of temperature.

Case a: ideal Bose gas.
($T_c = 3.14^\circ\text{K}$)

Case b: Bose liquid.
($T_c = 2.9^\circ\text{K}$)

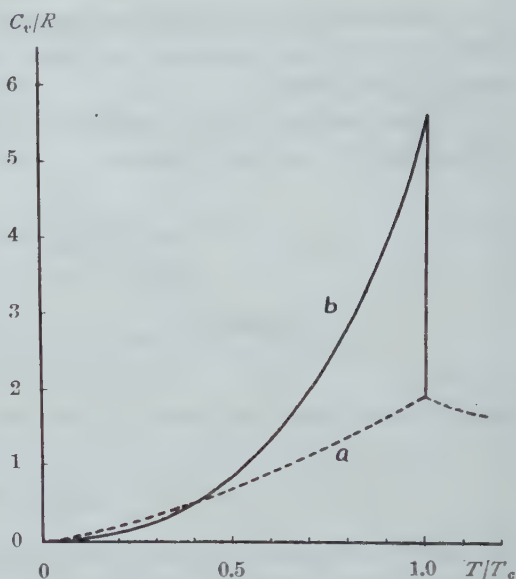


Fig. 3. Specific heat as a function of temperature.

Case a: ideal Bose gas.
Case b: Bose liquid.

a little nearer to the actual transition temperature (2.19°K) than the value given by the ideal gas model (3.14°K). Since our simplification of calculations seems to go too far, we must here be satisfied with a qualitative success of having obtained a discontinuity in the specific heat curve. One of the main obstacles which prevent us from obtaining a quantitative agreement is the temperature variation of Δ' . As may be seen from its definition, Δ' is nearly proportional to T^2 so that $e^{-\beta\Delta'}$ decreases with increasing temperature, whereas it behaves inversely if Δ' were a constant. It will be shown in the next section that the quantum effect make Δ' to tend to a constant value as temperature decreases, and that our quantitative disagreements are removed to some extent by this fact.

To sum up, it is clear that the molecular clustering as the consequence of the Bose statistics, or the "Bose-Einstein excitation," plays a predominant rôle near the lambda point. This clustering is expected to be remarkable down to the temperature where $1/\sqrt{\gamma}$, given by (4.10), reaches 1, that is,

$$0.21 \left(\frac{T_\lambda}{T} \right)^{1/2} \sim 1: T \sim 0.96^\circ\text{K}.$$

Although the quantum effect will have an influence on this excitation, we may say that 1°K is the lower limit of the "Bose-Einstein excitation."

§ 5. The quantum effect

It is generally recognized that the quantum effect of helium is already remarkable in gas phase. Uhlenbeck and Beth's calculation,⁹⁾ for instance, shows that the temperature at which the second virial coefficient of helium gas begins to deviate from its classical value is about 75°K. Accordingly, the neglect of the quantum effect in liquid helium is hardly allowable. One method of taking the quantum effect into consideration is, as often used for gas state, the expansion of the Bloch equation into a power series of \hbar . It corresponds to replacing the potential energy of the system, $\Phi(\mathbf{r}_1 \dots \mathbf{r}_N)$ by

$$\Phi^*(\mathbf{r}_1 \dots \mathbf{r}_N) = \Phi(\mathbf{r}_1 \dots \mathbf{r}_N) + \frac{\lambda^2}{24\pi} \left\{ \sum_{s=1}^N \nabla_s \nabla_s \Phi - \frac{1}{2kT} \sum_{s=1}^N (\nabla_s \Phi)^2 \right\} + \dots \quad (5.1)$$

when the statistics is not taken into consideration, where $\lambda^2 = \hbar^2 / 2\pi m k T$. For the interatomic potential between two atoms, it thus corresponds to replacing it by

$$\varphi^*(r) = \varphi(r) + \frac{\lambda^2}{12\pi} \left\{ \nabla \nabla \varphi - \frac{1}{2kT} (\nabla \varphi)^2 \right\} + \dots \quad (5.2)$$

Taking the potential (4.6), it proves that these series converges very slowly for the temperature range we are interested in. We therefore cannot use this method. In spite of this, we can get a rough idea of the quantum effect in the present case in the following way. Namely, according to (5.2), this effect causes the minimum of the potential to displace apparently to the side of larger r when the temperature is not too low. Consequently, if one follows the treatment of the preceding section using $\varphi^*(r)$ instead of $\varphi(r)$, Δ would become greater, and as this effect is more predominant the lower is the temperature, the decrease in Δ with decreasing temperature is partly cancelled. As mentioned at the end of the last section, this tendency proves to be convenient in explaining the anomalous properties of liquid helium near the lambda point.

It is a very difficult work to develop the quantum mechanics of liquid on the orthodox basis. However, the following method can be adopted for our purpose of the study of the quantum effect on the "Bose-Einstein excitation." Let the sum over states be expressed by (2.5)–(2.7), and the Hamiltonian by (4.1), where the kinetic energy and the potential energy are now non-commutable. In calculating $Z(m_1 \dots m_N)$, we shall classify the coordinates into sets in such a way that those and only those coordinates which are contained in the same set are interchanged by the permutation P , with an exception that m_1 isolated coordinates are gathered into a set. In this way we obtain $1 + m_2 + m_3 + \dots + m_N = M$ sets, which we shall enumerate as 1, 2, ..., M . Correspondingly, we divide the Hamiltonian into M commutable parts H_1, H_2, \dots, H_M which contain each the corresponding coordinates and momenta of the corresponding set, and the potential

energies V_{ij} between different sets ($i \neq j$, $i, j = 1, 2, \dots, M$):

$$H = H_1 + H_2 + \dots + H_M + \sum_{i>j} V_{ij}, \quad (5.3)$$

V_{ij} does not commute with H_i and H_j but commutes with the others. If we neglect as the first approximation the non-commutativity of V_{ij} with H_i and H_j , we obtain evidently

$$Z(m_1 \dots m_N) = \int \dots \int e^{-\beta \sum_{i>j} V_{ij}} \prod_{i=1}^M S_i(\mathbf{x}_i) \prod_i d\mathbf{x}_i, \quad (5.4)$$

$$S_i(\mathbf{x}_i) = \sum_{k_i} P \psi_{k_i}(\mathbf{x}_i) e^{-\beta H_i} \psi_{k_i}(\mathbf{x}_i). \quad (5.5)$$

Here $\{\psi_{k_i}(\mathbf{x}_i)\}$ is a complete set for the system of particles contained in the i -th set, and P is the identical permutation for the first set of m_1 isolated atoms and the cyclic permutation for other sets. Previously, $S_i(\mathbf{x}_i)$ ($i \neq 1$) was such a function that has a large contribution to the integral only for those configuration \mathbf{x}_i for which the particles in the i -th set form a "molecule." We expect this to hold even if the quantum effect is taken into account. A general proof of this expectation is difficult, but we shall show below that it is true in the case of the sets consisting of two particles. If this is allowed, we again arrive at the picture developed in section 4, with an exception that both the internal motions of the molecules and the motions of the molecules and the atoms as the whole are quantized.

We are thus perhaps lead to the conclusion that there are two types of excitation in Bose quantum liquid: one the molecular clustering or the Bose-Einstein excitation which is dominant at higher temperatures, and the other the excitation connected with the quantized thermal motions of the assembly of molecules and atoms, or which may be called "photons" and "rotons," the former prevailing at lower temperatures. This result may be considered as a compromise between Tisza's idea and Landau's. Indeed, the molecules are capable of rotating motion which may correspond to Landau's rotons. However, the two kinds of excitations are not independent of each other, and in case if the coupling among them could be properly treated, one would probably attain a real solution of the problem of the second sound velocity at lower temperatures.

In the last place, we shall study the properties of $S(x)$ for the system of two particles. The Hamiltonian of this system is

$$H' = \frac{1}{2m} p_1^2 + \frac{1}{2m} p_2^2 + \varphi(r), \quad r = |\mathbf{r}_1 - \mathbf{r}_2|, \quad (5.6)$$

which can be transformed, by using the coordinates of the center of gravity and the relative coordinate, into the following expression:

$$\mathbf{H} = \frac{1}{4m} (\mathbf{p}_1 + \mathbf{p}_2)^2 + \frac{1}{4m} (\mathbf{p}_1 - \mathbf{p}_2)^2 + \varphi(r) = \frac{1}{4m} (\mathbf{p}_1 + \mathbf{p}_2)^2 + \mathbf{H}. \quad (5.7)$$

Since we are not interested in the motion of the center of gravity, we set it aside and have

$$S(\mathbf{r}) = \sum_{E, l, m} P \phi_{lm}^*(E\mathbf{r}) e^{-\beta \mathbf{H}} \phi_{lm}(E\mathbf{r}), \quad (5.8)$$

where

$$\phi_{lm}(E, \mathbf{r}) = \frac{1}{r} v_l(E, r) Y_{lm}(\theta\varphi) \quad (5.9)$$

is the eigenfunction of \mathbf{H} and $v_l(E, r)$ is the solution of the Schrödinger equation

$$\frac{d^2 v_l}{dr^2} + \left[\frac{m}{\hbar^2} (E - \varphi(r)) - \frac{l(l+1)}{r^2} \right] v_l = 0. \quad (5.10)$$

Neglecting the existence of discrete levels, we normalize $v_l(r)$ in the way*

$$kr \gg 1, \quad v_l(kr) \sim \sin(kr + \delta_l(k)), \quad k^2 = \frac{m}{\hbar^2} E, \quad (5.11)$$

and noticing that $P \phi_{lm}(E, \mathbf{r}) = (-1)^l \phi_{lm}(E, \mathbf{r})$, we finally obtain

$$S(r) = \frac{1}{2\pi^2} \sum_{l=0}^{\infty} (-1)^l (2l+1) \frac{1}{r^2} \int_0^{\infty} dk e^{-(\lambda^2 k^2 / 2\pi)} v_l^2(kr) = S_{\text{even}}(r) - S_{\text{odd}}(r). \quad (5.12)$$

$S_{\text{even}}(r)$ is the sum over even l and corresponds to the Slater-sum of the system of two Bose particles constructed with symmetric wave functions, and $S_{\text{odd}}(r)$ is the sum over odd l and corresponds to the Slater-sum of the system of two Fermi particles. Owing to the normalization condition, $S_{\text{even}}(r) = S_{\text{odd}}(r)$ for $r \rightarrow \infty$, and $S(r)$ then vanishes. For small r both take small values owing to the interatomic repulsion, and it is expected that $S_{\text{even}}(r) > S_{\text{odd}}(r)$. Accordingly, $S(r)$ vanishes for $r \rightarrow 0$ and $r \rightarrow \infty$ and is expected to have a maximum at a certain finite value of r . This maximum should be very sharp at high temperatures but should become broad as temperature is lowered, owing to the quantum effect. If the mean breadth of this maximum is smaller or of the same order as the atomic radius a , it may be allowed to assume that the two particles form a "molecule." To see to what extent this concept of molecular bonding is valid, we shall carry out a computation by use of a rigid sphere model.

* The normalization here adopted is somewhat different from that used in the literature (9). In the case the quantum effect is neglected or there is no interatomic forces, it can be proved that (5.12) is equal to $S_{\text{class}}(r)$ given by (5.14).

Take as $\varphi(r)$ the following form:

$$\varphi(r) = \begin{cases} 0 & r > a \\ \infty & r < a. \end{cases}$$

The $v_l(kr)$ satisfying the normalization condition (5.11) is given by

$$v_l(kr) = \sqrt{\frac{\pi kr}{2}} \frac{J_{-l-(1/2)}(ka) J_{l+(1/2)}(kr) - J_{l+(1/2)}(ka) J_{-l-(1/2)}(kr)}{\{J_{l+(1/2)}^2(ka) + J_{-l-(1/2)}^2(ka)\}^{1/2}}. \quad (5.13)$$

By use of this function, we can calculate $S(r)$, Eq. (5.12), which we shall denote as $S_{qt}(r)$.

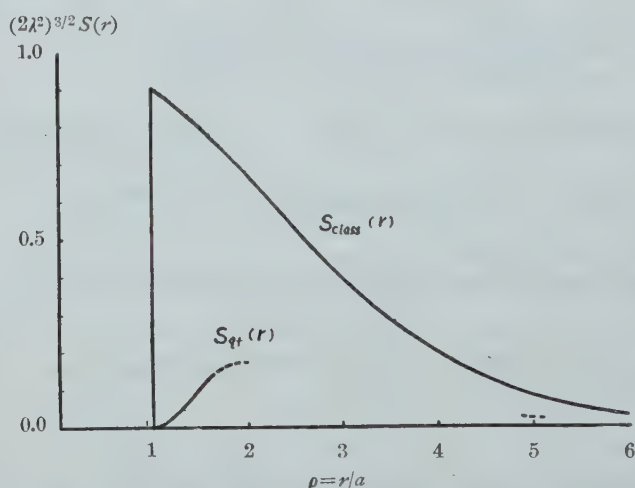


Fig. 4. Plots of functions $S_{\text{class}}(r)$ and $S_{qt}(r)$ for $T=0.3^\circ\text{K}$.

The actual calculation of this quantity is complicated, and we give in Fig. 4 only its value for $r \sim a$ and $r \gg a$ at a very low temperature ($\lambda \gg a$), with $a = 2 \times 10^{-8}$ cm. The corresponding classical quantity is given by

$$S_{\text{class}}(r) = \frac{V}{h^3} \int_{-\infty}^{\infty} e^{-\beta\varphi(r) - \frac{\beta}{m} \mathbf{p}^2 + \frac{2i}{\hbar} (\mathbf{p} \cdot \mathbf{r})} d\mathbf{p} = \frac{1}{(2\lambda^2)^{3/2}} e^{-\frac{2\pi r^2}{\lambda^2} - \beta\varphi(r)}, \quad (5.14)$$

and is plotted in the same figure. Although no detail of $S_{qt}(r)$ is obtained from this figure, yet we may say that our expectation is really the case. A measure to indicate to how extent the quantum effect is of importance is provided by the quantity

$$B = \lambda^3 \int_0^{\infty} 4\pi r^2 [S_{\text{class}}(r) - S_{qt}(r)] dr. \quad (5.15)$$

The calculation of this quantity can be executed in the same way as that of the second virial coefficient,^{9) 10)} and we have

$$B = -\frac{\lambda^3}{\sqrt{2}\pi} \left[\sum_{l=0}^{\infty} (-1)^l (2l+1) \Phi_l(s) - \frac{\sqrt{\pi}}{2} \int_0^s t^2 e^{-t^2} dt \right] \quad (5.16)$$

with

$$\Phi_l(s) = \frac{2}{\pi} \int_0^{\infty} \frac{dx}{x} \frac{e^{-x^2/s^2}}{J_{l+(1/2)}(x) + J_{-l-(1/2)}(x)}, \quad s^2 = \frac{2\pi a^2}{\lambda^2} \quad (5.17)$$

Taking the value of B for $s \gg 1$ and $s \ll 1$, and calculating it by interpolation for intermediate values of s , we obtain Fig. 5. The temperature at which B is very much different from zero corresponds to $s \sim 1$, or several degrees abs., which is very much lower than the temperature at which the virial coefficient begins to deviate from its classical value, namely, $s \sim 5$. This difference is based on the fact that although the quantum effect appears from very high temperatures for the quantities $S_{\text{even}}(r)$ and $S_{\text{odd}}(r)$, the main part of it cancels by taking their difference, until at very low temperatures. Although we are not sure to have $S_{\text{class}}(r) = S_{\text{qt}}(r)$ from $B=0$, it may fairly well be allowed to conclude that the model developed in section 4 can reasonably be accepted for liquid helium even below the lambda point. Furthermore, the quantity

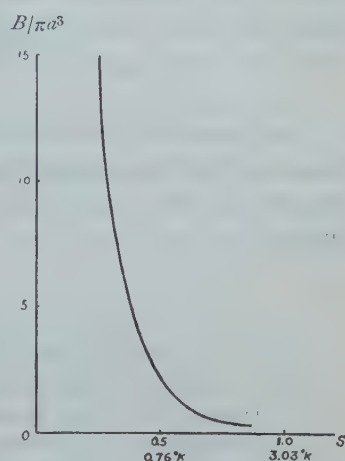


Fig. 5. Dependence of B on temperature.

$$\lambda^3 \int_0^{\infty} 4\pi r^2 S_{\text{class}}(r) dr$$

was computed by use of the method of steepest descent in the preceding section and we had for it $e^{-2\beta\Delta}$. The fact that B increases remarkably at very low temperatures implies the sharp decrease in

$$\lambda^3 \int_0^{\infty} 4\pi r^2 S_{\text{qt}}(r) dr = \lambda^3 \int_0^{\infty} 4\pi r^2 S_{\text{class}}(r) dr - B$$

and this in turn implies that, owing to the quantum effect, the quantity Δ tends to a constant value or even increases with decreasing temperature. From this fact we are led to expect that a quantitative explanation of the thermal properties of liquid helium is at hand by an extension of our theory more in detail.

§ 6. Conclusion

Our results may be summarized as follows:

(a) There exist two kinds of excitations in a Bose liquid; one the Debye-phonons and Landau-rotons, and the other an excitation corresponding to some molecular clustering, peculiar to an Bose assembly. The latter plays the essential rôle in the phase transition in liquid helium.

(b) As the result of the latter excitation, the Bose liquid has a character of a multi-component solution of associating and dissociating molecules. In the lower modification, there appears a parameter as a thermodynamical variable that describes the equilibrium between these components. Probably the peculiar properties of liquid helium in non-equilibrium would be accounted for by such a character of the system as a multi-component system.

In conclusion, the writer wishes to express his sincere thanks to Prof. T. Nagamiya for his continual interest and helpful advices on the present investigation, and to Dr. R. Kubo for his stimulating discussions during his stay at Tokyo University.

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On the Self-energies of Mesons

KAZUO YAMAZAKI and HIROSHI ENATSU

Department of Physics, Kyoto University

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§ 1. Introduction and summary

In the previous two papers* self-energies of nucleons have been investigated by making use of the method of mixing meson fields. In this note we shall apply the same method as that employed in the preceding ones to the cases of the self-energies of mesons. Contrary to the former cases, we could not find any simple relations which enable us to eliminate both the quadratic and the logarithmic divergences of them. As is well known in the computations of self-energies of vacuum polarization types, there appear several ambiguities which are concerned with the definition of a vacuum, the gauge invariancy and so forth.

Although many attempts have been made so far by various authors, it seems that we unfortunately have not yet succeeded in a satisfactory manner to derive a suitable formulation of the field theory free from the mentioned difficulties. We have not considered such ambiguities in detail and only performed formally the calculations as in the previous paper*, that is, according to Karplus and Kroll we have taken into account the effects of the surface integrals. It is the remarkable point that in the present case the surface terms will not contain any diverging integral, so that the divergent parts will provide the same form as that which could be derived by making use of the Schwinger's method. As to the surface terms themselves, the contributions which arise from the third and higher derivatives vanish entirely in the case of Boson field.

Throughout this paper we shall use the same notations as that adopted in I.

§ 2. Second order S-matrices

Using the same interaction Lagrangian densities as those given in I, the second order S-matrices which correspond to the process as shown in Fig. 1, are as follows.

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These papers will be referred to as I and II.

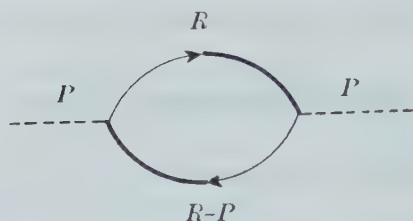


Fig. 1

Spin 0 mesons

$$S = \frac{1}{2!} \left(\frac{-i}{\hbar c} \right)^2 \left[\frac{1}{2} J_{S(S)} + \frac{1}{2} J_{S(V)} + J_{S(S, V)} + \frac{1}{2} J_{PS(PS)} + \frac{1}{2} J_{PS(PV)} + J_{PS(PS, PV)} \right]$$

$$J^0 = - \int (dk) S p \left[\frac{i(\gamma, k) - M}{k^2 + M^2} \Gamma^1_{\mu} \frac{i(\gamma, k-p) - M}{(k-p)^2 + M^2} \Gamma^{2\mu}_{\nu} \right. \\ \left. + \frac{i(\gamma, k) - M}{k^2 + M^2} \Gamma^2_{\mu} \frac{i(\gamma, k-p) - M}{(k-p)^2 + M^2} \Gamma^{1\mu}_{\nu} \right]$$

where

$$\left\{ \begin{array}{l} \Gamma^1_{S(S)} = \Gamma^2_{S(S)} = i f_1; \quad \Gamma^1_{S(V)} = \Gamma^2_{S(V)} = \frac{f_2}{x} i(\gamma, p); \\ \Gamma^1_{S(S, V)} = \frac{f_2}{x} i(\gamma, p), \quad \Gamma^2_{S(S, V)} = i f_1 \\ \Gamma^1_{PS(PS)} = \Gamma^2_{PS(PS)} = F_1 \gamma_5; \quad \Gamma^1_{PS(PV)} = \Gamma^2_{PS(PV)} = \frac{F_2}{x} i \gamma_5(\gamma, p) \\ \Gamma^1_{PS(PS, PV)} = \frac{F_2}{x} i \gamma_5(\gamma, p), \quad \Gamma^2_{PS(PS, PV)} = F_1 \gamma_5 \\ \Gamma^i_{\mu}(\not{p}) = \Gamma^i_{\mu}(-\not{p}) \quad (i=1, 2) \end{array} \right.$$

Spin 1 mesons

$$S = \frac{1}{2!} \left(\frac{-i}{\hbar c} \right)^2 \left[\frac{1}{2} J_{V(V)} + \frac{1}{2} J_{V(T)} + J_{V(V, T)} + \frac{1}{2} J_{PV(PV)} + \frac{1}{2} J_{PV(PT)} + J_{PV(PV, PT)} \right]$$

$$J^1 = - \int (dk) S p \left[\frac{i(\gamma, k) - M}{k^2 + M^2} \Gamma^1_{\mu} \frac{i(\gamma, k-p) - M}{(k-p)^2 + M^2} \Gamma^{2\mu}_{\nu} \right. \\ \left. + \frac{i(\gamma, k) - M}{k^2 + M^2} \Gamma^2_{\mu} \frac{i(\gamma, k-p) - M}{(k-p)^2 + M^2} \Gamma^{1\mu}_{\nu} \right]$$

where

$$\left\{ \begin{array}{l} \Gamma^1_{V(V)_{\mu}} = \Gamma^2_{V(V)_{\mu}} = g_1 \gamma_{\mu}; \quad \Gamma^1_{V(T)_{\mu}} = \Gamma^2_{V(T)_{\mu}} = g_2 \{ i(\gamma, p) \gamma_{\mu} - \gamma_{\mu} i(\gamma, p) \} \\ \Gamma^1_{V(V, T)_{\mu}} = g_2 \{ i(\gamma, p) \gamma_{\mu} - \gamma_{\mu} i(\gamma, p) \} \quad \Gamma^2_{V(V, T)_{\mu}} = g_1 \gamma_{\mu} \end{array} \right.$$

$$\left\{ \begin{array}{l} \Gamma_{PV(PV)\mu}^1 = \Gamma_{PV(PV)\mu}^2 = G_1 \gamma_5 \gamma_\mu; \quad \Gamma_{PV(PT)\mu}^1 = \Gamma_{PV(PT)\mu}^2 = G_2 \gamma_5 \{ i(\gamma, p) \gamma_\mu - \gamma_\mu i(\gamma, p) \} \\ \Gamma_{PV(PV, PT)\mu}^1 = G_2 \gamma_5 \{ i(\gamma, p) \gamma_\mu - \gamma_\mu i(\gamma, p) \}; \quad \Gamma_{PV(PV, PT)\mu}^2 = G_1 \gamma_5 \gamma_\mu \\ \Gamma^i(p) = \Gamma^i(-p) \quad (i=1, 2) \end{array} \right.$$

By making use of the well-known formula

$$\frac{1}{AB} = \int_0^1 du \frac{1}{[Au + B(1-u)]^2}$$

and changing the variables k_μ into $k_\mu + u p_\mu$, one obtains,

$$\begin{aligned} J = & -(dk) \int_0^1 du \left(1 - u p_\lambda \frac{\partial}{\partial k_\lambda} + \frac{u^2}{2!} p_\lambda p_\rho \frac{\partial^2}{\partial k_\lambda \partial k_\rho} \right) \frac{1}{\{k^2 + M^2 + p^2 u(1-u)\}^2} \\ & \times Sp \{ \{ i(\gamma, k + u p) - M \} \Gamma^1 \{ i(\gamma, k + (u-1)p) - M \} \Gamma^2 \\ & + \{ i(\gamma, k + u p) - M \} \Gamma^2 \{ i(\gamma, k + (u-1)p) - M \} \Gamma^1 \} \end{aligned}$$

In deriving them, none of the terms has been neglected since it can be easily shown that the terms which contain more than the third derivatives with respect to k vanish entirely.

Now we shall evaluate the spurs and work out the integrations on k . Further, in dealing with them it is necessary to use the symmetrical properties of integrations.

We use the following relations

$$I_1 = \int (dk) \int_0^1 du \frac{M^2}{\{k^2 + \Lambda(u)\}^2} = \frac{i\pi^2}{2} \left\{ 4M^2 \log \frac{K + K_0}{M} + 4M^2 L \right\}$$

$$\begin{aligned} I_2 = & \int (dk) \int_0^1 du \frac{p^2 u(1-u)}{\{k^2 + \Lambda(u)\}^2} \\ = & \frac{i\pi^2}{2} \left\{ \frac{2}{3} p^2 \log \frac{K + K_0}{M} - \frac{p^2}{9} - \frac{4M^2}{3} - \frac{2}{3} (2M^2 - p^2) L \right\} \end{aligned}$$

$$\begin{aligned} I_3 = & \int (dk) \int_0^1 du \frac{k_\mu k_\nu}{\{k^2 + \Lambda(u)\}^2} = -\frac{\delta_{\mu\nu}}{4} (I_1 + I_2) \\ & + \delta_{\mu\nu} \frac{i\pi^2}{2} \left\{ KK_0 - \left(M^2 + \frac{p^2}{6} \right) \log \frac{K + K_0}{M} - \frac{p^2}{18} - \frac{2M^2}{3} - \frac{p^2 + 4M^2}{6} L \right\} \end{aligned}$$

$$\begin{aligned} I_4 = & \int (dk) \int_0^1 du \frac{1}{k^2 + \Lambda(u)} \\ = & \frac{i\pi^2}{2} \left\{ KK_0 - \left(M^2 + \frac{p^2}{6} \right) \log \frac{K + K_0}{M} - \frac{p^2}{18} - \frac{2M^2}{3} - \frac{p^2 + 4M^2}{6} L \right\} \end{aligned}$$

$$I_5 = \int (dk) \int_0^1 du \, u p_\lambda \frac{\partial}{\partial k_\lambda} \frac{f(p, u, M) k_\mu}{\{k^2 + A(u)\}^2} = \frac{i\pi^2}{2} p_\mu \int_0^1 u f(p, u, M) du$$

$$I_6 = \int (dk) \int_0^1 du \, \frac{u^2}{2} p_\lambda p_\nu \frac{\partial^2}{\partial k_\lambda \partial k_\nu} \frac{f(p, u, M) k_\mu k_\nu}{\{k^2 + A(u)\}^2}$$

$$= \frac{2i\pi^2}{3} (p_\mu p_\nu + p^2 \delta_{\mu\nu}) \int_0^1 f(p, u, M) u^2 du.$$

where

$$A(u) \equiv M^2 + p^2 u(1-u)$$

$$L = -\frac{\sqrt{4M^2 + p^2}}{\sqrt{-p^2}} \tan^{-1} \frac{\sqrt{-p^2}}{\sqrt{4M^2 + p^2}}$$

$$\int_0^1 \frac{Au + B}{p^2 u^2 - p^2 u - M^2} du = \frac{2(2A + B)}{p^2 + 4M^2} L,$$

A, B : const.

and $f(p, u, M)$ is an arbitrary function not involving k . At the last step we substitute $-\mathbf{x}^2$ for p^2 .

§ 3. Self-energies of mesons

After some manipulations we attain to the following results.

Spin 0 mesons.

	$\frac{\partial}{\partial k}$	$\frac{\partial^2}{\partial k \partial k}$
$\frac{1}{2} J_{S(S)} = 8i\pi^2 f_1^2 \left\{ -KK_0 + \left(3M^2 - \frac{\mathbf{x}^2}{2} \right) \log \frac{K + K_0}{M} \right.$	$+$	$+$
$\left. + \frac{1}{2} (4M^2 - \mathbf{x}^2) L - \frac{\mathbf{x}^2}{8} \right\}$	$+\frac{1}{24} \mathbf{x}^2$	$+\frac{1}{12} \mathbf{x}^2$
$\frac{1}{2} J_{S(V)} = 8i\pi^2 f_2^2 \left\{ +\frac{KK_0}{2} - \frac{1}{2} M^2 - \frac{\mathbf{x}^2}{12} \right\}$	$+\frac{1}{24} \mathbf{x}^2$	$-\frac{1}{12} \mathbf{x}^2$
$J_{S(SV)} = 0$	0	0
$\frac{1}{2} J_{PS(PS)} = 8i\pi^2 F^2 \left\{ -KK_0 + \left(M^2 - \frac{\mathbf{x}^2}{2} \right) \log \frac{K + K_0}{M} \right.$	$-\frac{1}{24} \mathbf{x}^2$	$+\frac{1}{12} \mathbf{x}^2$
$\left. -\frac{1}{2} \mathbf{x}^2 L - \frac{\mathbf{x}^2}{8} \right\}$		

$$\frac{1}{2}J_{PS(PS)}=8i\pi^2F_2^2\left\{+\frac{KK_0}{2}-2M^2\log\frac{K+K_0}{M}\right. \\ \left.-2M^2L-\frac{1}{2}M^2-\frac{x^2}{12}\right\}$$

$$+\frac{1}{24}x^2 \quad -\frac{1}{12}x^2$$

$$J_{PS(PS, PV)}=8i\pi^2F_1F_2\left\{-2MK\log\frac{K+K_0}{M}-2MKL\right\}$$

$$0 \quad 0$$

Spin 1 mesons.

$$\frac{\partial}{\partial k} \quad \frac{\partial^2}{\partial k \partial k}$$

$$\frac{1}{2}J_{V(V)}=8i\pi^2g_1^2\left\{-\frac{1}{2}\delta_{\mu\nu}KK_0\right. \\ \left.-\frac{1}{3}(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)\log\frac{K+K_0}{M}\right. \\ \left.-\frac{1}{3}(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)\left(\frac{2M^2}{x^2}+1\right)L\right. \\ \left.+\frac{M^2}{6x^2}(4\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)+\frac{1}{18}x^2\delta_{\mu\nu}\right\}$$

$$\frac{1}{24}A \quad \frac{1}{36}B$$

$$\frac{1}{2}J_{V(\bar{V})}=8i\pi^2g_2^2\frac{(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)}{x^2}\left\{-\left(4M^2\right.\right. \\ \left.+\frac{2}{3}x^2\right)\log\frac{K+K_0}{M} \\ \left.-\frac{2}{3}(8M^2+x^2)L-\frac{4}{3}M^2-\frac{1}{9}x^2\right\}$$

$$\frac{1}{3}C \quad -\frac{1}{9}C$$

$$J_{V(V, \pi)}=8i\pi^2g_1g_2\frac{(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)}{x^2}\left\{4Mx\log\frac{K+K_0}{M}\right. \\ \left.+4MxL\right\}$$

$$0 \quad 0$$

$$\frac{1}{2}J_{PV(PV)}=8i\pi^2G_1^2\left\{-\frac{1}{2}\delta_{\mu\nu}KK_0\right. \\ \left.-\left(\frac{1}{3}(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)-2M^2\delta_{\mu\nu}\right)\log\frac{K+K_0}{M}\right. \\ \left.-\left(\frac{1}{3}(\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)\left(\frac{2M^2}{x^2}+1\right)\right.\right. \\ \left.\left.-2M^2\delta_{\mu\nu}\right)L\right. \\ \left.+\frac{M^2}{6x^2}(4\not{p}_\mu\not{p}_\nu+\delta_{\mu\nu}x^2)+\frac{1}{18}x^2\delta_{\mu\nu}\right\}$$

$$\frac{1}{24}A \quad \frac{1}{36}A$$

$\frac{1}{2}J_{PV(P\bar{T})} = 8i\pi^2 G_2^2 \frac{(\not{p}_\mu \not{p}_\nu + \partial_{\mu\nu} x^2)}{x^2}$ $- \left(4M^2 - \frac{2x^2}{3}\right) \log \frac{K+K_0}{M}$ $- \frac{2}{3}(4M^2 - x^2)L + \frac{4}{3}M^2 + \frac{1}{9}x^2 \}$ $J_{PV(PV, P\bar{T})} = 0$ <p>where</p> $L \equiv - \frac{\sqrt{4M^2 - x^2}}{x^2} \tan^{-1} \frac{x}{\sqrt{4M^2 - x^2}} \sim -1$ $+ \frac{1}{3} \frac{x^2}{4M^2 - x^2} - \dots$ $A = (2\not{p}_\mu \not{p}_\nu + x^2 \partial_{\mu\nu}), \quad B = (-\not{p}_\mu \not{p}_\nu + x^2 \partial_{\mu\nu}),$ $C = (\not{p}_\mu \not{p}_\nu + x^2 \partial_{\mu\nu})$ $K_0 = \sqrt{K^2 + M^2}$	<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 50%; text-align: center; vertical-align: top;"> $-\frac{1}{3}C$ 0 </td> <td style="width: 50%; text-align: center; vertical-align: top;"> $+\frac{1}{9}C$ 0 </td> </tr> </table>	$-\frac{1}{3}C$ 0	$+\frac{1}{9}C$ 0
$-\frac{1}{3}C$ 0	$+\frac{1}{9}C$ 0		

In order to make the effects of surface terms clear, their contributions are written on the right-hand side of each J. Therefore, for the case of Boson fields, contrary to the self-energies of Fermions, the contributions of the surface terms do not contain any diverging term.

In conclusion it may be inferred from our results obtained above that, when we assume the pseudoscalar theory for π mesons, we cannot remove all divergencies of the self-energies of them due to the Fermion field by the procedure of mixing various Fermion fields.

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The Scattering of Mesons by Nuclear Particles, I

R. C. MAJUMDAR, A. S. APTE* and M. K. SUNDARESAN

Department of Physics, University of Delhi, Delhi, India

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A general relativistic theory of scattering of mesons including the radiation damping has been worked out in which the incident pseudoscalar (vector) meson is scattered by nuclear particles into a pseudoscalar or a vector meson. The cross-sections for the scattering processes involving charged mesons only are given and it is shown that the integral equation of the radiation damping can be solved exactly in a very simple way. The extension of the present scheme of the scattering of mesons to that of Tomonaga-Schwinger-Feynman-Dyson's covariant formalism and the scattering processes involving neutral mesons together with discussions on the scattering experiments on π -meson will be subject matter of subsequent papers.

Introduction

The scattering of mesons by nucleons without radiation damping was worked out for the first time by Bhabha,¹⁾ Heitler,²⁾ and Yukawa and Tanikawa³⁾ from Yukawa-Proca wave formulation and subsequently by Wilson⁴⁾ from the Duffin-Kemmer Particle formulation of the meson equations. Although both the formulations are completely equivalent the calculations are, however, widely different, each having its own merits and demerits. In the wave formulation the calculations are to be separately carried out for the longitudinally and transversely polarised mesons and then finally summed or averaged over various polarisations. This makes the calculations extremely lengthy and laborious. In the particle formulation, on the other hand, although the distinction between the transversely or longitudinally polarised mesons does not arise explicitly, the knowledge of the correspondence between the particle and the wave aspect of the meson is required in order to introduce the nuclear interaction. This correspondence can only be established as shown by Wilson through a very cumbersome representation of Kemmer-Duffin β -matrices and as a result the calculations do not lead to any great simplification. It has been further shown by Ma and Hsüeh⁵⁾ that the integral equation giving the effect of the radiation reaction on the scattering of mesons can be easily solved on Kemmer-Wilson particle formulation.

An elegant approach to Duffin-Kemmer formulation has been recently made by Harish-Chandra⁶⁾ in which correspondence between the particle and wave aspect of the meson could be directly established by introducing one column

* At present at the C. W. I. N. Research Station, Poona.

vector matrix I^* satisfying certain algebraic relations. In this theory the correspondence between the wave and the particle formulations of the meson field is so close and complete that one can pass over easily from one formulation to the other at any stage of the calculation. The usual spur calculations are avoided and substituted by the repeated use of certain algebraic relations satisfied by the I^* matrices. The calculations of the meson scattering by the nucleon become extremely simplified and straightforward and the effect of radiation damping on the scattering of mesons can be included in a very simple way. In § 1 and § 2 we shall briefly state the necessary results of Harish-Chandra's formulation for the various types of meson interacting with the nuclear particles. In § 3 we shall apply the theory to calculate the relativistic nuclear scattering of mesons taking into consideration the reaction of the radiation. In the last section we shall calculate the general case of the scattering of mesons having spin zero (pseudoscalar) or spin one (vector) by the nuclear particles, the emitted mesons being however, either a pseudoscalar or a vector meson. In the present paper scattering processes involving the charged mesons only will be given, because in this case the integral equation for the radiation damping can be exactly solved. In the second part the whole scheme will be discussed from Tomonaga-Schwinger-Feynman-Dyson covariant Formalism. The scattering processes involving the emission of neutral mesons and the discussion on the scattering of π -meson will be reported in the third paper.

§ 1. The fundamental equation for mesons

The equation of the meson is in general given by

$$i\beta^\mu \partial_\mu \psi + \chi \psi = 0, \quad \chi = \frac{\mu c}{\hbar} \quad (1)$$

where μ is the mass of the meson and β^μ are Duffin-Kemmer matrices satisfying the following commutation relations

$$\beta^\mu \beta^\nu \beta^\rho + \beta^\rho \beta^\nu \beta^\mu = \beta^\mu g^{\nu\rho} + \beta^\rho g^{\nu\mu} \quad (2)$$

$g^{\mu\nu}$ are fundamental metric tensor

$$g^{\mu\nu} = 0 \quad \mu \neq \nu; \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1. \quad (3)$$

If $\beta^{\mu\dagger}$ is the Hermitian conjugate of β^μ , there exists a matrix A such that

$$A\beta^\mu A^{-1} = \beta^{\mu\dagger} \quad (4)$$

and

$$A^\dagger = A. \quad (5)$$

The Hermitian conjugate of (1) is

$$i\partial_\mu \psi^* \beta^\mu - \chi \psi^* = 0 \quad (6)$$

where we have put

$$\phi^* = \phi^\dagger \Lambda. \quad (7)$$

It has been shown by Kemmer that the equations (1) with the 10-row and 5-row representations of β matrices are equivalent to Proca's wave equations of meson for spin unity and spin zero respectively.

We now introduce with Harish-Chandra a one column vector matrix T_μ such that $\phi^* T_\mu$ transforms as a vector. Further putting

$$T_\mu^\dagger \Lambda = \Gamma_\mu^* \quad (8)$$

we obtain the following useful relations

$$\left. \begin{aligned} \Gamma_\mu^* T^\mu &= 4, & \Gamma_\mu^* T_\nu &= g_{\mu\nu}, & \Gamma_\mu^* \beta_\nu T_\rho &= 0 \\ \beta_\mu \beta_\nu T_\rho &= g_{\mu\nu} T_\rho - g_{\mu\rho} T_\nu, \\ \Gamma_\mu^* \beta_\nu \beta_\rho &= \Gamma_\mu^* g_{\nu\rho} - \Gamma_\nu^* g_{\mu\rho}, \\ \beta_\mu T_\nu &= -\beta_\nu T_\mu, & \Gamma_\mu^* \beta_\nu &= -\Gamma_\nu^* \beta_\mu \end{aligned} \right\} \text{Spin 1,} \quad (9)$$

and

$$\left. \begin{aligned} \beta_\mu \beta_\nu T_\rho &= T_\mu g_{\nu\rho}, \\ \Gamma_\mu^* \beta_\nu \beta_\rho &= \Gamma_\rho^* g_{\mu\nu}, \\ \beta_\nu T_\rho &= \frac{1}{4} g_{\nu\rho} \beta_\mu T^\mu, \\ \Gamma_\mu^* \beta_\nu &= \frac{1}{4} g_{\mu\nu} \Gamma_\rho^* \beta^\rho \end{aligned} \right\} \text{Spin 0.} \quad (10)$$

It is to be noted that if $\Lambda = 1 - 2\beta_0^2$ is taken as the reflection matrix then $\phi^* T_\mu$ is a vector for the 10-row representation and not a pseudovector, whereas the reflection matrix must be taken as $-\Lambda$ for the 5-row representation so that $\phi^* T_\mu$ may transform as a vector. This will always be implied in the following calculations.

Multiplying the equation (1) to the left by Γ_ν^* and by $\Gamma_\mu^* \beta_\nu$ respectively and putting

$$\Gamma_\mu^* \phi = U_\mu, \quad (11)$$

$$\frac{\chi}{i} \Gamma_\mu^* \beta_\nu \phi = G_{\mu\nu}, \quad (12)$$

we obtain with the help of (9)

$$\partial^\mu G_{\mu\nu} + \chi^2 U_\nu = 0, \quad (13)$$

$$\partial^\mu U_\nu - \partial_\nu U_\mu = G_{\mu\nu}. \quad (14)$$

Further it can be shown that

$$\partial^\mu U_\mu = 0, \quad (15)$$

$$\partial_\mu G_{\nu\rho} + \partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} = 0. \quad (16)$$

These are the usual wave equations for the meson field of spin unity.

Similarly by multiplying (1) by Γ_ν^* and $\Gamma_\mu^*\beta_\nu$ from left respectively and writing

$$\Gamma_\mu^*\phi = U_\mu, \quad (17)$$

$$\Gamma_\mu^*\beta_\nu\phi = -\frac{\chi}{2}g_{\mu\nu}U, \quad (18)$$

we obtain with the help of (10)

$$\partial_\mu U = U_\mu, \quad (19)$$

$$\partial_\mu U^\mu + \chi^2 U = 0, \quad (20)$$

which are again the usual equations for the scalar meson field.

Moreover, in this method, instead of calculating the spurs we have to evaluate the quantities of the form

$$\Gamma_{\mu_0}^*\beta_{\mu_1}\beta_{\mu_2}\dots\beta_{\mu_n}T_{\mu_{n+1}}.$$

If n is odd we have

$$\Gamma_{\mu_0}^*\beta_{\mu_1}\beta_{\mu_2}\dots\beta_{\mu_{2n-1}}T_{\mu_{2n}} = 0 \quad (21)$$

and if n is even we obtain

$$\Gamma_{\mu_0}^*\beta_{\mu_1}\beta_{\mu_2}\dots\beta_{\mu_{2n}}T_{\mu_{2n+1}} = \eta_{[\mu_0\mu_1]}\Gamma_{[\mu_2,\nu_1]}\eta_{[\nu_1,\mu_3]}\Gamma_{[\mu_4,\nu_2]}\dots\eta_{[\nu_{n-1},\mu_{2n-1}]}\Gamma_{[\mu_{2n},\mu_{2n+1}]}, \quad (22)$$

$$\eta_{[\mu,\nu][\rho,\lambda]} = g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}. \quad (23)$$

§ 2. Interaction of mesons with nuclear particles

The general scheme of the interaction Hamiltonian has been already discussed by Yachaspati⁷⁾. We give here the results only.

The matrix element describing the interaction of mesons with nucleons can be expressed as*

$$H^{MN} = \frac{1}{\mu c} [\epsilon v_p^\dagger(\vec{P})a_0A(\vec{p})u(\vec{p})v_N(\vec{P}')\delta(\vec{P}-\vec{P}'-\vec{p}) + \epsilon'v_N^\dagger(\vec{P})a_0u^\dagger(\vec{p})AB(\vec{p})\bar{v}_p(\vec{P}')\delta(\vec{P}-\vec{P}'+\vec{p})] \quad (24)$$

where $\epsilon = \epsilon' = 1$ for vector and scalar mesons and $\epsilon = -\epsilon' = i$ for pseudoscalar mesons and $u(\vec{p})$ denotes the amplitude of the meson wave function given by

$$u(\vec{p}) = \sum_{k=1}^3 a_k^*(\vec{p})u_k^{(+)}(\vec{p}) + \sum_{k=4}^6 b_k(\vec{p})u_k^{(-)}(\vec{p}) \quad (25)$$

$u^{(+)}(\vec{p})$ and $u^{(-)}(\vec{p})$ referring to the amplitudes for positive and negative mesons respectively satisfying the following orthogonal and normalisation relations

* The interaction Lagrangian given by Harish-Chandra for g_2 -part of pseudo-scalar and vector mesons is not Hermitian. The necessary changes have been made to meet this requirement.

$$\sum_{k,l=1}^6 (\beta_0^2 u_k) u_l \rho_{kl} (u_l^\dagger A \beta_0^2) = (\beta_0)_{\mu\nu} \quad (26)$$

and

$$\sum_{k=1}^3 \beta_0 u_k^{(+)}(\vec{p}) u_k^{(+)}(\vec{p})^\dagger A \beta_0 = \frac{1}{2E} \left(E \beta_0 + \frac{(\vec{p} \cdot \vec{\beta})^2}{\mu} - \mu c^2 \right) \beta_0^2 \quad (27)$$

where

$$\rho_{kl} = \begin{cases} +1 & \text{for } k=l=1, 2, 3, \\ -1 & \text{for } k=l=4, 5, 6 \end{cases} \quad (28)$$

and $E = c \sqrt{\vec{p}^2 + \mu^2 c^2}$. a_k^* , a_k , b_k^* and b_k are corresponding emission and absorption operators. $v(\vec{P})$ is the amplitude for the nucleon wave function, \vec{P} and \vec{p} are the momenta of nucleon and meson respectively and \vec{a} , a_0 are the usual Dirac matrices.

$A(\vec{p})$ and $B(\vec{p})$ are given by

$$\begin{aligned} A^s(\vec{p}) &= g_1^s \mu c \Gamma_0^* \beta_0 - g_2^s \{ (a\vec{p}) \Gamma_0^* \beta_0 + \mu c a_0 \Gamma_0^* \}, \\ A^p(\vec{p}) &= \gamma [g_1^p \mu c \Gamma_0^* \beta_0 - g_2^p \{ (a\vec{p}) \Gamma_0^* \beta_0 + \mu c a_0 \Gamma_0^* \}]; \quad \gamma = a_1 a_2 a_3 a_0 \\ A^v(\vec{p}) &= g_1^v [\mu c (a\Gamma^*) + a_0 (\vec{p}\Gamma^*) \beta_0] - 2g_2^v [(\vec{p}\Gamma^*) + (a\vec{p}) (a\Gamma^*) + \mu c a_0 (a\Gamma^*) \beta_0], \\ B^s(\vec{p}) &= g_1^s \mu c \beta_0 T_0 - g_2^s \{ (a\vec{p}) \beta_0 T_0 + \mu c a_0 T_0 \}, \\ B^p(\vec{p}) &= [g_1^p \mu c \beta_0 T_0 - g_2^p \{ (a\vec{p}) \beta_0 T_0 + \mu c a_0 T_0 \}] \gamma, \\ B^v(\vec{p}) &= g_1^v [\mu c (a\vec{I}) + a_0 \beta_0 (\vec{p}\vec{I})] + 2g_2^v [(\vec{p}\vec{I}) + (a\vec{p}) (a\vec{I}) + \mu c a_0 \beta_0 (a\vec{I})] \end{aligned} \quad (29)$$

where suffices s , p and v stand for scalar, pseudoscalar and vector mesons respectively. The coupling constants g_1 and g_2 will be expressed as

$$g_1 = \left(\frac{4\pi\hbar c}{\chi} \right)^{1/2} g, \quad g_2 = \frac{1}{2} \left(\frac{4\pi\hbar c}{\chi} \right)^{1/2} f. \quad (30)$$

where g and f have the dimensions of charge.

Further it can be easily seen

$$A^\dagger(\vec{p}) a_0 = a_0 A B(\vec{p}), \quad B(\vec{p})^\dagger A a_0 = a_0 A(\vec{p}). \quad (31)$$

§ 3. Scattering of mesons by nucleons

We consider the scattering of a positive meson of momentum \vec{p} by a neutron of momentum $-\vec{p}$ in a coordinate system in which the centre of mass is at rest. The scattering processes take place in two stages as shown in the following scheme:

$$Y^+(\vec{p}) + N(-\vec{p}) + [P(0)] \rightarrow \left[\begin{array}{l} P(0) + [P(0)] \\ Y^+(\vec{p}) + N(-\vec{p}) + Y^+(\vec{p}') + N(-\vec{p}') \end{array} \right] \rightarrow Y^+(\vec{p}') + N(-\vec{p}') + [P(0)] \quad (32)$$

where the square bracket [] indicates that the particle is in a state of negative energy. We consider the following types of mesons taking part in the scattering processes :

Case	Incident meson	Scattered meson
I	Scalar	Scalar
II	Pseudoscalar	Pseudoscalar
III	Vector	Vector

The differential cross-section for the scattering is given by

$$d\phi = \frac{1}{4\pi^2 \hbar^4 c^4} \frac{E^2 E_N^2}{E_i^2} |U_{if}|^2 d\Omega, \quad (33)$$

where E , E_N are energies of the incident meson, and the nucleon respectively given by $E = c\sqrt{p^2 + \mu^2 c^2}$, $E_N = c\sqrt{p^2 + M^2 c^2}$

$$\text{and} \quad E_s = E + E_N \quad (34)$$

$$\text{and} \quad U_{if} = H_{if} + i\pi \sum_{f'} \int U_{if'} H_{f'f} \rho_{f'} d\Omega_{f'}. \quad (35)$$

$\rho_{f'}$ is the density of the states f' per unit energy range and the summations are to be carried out over all directions and polarisations of states f' . We express

$$H_{if} = \frac{K^x(\vec{p}) (E_s - Mc^2 a_0) a_0 K(\vec{p}')}{\mu^2 c^2 (E_i^2 - M^2 c^4)}, \quad (36)$$

where

$$K(\vec{p}) = A(\vec{p}) u_i^+(\vec{p}) v_N(-\vec{p}) \quad (37)$$

and

$$K^+(\vec{p}) = K^x(\vec{p}) a_0 \quad (38)$$

and similar expressions are obtained for $H_{f'f}$.

It can be easily verified by applying the relations given by (9), (10) and (27) that

$$\sum_{p'} K(\vec{p}') K^x(\vec{p}') a_0 = \frac{4\pi \hbar^2 \mu^2 c^4}{4EE_N} \left(X - a_0 Y - \frac{a_0(a p')}{p'} Z \right), \quad (39)$$

where the suffices s , p and v will indicate the cases involving scalar, pseudoscalar and vector mesons respectively.

$$X^p = g^{v^2} E_N - g^p f^p \frac{EM}{\mu} + \frac{f^{p^2}}{4\mu^2 c^4} \{ E_N (E^2 + c^2 p^2) + 2Ec^2 p^2 \},$$

$$Y^p = -g^{v^2} Mc^2 + g^p f^p \frac{(EE_N + c^2 p^2)}{\mu c^2} - \frac{f^{v^2}}{4} Mc^2,$$

$$\begin{aligned}
Z^v &= c\dot{p} \left\{ g^{v2} + g^v f^v \frac{M}{\mu} - \frac{f^{v2}}{4\mu^2 c^4} (E^2 + c^2 \dot{p}^2 + 2EE_N) \right\}, \\
X^s &= g^{s2} E_N - g^s f^s \frac{EM}{\mu} + \frac{f^{s2}}{4\mu^2 c^4} \{ E_N (E^2 + c^2 \dot{p}^2) + 2Ec^2 \dot{p}^2 \}, \\
Y^s &= g^{s2} Mc^2 - g^s f^s \frac{(EE_N + c^2 \dot{p}^2)}{\mu c^2} - \frac{f^{s2}}{4} Mc^2, \\
Z^s &= c\dot{p} \left\{ g^{s2} + g^s f^s \frac{M}{\mu} - \frac{f^{s2}}{4\mu^2 c^4} (E^2 + c^2 \dot{p}^2 + 2EE_N) \right\}, \\
X^v &= g^{v2} \left(3E_N + 2E_i \frac{\dot{p}^2}{\mu^2 c^2} \right) + 6g^v f^v \frac{EM}{\mu} + f^{v2} \left(3E_N + 4E_i \frac{\dot{p}^2}{\mu^2 c^2} \right), \\
Y^v &= -3g^{v2} Mc^2 - 6g^v f^v \frac{EE_N + c^2 \dot{p}^2}{\mu c^2} - 3f^{v2} Mc^2, \\
Z^v &= c\dot{p} \left[g^{v2} \left(1 - \frac{2(EE_N + c^2 \dot{p}^2)}{\mu^2 c^4} \right) - 6g^v f^v \frac{M}{\mu} - f^{v2} \left(1 + \frac{4(EE_N + c^2 \dot{p}^2)}{\mu^2 c^4} \right) \right].
\end{aligned} \tag{40}$$

The equation (35) reduces to

$$\begin{aligned}
\mu^2 c^2 (E_i^2 - M^2 c^4) U_{ij} &= K^X(\dot{p}) (E_i - Mc^2 u_0) a_0 K(\vec{p}') \\
&\quad + i\pi\rho \sum_f \int U_{if} K^X(\dot{p}'') ([E_i - Mc^2 u_0) a_0 \vec{K}(\dot{p}') d\Omega_{f'}.
\end{aligned} \tag{41}$$

To solve the equation we multiply the equation by $K^X(\dot{p}') u_0$ and integrate over all the final states and introduce the abbreviation

$$I = \sum_f \int U_{if} K^X(\vec{p}') u_0 d\Omega_{f'}, \tag{42}$$

we thus obtain

$$I[1 - i\pi\rho(k_1 - k_2 u_0)] = K^X(\vec{p}') u_0 (k_1 - k_2 u_0), \tag{43}$$

where

$$k_1 = \frac{4\pi^2 c^2 \hbar^2}{EE_N} \frac{E_i X + Mc^2 Y}{E_i^2 - M^2 c^4}, \tag{44}$$

$$k_2 = \frac{4\pi^2 c^2 \hbar^2}{EE_N} \frac{Mc^2 X + E_i Y}{E_i^2 - M^2 c^4}. \tag{45}$$

Equation (43) gives immediately

$$I = K^X(\vec{p}') u_0 (k_1 - k_2 u_0) (F - G u_0), \tag{46}$$

$$\text{where} \quad [1 - i\pi\rho(k_1 - k_2 u_0)] (F - G u_0) = 1. \tag{47}$$

We therefore obtain finally

$$\mu^2 c^2 (E_i^2 - M^2 c^4) U_{ij} = K^X(\vec{p}') (F - G u_0) (E_i - Mc^2 u_0) a_0 K(\vec{p}'). \tag{48}$$

The cross-section is then expressed as

$$d\phi = \frac{1}{4\pi^2 \hbar^4 c^4} \frac{E^2 E_N^2}{\mu^4 c^4 (E_i^2 - M^2 c^4)^2 E_i^2} a \cdot Sp[a_0 K(\vec{p}') K^X(\vec{p}') (E_i - Mc^2 a_0) (F^\dagger - G^\dagger a_0) \times \\ \times a_0 K(\vec{p}) K^X(\vec{p}) (F - G a_0) (E_i - Mc^2 a_0)], \quad (49)$$

where,

$$a = a^s = a^p = \frac{1}{2}; \quad a^v = \frac{1}{6}, \quad (50)$$

$$F = \frac{1}{D} [1 + \pi^2 \rho^2 (k_1^2 + k_2^2) + i\pi \rho k_1 (1 + \pi^2 \rho^2 (k_1^2 - k_2^2))], \quad (51)$$

$$G = \frac{1}{D} [-2\pi^2 \rho^2 k_1 k_2 + i\pi \rho k_2 (1 - \pi^2 \rho^2 (k_1^2 - k_2^2))], \quad (52)$$

$$\rho = \frac{p E E_N}{(2\pi \hbar)^3 c^2 E_i}, \quad (53)$$

$$D = \{1 - \pi^2 \rho^2 (k_1^2 - k_2^2)\}^2 + 4\pi^2 \rho^2 k_1^2. \quad (54)$$

Evaluating the spur we obtain finally the differential cross-section

$$d\phi = \frac{a}{E_i^2 (E_i^2 - M^2 c^4)} \left[\frac{4\hbar^2 E_i^2}{p^2} (E_i^2 - M^2 c^4) \pi^2 \rho^2 \{k_1^2 + k_2^2 + \pi^2 \rho^2 (k_1^2 - k_2^2)\} + \right. \\ \left. + Z^2 \cos \theta \{1 + \pi^2 \rho^2 (k_1^2 - k_2^2)\} \right], \quad (55)$$

which give the scattering of incident scalar, pseudoscalar and vector mesons provided we write the corresponding suffices s , p and v to k_1 , k_2 and to X , Y and Z .

It is to be noted that the cross-sections given by (55) for the scattering of pseudoscalar and vector mesons agree with those already obtained by Ma and Hsüeh.

§ 4. We now work out the scattering given by the following general scheme

Case	Incident	Final
I	Pseudoscalar	Pseudoscalar or Vector
II	Pseudoscalar	Vector
III	Vector	Pseudoscalar or Vector
IV	Vector	Pseudoscalar

The scattering cross section is given by (33) as before where

$$|U_{if}|^2 = |U_{if^v}|^2 + |U_{if^p}|^2$$

with

$$U_{if^v} = H_{if^v} + i\pi \sum \int U_{if^v} H_{f'v} \rho_{f'} d\Omega_{f'} + i\pi \sum \int U_{if^p} H_{f'p} \rho_{f'} d\Omega_{f'} \quad (57)$$

and

$$U_{ifv} = H_{ifv} + i\pi \sum \int U_{if'v} H_{f'v} \rho_{f'} d\Omega_{f'} + i\pi \sum \int U_{if'v} H_{f'v} \rho_{f'} d\Omega_{f'}. \quad (58)$$

Proceeding in the same way as in the last section, we can easily solve the equations (57) and (58) and obtain for the incident pseudoscalar meson

$$U_{ifv} = -i \frac{K^X(\vec{p})^v u_0(\mathbf{F} - \mathbf{G} u_0)(E_i - Mc^2 u_0) K^v(\vec{p}')}{\mu^2 c^2 (E_i^2 - M^2 c^4)}, \quad (59)$$

$$U_{ifp} = \frac{K^X(\vec{p})^v u_0(\mathbf{F} - \mathbf{G} u_0)(E_i - Mc^2 u_0) K^p(\vec{p}')}{\mu^2 c^2 (E_i^2 - M^2 c^4)}. \quad (60)$$

where \mathbf{F} and \mathbf{G} are given by

$$[1 - i\pi \rho(\mathbf{K}_1 - \mathbf{K}_2 u_0)](\mathbf{F} - \mathbf{G} u_0) = 1 \quad (61)$$

with

$$\mathbf{K}_1 = k_1^p + k_1^v \quad \text{and} \quad \mathbf{K}_2 = k_2^p + k_2^v. \quad (62)$$

Thus \mathbf{F} and \mathbf{G} are still obtained from the corresponding expressions (51) and (52) with k_1 and k_2 being replaced by \mathbf{K}_1 and \mathbf{K}_2 as given by (62). Similarly integral equations for incident vector meson can be easily calculated.

The cross-section finally works out to be

$$d\phi = \frac{a}{E_i^2 (E_i^2 - M^2 c^4)} \mathbf{D} \left[\frac{4\hbar^2 E_i^2}{p^2} \pi^2 \rho^2 \{ \mathbf{K}_1 k_1^b (1 + \pi^2 \rho^2 (\mathbf{K}_1^2 - \mathbf{K}_2^2)) \right. \\ \left. + \mathbf{K}_2 k_2^b (1 - \pi^2 \rho^2 (\mathbf{K}_1^2 - \mathbf{K}_2^2)) \} (E_i^2 - M^2 c^4) + \mathbf{Z} \mathbf{Z}^b \cos \theta (1 + \pi^2 \rho^2 (\mathbf{K}_1^2 - \mathbf{K}_2^2)) \right], \quad (64)$$

where \mathbf{D} is still obtained from the previous expression (54) with k_1 and k_2 being replaced by \mathbf{K}_1 and \mathbf{K}_2 as given by (62) and

where	b	\mathbf{K}_1	\mathbf{K}_2	\mathbf{Z}	
I	p	\mathbf{K}_1	\mathbf{K}_2	$\mathbf{Z}^p + \mathbf{Z}^v$	
II	p	k_1^v	k_2^v	\mathbf{Z}^v	(65)
III	v	\mathbf{K}_1	\mathbf{K}_2	$\mathbf{Z}^p + \mathbf{Z}^v$	
IV	v	k_1^p	k_2^p	\mathbf{Z}^p	

It is interesting to study the cross sections in the extreme cases of non-relativistic and relativistic approximations. In the non-relativistic approximation when $p \ll \mu c$ (64) reduces to

$$d\phi = a \frac{4\hbar^2 \pi^2 \rho^2}{p^2 (1 + 4\pi^2 \rho^2 \mathbf{K}_1^2)} 2\mathbf{K}_1 k_1^b d\Omega, \quad (66)$$

and in the extreme relativistic approximation when $p \gg Mc$,

$$d\phi = a \frac{1}{(1 + \pi^2 \rho^2 \mathbf{K}_1^2)} \left[\frac{4\hbar^2 \pi^2 \rho^2 \mathbf{K}_1 k_1^b}{p^2} + \frac{\mathbf{Z} \mathbf{Z}^b \cos \theta}{16c^4 p^4} \right] d\Omega. \quad (67)$$

The results for the differential cross sections for the cases under consideration are tabulated below:

A. In the non-relativistic approximation in which $p \ll \mu c$

Differential cross sections $d\phi$

$$\left. \begin{array}{l} \text{I} \quad (f'^p - g^p)^2 \{ 3(f^v + g^v)^2 + (f'^p - g^p)^2 \} \\ \text{II} \quad 3[(f'^p - g^p)(f^v + g^v)]^2 \\ \text{III} \quad (f^v + g^v)^2 \{ 3(f^v + g^v)^2 + (f'^p - g^p)^2 \} \\ \text{IV} \quad [(f^v + g^v)(f'^p - g^p)]^2 \end{array} \right\} \times \left[\frac{M}{c^2(\mu + M)(\mu + 2M)} \right]^2 \frac{1}{1 + K_a^2} d\Omega. \quad (a = \text{I} \dots \text{IV})$$

Here

$$K_{\text{I}} = K_{\text{III}} = \frac{pM}{\hbar c^2} \cdot \frac{\{ 3(f^v + g^v)^2 + (f'^p - g^p)^2 \}}{(\mu + M)(\mu + 2M)}; \quad K_{\text{II}} = \frac{pM}{\hbar c^2} \cdot \frac{3(f^v + g^v)^2}{(\mu + M)(\mu + 2M)},$$

$$K_{\text{IV}} = \frac{pM}{\hbar c^2} \cdot \frac{(f'^p - g^p)^2}{(\mu + M)(\mu + 2M)}; \quad f'^p = \frac{1}{2} f^p,$$

B. In the relativistic case in which $\mu c \approx p \ll Mc$ and retaining terms up to p^2 only

Differential cross sections $d\phi$

$$\left. \begin{array}{l} \text{I} \quad \frac{f'^{p^2}}{\mu c^2} \cdot \frac{G^2}{\mu c^2} \\ \text{II} \quad \frac{f'^{p^2}}{\mu c^2} \frac{(g^{v^2} + 2f^{v^2})}{\mu c^2} \\ \text{III} \quad \frac{1}{3} \frac{(g^{v^2} + 2f^{v^2})}{\mu c^2} \cdot \frac{G^2}{\mu c^2} \\ \text{IV} \quad \frac{1}{3} \frac{(g^{v^2} + 2f^{v^2})}{\mu c^2} \frac{f'^{p^2}}{\mu c^2} \end{array} \right\} \times \left(\frac{p^2}{\mu E} \right)^2 \frac{1}{1 + L_a^2} d\Omega \quad (a = \text{I} \dots \text{IV})$$

$$L_{\text{I}} = L_{\text{III}} = \frac{p^3}{\hbar^3 \chi^2} \frac{G^2}{E}; \quad L_{\text{II}} = \frac{p^3}{\hbar^3 \chi^2} \frac{(g^{v^2} + 2f^{v^2})}{E}; \quad L_{\text{IV}} = \frac{p^3}{\hbar^3 \chi^2} \frac{f'^{p^2}}{E}; \quad G^2 = (f'^{p^2} + g^{v^2} + 2f^{v^2}),$$

C. In the extreme relativistic case in which $p \gg Mc$

Here the cross section is given by

$$d\phi = \left. \begin{array}{l} \text{Same expression} \\ \text{as in} \\ B \end{array} \right\} \frac{1}{2} \left(\frac{p}{\mu c} \right)^2 (1 + \cos \theta) \frac{1}{1 + \frac{1}{4} L_a^2} d\Omega \quad (a = \text{I} \dots \text{IV}).$$

The case I of (B) agrees with the expression already calculated by Heitler and Peng⁹⁾ and Majumdar.¹⁰⁾

In the non-relativistic limit it will be noticed that the cross sections in all the cases reduce to the Thomson like formula

$$\left(\frac{g^2}{Mc^2} \right)^2 \frac{1}{1 + K_a^2} d\Omega.$$

For not too high energies of the incident meson *i.e.* $\mu c \approx p \ll Mc$ the cross section first increases as $(p^2/E)^2$ and then ultimately decreases as $(1/p^2)$ due to radiation damping, whereas in the extreme relativistic region in which $E \gg Mc^2$ the cross section first increases as the second power of p , then finally decreases as $(1/p^2)$.

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Relativistic Two-Body Problem in Quantum Theory

Kiyoshi SAKUMA, Naomi SHÔNO and Tadashi OUCHI

Department of Physics, Hiroshima University

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A general method to treat the quantal two-body problem relativistically is developed on the line of Dirac's many-time theory. The case where two charged fermions interact with each other through the electromagnetic field is investigated as an example. We construct a canonical transformation which eliminates the electromagnetic field variables in the original Schrödinger equations, and derive the potential energy between two charged particles. Finally, some properties of the simultaneous wave equations for the system of two particles are discussed, and a method to treat the stationary state of that system is indicated.

§ 1. Introduction

Tomonaga et al. and Schwinger have succeeded in the extension of the field theory of Heisenberg-Pauli into a completely relativistic formalism. By means of this formalism, we may calculate the potential energy between two fermions coupled by the Bose field, but this formalism seems to give difficulty for the consistent treatment of the stationary state of the two-body system. On the other hand, the S -matrix theory of Heisenberg, which is another approach to the problem—integral formalism—, affords a hope to deal with the bound system as well as the scattering process. However, in the present state of the theory, few investigations have been tried on this problem, as far as we know. Thus we had better attempt to make another approach.

As early as 1932, Dirac proposed the many-time theory¹⁾ as a relativity-theoretical generalization of Heisenberg-Pauli's. But in this theory, it is to be noted, the second quantization was not applied to the matter field. Thus, for a system of many material particles, the same number of equations as that of particles was required to be considered simultaneously, while the Bose fields as the interaction between matter fields are the free fields. This theory is very useful for the two-body system, in that we can retain the particle picture throughout the calculations.

On the other hand, for the non-stationary problems, Feynman²⁾ has succeeded in solving the Schrödinger equations without the procedure of the second quantization. That is, he solves the Schrödinger equation with the aid of Green's function which plays the role of the D -function (or associated D -function \overline{D} , $D^{(1)}$ etc.) which appeared in the procedure of the second quantization. Therefore, by means of the many-time formalism, we also deal with the non-stationary state

for the many-body system, if the structure of the simultaneous equations is known. And we may add that the above structure has been partly discussed by Bloch,³⁾ on which more details will be given in the second section of this paper. Thus, it is with the many-time formalism, we assert, that both stationary state and non-stationary state can be attacked. It is especially to be noted that we can handle relativistically the stationary state of the two-body system in a consistent manner, preserving the particle aspect.

Recently Thirring⁴⁾ proposed a theory on the line of the many-time formalism, and discussed on the mass-defect of a bound system. But the foundation of his field equation is obscure, and our present investigation shows that the potentials in his equation do not seem to be adequate as far as the relativistic treatment of the two-body system is intended.

The second section of the present paper begins with a simple description of the many-time theory for the case where two charged fermions interact through the electromagnetic field. And the conditions for the integrability of equations of motion are investigated. The part (a) of the third section is devoted to a discussion on the covariant elimination of the longitudinal field which is replaced by the Coulomb potential. The part (b) concerns itself with the general derivation of the potential by performing a canonical transformation in the configuration space. The concept of potential seems to be valid to some extent even in the relativistic system. In the fourth section, we discuss the necessary and sufficient condition for the stationary state of the two-body system, retaining the relativistic formalism as perfectly as possible. Then, we construct the convenient equations to solve the stationary state, the potential energy being contained in the equation. This enables us to examine the structure of the Kemmer's⁵⁾ equation which he has taken for the deuteron problem.

The present theory can be applied without difficulty to other cases, for example, to the deuteron problem, where nucleons interact through the meson field.

§ 2. Fundamental equations of motion for two-body system

According to Dirac's many-time theory, the wave function $\Psi(x_I, x_{II})$ for the two-body system satisfies the following equations:*

$$F_I \Psi(x_I, x_{II}) = 0, \quad (1a)$$

$$F_{II} \Psi(x_I, x_{II}) = 0, \quad (1b)$$

where

$$F_I = i\gamma_\mu^I \left(p_\mu^I - \frac{e}{c} A_\mu(x_I) \right) + m_I c^2, \quad (2a)$$

* Hereafter Greek and Latin subscripts assume values ranging from 1 to 4 and from 1 to 3, respectively and a repeated index is to be so summed, and the symbol I or II signifies the particle I or II , respectively.

$$F_{II} = ic\gamma_{\mu}^{II} \left(\dot{p}_{\mu}^{II} - \frac{e}{c} A_{\mu}(x_{II}) \right) + m_{II}c^2, \quad (2b)$$

in the case where two charged fermions interact through the electromagnetic field. The notation γ_{μ} is used for the Hermitian matrices that obey the following commutation relations

$$\{\gamma_{\mu}^I, \gamma_{\nu}^I\} = 2\delta_{\mu\nu}, \quad \{\gamma_{\mu}^{II}, \gamma_{\nu}^{II}\} = 2\delta_{\mu\nu}. \quad (3)$$

The four dimensional coordinate vector x^{μ} is denoted by

$$x^{\mu} = (\mathbf{r}, x^4) = (\mathbf{r}, ix^0), \quad x^0 = ct,$$

and the energy-momentum four vector p_{μ} is represented by

$$p_{\mu} = -i\hbar \frac{\partial}{\partial x^{\mu}} = -i\hbar \partial_{\mu}.$$

The four vector potential A_{μ} of the electromagnetic field satisfies the relations

$$\square A_{\mu}(x) = 0, \quad [A_{\mu}(x), A_{\nu}(x')] = i\hbar c \delta_{\mu\nu} D(x - x'), \quad (4)$$

where D is the usual invariant D -function, and the following supplementary condition must be added to the equations of motion in order to be consistent with the commutation relations

$$[\partial_{\mu} A_{\mu}(x) - eD(x - x_I) - eD(x - x_{II})] \Psi = 0. \quad (5)$$

Now, the integrability conditions for the system of equations (1a) and (1b) become

$$[\gamma_{\mu}^I, \gamma_{\nu}^I] = 0, \quad \text{for all } \mu, \nu, \quad (6)$$

and

$$D(x_I - x_{II}) \Psi(x_I, x_{II}) = 0, \quad (7)$$

where the latter condition was formerly discussed in details by Bloch.

As is well known, the relations (3) and (6) are satisfied by 16 rows and columns matrices, but not by any 4 rows and columns matrix, and the representation of the former matrix will be conveniently given with the aid of Kronecker's symbol:⁽⁶⁾

$$\begin{aligned} \gamma_k^I &= \sigma_2 \times I \times I \times \sigma_k, & \gamma_k^{II} &= I \times \sigma_2 \times \sigma_k \times I, \\ \gamma_4^I &= \sigma_3 \times I \times I \times I, & \gamma_4^{II} &= I \times \sigma_3 \times I \times I, \end{aligned} \quad (8)$$

where σ 's and I are Pauli's 2-2 spin matrices and 2-2 unit matrix respectively.

As the fundamental relativistic equations for the stationary state of the two-body system, we will employ, for convenience's sake, the following equations instead of (1a) and (1b):

$$(\gamma_4^I F_I + \gamma_4^{II} F_{II}) \Psi = 0, \quad (9a)$$

$$(\gamma_4^I F_I - \gamma_4^{II} F_{II}) \Psi = 0. \quad (9b)$$

Then it becomes our task to solve the equations (9a) and (9b) simultaneously under the conditions of (4), (5) and (7). As will be shown later, when we take the time-points of two particles to be the same, (9b) may be regarded as an auxiliary equation which is identically satisfied as the relation between energies of two particles, and thus the equation (9a) corresponds to the equation of Heisenberg-Pauli or that of Kemmer.

§ 3. Canonical transformation in the many-time theory and the potential energy

(a) Separation of the Coulomb potential

Both in the classical and quantal electrodynamics, it was possible to eliminate the scalar potential and the longitudinal part of the vector potential, leaving only the Coulomb potential and the transverse vector potential. This conventional procedure will be preserved even in the many-time theory in a covariant manner. We can, after the procedure of Schwinger,⁷⁾ replace the electromagnetic field vector $A_\mu(x)$ by two scalar fields, $\Lambda(x)$ and $\Lambda'(x)$, together with a restricted vector field $\mathfrak{A}_\mu(x)$, in such a way that the supplementary condition (5) involves only the scalar fields, while the equations of motion contain only $\mathfrak{A}_\mu(x)$. The decomposition will be conveniently expressed with the aid of an arbitrary time-like unit vector $n_\mu (n_\mu^2 = -1)$;

$$A_\mu(x) = n_\mu n_\nu \partial_\nu \Lambda(x) - (\partial_\mu + n_\mu n_\nu \partial_\nu) \Lambda'(x) + \mathfrak{A}_\mu(x), \quad (10)$$

where

$$n_\mu \mathfrak{A}_\mu = 0, \quad \partial_\mu \mathfrak{A}_\mu(x) = 0.$$

The commutation relations of the three fields thus defined are the same as Schwinger's, for example :

$$[\mathfrak{A}_\mu(x), \mathfrak{A}_\nu(x')] = i\hbar c (\delta_{\mu\nu} + n_\mu n_\nu) D(x-x') - i\hbar c (\partial_\mu + n_\mu n_\lambda \partial_\lambda) (\partial_\nu + n_\nu n_\sigma \partial_\sigma) \mathfrak{D}(x-x'), \quad (11)$$

where $\mathfrak{D}(x)$ is defined by

$$(n_\mu \partial_\mu)^2 \mathfrak{D}(x) = D(x), \quad \square \mathfrak{D}(x) = 0, \quad (12)$$

and is an odd function of the coordinates.

The supplementary condition involves only the scalar fields and, in fact, only combination $\Lambda(x) - \Lambda'(x)$, since

$$[\Lambda(x) - \Lambda'(x) - e\mathfrak{D}(x-x_I) - e\mathfrak{D}(x-x_{II})] \Psi = 0. \quad (13)$$

Now we introduce the canonical transformation

$$\Psi = \exp(-iG) \Phi, \quad G = \frac{e}{\hbar c} \{ \Lambda'(x_I) + \Lambda'(x_{II}) \}, \quad (14)$$

here we must note the fact that

$$\exp(-iG) \neq \exp\left\{-\frac{ie}{\hbar c} A'(x_I)\right\} \times \exp\left\{-\frac{ie}{\hbar c} A'(x_{II})\right\}.$$

Therefore, we define the operator $\exp(-iG)$ in the following way:

$$\exp(-iG) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \overbrace{G \cdot G \cdot \dots \cdot G}^n, \quad (15)$$

with

$$G = \frac{e}{\hbar c} \{A'(x_I) + A'(x_{II})\}.$$

The new equations of motion and supplementary condition are

$$[\exp(iG) \cdot F_I \cdot \exp(-iG)]\Phi = 0, \quad (16a)$$

$$[\exp(iG) \cdot F_{II} \cdot \exp(-iG)]\Phi = 0 \quad (16b)$$

and

$$[\exp(iG) \{A(x) - A'(x) - e\mathcal{D}(x - x_I) - e\mathcal{D}(x - x_{II})\} \exp(-iG)]\Phi = 0. \quad (17)$$

Since the following formula is true for any operator F

$$\exp(iG) F \exp(-iG) = F + i[G, F] - \frac{1}{2!}[G, [G, F]] + \dots, \quad (18)$$

the supplementary condition (17) becomes simply

$$[A(x) - A'(x)]\Phi = 0 \quad (17')$$

and the equation of motion (16a)

$$\begin{aligned} & \left[ic\gamma_{\mu}^I \left\{ p_{\mu}^I - \frac{e}{c} \mathcal{A}_{\mu}(x_I) + \frac{e^2}{2c} \partial_{\mu}^I \mathcal{D}(x_{II} - x_I) + \frac{e^2}{c} n_{\mu} n_{\nu} \partial_{\nu}^I \mathcal{D}(x_{II} - x_I) \right\} \right. \\ & \quad + m_I c^2 + ic \lim_{x \rightarrow x_I} \left\{ \frac{e^2}{2c} \gamma_{\mu}^I \partial_{\mu}^I \mathcal{D}(x - x_I) + \frac{e^2}{c} \gamma_{\mu}^I n_{\mu} n_{\nu} \partial_{\nu}^I \mathcal{D}(x - x_I) \right\} \\ & \quad \left. - ic\gamma_{\mu}^I n_{\mu} n_{\nu} \partial_{\nu}^I \{A(x_I) - A'(x_I)\} \right] \Phi = 0 \end{aligned} \quad (16a')$$

and by interchanging I for II in (16a) and vice versa, we have the corresponding equation for (16b). In view of the supplementary condition (17'), Eq. (16a') is reduced to

$$\begin{aligned} & \left[ic\gamma_{\mu}^I \left\{ p_{\mu}^I - \frac{e}{c} \mathcal{A}_{\mu}(x_I) + \frac{e^2}{2c} \partial_{\mu}^I \mathcal{D}(x_{II} - x_I) + \frac{e^2}{c} n_{\mu} n_{\nu} \partial_{\nu}^I \mathcal{D}(x_{II} - x_I) \right\} \right. \\ & \quad \left. + m_I c^2 \right] \Phi = 0 \end{aligned} \quad (19a)$$

neglecting the static self-energy term $\lim_{x \rightarrow x_I} \{ \}$. We have thereby obtained the

equations of motion for Φ which no longer contain the electromagnetic field variables involved in the supplementary condition. The additional term thus introduced must correspond evidently to the covariant generalization of the Coulomb interaction between charged fermions. To exhibit this property somewhat more clearly, we choose such arbitrary time-like unit vector n_μ as normal to a certain space-like plane surface containing two space-like points x_I and x_{II} , then the following relations hold:

$$\begin{aligned} (\partial_\mu^I + n_\mu n_\nu \partial_\nu^I) \mathfrak{D}(x_I - x_{II}) &= 0, \\ n_\mu \partial_\mu^I \mathfrak{D}(x_I - x_{II}) &= \frac{1}{4\pi} [(x_I^\mu - x_{II}^\mu)^2]^{-\frac{1}{2}}. \end{aligned} \quad (20)$$

These relations enable the term containing \mathfrak{D} -function in the equation (19a) to be simplified, yielding

$$-\frac{ie^2 \gamma_\mu^I n_\mu}{8\pi} [(x_I^\mu - x_{II}^\mu)^2]^{-\frac{1}{2}} \equiv V_I. \quad (21)$$

The procedure of Heisenberg-Pauli theory corresponds to the special choice $n_\mu = (0, 0, 0, i)$. In this case $\gamma_\mu^I V_I$ becomes a half of the usual three dimensional Coulomb potential;

$$\gamma_\mu^I V_I = \frac{e^2}{8\pi r}, \quad r = |\mathbf{r}_I - \mathbf{r}_{II}|. \quad (22a)$$

The corresponding procedure is applied to Eq. (16b) and we get

$$\gamma_\mu^{II} V_{II} = \frac{e^2}{8\pi r}. \quad (22b)$$

Thus, when the time-points of two particles are taken to be the same, Eq. (9a) is entirely equivalent to the equation of Heisenberg-Pauli. As for Eq. (9b), its meaning will be clarified when the transverse potential \mathfrak{A}_μ is eliminated in the next subsection.

(b) Møller interaction

We show, in this subsection, the general method to derive the interaction potential, whose relativity-theoretical interpretation can be furnished more clearly, when we take the procedure of the canonical transformation in the many-time theory, than when the perturbation-theoretical one is taken.

In the previous subsection we derived the following equations:

$$[-i\hbar c \partial_0^I + ic\gamma_\mu^I \gamma_k^I p_k^I + \gamma_4^I m_I c^2 + \gamma_4^I V_I - ie\gamma_\mu^I \gamma_\mu^I \mathfrak{A}_\mu(x_I)]\Phi = 0, \quad (19a)$$

$$[-i\hbar c \partial_0^{II} + ic\gamma_\mu^{II} \gamma_k^{II} p_k^{II} + \gamma_4^{II} m_{II} c^2 + \gamma_4^{II} V_{II} - ie\gamma_\mu^{II} \gamma_\mu^{II} \mathfrak{A}_\mu(x_{II})]\Phi = 0. \quad (19b)$$

These equations are given in the Schrödinger representation. To derive the interaction potential by the method of the canonical transformation, we perform

a unitary transformation which changes the Schrödinger representation into the interaction representation. Namely, we transform the Schrödinger function $\Phi(x_I, x_{II})$ into $\tilde{\Phi}(x_I, x_{II})$ of the interaction representation by means of a unitary operator $U(x_I^0, x_{II}^0)$:

$$\Phi = U\tilde{\Phi} = U_I(x_I^0)U_{II}(x_{II}^0)\tilde{\Phi}, \quad (23)$$

where

$$U_I = \exp\left\{-\frac{i}{\hbar c}H_I^0x_I^0\right\}, \quad U_{II} = \exp\left\{-\frac{i}{\hbar c}H_{II}^0x_{II}^0\right\},$$

$$[U_I, U_{II}] = 0, \quad (24)$$

and

$$H_I^0 = ic\gamma_{kl}^I\gamma_k^I p_l^I + \gamma_4^I m_I c^2,$$

$$H_{II}^0 = ic\gamma_{kl}^{II}\gamma_k^{II} p_l^{II} + \gamma_4^{II} m_{II} c^2. \quad (25)$$

Then the transformed equations become

$$[-i\hbar c\partial_0^I - ie\tilde{\gamma}_4^I\tilde{\gamma}_\mu^I\tilde{\mathfrak{A}}_\mu(x_I) + \tilde{\gamma}_4^I\tilde{V}_I]\tilde{\Phi} = 0, \quad (26a)$$

$$[-i\hbar c\partial_0^{II} - ie\tilde{\gamma}_4^{II}\tilde{\gamma}_\mu^{II}\tilde{\mathfrak{A}}_\mu(x_{II}) + \tilde{\gamma}_4^{II}\tilde{V}_{II}]\tilde{\Phi} = 0, \quad (26b)$$

where the symbol \sim is used to denote the transformed quantities, and $\tilde{\gamma}$, $\tilde{\mathfrak{A}}(x)$ are defined by

$$\tilde{\gamma}_\mu^I = U_I^{-1}\gamma_\mu^I U_I, \quad \tilde{\mathfrak{A}}_\mu(x_I) = U_I^{-1}\mathfrak{A}_\mu(x_I)U_I = \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x_I^0), \quad (27a)$$

$$\tilde{\gamma}_\mu^{II} = U_{II}^{-1}\gamma_\mu^{II} U_{II}, \quad \tilde{\mathfrak{A}}_\mu(x_{II}) = U_{II}^{-1}\mathfrak{A}_\mu(x_{II})U_{II} = \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x_{II}^0). \quad (27b)$$

Now, the transformed $\tilde{\gamma}$'s are time-dependent—operators in the Heisenberg representation—, but the commutation relations between these operators retain the original form, that is:

$$\{\tilde{\gamma}_\mu^I(x_I^0), \tilde{\gamma}_\nu^I(x_I^0)\} = \{\tilde{\gamma}_\mu^{II}(x_{II}^0), \tilde{\gamma}_\nu^{II}(x_{II}^0)\} = 2\delta_{\mu\nu},$$

$$[\tilde{\gamma}_\mu^I(x_I^0), \tilde{\gamma}_\nu^{II}(x_{II}^0)] = 0, \quad \text{for all } \mu, \nu. \quad (28)$$

And it is easily shown that the time-dependence of γ 's is determined by the following equations:

$$i\hbar c \frac{d\tilde{\gamma}_I^I}{dx_I^0} = [\tilde{\gamma}_I^I, \tilde{H}_I^0], \quad i\hbar c \frac{d\tilde{\gamma}_I^I}{dx_{II}^0} = 0, \quad (29)$$

where

$$\tilde{H}_I^0 = U_I^{-1}H_I^0U_I = \hbar c\tilde{\gamma}_4^I\tilde{\gamma}_k^I\frac{\partial}{\partial\tilde{x}_I^k} + \tilde{\gamma}_4^Im_Ic^2,$$

and the corresponding relations for $\tilde{\gamma}^{II}$'s can be obtained by interchanging suffixes I and II in (29). On the other hand, the transformed $\tilde{\mathfrak{A}}_\mu(x)$ satisfies evidently

the following commutation relation :

$$\begin{aligned} [\tilde{\mathfrak{A}}_\mu(x_I), \tilde{\mathfrak{A}}_\nu(x_{II})] &= [\mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x_I^0), \mathfrak{A}_\nu(\tilde{\mathbf{r}}_{II}, x_{II}^0)] \\ &= i\hbar c (\delta_{\mu\nu} + n_\mu n_\nu) D(\tilde{\mathbf{r}}_I - \tilde{\mathbf{r}}_{II}, x_I^0 - x_{II}^0) \\ &\quad - i\hbar c (\tilde{\partial}_\mu^I + n_\mu n_\lambda \tilde{\partial}_\lambda^I) (\tilde{\partial}_\mu^I + n_\mu n_\sigma \tilde{\partial}_\sigma^I) \mathfrak{D}(\tilde{\mathbf{r}}_I - \tilde{\mathbf{r}}_{II}, x_I^0 - x_{II}^0). \end{aligned} \quad (30)$$

Now, our task is to solve the Eqs. (26a) and (26b) simultaneously under the conditions of (28), (29) and (30). This can be carried out quite analogously to the case of Tomonaga-Schwinger theory, though the equations to be solved in our case are simultaneous ones. Thus we perform the following canonical transformation :

$$\tilde{\psi} = \exp[-iS] \tilde{\phi} = \exp[-i(S_I + S_{II})] \tilde{\phi} \quad (31)$$

where

$$\begin{aligned} S_I &= -\frac{ie}{\hbar c} \int_{-\infty}^{x_I^0} \tilde{\gamma}_4^I \tilde{\gamma}_\mu^I \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x^0) dx^0, \\ S_{II} &= -\frac{ie}{\hbar c} \int_{-\infty}^{x_{II}^0} \tilde{\gamma}_4^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x^0) dx^0. \end{aligned} \quad (32)$$

Here, since S_I and S_{II} are not commutable, we define the operator $\exp[-iS]$ to be such as shown in (15). The transformed equations of motion for $\tilde{\psi}$ become

$$\left\{ -i\hbar c \partial_0^I + \frac{e}{2} [S, \tilde{\gamma}_4^I \tilde{\gamma}_\mu^I \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x_I^0)] + \tilde{\gamma}_4^I \tilde{V}_I \right\} \tilde{\psi} = 0, \quad (33a)$$

$$\left\{ -i\hbar c \partial_0^{II} + \frac{e}{2} [S, \tilde{\gamma}_4^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x_{II}^0)] + \tilde{\gamma}_4^{II} \tilde{V}_{II} \right\} \tilde{\psi} = 0 \quad (33b)$$

in which we have retained only the second order interactions. We neglect, hereafter, the terms $[S_I, \tilde{\gamma}_4^I \tilde{\gamma}_\mu^I \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I)]$ and $[S_{II}, \tilde{\gamma}_4^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II})]$ in Eqs. (33a) and (33b) respectively, which represent the self-energy. On the other hand, the interaction energy of Møller type (excluding Coulomb energy) becomes,

$$\frac{e}{2} [S_{II}, \tilde{\gamma}_4^I \tilde{\gamma}_\mu^I \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x_I^0)] \equiv B_I, \quad (34a)$$

$$\frac{e}{2} [S_I, \tilde{\gamma}_4^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x_{II}^0)] \equiv B_{II}. \quad (34b)$$

Although $\tilde{\gamma}$'s and $\tilde{\mathbf{r}}$ are time-dependent, the actual evaluation of this potential can be easily done with the aid of a method proposed by Nambu.⁸⁾ Here we describe only the result obtained, leaving the detailed calculations for Appendix at the end of this paper. Expanding (34a) and (34b) in powers of $(v_I/c)(v_{II}/c)$, we have, as is expected, a half of the Breit potential in the order of $(v_I/c)(v_{II}/c)$ respectively :

$$B_I = B_{II} = \frac{e^2}{16\pi} \frac{1}{\tilde{r}} \left\{ (\tilde{\gamma}_4^I \tilde{\gamma}_k^I) (\tilde{\gamma}_4^{II} \tilde{\gamma}_k^{II}) + \frac{(\tilde{\gamma}_4^I \tilde{\gamma}_i^I \tilde{r}_i) (\tilde{\gamma}_4^{II} \tilde{\gamma}_j^{II} \tilde{r}_j)}{\tilde{r}^2} \right\}, \quad (35)$$

where we assumed the time-points of both the particle I and II to be the same. Terms of the order of $(v_I/c)^2(v_{II}/c)^2$ representing the recoil effect, depend, as is natural, on the velocity of particles.

Further detailed discussion about the potential energy, for example about the fourth order potential of charge e , is left over. Our theory differs from Tomonaga-Schwinger's in the point that in the latter the second quantization is applied to the matter field, but not in the former. It may be interesting to investigate this difference in the potentials of the fourth order or higher with respect to e .

Finally, we return to the Schrödinger representation from the interaction representation by means of the inverse transformation of (23),

$$\tilde{\psi} = U^{-1}\psi = U_{II}^{-1}U_I^{-1}\psi. \quad (36)$$

Then the operators with the symbol \sim return to those without \sim and Eqs. (33a) and (33b) become as follows:

$$[-i\hbar c\partial_0^I + H_I^0 + \gamma_4^I V_I + B_I]\psi = 0, \quad (37a)$$

$$[-i\hbar c\partial_0^{II} + H_{II}^0 + \gamma_4^{II} V_{II} + B_{II}]\psi = 0. \quad (37b)$$

§ 4. Some properties of equations for the stationary state

When we denote the potentials derived in section 3 as J_I and J_{II} , the fundamental equations become:

$$[-i\hbar c\partial_0^I + H_I^0 + J_I]\psi = 0, \quad (37a')$$

$$[-i\hbar c\partial_0^{II} + H_{II}^0 + J_{II}]\psi = 0 \quad (37b')$$

where, for the time being, we take J_I and J_{II} as general potentials which are not restricted to the same time-points of two particles.

To solve these equations generally, it will be desired to separate the motion of particles into the motion of centre of gravity and the relative motion, the former showing the motion of the two-particle system as a whole, that is, the motion of a Bose particle. But, reserving this problem for future occasion, we will discuss here only the stationary state.

In our theory also, $\psi^*(x_I, x_{II})\psi(x_I, x_{II})$ has the meaning of the probability density, then we can take the following condition

$$\psi^*\psi = \text{constant}, \quad (38)$$

as the generalized definition for the stationary state. The wave function ψ which satisfies (38) can generally be given in the form:

$$\psi(x_I, x_{II}) = \varphi(\mathbf{r}_I, \mathbf{r}_{II}) \exp[iE(x_I^0, x_{II}^0)], \quad (39)$$

where E depends only on x_I^0 and x_{II}^0 explicitly. Substituting (39) into (37a'), we have

$$\hbar c \frac{\partial E}{\partial x_I^0} + \frac{(\varphi^* H_I^0 \varphi)}{(\varphi^*, \varphi)} = -\exp(-iE) \frac{\{(\varphi^* J_I \varphi) \exp(iE)\}}{(\varphi^*, \varphi)}. \quad (37a'')$$

In order that this equation may hold identically, J_I should be divided into two parts.

$$J_I^0 = J_I^0(\mathbf{r}_I, \mathbf{r}_{II}, \partial/\partial x_I^k, \partial/\partial x_{II}^k), \quad (40a)$$

$$J_I' = J_I'(x_I^0, x_{II}^0, \partial/\partial x_I^0, \partial/\partial x_{II}^0) \quad (40b)$$

where J_I^0 is a function only of $\mathbf{r}_I, \mathbf{r}_{II}, \partial/\partial x_I^k$ and $\partial/\partial x_{II}^k$, while J_I' depends only on $x_I^0, x_{II}^0, \partial/\partial x_I^0$ and $\partial/\partial x_{II}^0$. On the other hand, the potential derived in the previous section can be transformed into the form (40a) through a real transformation, but this is not the case for (40b). Thus we put $J_I' = 0$, and only such potentials as take the form (40a) survive. This means that, in the actual calculations, the potentials for the stationary state should be taken as those which correspond to the case where the time-points of the two particles are the same. Accordingly, the Eq. (37a'') becomes

$$-\hbar c \frac{\partial E}{\partial x_I^0} = \frac{(\varphi^* (H_I^0 + J_I^0) \varphi)}{(\varphi^*, \varphi)} \text{const.} = E_I, \quad (41)$$

whence we have

$$E = -\frac{E_I x_I^0}{\hbar c} + E'(x_{II}^0). \quad (42)$$

The analogous discussion can be made for (37b') and we have

$$-\hbar c \frac{\partial E'}{\partial x_{II}^0} = -\hbar c \frac{dE'}{dx_{II}^0} = \frac{(\varphi^* (H_{II}^0 + J_{II}^0) \varphi)}{(\varphi^*, \varphi)} = \text{const.} = E_{II}. \quad (43)$$

Thus, for the following solution of the stationary state;

$$\Phi = \varphi(\mathbf{r}_I, \mathbf{r}_{II}) \exp \left[-\frac{i}{\hbar c} (E_I x_I^0 + E_{II} x_{II}^0) \right]. \quad (44)$$

Eqs. (37a') and (37b') are reduced to

$$[-E_I + H_I^0 + J_I^0] \varphi = 0, \quad (45a)$$

$$[-E_{II} + H_{II}^0 + J_{II}^0] \varphi = 0. \quad (45b)$$

From Eqs. (45a) and (45b), we can construct the equations corresponding to Eqs. (9a) and (9b) as follows:

$$[-E_I - E_{II} + H_I^0 + H_{II}^0 + V^0 (= J_I^0 + J_{II}^0)] \varphi = 0, \quad (46a)$$

$$[-E_I + E_{II} + H_I^0 - H_{II}^0] \varphi = 0. \quad (46b)$$

Eq. (46b) can be reduced to

$$E_I - E_{II} = \pm \sqrt{c^2 p_I^2 + m_I^2 c^4} \mp \sqrt{c^2 p_{II}^2 + m_{II}^2 c^4} \quad (46b')$$

which is identically satisfied and indicates the possibility of the negative energy states for each particle.

The energy-momentum four vector of the centre of gravity is covariantly defined by

$$P_\mu = p_\mu^I + p_\mu^{II}.$$

Then the proper mass M_0 of the centre of gravity is defined, in the centre of gravity system ($\mathbf{P}=0$), by

$$P_\mu^2 = -(E_I + E_{II})^2/c^2 = -M_0^2 c^2$$

and Eq. (46a) in this system becomes

$$\pm M_0 c^2 \varphi = [\beta^I m_I c^2 + \beta^{II} m_{II} c^2 + c(a_k^I - a_k^{II})p_k + V^0]\varphi \quad (48)$$

with

$$p_k = p_k^I = -p_k^{II}; \quad a_k^\sigma = i\gamma_4^\sigma \gamma_k^\sigma, \quad \beta^\sigma = \gamma_4^\sigma \quad \sigma = 1, 2.$$

We can obtain the mass spectrum of the centre of gravity by solving the above equation. This form of equation had previously been proposed by Kemmer for the deuteron problems.

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Appendix

Here we calculate the Möller interaction energy of the following type:

$$\frac{e}{2} [S_I, \tilde{\gamma}_I^J \tilde{\gamma}_\mu^J \mathfrak{A}_\mu(\tilde{\mathbf{r}}_I, x_I^0)] = B_I, \quad (34a)$$

$$\frac{e}{2} [S_I, \tilde{\gamma}_I^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x_{II}^0)] = B_{II}. \quad (34b)$$

As S_I and S_{II} are operators given by (32), (34a) becomes as follows:

$$B_I = -\frac{ie^2}{2\hbar c} \int_{-\infty}^{x_{II}^0} [\tilde{\gamma}_I^{II} \tilde{\gamma}_\mu^{II} \mathfrak{A}_\mu(\tilde{\mathbf{r}}_{II}, x^0), \tilde{\gamma}_I^J \tilde{\gamma}_\nu^J \mathfrak{A}_\nu(\tilde{\mathbf{r}}_I, x_I^0)] dx^0. \quad (A1)$$

The integral $\int_{-\infty}^{x_{II}^0} dx^0$ can be written as $\frac{1}{2} \int_{-\infty}^{\infty} [1 + \epsilon(x_{II}^0, x^0)] dx^0$ where $\epsilon(x^0) = \pm 1$ according as $x_0 \geq 0$. We are concerned only with the phenomena in which any real process to emit or absorb the photon does not occur, so the integral $\int_{-\infty}^{\infty} dx^0$ can be omitted. Moreover, if the time-points of the two particles I and II are the same ($x_I^0 = x_{II}^0$ and $u_\mu = (0, 0, 0, i)$), (34a) is reduced, on account of (30), to the following:

$$B_I = H_I + K_I,$$

$$H_I = \frac{e^2}{2} \int_{-\infty}^{\infty} \overline{D}(\tilde{\mathbf{r}}, x^0 - x_I^0) \tilde{\gamma}_4^I \tilde{\gamma}_k^I \tilde{\gamma}_4^I \tilde{\gamma}_k^I dx^0, \quad \tilde{\mathbf{r}} = \tilde{\mathbf{r}}_I - \tilde{\mathbf{r}}_{II},$$

$$K_I = -\frac{e^2}{2} \partial_I^I \partial_k^I \int_{-\infty}^{\infty} \overline{\mathfrak{D}}(\tilde{\mathbf{r}}, x^0 - x_I^0) \tilde{\gamma}_4^I \tilde{\gamma}_k^I \tilde{\gamma}_4^I \tilde{\gamma}_k^I dx^0. \quad (\text{A2})$$

As the operator $\tilde{\gamma}$'s are time-dependent, the rigorous evaluation of these integrals is not easy. Accordingly, we take up the convenient procedure employed by Nambu,⁸⁾ then we have:

$$\begin{aligned} \int_{-\infty}^{\infty} \overline{D}(\tilde{\mathbf{r}}, x^0 - x_I^0) f(x^0) dx^0 &= \frac{1}{4\pi} \cosh\left(\frac{d}{dx^0} \tilde{\mathbf{r}}\right) \cdot \frac{f(x^0)}{\tilde{\mathbf{r}}} \Big|_{x^0=x_I^0} \\ &= \frac{1}{4\pi} \left\{ \frac{1}{\tilde{\mathbf{r}}} + \frac{1}{2!} \left(\frac{d}{dx^0}\right)^2 \tilde{\mathbf{r}} + \dots \right\} f(x^0) \Big|_{x^0=x_I^0}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \int_{-\infty}^{\infty} \overline{\mathfrak{D}}(\tilde{\mathbf{r}}, x^0 - x_I^0) f(x^0) dx^0 &= \frac{1}{4\pi} \left\{ \frac{\cosh\left(\frac{d}{dx^0} \tilde{\mathbf{r}}\right) - 1}{\left(\frac{d}{dx^0}\right)^2} \right\} \cdot \frac{f(x^0)}{\tilde{\mathbf{r}}} \Big|_{x^0=x_I^0} \\ &= \frac{1}{4\pi} \left\{ \frac{1}{2!} \tilde{\mathbf{r}} + \frac{1}{4!} \left(\frac{d}{dx^0}\right)^2 \tilde{\mathbf{r}}^3 + \dots \right\} f(x^0) \Big|_{x^0=x_I^0}. \end{aligned} \quad (\text{A4})$$

As is easily seen, this is no more than the form expanded in powers of (v/c) . Thus we have:

$$\begin{aligned} H_I &= \sum_{n=1}^{\infty} H_I^{(n)} = \frac{e^2}{8\pi} \left[\frac{1}{\tilde{\mathbf{r}}} + \frac{1}{2} \left(\frac{d}{dx^0}\right)^2 \tilde{\mathbf{r}} + \dots \right] \tilde{\gamma}_4^I \tilde{\gamma}_k^I \tilde{\gamma}_4^I \tilde{\gamma}_k^I, \\ K_I &= \sum_{n=1}^{\infty} K_I^{(n)} = -\frac{e^2}{16\pi} \partial_I^I \partial_k^I \left[\tilde{\mathbf{r}} + \frac{1}{12} \left(\frac{d}{dx^0}\right)^2 \tilde{\mathbf{r}}^3 + \dots \right] \tilde{\gamma}_4^I \tilde{\gamma}_k^I \tilde{\gamma}_4^I \tilde{\gamma}_k^I. \end{aligned} \quad (\text{A2}')$$

The first term of each equation corresponds to the order of $(v_I/c)(v_{II}/c)$, and $H_I^{(1)} + K_I^{(1)}$ is reduced to (35). For the other terms, on account of (29), we substitute the relations

$$\frac{d}{dx_I^0} = \frac{i}{\hbar c} [\tilde{H}_I^0, \quad], \quad \frac{d}{dx_{II}^0} = \frac{i}{\hbar c} [\tilde{H}_{II}^0, \quad] \quad (\text{A5})$$

in (A2'). If we put the time of the two particles in the integrand of B_I to be the same, Nambu's procedure can be applied to our case without contradicting the integrability condition for Eqs. (33a) and (33b). Moreover, we take into account the law of energy-conservation in the form:

$$\frac{d}{dx_I^0} = -\frac{d}{dx_{II}^0}$$

and the higher powers of temporal differential operator, we write for symmetry's sake:

$$\left(\frac{d}{dx_I^0}\right)^{2n} = \left(\frac{d}{dx_{II}^0}\right)^{2n} = (-1)^n \left(\frac{d}{dx_I^0}\right)^n \left(\frac{d}{dx_{II}^0}\right)^n. \quad (\text{A } 6)$$

Then we can obtain the explicit form of the term $H_I^{(2)} + K_I^{(2)}$ which corresponds to the order of $(v_I/c)^2(v_{II}/c)^2$ and depends on the velocity of particles. The calculation for the term B_{II} is carried out analogously to the case of B_I , and the result is obtained by exchanging the symbol I and II in B_I .

Thus we have the Möller interaction energy of the order of $(v_I/c)^2(v_{II}/c)^2$ which runs as follows:

$$\begin{aligned} & H_I^{(2)} + K_I^{(2)} + H_{II}^{(2)} + K_{II}^{(2)} \\ &= \frac{e^2}{8\pi} \left[\mathbf{x}_I \cdot \mathbf{x}_{II} r \left\{ 3(\gamma_i^I \gamma_i^{II}) - \frac{(\gamma_i^I \gamma_i)(\gamma_j^{II} \gamma_j)}{r^2} \right\} \right. \\ &+ \mathbf{x}_I \left\{ 3r(\gamma_i^I \gamma_i^{II})(\gamma_j^{II} \partial_j^{II}) - 3r(\gamma_i^I \partial_i^{II}) - \frac{1}{r}(\gamma_i^I \gamma_i)(\gamma_j^I \partial_j^{II}) + \frac{1}{r}(\gamma_i^{II} \partial_i^{II})(\gamma_j^I \gamma_j)(\gamma_k^{II} r_k) \right\} \\ &+ \mathbf{x}_{II} \left\{ 3r(\gamma_i^I \gamma_i^{II})(\gamma_j^I \partial_j^I) - 3r(\gamma_i^{II} \partial_i^I) - \frac{1}{r}(\gamma_i^{II} \gamma_i)(\gamma_j^I \partial_j^I) + \frac{1}{r}(\gamma_i^I \partial_i^I)(\gamma_j^{II} \gamma_j)(\gamma_k^I r_k) \right\} \\ &- \left\{ \frac{(\gamma_i^I \gamma_i^{II})^2 - 2}{r} - \frac{1}{r^3}(\gamma_i^I \gamma_i)(\gamma_j^{II} \gamma_j)(\gamma_k^I \gamma_k^{II}) \right\} \\ &3r \{ (\partial_i^I \partial_i^{II}) - (\gamma_i^I \partial_i^{II})(\gamma_j^I \partial_j^I) - (\gamma_i^{II} \partial_i^I)(\gamma_j^{II} \partial_j^{II}) + (\gamma_i^I \gamma_i^{II})(\gamma_j^I \partial_j^I)(\gamma_k^{II} \partial_k^{II}) \} \\ &+ \frac{1}{r} \{ 3(r_i \partial_i^I)(\gamma_j^I \partial_j^{II}) - (r_i \partial_i^I)(\gamma_j^{II} \partial_j^{II})(\gamma_k^{II} r_k) - (r_i \partial_i^{II})(\gamma_j^I \partial_j^I)(\gamma_k^I r_k) \\ &- (\gamma_i^I \gamma_i)(\gamma_j^{II} \gamma_j)(\gamma_k^I \partial_k^I)(\gamma_l^{II} \partial_l^{II}) \}, \end{aligned}$$

where

$$\mathbf{x}_I = \frac{m_I c}{\hbar}, \quad \mathbf{x}_{II} = \frac{m_{II} c}{\hbar}.$$

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Note added in poof

In our calculation of the potential in § 3 (b), the negative energy states are neglected. Araki* has recently showed that, in the calculation of the potential, the neglect of negative energy states is not adequate and that, if the negative states are taken into account, much better coincidence with experiment can be obtained in the case of hydrogen atom.

Accordingly, it may be necessary to show that our method adopted in this paper can also be applied to the case when the negative energy states are considered.

For this purpose, it is convenient to carry out the calculation of the potential in the momentum representation in the following.

Bringing the transformations (23), (31) and (36) into one, we can write it as

$$\begin{aligned} \phi = S\psi \equiv & \exp \left[-\frac{i}{\hbar c} (H_I^0 x_I^0 + H_{II}^0 x_{II}^0) \right] \\ & \times \exp [-i(\bar{S}_I + S_{II})] \times \exp \left[-\frac{i}{\hbar c} (H_I^0 x_I^0 + H_{II}^0 x_{II}^0) \right] \phi. \end{aligned} \quad (\text{N } 1)$$

This transformation can be written in the momentum representation as

$$\begin{aligned} \bar{S} = & (\mathbf{p}_n^I, \mathbf{p}_{n'}^{II}; N\mathbf{k} \pm 1 | S | \mathbf{p}_m^I, \mathbf{p}_{m'}^{II}; N\mathbf{k}) \\ = & \iint (\mathbf{p}_n^I, \mathbf{p}_{n'}^{II} | \mathbf{r}_I, \mathbf{r}_{II}) (N\mathbf{k} \pm 1; \mathbf{r}_I, \mathbf{r}_{II} | S | \mathbf{r}_I, \mathbf{r}_{II}; N\mathbf{k}) \\ & \times (\mathbf{r}_I, \mathbf{r}_{II} | \mathbf{p}_m^I, \mathbf{p}_{m'}^{II}) d\mathbf{r}_I d\mathbf{r}_{II}. \end{aligned} \quad (\text{N } 2)$$

Taking into account the relation

$$(\mathbf{r}_I, \mathbf{r}_{II} | \mathbf{p}^I, \mathbf{p}^{II}) = \chi^I(\mathbf{p}^I) e^{i\mathbf{p}^I \mathbf{r}_I / \hbar} \cdot \chi^{II}(\mathbf{p}^{II}) e^{i\mathbf{p}^{II} \mathbf{r}_{II} / \hbar},$$

(N 2) runs as follows:

$$\begin{aligned} \bar{S} = & \exp [-i\bar{S}_I(\bar{S}_I + \bar{S}_{II})], \\ (\mathbf{p}_n^I; N\mathbf{k} \pm 1 | \bar{S}_I | \mathbf{p}_m^I; N\mathbf{k}) \\ = & c \cdot \delta(\mathbf{p}_m^I - \mathbf{p}_n^I \pm \mathbf{k}) (\chi_n^{I*} \gamma_4^I \gamma_\mu^I \chi_m^I) (N\mathbf{k} \pm 1 | \mathfrak{A}_\mu^\pm(\mathbf{k}, x_I^0) | N\mathbf{k}) / E_m^I - E_n^I \mp \omega\mathbf{k}, \\ (\mathbf{p}_{n'}^{II}; N\mathbf{k} \pm 1 | \bar{S}_{II} | \mathbf{p}_{m'}^{II}; N\mathbf{k}) \\ = & c \delta(\mathbf{p}_{m'}^{II} - \mathbf{p}_{n'}^{II} \mp \mathbf{k}) (\chi_{n'}^{II*} \gamma_4^{II} \gamma_\mu^{II} \chi_{m'}^{II}) (N\mathbf{k} \pm 1 | \mathfrak{A}_\mu^\pm(\mathbf{k}, x_{II}^0) | N\mathbf{k}) / E_{m'}^{II} - E_{n'}^{II} \mp \omega\mathbf{k}. \end{aligned} \quad (\text{N } 3)$$

(N 3) is just the transformation used by van Hove** if we put $x_I^0 = x_{II}^0$.

As is well known, van Hove's procedure can also be applied to the calculation of the potential when the negative energy states are considered, and thus this is also true in our method. But it may need further investigation whether the consideration of the negative energy states in the calculation of the interaction energy is also necessary in meson field as well as in electromagnetic field.

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On the Path Integral and Its Application

Yasuhisa MURAI

Department of Physics, Saitama University

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The exact form of the path integral has been obtained by the method of ordered operators. For illustration the result has been applied to the free Dirac particle. It has been further shown that the electromagnetic mass of the electron appears in the path integral in a natural way when the interaction with the self-field is considered.

§ 1. Introduction

In this paper we show that the exact form of the path integral can be deduced from the usual wave equation, using the method of ordered operators given by Feynman.¹⁾ We use something like a proper time to denote an ordering parameter. The final result coincides with that given by Morette²⁾; but in her case the meaning of the path integral was not clear until the appropriate Lagrangian for a problem was constructed. Here this is clarified during the course of calculation. To introduce the proper time we adopted the process first introduced by Fock³⁾. We shall omit the explanation about the rôle of the proper time in the quantum electrodynamics as this has been clearly stated by Nambu⁴⁾. The result is applied to the free Dirac particle and then to the case where the interaction with its own field is taken into account. The problem of the vacuum polarization and others will be treated in the succeeding papers.

§ 2. The path integral

The solution of the Dirac equation

$$[\gamma_\mu(\not{p}_\mu - e/cA_\mu) + imc]\psi = 0, \quad (2, 1)$$

can be expressed as

$$\psi = [\gamma_\mu(\not{p}_\mu - e/cA_\mu) - imc]\Psi, \quad (2, 2)$$

where Ψ is the function which satisfies the following equation;

$$\begin{aligned} & [\gamma_\mu(\not{p}_\mu - e/cA_\mu) + imc][\gamma_\mu(\not{p}_\mu - e/cA_\mu) - imc]\Psi \\ &= [(\not{p}_\mu - e/cA_\mu)^2 + m^2c^2 + \hbar e/2ic \cdot \gamma_\mu \gamma_\nu F_{\mu\nu}]\Psi \equiv \hbar^2 A \Psi = 0. \end{aligned} \quad (2, 3)$$

According to the proposition made by Fock³⁾, we shall make use of the auxiliary

function F , which is the function of five variables x_1, x_2, x_3, x_4 and τ , and satisfies

$$i\hbar \frac{\partial F}{\partial \tau} = \frac{\hbar^2}{2m} \Delta F. \quad (2, 4)$$

Then Ψ can be put in the form

$$\Psi = \int F d\tau. \quad (2, 5)$$

Now we shall solve Eq. (2, 4) not restricting ourselves to the form of Δ given above. We assume that the successive events are ordered by τ , and shall follow the procedure given by Feynman¹⁾ concerning the ordered operators. The formal solution of Eq. (2, 4) can be written as

$$F(\mathbf{x}, \tau) = \exp\left(-i\hbar/2m \int_0^\tau \Delta[\mathbf{p}(\sigma), \mathbf{x}(\sigma)] d\sigma\right) F(\mathbf{x}, 0). \quad (2, 6)$$

Here $\mathbf{p}(\sigma)$, for example, means that \mathbf{p} acts at the "time" σ . Using a Fourier representation we write the exponential function as follows;

$$\begin{aligned} & \exp\left(-i\hbar/2m \int_0^\tau \Delta[\mathbf{p}(\sigma), \mathbf{x}(\sigma)] d\sigma\right) \\ &= \int \cdots \int \exp\left(i/\hbar \int_0^\tau \boldsymbol{\beta}(\sigma) \cdot \mathbf{p}(\sigma) d\sigma\right) G[\boldsymbol{\beta}(\sigma), \mathbf{x}(\sigma)] d[\boldsymbol{\beta}(\sigma)]. \end{aligned} \quad (2, 7)$$

The inverse transformation is

$$\begin{aligned} G[\boldsymbol{\beta}(\sigma) \mathbf{x}, (\sigma)] &= \int \cdots \int \exp\left(-i/\hbar \int_0^\tau \boldsymbol{\beta}(\sigma) \cdot \mathbf{p}(\sigma) d\sigma\right) \\ &\cdot \exp\left(-i\hbar/2m \int_0^\tau \Delta[\mathbf{p}(\sigma), \mathbf{x}(\sigma)] d\sigma\right) A(\sigma) d[\mathbf{p}(\sigma)], \end{aligned} \quad (2, 8)$$

where $A(\sigma)$ is inserted to secure the normalization condition

$$\int \cdots \int \exp\left(-i/\hbar \int_0^\tau \boldsymbol{\beta}(\sigma) \cdot \mathbf{p}(\sigma) d\sigma\right) A(\sigma) d[\mathbf{p}(\sigma)] = \delta[\boldsymbol{\beta}(\sigma)], \quad (2, 9)$$

the delta functional $\delta[\boldsymbol{\beta}]$ being defined as follows;

$$\delta[\boldsymbol{\beta}(\sigma)] = 0 \quad \text{if } \boldsymbol{\beta}(\sigma) \neq 0, \quad \text{and} \quad \int \cdots \int \delta[\boldsymbol{\beta}(\sigma)] d[\boldsymbol{\beta}(\sigma)] = 1.$$

The formal expression on the right hand side of (2, 7) is easily transformed into the usual one, that is the expression in which the ordering of operators is designated by their spatial position, Putting

$$\mathbf{B}(\sigma) = \int_0^\tau \boldsymbol{\beta}(\sigma) d\sigma, \quad \dot{\mathbf{B}}(\sigma) = -\boldsymbol{\beta}(\sigma),$$

we have

$$\begin{aligned} F(\mathbf{x}, \tau) &= \int \cdots \int G[\beta(\sigma), \mathbf{x} + \mathbf{B}(\sigma)] \exp(i/\hbar \cdot \mathbf{B}(0) \cdot \mathbf{p}) d[\beta(\sigma)] F(\mathbf{x}, 0) \\ &= \int \cdots \int G[-\dot{\mathbf{B}}(\sigma), \mathbf{x} + \mathbf{B}(\sigma)] d[\dot{\mathbf{B}}(\sigma)] F(\mathbf{x} + \mathbf{B}(0), 0) d\mathbf{B}(0). \end{aligned}$$

Changing $\mathbf{B}(\sigma)$ to $\mathbf{x}(\sigma) = \mathbf{x} + \mathbf{B}(\sigma)$ and making use of the inverse transformation (2, 8), we get

$$F(\mathbf{x}(\tau), \tau) = \int K(\mathbf{x}, \tau; \mathbf{x}^0, 0) F(\mathbf{x}^0, 0) d\mathbf{x}^0, \quad (2, 10)$$

where

$$\begin{aligned} K(\mathbf{x}, \tau; \mathbf{x}^0, 0) &= \int \cdots \int \exp\left[\frac{i}{\hbar} \int_0^\tau \{\dot{\mathbf{x}}(\sigma) \cdot \mathbf{p}(\sigma) - \hbar^2/2m \cdot A[\mathbf{p}(\sigma), \mathbf{x}(\sigma)]\} d\sigma\right] \\ &\quad \times A(\sigma) d[\dot{\mathbf{x}}(\sigma)] d[\mathbf{p}(\sigma)]. \end{aligned} \quad (2, 11)$$

We shall now make the form of the kernel $K(\mathbf{x}, \tau; \mathbf{x}^0, 0)$ neater:

At first we shall integrate with respect to $\mathbf{p}(\sigma)$. Let $\mathbf{p}_0(\sigma)$ be those values of $\mathbf{p}(\sigma)$ which fulfil

$$\dot{\mathbf{x}}_\mu = \frac{\hbar^2}{2m} \frac{\partial A[\mathbf{p}(\sigma), \mathbf{x}(\sigma)]}{\partial \mathbf{p}_\mu}, \quad (2, 12)$$

then we can write

$$\mathbf{p}_0 \cdot \dot{\mathbf{x}} - \hbar^2/2m \cdot A[\mathbf{p}_0, \mathbf{x}] = L[\mathbf{x}, \dot{\mathbf{x}}]. \quad (2, 13)$$

Putting $\mathbf{p} = \mathbf{p}_0 + \mathbf{p}'$, and expanding $\dot{\mathbf{x}} \cdot \mathbf{p} - \hbar^2/2m \cdot A[\mathbf{p}, \mathbf{x}]$ around \mathbf{p}_0 , we get

$$\dot{\mathbf{x}} \cdot \mathbf{p} - \frac{\hbar^2}{2m} A[\mathbf{p}, \mathbf{x}] = L[\mathbf{x}, \dot{\mathbf{x}}] - \frac{\hbar^2}{4m} \frac{\partial^2 A[\mathbf{p}_0, \mathbf{x}]}{\partial p_{0\mu} \partial p_{0\nu}} p'_\mu p'_\nu, \quad (2, 14)$$

the linear term in p' being dropped on account of (2, 12). We shall consider the case where Eq. (2, 14) stands rigorously. Performing the integration in the interval $(\sigma, \sigma + \Delta\sigma)$ we have

$$\int \exp\left(-\frac{i\hbar\Delta\sigma}{4m} A_{\mu\nu} p'_\mu p'_\nu\right) A d\mathbf{p}' = \left(\frac{4\pi m}{i\hbar\Delta\sigma}\right)^2 A |A_{\mu\nu}|^{-\frac{1}{2}}. \quad (2, 15)$$

From the condition (2, 9) $A(\sigma)$ can be decided and found to be $(\Delta\sigma/2\pi\hbar)^4$. Inserting these results into (2, 11), this can be written as follows;

$$K(\mathbf{x}, \tau; \mathbf{x}^0, 0) = \Pi \int \exp\left(\frac{i}{\hbar} L[\mathbf{x}, \dot{\mathbf{x}}] \Delta\sigma\right) \cdot \left(\frac{m\Delta\sigma}{i\pi\hbar^3}\right)^2 |A_{\mu\nu}|^{-\frac{1}{2}} d\dot{\mathbf{x}}(\sigma). \quad (2, 16)$$

Now, as $\frac{\partial A[\mathbf{p}_0, \mathbf{x}]}{\partial \mathbf{p}_{0\mu}} = \frac{2m}{\hbar^2} \dot{x}_\mu$ and $|A_{\mu\nu}| = \left| \frac{\partial^2 A}{\partial \mathbf{p}_{0\mu} \partial \mathbf{p}_{0\nu}} \right| = \left(\frac{2m}{\hbar^2} \right)^4 \frac{\partial(\dot{x})}{\partial(\mathbf{p}_0)} = \left(\frac{2m}{\hbar^2} \right)^4 \left(\frac{\partial(\mathbf{p}_0)}{\partial(\dot{x})} \right)^{-1}$,

$$\begin{aligned} \left(\frac{m \Delta \sigma}{i \pi \hbar^3} \right)^2 |A_{\mu\nu}|^{-\frac{1}{2}} d\dot{x} &= \left(\frac{\Delta \sigma}{2 \pi \hbar i} \right)^2 \left(\frac{\partial(\mathbf{p}_0)}{\partial(\dot{x})} \right)^{\frac{1}{2}} d\dot{x} \\ &= \left(\frac{\partial(\mathbf{p}_0/2\pi\hbar i)}{\partial(\dot{x})} \frac{\partial(\dot{x})}{\partial(x)} \right)^{\frac{1}{2}} \left(\frac{\partial(\dot{x} \Delta \sigma)}{\partial(x)} \right)^{\frac{1}{2}} dx. \end{aligned}$$

Therefore,

$$\begin{aligned} K(x, \tau; x^0, 0) &= \Pi \int \exp \left(\frac{i}{\hbar} L[x, \dot{x}] \Delta \sigma \right) (\partial(\mathbf{p}_0/2\pi\hbar i) / \partial(x))^{\frac{1}{2}} d\mathbf{x}(\sigma) \\ &= \int \dots \int \exp \left(\frac{i}{\hbar} \int_0^\tau L[x(\sigma), \dot{x}(\sigma)] d\sigma \right) (\partial(\mathbf{p}_0/2\pi\hbar i) / \partial(x))^{\frac{1}{2}} d[x(\sigma)]. \end{aligned} \quad (2, 17)$$

This is the final expression for the kernel and coincides with that given by Morette⁹⁾. She has obtained this result starting from Feynman's definition $K(x^{k+1}, x^k) = \exp(i/\hbar S_{cl}(x^{k+1}, x^k)) 1/c^{k+1,k}$, where S_{cl} is the classical action. Our result is deduced from the "Schrödinger" equation (2, 4) and the meaning of $K(x, \tau; x^0, 0)$ is evident when we see Eqs. (2, 10), (2, 5) and (2, 2). When "Hamiltonian" A is initially known, Lagrangian L can be formed as shown in Eqs. (2, 12) and (2, 13).

§ 3. Free Dirac particle

We shall begin with the application of the above result to the free Dirac particle. (Needless to say that this case includes the Bose particle as one will see when the procedure (2, 2) is omitted and Ψ itself is regarded as a wave function.) In this case

$$A = 1/\hbar^2 (\mathbf{p}^2 + m^2 c^2), \quad (3, 1)$$

$$L = -mc^2/2 + m\dot{x}^2/2. \quad (3, 2)$$

Introducing the classical path $x_{cl}(\sigma)$ determined from $\delta \int_0^\tau L d\tau = 0$, we write

$$x(\sigma) = x_{cl}(\sigma) + x'(\sigma). \quad (3, 3)$$

In the envisaged case

$$x_{cl}(\sigma) = \text{const.}, \quad x_{cl}(\sigma) = x^0 + (x - x^0)\sigma/\tau, \quad (3, 4)$$

and

$$\int_0^\tau L(\sigma) d\sigma = -mc^2\tau/2 + m(\mathbf{x} - \mathbf{x}^0)^2/2\tau + m/2 \int_0^\tau \dot{\mathbf{x}}'^2(\sigma) d\sigma, \quad (3, 5)$$

as $\mathbf{x}'(\tau) = \mathbf{x}'(0) = 0$. Therefore (2, 17) gives for the free kernel

$$K(\mathbf{x}, \tau; \mathbf{x}^0, 0) = \exp \left[\frac{im}{2\hbar} \left\{ -c^2\tau + \frac{(\mathbf{x} - \mathbf{x}^0)^2}{\tau} \right\} \right] \int \dots \int \exp \left\{ \frac{im}{2\hbar} \int_0^\tau \dot{\mathbf{x}}'^2(\sigma) d\sigma \right\} \left(\frac{\partial(\mathbf{p}_0/2\pi i\hbar)}{\partial(\mathbf{x}(u))} \right)^{\frac{1}{2}} d[\mathbf{x}'(\sigma)]. \quad (3, 6)$$

The involved integral is computed by dividing the interval τ into n parts of equal length ϵ , as follows;

$$\begin{aligned} & \int \dots \int \exp \left\{ \frac{im}{2\hbar\epsilon} [\mathbf{x}_1^2 + (\mathbf{x}_2 - \mathbf{x}_1)^2 + \dots + (\mathbf{x}_{n-1} - \mathbf{x}_{n-2})^2 + \mathbf{x}_{n-1}^2] \right\} \\ & \quad (m/2\pi\hbar i\epsilon)^{n-1} \prod_{k=1}^{n-1} (m/2\pi\hbar i\epsilon)^2 d\mathbf{x}_k \\ &= \int \dots \int \exp \left\{ \frac{im}{2\hbar\epsilon} \sum_{k=1}^{n-1} \frac{k+1}{k} \left(\mathbf{x}_k - \frac{2k}{k+1} \mathbf{x}_{k+1} \right)^2 \right\} (m/2\pi\hbar i\epsilon)^2 \prod_{k=1}^{n-1} (m/2\pi\hbar i\epsilon)^2 d\mathbf{x}_k \\ &= (m/2\pi\hbar i\epsilon)^{n-1} \prod_{k=1}^{n-1} (2\hbar\epsilon k i\pi / m(k+1))^2 (m/2\hbar i\epsilon\pi)^2 = (m/2\pi i\hbar\tau)^2. \end{aligned}$$

And we have finally

$$K(\mathbf{x}, \tau; \mathbf{x}^0, 0) = -(\mathbf{x}/2\pi c)^2 1/\tau^2 \{ i\mathbf{x}/2[-c\tau + (\mathbf{x} - \mathbf{x}^0)^2/c\tau] \}. \quad (3, 7)$$

This kernel being obtained, the desired solution ψ can be found according to the prescriptions (2, 2), (2, 5) and (2, 6). At first we have

$$F(\mathbf{x}, \tau) = -(\mathbf{x}/2\pi c)^2 \int 1/\tau^2 \exp \{ i\mathbf{x}/2[-c\tau + (\mathbf{x} - \mathbf{x}^0)^2/c\tau] \} F(\mathbf{x}^0) d\mathbf{x}^0.$$

Now we shall integrate $F(\mathbf{x}, \tau)$ with respect to τ from 0 to ∞ , as this definite integration furnishes the positron-theoretical result,

$$\Psi(\mathbf{x}) = -i\mathbf{x}/c \int \Delta_{\mathbf{r}}(\mathbf{x} - \mathbf{x}^0) F(\mathbf{x}^0) d\mathbf{x}^0.$$

Therefore if we set as an initial condition for $F(\mathbf{x}, \tau)$

$$F(\mathbf{x}^0, 0) = -2c/\mathbf{x} \cdot \gamma_\mu n_\mu \delta[\mathbf{x}^0 \text{ on } C] \psi^0(C), \quad (3, 8)$$

we can get the well known solution for the problem of finding the Dirac wave function when on some space-like surface C a state $\psi^0(C)$ is initially given.

§ 4. Interaction with self-field

We shall now write down the kernel for the Dirac particle interacting with the electromagnetic field. As the case of the constant field was already treated by many authors⁶⁾ we shall now consider the case of the interaction with the self-field. Because of the usage of the proper time representation, no pair creation and pair annihilation in the ordinary sense occur, and so we use the second quantization only for the photon field.

The Hamiltonian H is given by (2, 3) and from this we have for the momentum and the Lagrangian L ,

$$p_{0\mu} = m\dot{x}_\mu + e/c A_\mu, \quad (4, 1)$$

$$L = L^0 + L' \quad (4, 2)$$

$$L^0 = m\dot{x}^2/2 - mc^2/2 \quad (4, 3)$$

$$L' = e/c (A \cdot \dot{x} + i\hbar/4m \cdot \gamma_\mu \gamma_\nu F_{\mu\nu}). \quad (4, 4)$$

We have to compute the path integral,

$$\int \dots \int \exp \left\{ \frac{i}{\hbar} \int_0^\tau (L^0 + L') d\tau \right\} \left(\frac{\partial(\mathbf{p}_0/2\pi i\hbar)}{\partial(\mathbf{x})} \right)^{\frac{1}{2}} d[\mathbf{x}(\tau)]. \quad (4, 5)$$

We shall use the perturbation method and expand the exponential function as follows;

$$\begin{aligned} & \exp \left\{ \frac{i}{\hbar} \int_0^\tau (L^0 + L') d\tau \right\} \\ &= \exp \left(\frac{i}{\hbar} \int_0^\tau L^0 d\tau \right) \left\{ 1 + \frac{i}{\hbar} \int_0^\tau L'(\sigma) d\sigma + \frac{1}{2} \left(\frac{i}{\hbar} \right)^2 \int_0^\tau L'(\sigma) d\sigma \int_0^\tau L'(\sigma') d\sigma' + \dots \right\} \\ &= \exp \left(\frac{i}{\hbar} \int_0^\tau L^0 d\tau \right) + \frac{i}{\hbar} \int_0^\tau e^{\frac{i}{\hbar} \int_0^\tau L^0 d\tau} L'(\sigma) e^{-\frac{i}{\hbar} \int_0^\sigma L^0 d\tau} d\sigma \\ & \quad + \left(\frac{i}{\hbar} \right)^2 \int_0^\tau \int_0^\sigma e^{\frac{i}{\hbar} \int_0^\tau L^0 d\tau} L'(\sigma) e^{-\frac{i}{\hbar} \int_0^\sigma L^0 d\tau} L'(\sigma') e^{-\frac{i}{\hbar} \int_0^{\sigma'} L^0 d\tau} d\sigma d\sigma' + \dots, \end{aligned}$$

remembering the meaning of the ordering parameter σ . If we consider the case where the initial and the final states have no photon, the kernel in the second order approximation comes to be as follows;

$$K^2(\mathbf{x}, \tau; \mathbf{x}^0, 0) = K_0(\mathbf{x}, \tau; \mathbf{x}^0, 0) + K_2^2(\mathbf{x}, \tau; \mathbf{x}^0, 0),$$

K_0 being given by (3, 7) and

$$\begin{aligned}
K_2(x, \tau; x^0, 0) = & \left(\frac{ie}{\hbar c} \right)^2 \int_0^\tau d\sigma \int_0^\sigma d\sigma' \int \cdots \int d[x] \exp \left(\frac{i}{\hbar} \int_\sigma^\tau L^0 d\tau \right) \\
& \times \left(\dot{x}(\sigma) \cdot A(x(\sigma)) + \frac{i\hbar}{4m} \gamma_\mu \gamma_\nu F_{\mu\nu}(x(\sigma)) \right) \cdot \exp \left(\frac{i}{\hbar} \int_{\sigma'}^\sigma L^0 d\tau \right) \\
& \times \left(\dot{x}(\sigma') \cdot A(x(\sigma')) + \frac{i\hbar}{4m} \gamma_\mu \gamma_\nu F_{\mu\nu}(x(\sigma')) \right) \exp \left(\frac{i}{\hbar} \int_0^{\sigma'} L^0 d\tau \right).
\end{aligned} \quad (4, 5)$$

In evaluating (4, 5) we must express $\dot{x}(\sigma)$ which multiplies $A(x(\sigma))$ as follows,

$$\dot{x}(\sigma) = \{ (x(\sigma + \varepsilon) - x(\sigma)) + (x(\sigma) - x(\sigma - \varepsilon)) \} / 2\varepsilon. \quad (4, 6)$$

This expression being inserted into (4, 5), the elementary Gaussian integral yields,

$$\begin{aligned}
& \left(\frac{ie}{\hbar c} \right)^2 \int \cdots \int d\sigma d\sigma' dx^1 dx^2 K_0(x, \tau; x^1, \sigma) A(x^1) \cdot \left\{ \frac{1}{4} \left(\frac{x - x^1}{\tau - \sigma} + \frac{x^1 - x^2}{\sigma - \sigma'} \right) \right. \\
& \times \left(\frac{x^1 - x^2}{\sigma - \sigma'} + \frac{x^2 - x^0}{\sigma'} \right) + \frac{c}{i\varepsilon} \frac{1}{\sigma - \sigma'} \left. \right\} K_0(x^1, \sigma; x^2, \sigma') \cdot A(x^2) K_0(x^2, \sigma'; x^0, 0) \\
& + \left(\frac{ie}{\hbar c} \right)^2 \left(\frac{i\hbar}{4m} \right)^2 \int \cdots \int d\sigma d\sigma' dx^1 dx^2 K_0(x, \tau; x^1, \sigma) \gamma_\mu \gamma_\nu F_{\mu\nu}(x^1) \\
& \times K_0(x^1, \sigma; x^2, \sigma') \gamma_\mu \gamma_\nu F_{\mu\nu}(x^2) K_0(x^2, \sigma'; x^0, 0).
\end{aligned} \quad (4, 7)$$

The cross term between $\dot{x} \cdot A$ and $\gamma_\mu \gamma_\nu F_{\mu\nu}$ vanishes after integration, because of the form of \dot{x} , and $\dot{x}(\sigma)$ is nearly equal to the average of the classical velocities before and after the "time" σ ; the additional term $4c/i\varepsilon(\sigma - \sigma')$ is the supplement to the square term $(x^1 - x^2/\sigma - \sigma')^2$. We replace $A_\mu(x^1)A_\nu(x^2)$ by its vacuum expectation value

$$\langle A_\mu(x^1) A_\nu(x^2) \rangle_0 = \hbar c \partial_{\mu\nu} D_F(x^1 - x^2). \quad (4, 8)$$

To justify this procedure, of course, we should have added the Lagrangian for the electromagnetic field to (4, 2) and then eliminated it.

We shall begin with the first term of (4, 7). The result of the integration with respect to x^1 and x^2 is

$$\begin{aligned}
& - \frac{i\varepsilon e^2}{16\pi^2 \hbar c^2} \left(\frac{\varepsilon}{2\pi c} \right)^2 \int_0^\infty d\beta \int_0^\tau d\sigma \int_0^\sigma d\sigma' \left[\frac{4c}{i\varepsilon} \frac{\tau + 4\beta}{\{(\beta + \sigma - \sigma')\tau - (\sigma - \sigma')^2\}^3} \right. \\
& \left. + \frac{(2\beta + \sigma - \sigma')^2 (x - x^0)^2}{\{(\beta + \sigma - \sigma')\tau - (\sigma - \sigma')^2\}^4} \right] \cdot \exp \left[\frac{i\varepsilon}{2} \left\{ \frac{\beta + \sigma - \sigma'}{(\beta + \sigma - \sigma')\tau - (\sigma - \sigma')^2} \frac{(x - x^0)^2}{c} - c\tau \right\} \right]
\end{aligned}$$

(β comes from the integral representation of $D_F(x^1 - x^2)$). Or,

$$-\frac{i\alpha c^2}{16\pi^2\hbar c^2}\left(\frac{\alpha}{2\pi c}\right)^2\int_0^\infty d\beta\int_0^\tau d\xi\left[\frac{4c}{i\alpha}\frac{(\tau+4\beta)(\tau-\xi)}{\{(\beta+\xi)\tau-\xi^2\}^3}+\frac{(2\beta+\xi)^2(\tau-\xi)}{\{(\beta+\xi)\tau-\xi^2\}^4}(\alpha-\alpha^0)^2\right] \\ \times \exp\left[\frac{i\alpha}{2}\left\{\frac{(\beta+\xi)(\alpha-\alpha^0)^2}{((\beta+\xi)\tau-\xi^2)c}-c\tau\right\}\right].$$

We change the form further in the following way,

$$-\frac{e^2}{8\pi^2\hbar c}\left(\frac{\alpha}{2\pi c}\right)^2\int_0^\infty d\beta\int_0^\tau d\xi\left[\frac{4(\tau-\xi)}{\xi\{(\beta+\xi)\tau-\xi^2\}^2}-\frac{\partial}{\partial\beta}\cdot\frac{(\tau-\xi)(2\beta+\xi)^2}{\xi^2\{(\beta+\xi)\tau-\xi^2\}^2}\right] \\ \times \exp\left[\frac{i\alpha}{2}\left\{\frac{(\beta+\xi)(\alpha-\alpha^0)^2}{((\beta+\xi)\tau-\xi^2)c}-c\tau\right\}\right] \\ =-\frac{ie^2}{8\pi^2\hbar c(2\pi)^4}\int(dp)e^{-\frac{ic}{2\alpha}(k\mu^2+\alpha^2)\tau}e^{ik\mu(x\mu-x\mu^0)} \\ \times \int d\beta\int d\xi\left[\frac{4(\tau-\xi)}{\xi(\beta+\xi)^2}-\frac{\partial}{\partial\beta}\cdot\frac{(\tau-\xi)(2\beta+\xi)^2}{\xi^2(\beta+\xi)^2}\right]e^{\frac{ic}{2\alpha}\frac{\xi^2}{\beta+\xi}k\mu^2} \quad (4,9)$$

where $\left[A+\frac{\partial}{\partial\beta}\cdot B\right]C$ means $AC+\frac{\partial}{\partial\beta}(BC)$. In the same way we can calculate the second term of (4, 7). $\gamma_\mu\gamma_\nu F_{\mu\nu}(\alpha^1)\gamma_\mu\gamma_\nu F(\alpha^2)$ is to be replaced by its vacuum expectation value, which is obtained from (4, 8) and found to be

$$\langle\gamma_\mu\gamma_\nu F_{\mu\nu}(\alpha^1)\gamma_\rho\gamma_\sigma F_{\rho\sigma}(\alpha^2)\rangle_0=12\hbar c\Box D_F(\alpha^1-\alpha^2)=24i\hbar c\delta(\alpha^1-\alpha^2).$$

The calculation gives for the second term of (4, 7),

$$-\frac{3e^2}{8\pi^2\hbar c}\left(\frac{\alpha}{2\pi c}\right)^2\int_0^\tau d\xi\frac{1}{\xi^2(\tau-\xi)}\exp\left[\frac{i\alpha}{2}\left\{\frac{(\alpha-\alpha^0)^2}{(\tau-\xi)c}-c\tau\right\}\right] \\ =-\frac{3ie^2}{8\pi^2\hbar c(2\pi)^4}\int(dk)e^{-\frac{ic}{2\alpha}(k\mu^2+\alpha^2)\tau}e^{ik\mu(x\mu-x\mu^0)}\cdot\int_0^\tau d\xi\frac{\tau-\xi}{\xi^2}e^{\frac{ic}{2\alpha}\frac{\xi^2}{\tau-\xi}k\mu^2}. \quad (4,10)$$

(4, 9) and (4, 10) together give

$$K_2(\alpha, \tau; \alpha^0, 0)=-\frac{ie^2}{2\pi^2\hbar c(2\pi)^4}\int(dk)e^{-\frac{ic}{2\alpha}(k\mu^2+\alpha^2)\tau}e^{ik\mu(x\mu-x\mu^0)} \\ \cdot\int_0^\tau d\xi\left\{\frac{(\tau-\xi)}{\xi^2}e^{\frac{ic}{2\alpha}k\mu^2\xi}-\frac{(\tau-\xi)}{\xi^2}+\int_0^\infty\frac{\tau-\xi}{\xi(\beta+\xi)^2}e^{\frac{ic}{2\alpha}\frac{\xi^2}{\beta+\xi}k\mu^2}d\beta\right\}.$$

At this stage we can not show that $K_0(\alpha, \tau; \alpha^0, 0)+K_2(\alpha, \tau; \alpha^0, 0)$ is the free kernel for the particle of mass $m+\delta m$. The tentative of making (4,11) in this form was unsuccessful. Only after the integration with respect to τ we can find the electromagnetic mass of the Dirac particle. As $\int_0^\infty d\tau\int_0^\tau d\xi=\int_0^\infty d\xi\int_\xi^\infty d\tau$ and

$$\begin{aligned} \int_{\xi}^{\infty} (\tau - \xi) e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \tau} d\tau &= \left(\frac{2\alpha}{c} \right)^2 e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \xi} \int_0^{\infty} \eta e^{-i(k_{\mu}^2 + \alpha^2) \eta} d\eta \\ &= \frac{2\pi i \alpha}{c^2} e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \xi} \frac{\partial}{\partial \alpha} \delta_+(k_{\mu}^2 + \alpha^2), \end{aligned}$$

we get

$$\begin{aligned} \int_0^{\infty} K_2(x, \tau; x^0, 0) d\tau &= \frac{\alpha c^2}{\pi \hbar c^3 (2\pi)^4} \int (dk) \frac{\partial}{\partial \alpha} \delta_+(k_{\mu}^2 + \alpha^2) e^{ik_{\mu}(x_{\mu} - x_{\mu}^0)} \\ &\cdot \int_0^{\infty} d\xi \left\{ \frac{2}{\xi^2} e^{-\frac{ic\alpha}{2}\xi} - \frac{1}{\xi^2} e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \xi} + \int_0^{\infty} \frac{d\beta}{\xi(\beta + \xi)^2} e^{-\frac{ic}{2\alpha} \left(\frac{\beta \xi}{\beta + \xi} k_{\mu}^2 + \xi \alpha^2 \right)} \right\}. \end{aligned} \quad (4, 12)$$

Comparing with

$$\int_0^{\infty} K_0 d\tau = -\frac{2\pi i \alpha}{c(2\pi)^4} \int (dk) \delta_+(k_{\mu}^2 + \alpha^2) e^{ik_{\mu}(x_{\mu} - x_{\mu}^0)}, \quad (4, 13)$$

we see that $\delta\alpha$ is given by

$$\delta\alpha = \frac{ie^2}{2\pi^2 \hbar c^3} \int_0^{\infty} d\xi \left[\frac{1}{\xi^2} e^{-\frac{ic\alpha}{2}\xi} - \frac{1}{\xi^2} + \int_0^{\infty} \frac{d\beta}{\xi(\beta + \xi)^2} e^{-\frac{ic\alpha}{2} \frac{\xi^2}{\beta + \xi}} \right] \quad (4, 14)$$

$$\alpha = \frac{3a}{2\pi} \left(\int_0^{\infty} \frac{1}{\xi} e^{-\frac{ic\alpha}{2}\xi} d\xi + \frac{7}{6} + \frac{2}{c\alpha\xi_0} \right). \quad (4, 14')$$

Here ξ_0 means the minimum "time" between the emission and the absorption of the virtual photon. And as a result we have

$$\int_0^{\infty} K(x, \tau; x^0, 0) d\tau = -\frac{i\alpha}{c(2\pi)^3} \int (dk) e^{ik_{\mu}(x_{\mu} - x_{\mu}^0)} \left(1 + \delta\alpha \frac{\partial}{\partial \alpha} \right) \delta_+(k_{\mu}^2 + \alpha^2)$$

and the α in the coefficient being cancelled with the α in (3, 8),

$$\Psi(x) = \Psi_0(x) + \frac{\partial \Psi_0(x)}{\partial \alpha} \delta\alpha.$$

If we want to see the change of mass during the course of mutation, we have available Eq. (2, 4). Using this equation we see that $\delta\alpha(\tau)$ is given by

$$\begin{aligned} \delta\alpha(\tau) \int (dk) e^{ik_{\mu}(x_{\mu} - x_{\mu}^0)} \frac{k_{\mu}^2 - \alpha^2}{\alpha} e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \tau} \\ = -\frac{ie^2}{\pi^2 \hbar c^2} \int (dk) e^{ik_{\mu}(x_{\mu} - x_{\mu}^0)} e^{-\frac{ic}{2\alpha} (k_{\mu}^2 + \alpha^2) \tau} \int_0^{\tau} d\xi \left[\frac{1}{\xi^2} e^{\frac{ic}{2\alpha} k_{\mu}^2 \xi} - \frac{1}{\xi^2} \right. \\ \left. + \int_0^{\infty} \frac{d\beta}{\xi(\beta + \xi)^2} e^{\frac{ic}{2\alpha} \frac{\xi^2}{\beta + \xi} k_{\mu}^2} \right]. \end{aligned}$$

Even when we put $k_\mu^2 + \alpha^2 = 0$, though it is not allowable now as there is no delta function in the integrand, we have

$$\delta \alpha(\tau) = \frac{ie^2}{2\pi^2 \hbar c^2} \int_0^\tau d\xi \left[\frac{1}{\xi^2} e^{-\frac{ic\alpha}{2}\xi} - \frac{1}{\xi^2} + \int_0^\infty \frac{d\beta}{\xi(\beta+\xi)^2} e^{-\frac{ic\alpha}{2}\frac{\xi^2}{\beta+\xi}} \right], \quad (4, 15)$$

that is such $\delta \alpha(\tau)$ as $\lim_{\tau \rightarrow \infty} \delta \alpha(\tau) = \delta \alpha$. Therefore, during the course of travel of the imaginary particle, represented by F , the additional mass is not constant and unlike the usual procedure of the mass renormalization this $\delta \alpha(\tau)$ can not be simply absorbed in the Lagrangian of the free particle.

When we consider the Bose particle, the corresponding Lagrangian, *mutatis mutandis*, is (4, 2), the term containing $\gamma_\mu \gamma_\nu F_{\mu\nu}$ being omitted. Therefore from (4, 7) the second term disappears and we have for $\delta \alpha$ (4, 14), of which the first term in the parenthesis is multiplied by 1/4.

In concluding we say that the method of the path integral seems not to be appropriate for such a problem. Only because of commonness we have chosen this self-energy problem and shown that $\delta \alpha$ appears in an intuitive way in the kernel. As to the expression for $\delta \alpha$, the linearly divergent imaginary term may be due to the inadequacy of the starting second order Dirac Hamiltonian. Or else this divergent term may be arisen from the definiteness of the initial and final boundary contrary to the usual Feynman-Dyson diagram method. Furthermore, from the outset we have omitted the case $\sigma = \sigma'$, and so even for the Klein-Gordon equation no quadratically divergent term has appeared.

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The Phenomenological Analyses of Mesonic Processes

Yoshio YAMAGUCHI

Department of Physics, Osaka City University

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This paper is devoted to qualitatively discuss the available data concerning artificially produced π -mesons:

- the photo-meson production from nucleon and deuteron,
- the scattering of meson by nucleon,
- the meson production by nucleon-nucleon collision,
- and the slow π^- -meson capture by hydrogen and deuterium.

We want to adopt here the phenomenological considerations rather than the current meson theoretical calculations. We shall predict some new results not yet observed, which are expected to determine whether our view-points are correct or not.

§1. Introduction

Many experiments concerning π -mesons have been performed by American physicists in a recent few years. Most of the experimental confirmations give strong evidences against the lowest order perturbational predictions based on meson theories, which is, however, believed to be at least qualitatively correct in future. Particularly the photo-production of π^0 -mesons from hydrogen shows the most definite discrepancy with the lowest order perturbational result. Therefore, we are forced to analyze the mesonic phenomena upon the basis of phenomenological theory. The methods used in our discussions are borrowed from the nuclear physics: the statistical theory of nuclear reactions (mainly, detailed balancing) and the partial wave analysis. Because our descriptions have no direct connection with "meson theories plus perturbation", our results do not always agree with the perturbational predictions.

In §2 we want to summarize some fundamental properties of π -mesons, which serve to conveniently simplify our following considerations. §3 is devoted to the general description of "two-body mesonic processes", i.e.,

$$\gamma + N \rightleftharpoons N + \pi \quad (\text{meson production, capture and scattering})$$

and

$$\pi + N \rightarrow N + \pi \quad (\text{scattering and capture})$$

where γ , N and π denote photon, nucleon and π -meson*. The usefulness of the

* If necessary, we use the more refined notations:

n neutron,
 p proton,
 π^+ positively charged π -meson,
 π^- negatively charged π -meson

detailed balancing and (probably) of the so-called dispersion formulas are shown in this chapter. Finally we give in §4 the qualitative interpretation of the processes;

$$N+N \rightleftharpoons N+N+\pi.$$

Here the partial wave analysis is used as the basis of our explanation.

§ 2. Preliminaries

As is well-known, all experiments suggest the boson character of π^\pm - and π^0 -mesons. Particularly, the two- γ decay of π^0 offers the definite support of zero spin¹⁾. And the zero spin of π^\pm -meson was established by recent experiments²⁾ which compared the two processes:

$$\pi^+ + d \rightleftharpoons p + p.$$

Further experimental result of slow π^- -meson capture by deuterium, i.e., the existence of $\pi^- + d \rightarrow n + n^{3)}$, is sufficient to determine the parity of π^\pm as odd. In the following we will, therefore, analyze the available data assuming that π^\pm is pseudo-scalar and π^0 is scalar or pseudo-scalar*, as far as without a proviso.

§3. General considerations of two-body mesonic processes

1. Some features of photo-meson production from hydrogen

The excitation functions^{4),5)} of

$$\gamma + p \rightarrow n + \pi^+ \quad (1.1)$$

$$\text{and } \gamma + p \rightarrow p + \pi^0 \quad (1.2)$$

and π^0 neutral π -meson.

Furthermore, we adopt in this paper the natural unit ($\hbar=c=1$) and following notations:

	mass		momentum (laboratory system)	energy (laboratory system)
photon	0		k ($ k =k$)	k
meson	μ		q ($ q =q$)	$q_0 = \sqrt{q^2 + \mu^2}$
nucleon	M	initial	0	M
		final	P_f ($ P_f =P_f$)	$E_f = \sqrt{P_f^2 + M^2}$

The corresponding quantities in the center of mass system are specified by asterisk * (for instance, k^* , etc.)

* Because there are no definite experimental conclusions about the parity of π^0 . (The polarization-correlation of two γ 's from decay of π^0 is one of the most promising methods to determine the parity of π^0 . But at present this experiment seems too difficult to get any definite conclusion.)

near threshold show that photo- π^+ -mesons are emitted in s - and p -wave (thus it turns out that the π^\pm -meson-nucleon coupling must be, at least partly, spin-dependent) while π^0 mainly in p -wave (the contribution from s -wave is not larger than 10%). This marked difference and the different angular distribution of photo- π^\pm from that of photo- π^0 must be attributed to the difference in their production mechanism (electric or magnetic, spin-dependent or not, and so on) and/or the opposite parity of π^\pm and π^0 . Let us now check the latter statement.

As was mentioned above, the observed relative cross section of (1.2) to (1.1) does definitely disagree with the lowest order perturbation results. This important fact compelled us either to adopt the opposite approximation of strong coupling⁷⁾ or to examine the higher order perturbation⁸⁾. It is, however, regrettable that the validity of these procedures cannot be regarded as well-established so far. Moreover, the π^- -meson scattering by hydrogen⁹⁾ shows that the meson-nucleon interaction is neither weak nor strong enough to justify the weak or strong coupling approximation, respectively. Thus it seems better to analyze the processes (1.1) and (1.2) in a phenomenological way. The investigation of possible (i.e., phenomenologically allowed types) of transition matrices for photo-meson production shows* that the parity of π^0 is hardly determined from the photo-meson production (its excitation function and angular distribution of meson). A brief proof is given as follows*:

Let Ψ_i , Ψ_f , φ and $F_{\alpha\beta}$ be the initial and final nucleon wave function, meson wave function and the electro-magnetic field strength, respectively. Then the possible types of transition matrices are

$$\text{and} \quad \left. \begin{aligned} & \int \bar{\varphi} \bar{\Psi}_f \beta O_{\alpha\beta} \Psi_i F_{\alpha\beta} \\ & \int \bar{\varphi} \bar{\Psi}_f \beta \gamma_5 O_{\alpha\beta} \Psi_i \mathfrak{F}_{\alpha\beta} \end{aligned} \right\} \quad \text{for scalar meson,} \quad (1.3)$$

$$\text{and} \quad \left. \begin{aligned} & \int \bar{\varphi} \bar{\Psi}_f \beta \gamma_5 O_{\alpha\beta} \Psi_i F_{\alpha\beta} \\ & \int \bar{\varphi} \bar{\Psi}_f \beta O_{\alpha\beta} \Psi_i \mathfrak{F}_{\alpha\beta} \end{aligned} \right\} \quad \text{for pseudoscalar meson;} \quad (1.4)$$

where $\bar{}$ means complex conjugate (and transposed), $\mathfrak{F}_{\alpha\beta}$ is the dual tensor of $F_{\alpha\beta}$ and $O_{\alpha\beta}$ means the appropriate second rank tensor generated from Dirac matrices γ , photon momentum k , meson momentum q and initial (or final) nucleon momentum p . From various theoretical requirements (energy-momentum conservation, Lorentz invariance, gauge invariance and Lorentz condition for electro-magnetic potential) there are only four types of possible $O_{\alpha\beta}$;

$$\gamma_\alpha \gamma_\beta, \quad p_\alpha \gamma_\beta, \quad \gamma_\alpha q_\beta \quad \text{and} \quad p_\alpha q_\beta \quad (1.5)$$

* The author expresses his hearty thanks to Associate Prof. Utiyama and Messrs. Nishijima, Kotani and Watanabé for their valuable discussions on this proof. The similar argument was also given independently by Koba, Kotani and Nakai (ref. 8).

multiplied by arbitrary scalar function constructed from k , q and p . Of course the eight types of (1.3) or (1.4) are not linearly independent, and the scalar function appearing in $O_{\alpha\beta}$ is further reducible. We do not enter here into the details of such discussions, but merely note that *there are not essential differences between (1.3) and (1.4) with respect to either excitation function or angular distribution of mesons*. Therefore we will hereafter consider the both cases of even and odd parity of π^0 .

2. Some considerations about photo-mesons

It is worth while to remark the very good efficiency of photo-mesons as compared with other modes of photo-reactions (also cf. the meson production by nucleon-nucleon collision). The large photo-meson yield is readily understood by, e.g., the sum-rule method¹⁰⁾ under the appropriate assumptions about the linear dimension of meson cloud and the effective number of virtual mesons around the nucleon (these quantities are estimated from the anomalous magnetic moments of nucleons, electron-neutron scattering¹¹⁾ and so on).

Moreover the large yield of photo-mesons from deuteron as compared with the simple photo-disintegration are expected for hard γ -rays ($\gtrsim 160$ Mev for π^+ and $\gtrsim 170$ Mev for π^0). This fact suggests the usefulness of

$$\gamma + d \rightarrow p + p + \pi^- \quad \text{and} \quad \gamma + d \rightarrow p + n + \pi^0 \quad (2.1)$$

as a source of informations concerning the processes

$$\gamma + n \rightarrow p + \pi^- \quad \text{and} \quad \gamma + n \rightarrow n + \pi^0. \quad (2.2)$$

Hence the accurate calculations of nuclear binding effects in (2.1) are necessary in order to derive the definite conclusions about (2.2). For high energy γ -rays the largeness of deuteron radius may allow the impulse approximation for the photo-meson production from deuterium. Thus as for π^0 , the relative yield of four types of the final nucleon system (i.e., singlet even, singlet odd, triplet even and triplet odd) is just sufficient to determine the relative importance of spin- and charge-dependent parts and independent parts of transition matrices. Quite similarly, the relative frequency of different spin states in final two (like) nucleon system serves to determine the spin-dependence of π^\pm -production. More extensive arguments of (2.1) are given in another paper¹²⁾.

3. The dispersion formula

Next we shall proceed the general discussion of the two-body processes concerning meson:

$$\begin{cases} \gamma + N \rightleftharpoons N + \pi & (\text{scattering or capture}), \\ \pi + N \rightarrow N + \pi & (\text{scattering or capture}). \end{cases} \quad (3.1)$$

For brevity we want to describe these processes by dispersion formulas with one excited level E_r . The total cross section of, e.g., $\gamma + p \rightarrow n + \pi^+$ is expressed in terms of γ -ray width Γ_γ^p , and meson widths $\Gamma_{\pi^+}^p$ and $\Gamma_{\pi^0}^p$:

$$\frac{1}{2} \frac{\pi}{k^{*2}} \cdot \frac{\Gamma_\gamma^p \Gamma_{\pi^+}^p}{(E^* - E_r)^2 + \frac{1}{4} (\Gamma_\gamma^p + \Gamma_{\pi^+}^p + \Gamma_{\pi^0}^p)^2}, \quad (3.2)$$

where

$$\begin{aligned} E^* + M &= k^* + \sqrt{k^{*2} + M^2} \\ &= q_0^* + \sqrt{q^{*2} + M^2}. \end{aligned}$$

We shall assume that the meson widths $\Gamma_{\pi^-}^n$ and $\Gamma_{\pi^0}^n$ appearing in $\gamma + n \rightarrow p + \pi^-$ and $n + \pi^0$ are almost equal to $\Gamma_{\pi^-}^p$ and $\Gamma_{\pi^0}^p$, respectively (charge independent hypothesis). But the widths $\Gamma_{\pi^+}^n$ and $\Gamma_{\pi^-}^p$ appearing in the scattering processes

$$\pi^+ + p \rightarrow p + \pi^+ \quad \text{and} \quad \pi^- + n \rightarrow n + \pi^- \quad (3.3)$$

may be different from $\Gamma_{\pi^+}^p = \Gamma_{\pi^-}^n$, and we can say nothing about $\Gamma_{\pi^+}^n$ and $\Gamma_{\pi^-}^p$, because there are no experimental informations about (3.3).

If the change of denominator of (3.2) is not so rapid, the meson widths are responsible for the behavior of excitation functions near threshold, and thus we may put

$$\Gamma_{\pi^+}^p = c_{ch} \left[\frac{q^*}{q_0^*} + \left(\frac{q^*}{q_0^*} \right)^3 \right] \quad (3.4)$$

and

$$\Gamma_{\pi^0}^p = c_n \left(\frac{q^*}{q_0^*} \right)^3, \quad (3.5)$$

where c 's have a dimension of energy and may be regarded as nearly energy independent. As for Γ_γ^p , we assume only the " p -wave" contribution:

$$\Gamma_\gamma^p = c_\gamma \left(\frac{k^*}{\mu} \right)^{2l+1}, \quad l=1. \quad (3.6)$$

But it is noted that the results (2.7) and (3.8) stated below do not depend on the choice of l in (3.6).

The process $\gamma + p \rightarrow n + \pi^+$ has a larger yield than that of $\gamma + p \rightarrow p + \pi^0$ by a factor of ~ 3 at $k=250$ Mev. This leads to the relation:

$$c_{ch} \approx c_n. \quad (3.7)$$

Assuming the charge independence, $\Gamma_{\pi^0}^p = \Gamma_{\pi^0}^n$, (which must be checked by $\gamma + d \rightarrow p + n + \pi^0$, as was stated in 2 of this chapter), the data of slow π^- -meson capture by hydrogen ($\pi^- + p \rightarrow n + \pi^0$ and $n + \gamma$) is sufficient to determine the ratio of c_n to c_γ ; (we also use the relation $\Gamma_\gamma^p = \Gamma_\gamma^n$);

$$c_n \approx 50 c_\gamma. \quad (3.8)$$

This result seems quite reasonable.

The above considerations correspond to the even parity hypothesis of π^0 , because of p -wave nature of π^0 emitted after the slow π^- -meson capture by hydrogen. If we assume the parity of π^0 is odd, (3.5) must be replaced by a more refined form:

$$\Gamma_{\pi^0}^p = c_n^0 \frac{q^*}{q_0^*} + c_n \left(\frac{q^*}{q_0^*} \right)^3. \quad (3.5')$$

The shape of excitation function of $\gamma + p \rightarrow p + \pi^0$ gives the upper limit of c_n^0 :

$$c_n^0 \lesssim 0.1 c_n. \quad (3.9)$$

In this case the photo- π^0 -mesons are mainly produced in p -wave while π^0 -emission after π^- -capture by proton occurs in s -wave. The competition of two capture modes $\pi^- + p \rightarrow n + \pi^0$ and $n + \gamma$ gives

$$c_n^0 \approx 4 c_\gamma. \quad (3.10)$$

(3.9) and (3.10) do not contradict (3.8). Also the lowest order perturbation supports (3.9). Thus we may use (3.7) and (3.8) irrespective of the parity of π^0 .

4. The scattering of meson and γ -ray by nucleon

If we use (3.4)—(3.8), we can calculate the relative frequencies of various modes of meson scattering; e.g., it turns out for 85 Mev

$$\pi^- + p \rightarrow \begin{cases} p + \pi^- & 73\%, \\ n + \pi^0 & 24\%, \\ n + \gamma & 3\%. \end{cases} \quad (4.1)$$

In fact this result is practically independent of the assumed forms of meson widths (3.4) and (3.5), because the value of q^* appearing in the calculations of (4.1) is almost equal to that used to evaluate (3.7) and (3.8). For example, if we use the expression

$$\Gamma_{\pi^+}^p = c'_{ch} \cdot \frac{q^*}{q_0^*}, \quad c'_{ch} = \text{const.}, \quad (4.2)$$

instead of (3.4), the relative frequencies suffer only slight changes:

$$\pi^- + p \rightarrow \begin{cases} p + \pi^- & 74\%, \\ n + \pi^0 & 23\%, \\ n + \gamma & 3\%. \end{cases} \quad (4.1')$$

Secondly let us consider the elastic scattering of γ -ray by proton. Assuming (3.8), abnormal scattering of γ -ray by proton has the total cross section σ_{tr} equal to that of $(\gamma + p \rightarrow p + \pi^0)$ multiplied by

$$\sim \frac{1}{50} \left(\frac{k^* q_0^*}{\mu q^*} \right)^3.$$

Thus $\sigma_{\gamma\pi}$ at $k=250$ Mev turns out to be $\sim 6.7 \times 10^{-30} \text{cm}^2$. Note that $\sigma_{\gamma\pi}$ is proportional to k^4 near $k=250$ Mev. The value of $\sigma_{\gamma\pi}$ is large enough to be observed by present experimental technique. (Cf., the total cross sections of Thomson scattering are

$$8.6 \times 10^{-30} \text{cm}^2 \text{ for } \pi^\pm\text{-meson,}$$

$$\text{and } 1.9 \times 10^{-31} \text{cm}^2 \text{ for proton,}$$

respectively.)

In the above discussions, we did not make full use of the dispersion formula, but it sufficed merely to assume the factorizability of transition probability for each process into two terms and the applicability of detailed balancing to these individual terms. The validity of these assumptions must be checked by the two predictions given in this section.

Finally let us tentatively determine the resonance level E_r and the absolute values of widths. The total cross sections of $\gamma + p \rightarrow n + \pi^+$ at $k=250$ Mev⁴⁾ and of 85 Mev π^- -meson scattering by proton⁹⁾ give two set of the values:

$$E_r \doteq 310 \text{ Mev}, \quad c_\pi \doteq 0.63 \text{ Mev}; \quad (4.3)$$

$$\text{and } E_r \doteq 220 \text{ Mev}, \quad c_\pi \doteq 0.037 \text{ Mev}. \quad (4.4)^*$$

Our previous assumption that the denominator of (3.2) do not appreciably change with k^* or q^* is consistent with the set (4.3) rather than (4.4). And the extrapolation of the γ -ray width of nucleus favours also (4.3).

We may therefore conclude that our results are nearer to the strong coupling theories⁷⁾ rather than the weak coupling theories. Moreover this fact may be seen from the abnormal scattering of γ -rays by protons when our result is compared with that of Sachs and Foldy¹³⁾ (also see Concluding Remarks).

Our considerations are based on one level formula with simple assumptions about widths. We can, of course, generalize our treatment to many level formula with more refined assumptions concerning widths. These possibilities will be discussed, if necessary, in future**.

§4. The mesonic processes in two nucleon system

1. General discussion***

Besides the energy-momentum and the angular momentum, the two nucleon

* The set (4.4) is somewhat attractive, after some suitable modifications of forms of widths, because the value of $E_r + M$ is close to the mass of V -particles¹⁴⁾, although there remains the interpretation of their long lives.

** Also the photo-mesons from nuclei can be treated in similar (phenomenological) way, if we take into account the reabsorption of produced mesons¹⁵⁾. But we do not discuss their details in this paper.

*** Also see ref. 24).

system has two constants of motion:

- i) the symmetry property with respect to interchange of spin coordinates, and
- ii) the parity.

Thus it seems appropriate to describe the mesonic processes

$$N+N \rightleftharpoons N+N+\pi \quad (1.1)$$

in terms of selection rule in regard to the spin state or parity of two nucleon system. Such a statement is desirable, because we think that we may succeed in deriving the "selection rule" from the meson theories as in the case of $\pi^0 \rightarrow \gamma + \gamma$, in spite of the great difficulties to obtain any finite numerical results at present.

Considering the smallness of the relative energy in the final nuclear system in (1.1), it is readily seen that the partial wave analysis will be very useful for our discussions. Thus we can assume that the smaller the relative (angular) momentum of final nucleon system the larger the probability of its occurrence in the final state; i.e., the π -yield accompanying with the nucleon system in S -state is much larger than that with nuclear P -state, as far as there is no any effective selection rule.

These arguments are the foundation of our following analyses of (1.1).

2. Fundamental data

The very marked features of recent experiments are as follows:

$$(a) \quad p+p \rightarrow \begin{cases} p+n+\pi^+, \\ d+\pi^+. \end{cases}$$

The final $(p+n)$ is mainly in 3S -state¹⁶⁾.

$$(b) \quad p+p \rightarrow \begin{cases} p+n+\pi^+, \\ p+p+\pi^0. \end{cases}$$

The π^0 -yield is very small compared with π^+ -yield¹⁷⁾.

$$(c) \quad p + (\text{nucleus}) \rightarrow \begin{cases} \pi^+ + (\text{nucleus} + \text{other nucleons}), \\ \pi^- + (\text{nucleus} + \text{other nucleons}). \end{cases}$$

The π^+ -yield from proton bombardment is large (by a factor of $5 \sim 10$) compared with π^- -yield¹⁸⁾. Analogously the π^- -yield from neutron bombardment is large compared with the π^+ -yield¹⁹⁾. There are two possible interpretations of (c):

- (c1) The fact (c) is mainly attributed to the elementary processes; i.e., the π^- -yield from proton bombardment on neutron is small compared with the π^+ -yield from proton bombardment on proton.
- (c2) (c) is due to the effect of exclusion principle in the final nuclear state²⁰⁾. The π^\pm -yield from $p+n$ is not necessarily smaller than the π^+ -yield from $p+p$.

Next we will show that these three data can be explained by the simple "selection rule" (no singlet \longleftrightarrow triplet transition or no parity change).

Let us consider the process

$$p + p \rightarrow p + n + \pi^+. \quad (2.1)$$

First of all, transitions of nuclear system:

$$^1S_0 \rightarrow ^1S_0, \quad ^3P_1 \rightarrow ^1S_0 \quad \text{and} \quad ^3P_0 \rightarrow ^3S_1 \quad (2.2)$$

are forbidden by the parity and total angular momentum conservation law (because π^\pm is pseudoscalar). The fact (a) demands that

$$^3P_0 \rightarrow ^1S_0 \quad \text{and} \quad ^3P_2 \rightarrow ^1S_0$$

must be forbidden. This requirement is fulfilled by assuming either the selection rule (t);

"the triplet singlet transitions must be forbidden,"

or the selection rule (p):

"the parity must not change"

with regard to the nucleon state.

Secondly we will examine the processes

$$p + n \rightarrow \begin{cases} p + p + \pi^-, \\ n + n + \pi^+. \end{cases} \quad (2.2)$$

The transitions with final nuclear S -state are

$$\begin{cases} ^3S_1 \rightarrow ^1S_0, \\ ^3P_0 \rightarrow ^1S_0, \end{cases} \quad ^3P_2 \rightarrow ^1S_0. \quad (2.3)$$

If we adopt the view-point of (c1), we must forbid these transitions (2.3). Assuming the validity of the selection rules irrespective of the charge of nuclear system*, (the charge independences are often noticed in the nuclear physics), then only the selection rule (t) is permissible. On the other hand, if we use the interpretation (c2), it need not forbid (2.3) and the selection rule (p) is now adoptable.

Furthermore if we assume π^0 is pseudoscalar**, the main processes $^3P_0 \rightarrow ^1S_0$ and $^3P_2 \rightarrow ^1S_0$ of $p + p \rightarrow p + p + \pi^0$ must be forbidden and (b) is readily explained by either (t) or (p), showing the usefulness of charge independent hypothesis (in π^\pm - and π^0 -production)*.

Thus we see that the selection rules (t) and (p) are equally useful to account for the observations (a), (b) and (c).

It need not, of course, require the absolute validity of these selection rules but it merely suffices to be valid in good approximation at low energy regions.

* Though the charge independence is assumed for both nucleons and mesons, it does not necessarily mean that only π^0 of the "symmetrical" (i.e., $((\tau_3))$; see below) type should be used; the "neutral" (i.e., $((1))$) type interaction is also permissible.

** The scalar π^0 case will be discussed in § 5.

3. Some consequences of the selection rule (t)

In the first place we consider the case of (t). Assuming π^0 is pseudo-scalar, the main processes of meson production are as follows*:

$$\begin{aligned}
 p+p &\rightarrow p+n+\pi^+, & {}^3(\text{odd}) &\rightarrow {}^3S+(\bar{s}+\bar{d}+\dots), \\
 p+p &\rightarrow p+p+\pi^0, & {}^3(\text{odd}) &\rightarrow {}^3P+(\bar{p}+\dots), \\
 p+n &\rightarrow n+n+\pi^+ \\
 p+n &\rightarrow p+p+\pi^- \quad \left. \vphantom{\begin{matrix} p+n \\ p+n \end{matrix}} \right\} \begin{cases} {}^3(\text{even}) \rightarrow {}^3P+(\bar{s}+\bar{d}+\dots), \\ {}^3(\text{odd}) \rightarrow {}^3P+(\bar{p}+\dots), \end{cases} \\
 p+n &\rightarrow p+n+\pi^0. \quad \left\{ \begin{array}{l} {}^3(\text{even}) \rightarrow {}^3S+(\bar{p}+\dots), \\ {}^3(\text{odd}) \rightarrow {}^3S+(\bar{s}+\bar{d}+\dots). \end{array} \right. \quad \begin{matrix} ((1)) \\ ((\tau_3)) \end{matrix}
 \end{aligned}$$

where small letter s, p, \dots in parenthesis denote the s -, p -, ... wave of produced meson, and $\bar{}$ is added to clearly show the odd parity of π -mesons.

We can see the following facts without any detailed calculations.

(t-1) The angular distribution of π^+ -meson from $p+p$ (proton bombardment on proton). The main processes of the reaction $p+p \rightarrow p+n+\pi^+$ are

$$\begin{cases} {}^3P_1 \rightarrow {}^3S_1+(\bar{s}+\bar{d}) & (J=1), \\ {}^3P_2 \rightarrow {}^3S_1+(\bar{d}) & (J=2). \end{cases}$$

We must therefore consider that the observed angular distribution ($\sim \cos^2\theta d\Omega$, in the center of mass system for 343 Mev proton¹⁶⁾; where θ is the angle between \mathbf{q}^* and the incident beam, $d\Omega$ is an element of solid angle into which the meson is emitted) is due to the suitable superposition of s - and d -wave of mesons.

(t-2) The ratio of π^0 -yield to π^+ -yield (or π^- -yield) from proton bombardment on neutron (we shall denote this long statement briefly as " π^0/π^+ (or π^0/π^-) from $p+n$ ") is nearly equal to π/π^0 from $p+p$. In the deuteron the nuclear binding effects will not be so appreciable, that π^+/π^- from $p+d$ or π^-/π^+ from $n+d$ must be close to π^+/π^0 from $p+p$, while π^+/π^0 from $p+d$ or π^-/π^0 from $n+d$ is nearly equal to unity.

(t-3) The angular distribution of π^0 from $p+n$. If the transitions are of the type $((\tau_3))$, the main transition of π^0 -production from $p+n$ is of the same type as π^+ -production from $p+p$. Thus in this case the angular distribution of π^0 from $p+n$ will be the same as that of π^+ from $p+p$. However the existence of $((1))$ -type transitions may deform the π^0 -angular distribution in the direction of isotropic distribution. As in the case $p+p \rightarrow p+n+\pi^{+16)}$ a large part of π^0

* In the π^0 -production from $p+n$, the symmetry property of isotopic-spin eigen-function of nuclear state will change in some cases (which are specified by " $((\tau_3))$ ") and will not change in other cases (" $((1))$ "). The difference between $((\tau_3))$ and $((1))$ does not appear in π^0 -production from $p+p$ or $n+n$.

from $p+n$ will be accompanied by deuteron (i.e., $p+n \rightarrow d+\pi^0$, but no 1S -resonance).

(t-4) We discuss π^+/π^0 from $p+p$ as an example (other cases can be treated quite similarly). If we fix the incident energy of proton, π^+/π^0 increases with increasing meson energy. Because; the larger the meson energy the smaller the relative energy of final nucleon system, the latter means that P -(or higher angular momentum) state contribution becomes the smaller with the result that π^+/π^0 increases. The same argument shows that, fixing the meson energy, π^+/π^0 increases as decreasing incident proton energy.

In the above considerations we neglected the Coulomb effect. In order to refine our arguments this effect must be taken into account.

(t-5) π^+/π^- from $p+(\text{nucleus})$ may be smaller than the ratio $Z \cdot (\pi^+/\pi^-)$ yielded from $p+p$ to $N \cdot (\pi^-/\pi^+)$ (yield from $p+n$) at the same incident proton energy, because of the combined effect of next reasons (Z, N =number of proton and neutron in the nucleus);

- i) π^- has larger wave function than π^+ near the target nucleus owing to the Coulomb field of the nucleus;
- ii) nuclear binding effects:
 - ii-1) the internal motion of target nucleon inside the nucleus makes the incident proton energy effectively higher,
 - ii-2) the final nuclear system with higher angular momenta may easily be realized as compared with the case of $p+(\text{nucleon})$.
 - ii-3) the exclusion principle forbids the final states the more seriously the smaller the energy of final nuclear system; the smallness of energy may more or less correspond to the smallness of relative angular momenta of final nuclear states.

All of these effects reduce the π^+/π^- from $p+(\text{nucleus})$.

π^-/π^+ from $n+(\text{nucleus})$ can be discussed analogously, but in this case the effect i) and ii) act on π^-/π^+ in the opposite directions.

4. Some consequences of the selection rule (p)

Next we want to examine the case of (p), in which the interpretation (c2) must be adopted. π^0 is assumed here to be pseudoscalar. The main production processes are

$$\begin{array}{ll}
 p+p \rightarrow p+n+\pi^+, & {}^1S_0, {}^1D_2 \rightarrow {}^3S_1 + (\bar{p}), \\
 p+p \rightarrow p+p+\pi^0, & {}^3P \rightarrow {}^3P + (\bar{p}), \\
 \left. \begin{array}{l} p+n \rightarrow n+n+\pi^+ \\ p+n \rightarrow p+p+\pi^- \end{array} \right\}, & {}^3S_1, {}^3D_1 \rightarrow {}^3S_1 + (\bar{p}), \\
 p+n \rightarrow p+n+\pi^0, & \left\{ \begin{array}{l} {}^3S_1, {}^3D_1 \rightarrow {}^1S_0 + (\bar{p}), \\ {}^1S_0, {}^1D_2 \rightarrow {}^3S_1 + (\bar{p}), \\ {}^3S_1, {}^3D_{1,2} \rightarrow {}^3S_1 + (p). \end{array} \right\} \quad ((\tau_3)) \quad (1)
 \end{array}$$

Some qualitative discussions as in 3 are possible.

(p-1) The angular distribution and excitation function of π^+ from $p+p$. The angular distribution (in the center of mass system) of π^+ -meson is

$$\sim d\Omega \quad \text{for} \quad {}^1S_0 \rightarrow {}^3S_1 + (\bar{p}) \quad (J=0), \quad (4.1a)$$

$$\text{and} \quad \sim \cos^2\theta d\Omega \quad \text{for} \quad {}^1D_2 \rightarrow {}^3S_1 + (\bar{p}) \quad (J=2). \quad (4.1b)$$

The proton-proton scattering at 340 Mev shows the strong interactions in P - and D -state as well as S -state. Therefore it is not unreasonable to assume that (4.1b) has much larger probability (at least by a factor ~ 5 , as can be seen from the statistical weight) than that of (4.1a). Then the $\cos^2\theta$ angular distribution does agree very well with the observation at 343 Mev proton. The angular distribution of π^+ for 381 Mev $p+p$ is also expected to be $\sim \cos^2\theta d\Omega$; and thus the ratio of π^+ -yield from $p+p$ at 343 Mev to 381 Mev turns out to be 2.5 ± 0.8^{21} . This figure is also consistent with the p -wave of π^+ -meson (excitation function is roughly proportional to $(E-\mu)^{l+\frac{1}{2}}$, where E is the incident proton energy and l is the angular momentum of a meson). These explanations are more natural than the case of (t) (cf. (t-1)).

(p-2) π^\pm - and π^0 -yield from $p+n$ are nearly equal to π^+ -yield from $p+p$. This is the marked difference between (t) and (p), and may easily be checked by experiments.

(p-3) The energy distribution of π^0 from $p+n$ is expected to have two peaks corresponding to 1S -resonance and deuteron state of the final nucleon system. The angular distribution of π^0 from $p+n$ is hard to predict only from the qualitative consideration, but it seems to be rather collimated in the direction of incident beam.

(p-4) π^+/π^0 from $p+p$ and others show the same behaviour as described in (t-4).

As was stated above, a definite determination of superiority between (t) and (p) must be done by comparison (t-2) and (p-2) with observation.

5. The case of scalar π^0

The selection rule (t) or (p) may be used for pseudoscalar π^\pm -meson. (b) in § 2 leads to the selection rule for production of scalar π^0 :

the selection rule (t')

"only the triplet triplet transitions can be allowed"

Assuming the validity of this selection rule irrespective of the charge of nuclear system, the main processes of π^0 -production are as follows:

$$\begin{array}{lll} p+p \rightarrow p+p+\pi^0, & {}^3P \rightarrow {}^3P+(s) & \sim d\Omega, \\ p+n \rightarrow p+n+\pi^0, & \left\{ \begin{array}{ll} {}^3S \rightarrow {}^3S+(s) & ((1)) \sim d\Omega, \\ {}^3P \rightarrow {}^3P+(p) & ((\tau_3)) \sim \cos^2\theta d\Omega. \end{array} \right. \end{array}$$

Considering the case of $p+p \rightarrow p+p+n+\pi^+$ a majority of produced π^0 will be accompanied by deuteron state,

$$p+n \rightarrow d+\pi^0.$$

It is further noted that the charge independent (in a wide sense) selection (t') is also permissible for pseudo-scalar π^\pm and scalar π^0 -meson, because in the pseudo-scalar case (t) is equivalent to (t') at low energy regions.

6. Slow π^- -meson capture by deuterium

As was mentioned in § 2, the existence of capture mode

$$\pi^- + d \rightarrow n + n$$

determined the parity of π^\pm as odd. This transition is allowed from the selection rule (t) while of the first forbidden from the selection rule (p). According to the detailed balancing the relative frequency of two capture modes

$$\pi^- + d \rightarrow \begin{cases} n + n, \\ n + n + \gamma. \end{cases}$$

can be calculated from the observed cross section of inverse processes (assuming charge independence)⁽²²⁾:

$$\begin{cases} p + p \rightarrow d + \pi^+, \\ \gamma + p \rightarrow n + \pi^+. \end{cases}$$

If we adopt the selection rule (t) (or (p)), one finds

$$(n+n) : (n+n+\gamma) = 4:1 \text{ (or } 2:1). \quad (6.1)$$

Because we do not know the precise values of cross section of $n+n \rightarrow d+\pi^-$ and $\gamma+n \rightarrow p+\pi^-$ the result (6.1) must not be understood as conclusive. However, this result seems to favour (p) over (t).

In the process $\gamma+p \rightarrow n+\pi^+$, the π^+ -production of, at least, s -wave meson must be spin-dependent. Therefore the capture process $\pi^- + d \rightarrow n+n+\gamma$ is also certainly spin-dependent, and the final $(n+n)$ must be in 1S -state. The energy spectrum of final γ -rays will be very useful to know the neutron-neutron interaction^{(23), (24)}.

Finally let us consider the process $\pi^- + d \rightarrow n+n+\pi^0$ (kinetic energy of $\pi^0 \approx 1.9 \pm 1.5$ Mev)⁽²⁴⁾. If π^0 is pseudoscalar, this process has negligible probability as compared with other capture modes, because the main capture process turns out to be

$$\pi^- + d \rightarrow n+n+\pi^0, \quad (\bar{s}) + {}^3S_1 \rightarrow {}^3P + (\bar{p}).$$

On the other hand, for scalar π^0 the main capture processes is

$$\pi^- + d \rightarrow n + n + \pi^0, \quad (\bar{s}) + {}^3S_1 \rightarrow {}^1S_0 + (\bar{p}), \quad (6.1)$$

If we compare this with the process

$$\begin{array}{ccc} \pi^- + p \rightarrow n + \pi^0 & (J=1/2, \text{ odd}) & (6.2) \\ (\bar{s}) & (p) & (\text{kinetic energy of } \pi^0 = 5.4 \pm 1.0 \text{ Mev}). \end{array}$$

The relative frequency of these two processes is given by

$$\frac{(\pi^- + d \rightarrow n + n + \pi^0)}{(\pi^- + p \rightarrow n + \pi^0)} \approx \left(\frac{1.9 \pm 1.0}{5.4 \pm 1.0} \right)^{3/2} = 0.2 \pm 0.1.$$

Thus we find the relative rate

$$\pi^- + d \rightarrow \begin{cases} n + n + \gamma & 1, \\ n + n + \pi^0 & \lesssim 0.2 \pm 0.1. \end{cases}$$

If we use the experimental result³⁾

$$\pi^- + d \rightarrow \begin{cases} n + n & 7, \\ n + n + \gamma & 3. \end{cases}$$

the final state $(n + n + \pi^0)$ occurs with probability not larger than $(6 \pm 3)\%$. Thus the even parity of π^0 can not be excluded by the present accuracy of experiment.

7. Slow π^- -meson capture and fast π^- -meson scattering by nuclei

A) π^- -meson capture by nucleus

Because of the internal motion of nucleons inside the nucleus the main capture mode is attributed to

$$\pi^- + p + N \rightarrow n + N. \quad (7.1)$$

In the case of (p), (7.1) is now the allowed transition though it was first forbidden in the deuteron case $\pi^- + d \rightarrow n + n$. Therefore the great majority of "stars" produced by π^- -meson capture by nucleus is easily understood. It should be noted that in the case of (t) similar situations occur, though in this case (7.1) has somewhat smaller probability than that in the case of (p), providing further evidence favoring (p).

B) Fast π^- -meson scattering by nucleus

The detailed balancing predicts that the major parts of π^- is captured through the process

$$\pi^- + p + N \rightarrow n + N$$

and the total cross section may be close to nuclear area. This was confirmed by recent experiment⁹⁾.

The more quantitative discussions of the contents of this chapter are possible on the basis of phenomenologically constructed transition matrices (cf. § 3, 3). The preliminary results of such a treatment have already been published by Fujimoto and Yamaguchi²⁴⁾. Further details will be given in near future.

§ 5. Concluding Remarks

We have analyzed the mesonic phenomena in somewhat different manners as compared with the viewpoint of current meson theories.

The rather strong interaction between meson and nucleon may allow us to adopt the dispersion theory (§ 3, 3 and 4). Of course this does not necessarily mean the usefulness of strong coupling theory; instead the concept of dispersion theory (in a generalized form) has much wider validity than the so-called strong coupling theory has. We could derive some results concerning the meson-nucleon and photon-proton scattering. In fact these predictions were based on the simplified assumptions of "width" (factorizability and charge independence (i.e., $\Gamma_{\pi^+}^p = \Gamma_{\pi^-}^n$, $\Gamma_T^p = \Gamma_T^n$, etc.)), which may not be correct. We think, however, that these crude assumptions are sufficient for our preliminary considerations; and the check of our assumptions can be done by the experimental results of photo-meson production from deuterium (§ 3, 2) and π -or γ -proton scattering*, etc. (§ 3, 4).

Furthermore we discussed the meson production by nucleon-nucleon collision. This process can be qualitatively interpreted by the selection rule (t) or (p). While some evidences favouring the latter (p) were stated (§ 4, 4 (p-1), 6 (6.1)), the conclusive discrimination will only be established by the meson production by nucleon-deuteron collision (§ 4, 4). The close connection between the dispersion formulas for "two-body mesonic processes" and the selection rule in the meson production from nucleon nucleon collision should be established, but it was not discussed in this paper.

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Addendum The content of §4 was read in REKS held at Kyoto University on 24 July in 1951. Just after the author had finished the § 4 of this paper, the issue of the Physical Review, vol. 83, no. 1 (1951) arrived, in which an excellent paper of Watson and Brueckner (who adopted the case (p) with $((\tau_3))$ for π^0 -production) dealing with the same problem as in § 4 was published.

Since the perturbational results were not used in this paper, many papers written on the basis of perturbation theory were not cited.

* E. g., the energy dependence of σ_{TT} will give the correct form of Γ_T^p (i. e., c_T and l in eq. (3.6)) providing our description is correct (§3, 3 and 4).

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Nuclear Interaction of μ -Meson

Hiroshi FUKUDA, Yoichi FUJIMOTO and Masatoshi KOSHIBA

Department of Physics, Tokyo University

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The experimental results concerning nuclear events in cosmic-ray underground, obtained by E. P. George and J. Evans by means of nuclear emulsions, were found to be in rather good agreement with the theoretical calculation, where the electromagnetic interaction of μ -meson with nucleon were treated in detail by Feynman-Dyson's procedure.

§ 1. Introduction

Recent experimental analysis of nuclear events in cosmic-ray underground¹⁾ indicates that these are mainly caused by μ -mesons. The direct interaction of μ -meson with nucleon, estimated from μ -meson capture, is too weak to account for the frequency of these nuclear events. One of the most probable process will be the electromagnetic effects of μ -meson, which we shall consider in this paper.

When a μ -meson passes by or through a nucleus, some proton in the nucleus gets a momentum transferred electromagnetically. The cross section of this process is of order $4 \cdot 10^{-31}$ cm² per nucleon, if we regard a proton as a point charge. However, in reality, a proton is not a point charge but is accompanied by a meson cloud around it, which will play the most significant role in this process. This situation is similar to that of photo-meson production, where we found a rather good agreement between the theoretical prediction and the experimental results.

The object of this paper is to clarify the above consideration and to get the cross section for the process, where an incident μ -meson collides with a nucleon to make it eject a π -meson. In order to apply our results to the nuclear events underground, we must consider the interference effect with other nucleons in the nucleus, which was analysed by Amaldi²⁾ in detail. When the momentum transfer is greater than the mean kinetic energy of nucleon, this effect will be negligible. The π -meson which is produced in a nucleus will again be absorbed by the nucleon whence it has come or by others and give rise to star.

In § 3-1 the general cross section will be given with some brief discussions about it. In § 3-2 the cross section in the case of the threshold energy will be given. In § 3-3 the extreme relativistic case will be treated in detail and the physical interpretations will also be made. In § 4 the average energy loss of μ -meson and average deflection angle of μ -meson and of the recoil nucleon will be given.

Considering the Coulomb field of the incident μ -meson as a set of virtual photon, our process has a close relationship with the artificial production of π -meson by γ -ray, which gave results compatible with theoretical predictions, although the validity of perturbation method is not established. This problem will be discussed in § 5.

§ 2. Notations and units

Throughout this paper we use the natural unit, $\hbar=c=1$, and the following notations;

- $P_0(\mathbf{p}_0, iE_0)$, $P(\mathbf{p}, iE)$; the energy-momentum 4 vector of initial and final nucleon, respectively;
- $K_0(\mathbf{k}_0, i\epsilon_0)$, $K(\mathbf{k}, i\epsilon)$; the energy-momentum 4 vector of initial and final μ -meson, respectively;
- $q(\mathbf{q}; i\epsilon)$; the energy-momentum 4 vector of the created π -meson;
- Ψ , φ , and ϕ ; the wave function of nucleon, μ -meson, and π -meson, respectively;

for which we have,

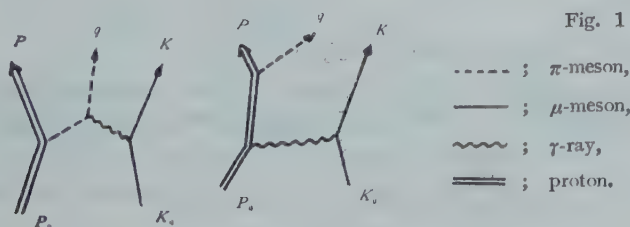
$$(i\gamma P + M)\Psi(P) = 0, \quad (i\gamma K + \mu)\varphi(K) = 0, \quad (q^2 + \pi^2)\phi(q) = 0, \quad (1)$$

where M , μ , and π mean the masses of nucleon, μ -meson, and π -meson.

Calculation will be performed in center of mass system.

§ 3-1. General formula for the cross section

The Feynman-Dyson diagrams for this process consist of the following two diagrams, which, combined together, constitute a gauge invariant set.



In Fig. 1 we can interpret the process as that the proton or π -meson are scattered by the external field of μ -meson; however it must be emphasised that each of these two diagrams cannot have rigorous physical meanings as they are, when separated, not gauge-invariant. The matrix element M of this process are of the same feature as that of photo-meson production:

$$= \overset{\circ}{A}_\mu M_\mu \quad (2)$$

where $\overset{\circ}{A}_\mu$ means the external electromagnetic field of μ -meson instead of free photon and is given as follows,

$$\overset{\circ}{A}_\mu = i e \frac{\bar{\varphi} \gamma_\mu \varphi}{\square} \quad (3)$$

and M_μ is

$$M_\mu = f \bar{\Psi}(P) \gamma'_5 \left\{ \frac{i \gamma (P_0 + q) - M}{(P_0 + q)^2 + M^2} \gamma_\mu + \frac{2 i q_\mu}{(P - q)^2 + x^2} \right\} \Psi(P_0) \phi^*(q), \quad (4)$$

$$\gamma'_5 = \begin{cases} 1 & \text{for scalar meson,} \\ i \gamma_5 & \text{for pseudoscalar meson.} \end{cases}$$

The cross section is given by,

$$\begin{aligned} \sigma &= e^4 f^2 \frac{1}{(2\pi)^9 B} \int \frac{d\mathbf{p}}{2E} \frac{d\mathbf{q}}{2\epsilon} \frac{d\mathbf{K}}{2\epsilon} (2\pi)^4 \delta(P + K + q - P_0 - K_0) \Phi, \\ \Phi &= \frac{1}{16} \sum_{pol.} |M|^2, \\ B &= \sqrt{(P_0 K_0)^2 - P_0^2 K_0^2}, \end{aligned} \quad (5)$$

in center of mass system. This is rewritten as,

$$\sigma = \left(\frac{e^2}{4\pi} \right)^2 \cdot \left(\frac{f^2}{4\pi} \right) \cdot \frac{2}{k_0(E_0 + \epsilon_0)} \int d\epsilon \int dE \frac{d(\cos \theta) d\varphi}{4\pi} \Phi, \quad (6)$$

where θ means the deflection angle of μ -meson and φ is the angle between the plane of incident and scattered μ -meson and the plane of nucleon and μ -meson after the scattering.

Using $\partial_\mu \overset{\circ}{A}_\mu = 0$ (charge conservation) and energy momentum conservation, Φ is rewritten as,

$$\Phi = B_{\mu\nu} C_{\mu\nu}, \quad (7)$$

$$\text{where } B_{\mu\nu} = B_{\mu\nu}^S + B_{\mu\nu}^C, \quad C_{\mu\nu} = C_{\mu\nu}^C + C_{\mu\nu}^S + C_{\mu\nu}^{S'}, \quad (8)$$

$$\begin{aligned} B_{\mu\nu}^C &= \frac{1}{2} \frac{(K'_0 + K)_\mu (K_0 + K)_\nu}{(K'_0 - K)^4}, \quad B_{\mu\nu}^S = \frac{1}{2(K'_0 - K)^2} \left(\delta_{\mu\nu} - \frac{(K_0 - K)_\mu (K_0 - K)_\nu}{(K_0 - K)^2} \right), \\ C_{\mu\nu}^C &= 2 \left\{ \binom{0}{4} M^2 - x^2 \right\} \left(\frac{P_{0\mu}}{(P_0 + K'_0 - K)^2 + M^2} + \frac{q_\mu}{(P_0 - P)^2 + x^2} \right) \\ &\quad \left(\frac{P_{0\nu}}{(P_0 + K'_0 - K)^2 + M^2} + \frac{q_\nu}{(P_0 - P)^2 + x^2} \right), \\ C_{\mu\nu}^S &= \frac{-2q_\mu q_\nu (K'_0 - K)^2}{[(P_0 - P)^2 + x^2][(P_0 + K'_0 - K)^2 + M^2]}, \\ C_{\mu\nu}^{S'} &= \delta_{\mu\nu} \left[\frac{(P_{00}) + \frac{x^2}{2}}{(P_0 + K'_0 - K)^2 + M^2} + \left\{ \binom{0}{4} M^2 - x^2 \right\} \cdot \frac{1}{2} \cdot \frac{(K'_0 - K)^2}{(P_0 + K'_0 - K)^2 + M^2} \right] \end{aligned} \quad (9)$$

Here $\binom{0}{4}$ indicates that we take 0 (or 4) for pseudoscalar (or scalar) meson. $B_{\mu\nu}^S$ does not appear if μ -meson is a scalar particle, thus it can be interpreted as the spin effect of μ -meson. This term is negligible compared with $B_{\mu\nu}^C$, the difference of spin 0 and 1/2 being not essential as in the case of the high energy bremsstrahlung.³⁾

$C_{\mu\nu}^S$ and $C_{\mu\nu}^{S'}$ also indicate the spin effects of nucleon and is absent in the case of scalar nucleon (but this is true only for scalar π -meson, and if π -meson is a pseudoscalar meson there are no interaction between nucleon and meson).

$C_{\mu\nu}^{S'}$ offers smaller contributions than the others and may be neglected. $C_{\mu\nu}^S$ and $C_{\mu\nu}^C$ must give the same energy dependence to the cross section, which can be easily seen from that $C_{\mu\nu}^C$ is negative in the case of pseudoscalar meson, so that it cannot have any higher dependences to the energy of the incident μ -meson than that of $C_{\mu\nu}^S$. In the case of pseudoscalar meson $C_{\mu\nu}^S$ is the most dominant term, while in the case of scalar meson $C_{\mu\nu}^C$ plays the role for it, similar to the photo-meson production, therefore we can interpret them as current-type and magnetic-moment type interaction, respectively.

§ 3-2. The threshold energy region

Near the threshold energy region, where the momenta of the final particles can be ignored compared with nucleon mass, the cross section is given by

$$\sigma = \left(\frac{e^2}{2\pi}\right)^2 \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{x}{M} \cdot \frac{\bar{\epsilon}^2}{k_0(E_0 + \epsilon_0)} (I_C + I_S + I_{S'}), \quad (10)$$

$$\text{where } \left. \begin{aligned} I_C &= \left\{ \binom{0}{4} M^2 - x^2 \right\} \frac{\mu(x+2\mu)}{8x^4(x+\mu)^2}, & I_S &= \frac{x+2\mu}{4x^3(x+\mu)}, \\ I_{S'} &= \frac{x-\mu}{2x^2\mu} + \frac{x^2-2x\mu-2\mu^2}{8Mx^2\mu}, \end{aligned} \right\} \quad (11)$$

and $\bar{\epsilon}$ is the maximum kinetic energy of final μ -meson.

I_C , I_S and $I_{S'}$ come from $C_{\mu\nu}^C$, $C_{\mu\nu}^S$ and $C_{\mu\nu}^{S'}$, respectively.

In the case of scalar meson the most dominant term is I_C , as was discussed in the preceding section.

$$\begin{aligned} \sigma_S &= \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{x^2} \cdot \frac{M}{x} \cdot \frac{\bar{\epsilon}^2}{k_0(E_0 + \epsilon_0)} \cdot \frac{1}{2} \cdot \frac{\mu(x+2\mu)}{(x+\mu)^2} \\ &\sim 0.35 \cdot 10^{-30} \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{\bar{\epsilon}^2}{x\epsilon_0} \text{ cm}^2. \end{aligned} \quad (12)$$

In the case of pseudoscalar meson with pseudoscalar coupling, I_C , I_S and $I_{S'}$ are all of the same order of magnitude and

$$\sigma_{PS} = \left(\frac{e^2}{4\pi}\right)^2 \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{Mx} \cdot \frac{\bar{\epsilon}^2}{k_0(E_0 + \epsilon_0)} \cdot \frac{\mu(x+2\mu)(3x+2\mu)}{8x(x+\mu)^2} \\ \sim 0.9 \times 10^{-32} \left(\frac{\bar{\epsilon}^2}{xk_0}\right) \cdot \left(\frac{f^2}{4\pi}\right). \quad (13)$$

In the case of pseudovector coupling, the equivalence theorem holds, and the cross section is given by

$$\sigma_{PV} = \left(\frac{2M}{x}\right)^2 \sigma_{PS} = 1.6 \cdot 10^{-30} \cdot \left(\frac{\bar{\epsilon}^2}{xk_0}\right) \cdot \left(\frac{f^2}{4\pi}\right) \text{ cm}^2. \quad (14)$$

These cross sections are proportional to $1/x^2$ rather than $1/M^2$, which shows us that proton is not a point charge but has a meson cloud around it.

§ 3-3. Extreme relativistic case

In the extreme relativistic case, where π - and μ -meson masses are much smaller than the incident μ -meson energy in center of gravity system, the main contribution comes from $B_{\mu\nu}^C C_{\mu\nu}^C$ and $B_{\mu\nu}^C B_{\mu\nu}^S$. This is verified as follows.

In center of mass system we have

$$\left. \begin{aligned} (P_0 + K_0 - K)^2 + M^2 &= -2(E_0 + \epsilon_0)(\epsilon_0 - \epsilon), \\ (K_0 - K)^2 &= 2(\epsilon_0\epsilon - \mu^2 - k k_0 \cos \theta), \\ (P_0 - P)^2 + x^2 &= 2E_0 E - 2M^2 + x^2 + 2p_0 p (\cos u \cos \theta + \sin u \sin \theta \cos \varphi), \end{aligned} \right\} > x^2 \quad (15)$$

These factors are in the denominators of the matrix elements, so that we can see by inspection that the integrands have strong maxima near $\epsilon \sim \epsilon_{max}$ and $\cos \theta \sim 1$, $\cos u \sim -1$, showing the validity of Weizsäcker-Williams approximation. Therefore we can say that the incident μ -meson does not suffer several energy loss, while the energy of the incident nucleon are distributed to the scattered nucleon and the created π -meson so as to give them the same velocity in center of gravity system.

The total cross section is given by

$$\sigma = \left(\frac{e^2}{4\pi}\right)^2 \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{2}{k_0(E_0 + \epsilon_0)} \cdot (\Phi_C + \Phi_S + \Phi_{S'}) \quad (16)$$

Here Φ_C , Φ_S and $\Phi_{S'}$ are the contributions from $C_{\mu\nu}^C$, $C_{\mu\nu}^S$ and $C_{\mu\nu}^{S'}$, respectively. $\Phi_{S'}$ can easily be integrated. Neglecting x^2 , and μ^2 compared with M^2 , we get

$$\Phi_{S'} = \int_{\mu}^{\epsilon_{max}} k d\epsilon A \left\{ \frac{u + M^2}{2u + M^2} \left[\frac{1}{4(\epsilon_0 - \epsilon)^2} - \frac{1}{4kk_0} \ln \frac{\epsilon_0\epsilon - \mu^2 + kk_0}{\mu(\epsilon_0 - \epsilon)} \right] \right. \\ \left. + \left\{ \binom{0}{4} M^2 - x^2 \right\} \frac{1}{8u^2} \frac{x^2}{kk_0} \left\{ \ln \left(\frac{\epsilon_0\epsilon - \mu^2 + kk_0}{\mu(\epsilon_0 - \epsilon)} \right) - 1 \right\} \right\},$$

$$u = (E_0 + \epsilon_0)(\epsilon_0 - \epsilon), \quad A = \sqrt{\left(u - \frac{x^2}{2}\right)^2 - M^2 x^2} / \left(u + \frac{M^2}{2}\right). \quad (17)$$

This gives a constant term to the cross section in high energy region, that is a smaller contribution than from Φ_c and Φ_s .

The integrations of Φ_c and Φ_s require rather tedious calculations and the terms which, at first sight, appeared most dominant cancel between them, so that we must be careful in carrying the integrations.

After the cancellation of the highest order terms we have

$$\begin{aligned} \Phi_c = & \left\{ \left(\frac{0}{4} \right) M^2 - x^2 \right\} \left\{ - \int k_A d\epsilon \left[\frac{(E_0 + \epsilon_0)^2 - \mu^2}{8u^2 k_0 k} \ln \frac{\epsilon_0 \epsilon - \mu^2 + k_0 k}{\mu(\epsilon_0 - \epsilon)} \right] \right. \\ & + \iiint d\epsilon dE d(\cos \theta) d\varphi \left\{ - \frac{(E_0 + \epsilon_0)(E_0 - E)}{(\epsilon_0 \epsilon - \mu^2 - k k_0 \cos \theta)(x^2 + (P_0 - P)^2)} \right. \\ & + \frac{(E_0 + \epsilon_0)(E_0 + \epsilon_0 - E) - \mu^2}{2u(\epsilon_0 \epsilon - \mu^2 - k_0 k \cos \theta)(x^2 + (P_0 - P)^2)} \\ & \left. \left. + \frac{1}{4(\epsilon_0 \epsilon - \mu^2 - k_0 k \cos \theta)^2} \left[\frac{(E_0 + \epsilon_0)\epsilon - \mu^2}{u} - \frac{A}{x^2 + (P_0 - P)^2} \right] \right\} \right\}, \\ \Phi_s = & - \iint d\epsilon dE \frac{(E_0 + \epsilon_0)E - M^2}{8(E_0 + \epsilon_0)(\epsilon_0 - \epsilon)p_0 p} \ln \frac{x^2 + 2E_2 E - 2M^2 + 2p p_0}{x^2 + 2E_0 E - 2M^2 - 2p p_0} \\ & + (M^2 - x^2) \int d\epsilon \frac{(E_0 + \epsilon_0)\epsilon - \mu^2}{4u(2u + M^2)} \frac{A}{k_0} \ln \frac{\epsilon_0 \epsilon - \mu^2 + k_0 k}{\mu(\epsilon_0 - \epsilon)} \\ & + \iiint d\epsilon dE d(\cos \theta) d\varphi \frac{A}{4u} \frac{(E_0 + \epsilon_0)(2E + \epsilon + \epsilon_0 - 2E_0) + x^2 - 2\mu^2}{(\epsilon_0 \epsilon - \mu^2 - k_0 k \cos \theta)(x^2 + (p_0 p)^2)}. \quad (18) \end{aligned}$$

The first and the second terms in Φ_s contribute only $1/\epsilon_0^2$ energy dependence in high energy region to the cross section. Integrating exactly with respects to $d\varphi$ and $d(\cos \theta)$, the last term becomes

$$\begin{aligned} \Phi_s^3 = \Phi_s = & \iint d\epsilon dE \frac{A \{ (E_0 + \epsilon_0)(2E + \epsilon - \epsilon_0) + x^2 - 2M^2 \}}{4(E_0 + \epsilon_0)(\epsilon_0 - \epsilon)^2 \sqrt{A^2 - 4\mu^2 x^2}} \\ & \times \ln \left[\frac{(\epsilon_0 \epsilon - \mu^2 + k_0 k)}{(\epsilon_0 \epsilon - \mu^2 - k_0 k)} \left\{ \frac{B_+ (C + \sqrt{C^2 + D}) + 4p_0^2 p^2 \sin^2 u (\epsilon_0 \epsilon - \mu^2 - k k_0)}{B_- (C + \sqrt{C^2 + D}) - 4p_0^2 p^2 \sin^2 u (\epsilon_0 \epsilon - \mu^2 - k k_0)} \right\} \right] \quad (19) \end{aligned}$$

where $(u$ is the angle between \mathbf{p} and $\mathbf{k})$

$$B_{\pm} = x^2 + 2E_0 E - 2M^2 \pm 2p p_0 \cos u,$$

$$C = \frac{p_0}{k} (\epsilon_0 - \epsilon) [\epsilon A - 2\mu^2 \epsilon],$$

$$C^2 + D = P_0^2 (\epsilon_0 - \epsilon)^2 [A^2 - 4x^2 \mu^2], \quad (20)$$

$$A = 2(E_0 + \epsilon_0)(E_0 - E) - x^2.$$

The integrand has a maximum near $\epsilon \sim \epsilon_{max}$. In this energy region we can approximate the logarithmic factor as follows,

$$\begin{aligned}
 \sqrt{C^2 + D} &\sim C \sim P_0(\epsilon_0 - \epsilon)A, \\
 B_+ &\sim \frac{k_0 - k}{k}A, \\
 B_+(C + \sqrt{C^2 + D}) - 4P^2P_0^2 \sin^2 a(\epsilon_0\epsilon - \mu^2 - k_0k) &\sim 2(\epsilon_0 - \epsilon)^2M^2, \\
 B_-(C + \sqrt{C^2 + D}) - 4P^2P_0^2 \sin^2 a(\epsilon_0\epsilon - \mu^2 - k_0k) &\sim \{x^4 + 4x^2(E_0E - M^2) + 4M^2(E_0 - E)^2\}, \\
 \ln. \text{ factor in } \phi_s &\sim \ln \frac{(\epsilon_0\epsilon - \mu^2 + k_0k)^2}{\mu^2(\epsilon_0 - \epsilon)^2} \frac{2(\epsilon_0 - \epsilon)^2 \cdot 8 \cdot (E_0 + \epsilon_0)^2(E_0 - E)^2}{2kk_0\{x^4 + 4x^2(E_0E + M^2) + 4M^2(E_0 - E)^2\}} \\
 &\sim \ln \frac{4\epsilon_0^2(E_0 + \epsilon_0)^2}{\mu^2M^2} = 2 \ln \frac{2\epsilon_0^*}{\mu},
 \end{aligned}$$

where ϵ_0^* is the incident μ -meson energy in laboratory system.

Therefore, we have

$$\begin{aligned}
 \phi_s &= \int d\epsilon \frac{k}{4(\epsilon_0 - \epsilon)^2} A \ln \frac{4\epsilon_0^{*2}}{\mu^2} = \frac{(E_0 + \epsilon_0)\epsilon_0}{4M^2} \left(\ln \frac{2\epsilon_0^*}{\mu} - \frac{1}{2} \right) \\
 &\quad \times \left\{ 2 \ln \frac{M}{x} + \frac{x}{4M} \cdot 5 \cdot \frac{\pi}{2} \right\}. \quad (21)
 \end{aligned}$$

(a detailed evaluation shows us that this estimation is correct.)

The evaluation of ϕ_c term is more tedious than the above. However, as stated before, the contribution from ϕ_c is negative in pseudoscalar meson case and is $-\frac{x^2}{4M^2}\phi_c$ (scalar), which is quite clear from the factor $\left\{ \binom{0}{4}M^2 - x^2 \right\}$, so that ϕ_c must not have any higher dependences than ϕ_s to the incident energy; that is, at most $\ln \left(\frac{\epsilon_0^*}{\mu} \right)$.

Because of long and tedious calculation we give only the results,

$$\begin{aligned}
 \phi_c &= \frac{\left\{ \binom{0}{4}M^2 - x^2 \right\}}{24x^2} \left\{ \frac{2(1 - 4a + a^2)}{1 - \frac{a}{4}} + \frac{x}{2M \left(\sqrt{1 - \frac{a}{4}} \right)^3} \left(\frac{\pi}{2} - \sin^{-1} \frac{x}{2M} \right) \right. \\
 &\quad \left. + 4(3 - 2a)a \ln \frac{M}{x} \right\} \left(\ln \frac{2\epsilon_0^*}{\mu} - \frac{1}{2} \right), \\
 a &= \frac{x^2}{M^2}. \quad (22)
 \end{aligned}$$

Collecting our results and neglecting the terms which vanish in high energy

region and then keeping only the terms of lower order in the expansion with respects to x^2/M^2 , we have

$$\begin{aligned}\sigma_s &= \left(\frac{\epsilon^2}{4\pi}\right)^2 \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{x^2} \left(2 \ln \frac{\epsilon_0^*}{\mu} - 1\right) \left[\frac{2}{3} + \frac{13}{3} \frac{x^2}{M^2} \ln \frac{M}{x} - \frac{5x}{6M} \pi\right], \\ \sigma_{ps} &= \left(\frac{\epsilon^2}{4\pi}\right)^2 \cdot \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{M^2} \left(2 \ln \frac{\epsilon_0^*}{\mu} - 1\right) \left[\ln \frac{M}{x} - \frac{5}{3} + \frac{35x}{24M} \pi\right].\end{aligned}\quad (23)$$

From the above expressions for the total cross sections in two cases, we can deduce following conclusions:

(1) In both cases, it is almost independent of the mass of the incident μ -meson, which is just expected from the beginning, for the incident μ -meson can be considered as merely the source of virtual photons.

(2) In the case of scalar meson, it does not depend on the nucleon mass, too, and is proportional to $1/x^2$, which can be understood by the simple or rather semi-quantum-mechanical consideration that a nucleon is accompanied by its virtual meson cloud, the diametre of which is of the order of π -meson Compton wave length.

(3) In the case of pseudoscalar meson, contributions from current and magnetic moment are of the same order of magnitude, $1/M^2$. We have γ_5 in the pseudoscalar coupling which implies that the transition from a positive energy state to a negative one of the nucleon and vice versa is to be performed, so that the nucleon must be virtually in one of its negative energy state. Out of the domain of order of nucleon Compton wave length, $1/M$, in diameter, there can hardly be expected such a transition, for the nucleon in this region can be considered as a classical particle. Therefore, it is rather a matter of course that we had the cross section of order of $1/M^2$. The cross section in the case of pseudo-vector coupling becomes $4M^2/x^2$ times that of pseudoscalar coupling (the equivalence theorem holds between pseudo-vector and -scalar coupling) and is proportional to $1/x^2$, comparable to that of scalar meson case.

If we substitute $\epsilon_0^* = 14$ Bev. (the mean energy of μ -meson underground)

$$\sigma_s = \left(\frac{f^2}{4\pi}\right) \times 4 \cdot 10^{-30} \text{ cm}^2, \quad \sigma_{pv} = \left(\frac{f^2}{4\pi}\right) \times 26 \cdot 10^{-30} \text{ cm}^2. \quad (24)$$

§ 4. Energy loss and deflection angle of μ -meson

Here in this section we will estimate the average energy loss and the average deflection angle of μ -meson. To evaluate them, it is convenient to make use of the following Lorentz invariant relations

$$M(\epsilon_0^* - \epsilon^*) = -(P_0^*, K_0^* - K^*) = -(P_0, K_0 - K) = (E_0 + \epsilon_0)(\epsilon_0 - \epsilon) + \frac{(K_0 - K)^2}{2},$$

$$M(\overline{\epsilon_0^* - \epsilon^*}) = (E_0 + \epsilon_0)(\overline{\epsilon_0 - \epsilon}) + \frac{(\overline{K_0 - K})^2}{2}. \quad (25)$$

Neglecting the terms which vanish in the high energy limit, we obtain

$$\int (E_0 + \epsilon_0) (\epsilon_0 - \epsilon) d\sigma = \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{2} \ln \frac{2\epsilon_0^*}{\mu} \times \left[\ln \frac{2\epsilon_0^*}{M} + \left\{ \left(\frac{0}{4}\right) - \frac{x^2}{M^2} \right\} \right. \\ \left. \left(\frac{M}{x} \pi + 2 - 4 \ln \frac{M}{x} \right) \right], \\ \int \frac{(K_0 - K)^2}{2} d\sigma = \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{f^2}{4\pi}\right) \cdot \frac{1}{4} \ln \frac{M\epsilon_0^*}{x^2} \times \left[\ln \left(1 + \frac{2\epsilon_0^*}{M}\right) - 2 + \frac{5}{4} \left\{ \left(\frac{0}{4}\right) - \frac{x^2}{M^2} \right\} \right], \quad (26)$$

$$\overline{\epsilon_0^* - \epsilon_0} = \begin{cases} \frac{x^2}{M} \frac{1}{2} \left\{ \frac{3}{2} \ln \frac{2\epsilon_0^*}{M} + 60 \right\}; & \text{scalar meson,} \\ \frac{M}{3.2} \left\{ \ln \frac{2\epsilon_0^*}{M} + \frac{1}{2} \ln \left(1 + \frac{2\epsilon_0^*}{M}\right) \right\}; & \text{pseudoscalar meson.} \end{cases} \quad (27)$$

(These energy losses in two cases do not depend on the coupling constants.)

The average energy loss of μ -meson in laboratory system increases only logarithmically with the increasing incident μ -meson energy. This logarithmic term comes from the spin effects of nucleon, while the constant terms from charge current of both nucleon and π -meson.

The average deflection angle of the incident μ -meson can be obtained from the relation

$$(\epsilon_0 \epsilon - \mu^2 - k_0 k \cos \theta) = \frac{(K_0 - K)^2}{2} = \frac{(K_0^* - K^*)^2}{2} = (\epsilon_0 \epsilon^* - \mu^2 - k_0^* k^* \cos \theta^*). \quad (28)$$

The main contribution comes from $\epsilon^* \sim \epsilon_0^*$, so that we have approximately

$$\epsilon_0^2 \frac{\bar{\theta}^2}{2} = \epsilon_0^{*2} \frac{\bar{\theta}^{*2}}{2} = \frac{\int d\sigma \frac{(K_0 - K)^2}{2}}{\sigma}. \quad (29)$$

The results are

$$\sqrt{\bar{\theta}^{*2}} = \begin{cases} \frac{x}{\epsilon_0^*} \sqrt{\ln \frac{2\epsilon_0^*}{M} + 5}; & \text{scalar meson,} \\ \frac{M}{\epsilon_0^*} \sqrt{\frac{2}{3} \ln \left(1 + \frac{2\epsilon_0^*}{M}\right)}; & \text{pseudoscalar meson.} \end{cases} \quad (30)$$

The fact that the average energy loss and average deflection angle are small expresses the validity of Weizsäcker-Williams approximation.

In Figs. 2 and 3, the above calculated energy loss and the deflection angle (both average value) are plotted against the energy of the incident μ -meson. The average energy loss and the average deflection angle of μ -meson do not depend on the magnitude of the coupling constant between nucleon and π -meson, as far as we confine ourselves to the second order approximation. For the

convenience of comparison, experimental data obtained by E. P. George and J. Evans are also listed in them.

The probable value of α , angle between scattered μ -meson and nucleon, can also be evaluated easily by means of the following equation (see Fig. 4)

$$\frac{\partial}{\partial E}(\cos \alpha)|_{\epsilon=\bar{\epsilon}} = 0.$$

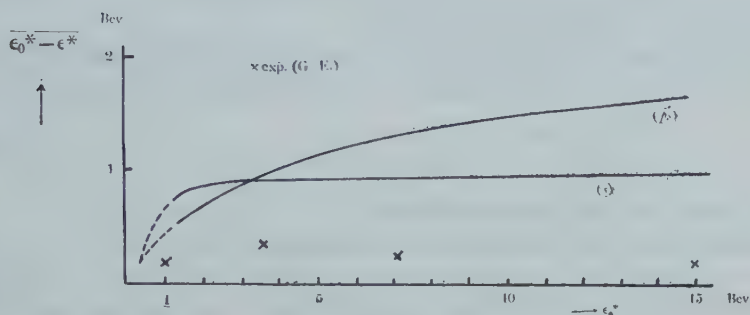


Fig. 2

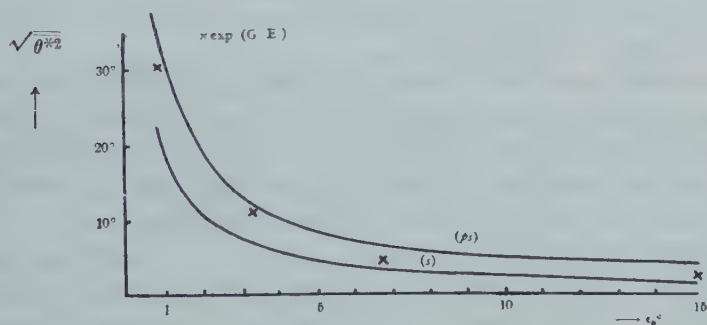


Fig. 3

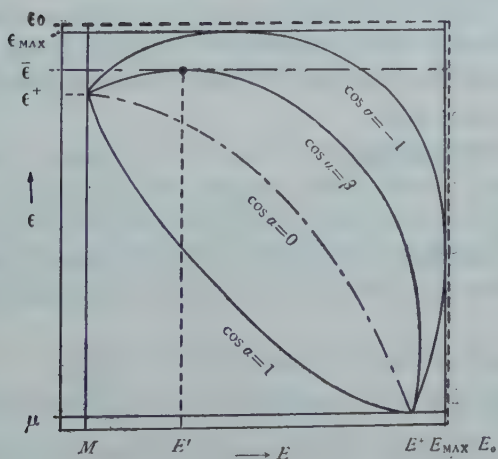


Fig. 4

$$E^+ = \frac{(W_0 - M)^2 + \mu^2 - x^2}{2(W_0 - M)},$$

$$E^+ = \frac{(W_0 - \mu)^2 + M^2 - x^2}{2(W_0 - M)},$$

$$E' = \frac{W_0(\epsilon_0 - \bar{\epsilon}) + M^2 - \frac{x^2}{2}}{(W_0 - \epsilon)},$$

$$\beta = -1 + \frac{M \ln \left(\frac{W_0^2}{M\mu} \right)}{W_0 L},$$

$$W_0 = E_0 + \epsilon_0,$$

$$L = \begin{cases} 1.48 & \text{for PS,} \\ 1.1 \times \frac{M^2}{x^2} & \text{for S.} \end{cases}$$

§ 5. Discussions*

(1) *Groundmaking of our calculations*

We have dealt with the third order perturbation only, so that some doubt may arise concerning the perturbation calculation itself, as it has been in several cases.

However, the following considerations will serve some basis to our results which are of scalar or of pseudoscalar meson.

According to Weizsäcker-Williams, the Coulomb field of the incident μ -meson can be replaced by a set of virtual photons with the intensity distribution

$$\frac{2}{\pi} \left(\frac{e^2}{4\pi} \right) \frac{dK}{K} \log \frac{\epsilon_0^*}{K}, \quad (32)$$

where K is the energy of the virtual photon.

Then we have for the cross section of μ - π process

$$\sigma_{\mu \rightarrow \pi} = \frac{2}{\pi} \left(\frac{e^2}{4\pi} \right) \cdot \int dK \cdot \sigma_{\gamma \rightarrow \pi} \cdot \frac{1}{K} \cdot \log \frac{\epsilon_0^*}{K}. \quad (33)$$

If we put the perturbation of lower order for $\sigma_{\gamma \rightarrow \pi}$, the above formula agrees with ours as will be seen in Appendix.

Therefore, our results share their destiny with the second order perturbation $\sigma_{\gamma \rightarrow \pi}$. Moreover, in the case of scalar or pseudoscalar meson the latter is trustable, at least, in the low frequency region of γ -ray, (the fourth order contribution to this γ - π process which was calculated by Osaka group being small.⁴⁾) In addition, that lower energy photons will contribute greatly is suggested by the above formula, while we know that in the low energy region the second order perturbation with pseudoscalar meson results in a rather good agreement with the experiment at Berkeley. Even if we do not have any conceivable ground to speak of the validity of the perturbation method in the high energy region, we would be supported by the small intensity of high frequency γ -ray, as was shown by Weizsäcker-Williams approximation. For example, E. P. George and J. Evans have substituted for $\sigma_{\gamma \rightarrow \pi}$ its experimental value, 10^{-28} cm². (McMillan et. al.) and got a result not much different from ours.

If the cross section $\sigma_{\gamma \rightarrow \pi}$ increases with increasing photon energy, which can be expected in cases of vector and pseudovector meson, the results may have a quite different feature. However, we need not be afraid of this point as far as we deal with scalar or pseudoscalar mesons which will not couple strongly with the high frequency γ -ray. These considerations will give some justification to our results.

(2) *The effect of the fact that nucleons are confined in a nucleus*

Until now we have considered the nucleon as being free. Now we examine

* I. N. Sneddon and B. F. Toucek have carried out the similar calculations, Proc. Roy. Soc. 199 (1949), 352.

to what extent this assumption can be fulfilled.

In the Weizsäcker-Williams approximation described in §5-1, the lower limit of K is taken to be π -meson mass κ . The wave length corresponding to this energy is of the same order as the nuclear force range. To estimate the magnitude of the mutual interference of nucleons we refer to the experiment of π -meson production by γ -ray. There was a variation by about factor 1/3 between carbon and hydrogen target,⁵⁾ which we suppose will be due to this interference effect. In the region where the γ -ray wave length is smaller, this effect will become much more reduced. Therefore, our assumption that the nucleon is isolated will not be very rough one.

(3) Comparison with experiment

E. P. George and J. Evans observed stars in cosmic ray underground by means of nuclear emulsion and gave 4×10^{-30} cm² as the cross section of star production. Meanwhile, our cross section concerns to the single π -meson production. However, we can compare our results with the experimental data of E. P. George and J. Evans considering as; once the incident μ -meson has produced in a nucleus a π -meson, which has a strong coupling to nucleon, the created π -meson will interact with neighbouring nucleons and some additional particle will be ejected to make the prongs of a star.

It does not seem us necessary to consider the pure multiple process. On this standpoint we have compared our results directly with their experimental data and obtained a fine consistency between them taking $f^2/4\pi$ to be ~ 0.2 , the value which is consistent with that from γ - π production.

The variation due to whether the spin of μ -meson is 1/2 or 0 is negligible as it was in the case of bremsstrahlung calculated by Christy and Kusaka.³⁾

Acknowledgement

We should like to express our gratitude to Professor T. Miyazima and to Mr. S. Ogawa for their kind interest and helpful discussions.

Appendix**

We consider the problem by the aid of Weizsäcker-Williams method. The cross section of π -meson production by a fast charged particle can be written as follows using the cross section by γ -ray,

$$\sigma_{\mu \rightarrow \pi^+} = \int \sigma_{\gamma \rightarrow \pi^+} \times (\text{intensity of virtual photons}).$$

In the above formula, we can replace $\sigma_{\gamma \rightarrow \pi^+}$ by its experimental value or by the lower order perturbation, of which the first was carried out by E. P. George and

** We are indebted to Prof. J. Oppenheimer for his helpful discussions to one of us (H. F.).

J, Evans and will perform in this Appendix the second approach.

G. Araki⁶⁾ has calculated the cross section of $\sigma_{\tau \rightarrow \pi^+}$ process, and we can immediately use his result, which can be approximated as follows,

$$\sigma_{\tau \rightarrow \pi^+}(P.S.) = \left(\frac{e^2}{4\pi}\right) \cdot \left(\frac{f^2}{4\pi}\right) \cdot \pi \frac{(M+K)}{M(M+2K)^2}.$$

The Araki's above cross section is to multiplied by the intensity of virtual photons,

$$\frac{2}{\pi} \left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{K} dK \ln \frac{\epsilon_0^*}{K}$$

to give the cross section for our $\mu \rightarrow \pi^+$ process.

We had the following result

$$\sigma_{\mu \rightarrow \pi^+}(P.S.) = \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{f^2}{4\pi}\right) \frac{2}{M^2} \ln \frac{\epsilon_0^*}{x} \left[\ln \frac{M}{x} - \frac{1}{2} \right].$$

Next we can similarly calculate the average loss of the incident μ -meson,

$$\langle \epsilon_0^* - \epsilon \rangle = \langle K \rangle = \frac{M}{6.8} \ln \left(\frac{2\epsilon_0^*}{M} + 1 \right).$$

For the sake of comparison, the pseudoscalar meson case only is treated.

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Deviations of Nuclear Magnetic Moments from the Schmidt Lines

Hironari MIYAZAWA

Department of Physics, University of Tokyo

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The anomalous magnetic moment of a nucleon bound in a nucleus differs from the value when it is free because of the modification of virtual meson cloud around the nucleon, due to the effect of other nucleons. This modified value is calculated and shown to be about half of the value when the nucleon is free. With this value of the magnetic moment, the deviations of nuclear magnetic moments from the Schmidt lines can be explained. It is shown that this effect is equivalent to the exchange magnetic moment.

§ 1. Introduction

The single particle model, which describes the nucleus in terms of stationary states of the individual nucleons in an averaged nuclear field, has achieved a remarkable success in explaining many nuclear properties, such as shell structure in nuclei, isomerism and beta-decay probabilities. According to this model, the magnetic moments of odd nuclei can be computed, under the assumption that they are solely due to one extra odd nucleon. The two possible cases, $I = I - \frac{1}{2}$ and $I = I + \frac{1}{2}$ for given total spin I , lead to two lines in a plot of magnetic moment against I , which are known as Schmidt lines. Experimentally-determined moments lie between these lines, but do not coincide with them exactly. They form two bands roughly paralleling the Schmidt lines. It is one of the success of the single particle model that it can account for the general dependence of the moments on I . But it fails to explain why the observed values of moments do not coincide with those obtained by the single particle model.

Deviations from the Schmidt lines indicate the crudity of the model. Foldy and Milford¹⁾, and A. Bohr²⁾ suggested that the deviation could be accounted for by assuming an exchange of angular momentum between the extra odd nucleon and the core. Foldy and Milford's calculation shows that in a majority of cases the theoretically predicted deviations are in the right direction and proper order of magnitude. One must admit, however, that for $I = \frac{1}{2}$ their model cannot predict any deviations, in disagreement with a considerable number of observed data, and for nuclei with $I = l - \frac{1}{2} > 5/2$ the model predicts deviations in the wrong direction resulting in further discrepancies. There must be a considerable

amount of mixing of states with $l=I-\frac{1}{2}$ and $l=I+\frac{1}{2}$ to account for experiments, which seems, however, hardly acceptable, in view of the wide separation of these two states caused by the strong spin-orbit coupling.

One will easily observe that the deviations are in the opposite direction for nuclei with odd nucleon spin parallel or anti-parallel to total spin, i.e. for odd proton nucleus, the deviation is minus if the spin of the extra proton points upwards, and is plus if it points downwards. One thus arrives at the simple explanation for the deviation if he assumes that the intrinsic magnetic moment of the proton is diminished inside a nucleus. This view of intrinsic moment is by no means unreasonable because, a nucleon, when bound in a nucleus, is exerted a strong influence by other densely packed nucleons which gives rise to the modification of the meson cloud around it. And the anomalous magnetic moment of a nucleon, which is supposed to arise from the virtual meson cloud around it, undergoes a change in its magnitude if brought into a nucleus.

In this paper the calculation will be carried out of this change of magnetic moment, and we shall see that this effect just accounts for the deviation in question. In § 2, the origin and the order of magnitude of this change are discussed. In § 3, the change is calculated with the aid of meson theory. It is shown in § 4 that this effect is equivalent to the exchange magnetic moment. Results are compared with experiments in § 5.

§ 2. Change of magnetic moment of a nucleon in a nucleus

The anomalous magnetic moment or the virtual meson cloud of a nucleon in a nucleus differs from that of a free one on account of following two reasons:

Firstly, the nucleon is exerted by nuclear forces, and its motion in the intermediate as well as in the initial or final state is not free but distorted. Secondly, the exclusion principle operates; the nucleon is prohibited to make (virtual) transitions leading to states already occupied by other nucleons.

For heavy nucleus a nucleon can in good approximation be assumed to move in an averaged field, and the potential may be taken as a spherical box with constant depth. If the radius of this box is sufficiently large (as compared with meson Compton wave length, which is the extension of the meson cloud around a nucleon), the effect of nuclear forces on the modification of meson cloud will be negligible. So in this paper, we shall set this effect aside and concern ourselves mainly with the second point mentioned above. This assumption is certainly invalid for lightest nuclei such as tritium or He^3 , less wrong for medium nuclei, and would be sufficiently good for heavy nuclei, probably with mass number > 50 . For light nuclei a separate treatment is required.

The above assumption once admitted, the size of nuclear box is irrelevant for the evaluation of the change of magnetic moment. We may therefore take

a nucleus as a Fermi gas of infinitely large dimension. The experimental fact that nuclear moments are determined almost only by their spin and nearly independent of details of individual properties such as their mass number or shell structure, indicates that such a simplified treatment for nucleus is sufficient for our purpose.

Consider a proton with momentum zero in a Fermi gas composed of nucleons up to the momentum P . P can be determined from the observed nuclear radii and is known to be about 210 Mev.*, independent of elements. In this nucleus the virtual process $p \rightarrow n + \pi^+$ is forbidden when leading to neutron states with momentum $< P$. From momentum balance (remember that we have neglected the effect of nuclear forces which would partake the momentum balance) this nucleon cannot emit a meson with momentum less than P , that is, a nucleon takes off a part of meson cloud inside a nucleus which it wore outside. Anomalous magnetic moment μ_p of a proton is expressed as a sum of contributions from each virtual mesons. Thus μ_p is expressed as $\int \dots d\mathbf{k}$ where the integrand is a positive definite function of momentum \mathbf{k} . The lack of virtual mesons with momentum $< P$ implies the diminution of the domain of this integral, thereby results in the decrease of the magnetic moment.

Also the presence of this proton in question forbids the transition $n \rightarrow p + \pi^-$ to this state of all neutrons in the nucleus. One can see that this effect just doubles the decrease of moment mentioned above. To see this it is more appropriate to use a representation in which angular momenta of particles are diagonal. A proton with upward spin makes a transition to a state of a neutron with downward spin and a positive meson with angular momentum $+1$ ($p\uparrow \rightarrow n\downarrow + \pi^+\uparrow$). The probability of duration of this state is

$$\frac{|(n + \pi | V | p)|^2}{(\omega + E_n - E_p)^2} \quad (1)$$

where V represents the meson-nucleon interaction and ω is the energy of the emitted meson. This state contributes to the positive magnetic moment and prohibition of this state decreases the moment. Similarly, the presence of $p\uparrow$ prevents the transition $n\downarrow \rightarrow p\uparrow + \pi^-\downarrow$ which would have given the positive moment, resulting in further decrease of moment. The dissociation probability of this state is again expressed as

$$\frac{|(n | V | p + \pi)|^2}{(\omega + E_p - E_n)^2}$$

in which the numerator is the same as that of (1), and the energy denominators of both expressions are also equal, if nucleon energy difference $E_p - E_n$ is neglected as compared with the meson energy ω . Thus both effects decrease equal amount

* Hereafter we use natural units, i.e., \hbar and $c=1$.

of moment. Processes such as $p\uparrow \rightarrow n\uparrow + \pi^+ \rightarrow$ have little to do with the spin magnetic moment*.

The magnitude of this decrease can only be calculated by meson theory, but we can estimate the order of its magnitude as follows:

Let a be the fraction of time which a proton (neutron) spends as a neutron (proton) and a meson. Then the magnetic moments of the proton and neutron are roughly given by³⁾

$$\mu_p = (M/\omega^*)a - a,$$

$$\mu_n = -(M/\omega^*)a - a$$

where M is the mass of the nucleon, and ω^* is a certain mean value of the inertial mass (energy) of the virtual mesons. The first term comes from meson's charge and the second term comes from proton's magnetic moment. In order that μ_p is almost equal to $-\mu_n$, M/ω^* must be large, i.e., momenta of virtual mesons that mainly contribute to the moment must not be appreciably larger than the meson rest mass κ , probably they must be of the order of κ . On the other hand, mesons with momentum $< P = 1.5 \kappa$ are excluded in a nucleus. Therefore a considerable amount is expected to decrease in a nucleus.

We have up to now considered a nucleon with momentum zero. The extra odd nucleon outside the nuclear core, in which we are mainly interested, occupies the highest energy level, so has $|\text{momentum}| = P$. It situates in momentum space on the surface of the sphere with radius P , in which nucleons are densely packed. The nucleon is free to move outwards, only prohibited to move inwards. Therefore the prohibition is only half as effective as it were placed at the origin. So the decrease never exceeds half of the free nucleon magnetic moment. However, the total decrease that is observed is twice as large. Thus the decrease is equal at most to the free nucleon moment, and to zero at least. As we shall see later, this decrease is about half of the free nucleon moment. Thus the anomalous magnetic moment of a odd nucleon in a nucleus is not equal to but about half of the value when it is free. This is the main conclusion of this paper.

§ 3. Change of the magnetic moment: Calculation

The change of the magnetic moment explained in the previous section can be calculated by meson theory. Unfortunately the meson theory in its present form is not always reliable. This situation prevents us from predicting an exact value of the magnetic moment. Nevertheless in view of the fair successes which the meson theory has displayed in various nuclear phenomena, we may be safe to say that it can predict the correct order of magnitude. Moreover, by compar-

* However, they contribute to the space exchange magnetic moment. See § 4.

ing with the free nucleon moments, we are free from uncertainties of the coupling constant.

We shall employ the perturbation method of Feynman⁴.

A free nucleon is in reality immersed in the sea of negative energy nucleons. A nucleon bound in a nucleus is thus soaped in the negative energy sea and the Fermi gas. Therefore, the magnetic moment of the bound nucleon can be calculated as if it were free, with the proviso that now 'vacuum' should mean 'vacuum + Fermi gas'. The quantity

$$\langle P(\bar{\psi}(x), \psi(y)) \rangle_{\text{vac.}} = \frac{1}{2} \epsilon(x, y) S_F(x, y)$$

used for free nucleon problem is now to be replaced by

$$\langle P(\bar{\psi}(x), \psi(y)) \rangle_{\text{vac.} + \text{Fermi gas}} = \frac{1}{2} \epsilon(x, y) S'_F(x, y),$$

where*

$$S'_F(x, y) = S_F(x, y) + \Delta S(x, y),$$

$$\Delta S(x, y) = 2 \sum_{\text{nucleus}} \bar{\psi}(x) \psi(y) = 2\rho(x, y).$$

ρ represents (for $x_4 = y_4$) the density matrix of the nucleus. For Fermi gas model, which is to be employed in the following,

$$\rho(x, y) = \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < P} \exp(iM(x_0 - y_0) - i\mathbf{k}(\mathbf{x} - \mathbf{y})) d\mathbf{k}. \quad (2)$$

One of the advantage of Feynman's method is that S_F automatically takes into account the effect that any negative energy nucleon cannot make transition to the state occupied by the nucleon in consideration. Quite similarly, our modified S'_F automatically reckons the prohibition of the process $n \rightarrow p + \pi^-$ of all neutrons in the nucleus, as well as, of course, the prohibition of $p \rightarrow n + \pi^+$. We have thus achieved in reducing the problem to single particle problem. The magnetic moment of a bound nucleon is calculated with this S'_F just as if it were free. ΔS causes the change of the magnetic moment, which is in question.

We take pseudoscalar symmetrical theory which gives correct sign and right order of magnitude of the magnetic moment of the proton and neutron⁵. Experimentally μ_p is almost equal to $-\mu_n$: this means that meson current which gives equal but opposite sign contribute a bulk, and nucleon current is negligible. So we need only to calculate the change of magnetic moment due to meson current. Calculation⁵ shows, however, that by pseudoscalar theory, the contribution of nucleon current is by no means small but even larger than that of meson

* This modified S'_F function was first obtained by Y. Nambu.

current. This serious situation is intimately connected with the r^{-3} difficulty of nuclear forces, which gives too large binding energy of the deuteron. Both of these difficulties arise from too large contribution of high momentum virtual mesons, owing to the derivative coupling of the pseudoscalar theory. Only way known for us to avoid these difficulties is to cut off these high momentum mesons. Cut off radius of nuclear force adjusted by the deuteron problem is known to be from $1/2\lambda$ to $1/3\lambda^{(6)}$. For such small region the higher order effects are more predominant and the validity of the lowest order perturbation is restricted to outer regions. Therefore the high momentum of the order of $2 \sim 3\lambda$ must be cut off⁷⁾. In low momentum region the effect of proton's charge can be neglected as compared to meson current. Unfortunately, we cannot fit both μ_p and μ_n quantitatively by choosing the cut off momentum in any way. So the cut off cannot be uniquely determined from the data of the free nucleon magnetic moment.

The interactions that are responsible for the magnetic moment will be listed below ;

$$i \frac{g}{x} \bar{\psi} \gamma_5 \gamma_\nu \tau_i \psi \partial_\nu \phi_i, \quad (3)$$

$$-e(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1) A_\mu, \quad (4)$$

$$i \frac{g^2}{x} \bar{\psi} \gamma_5 \gamma_\mu \psi (\tau_1 \phi_2 - \tau_2 \phi_1) A_\mu. \quad (5)$$

Energy of a nucleon in a magnetic field, modified by the second order meson cloud is given by (from the interaction (4))

$$E = \frac{ie g^2}{2x^2} \iint dx_1 dx_2 d\mathbf{x} \bar{\psi}(x_1) \tau_3 \gamma_5 \gamma_\nu S'_\nu(x_2 - x_1) \times \\ \times \gamma_5 \gamma_\sigma \psi(x_2) \partial_\nu A_\sigma(x - x_1) \partial_\sigma \partial_\mu A_\mu(x_2 - x) A_\mu(x). \quad (6)$$

Change ΔE of this quantity from the value when the nucleon is free, is obtained by inserting ΔS for S'_ν in the above expression. As only momenta below P are concerned, following approximation may be permitted: $\bar{\psi}$ and ψ depend on time as $\exp(\pm iMt)$ (in (2) this has already been done), and $\gamma_4 = 1$ etc.. Inserting the Fourier transform of various quantities, the time integration can readily be performed with the result

$$\Delta E = - \frac{eg^2}{(2\pi)^3} \frac{4}{x^2} \int_{|\mathbf{k}| < P} d\mathbf{k} d\mathbf{p} d\mathbf{q} \frac{\bar{\psi}(\mathbf{p}) \tau_3 \sigma_\nu \sigma_\lambda (\mathbf{p} - \mathbf{k})_\nu (\mathbf{q} - \mathbf{k})_\lambda (\mathbf{q} - \mathbf{k})_\mu \psi(\mathbf{q})}{((\mathbf{p} - \mathbf{k})^2 + x^2)((\mathbf{q} - \mathbf{k})^2 + x^2)} A_\mu(\mathbf{q} - \mathbf{p}) \quad (7)$$

where

$$A_\mu(\mathbf{q} - \mathbf{p}) = \frac{1}{(2\pi)^3} \int A_\mu(\mathbf{x}) e^{i\mathbf{x}(\mathbf{q} - \mathbf{p})} d\mathbf{x}.$$

Of course, one arrives at the same expression by the Schrödinger's perturbation method.

Define the axis in the direction of constant magnetic field. Choosing the gauge such that

$$A_{\mu}(x) = \frac{1}{2}H(-y, x, 0, 0)$$

i.e.

$$A_{\mu}(\mathbf{q}-\mathbf{p}) = \frac{i}{2}H\left(\frac{\partial}{\partial q_2}\delta(\mathbf{q}-\mathbf{p}), -\frac{\partial}{\partial q_1}\delta(\mathbf{q}-\mathbf{p}), 0, 0\right)$$

we can readily obtain

$$\begin{aligned} \Delta E_1 = & -H \frac{2ieg^2}{x^5} \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < P} d\mathbf{k} \frac{\bar{\psi}(\mathbf{p}) \tau_3 (\mathbf{p}-\mathbf{k})^2 ((\mathbf{p}-\mathbf{k})_2 \partial_1 - (\mathbf{p}-\mathbf{k})_1 \partial_2) \psi(\mathbf{p})}{((\mathbf{p}-\mathbf{k})^2 + x^2)^2} d\mathbf{p} \\ & -H \frac{2eg^2}{x^5} \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < P} d\mathbf{k} \frac{\bar{\psi}(\mathbf{p}) \tau_3 \sigma_3 (\mathbf{p}-\mathbf{k})^2 \psi(\mathbf{p})}{((\mathbf{p}-\mathbf{k})^2 + x^2)^2} d\mathbf{p} \\ & +H \frac{2ig^2}{x^5} \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < P} d\mathbf{k} \frac{\bar{\psi}(\mathbf{p}) \tau_3 (\mathbf{p}-\mathbf{k})_3 (\sigma(\mathbf{p}-\mathbf{k})) \psi(\mathbf{p})}{((\mathbf{p}-\mathbf{k})^2 + x^2)^2} d\mathbf{p}. \end{aligned} \quad (8)$$

Quite similarly, from (5)

$$\Delta E = H \frac{2ieg^2}{x^5} \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < P} d\mathbf{k} \frac{\bar{\psi}(\mathbf{p}) \tau_3 ((\mathbf{p}-\mathbf{k})_2 \partial_1 - (\mathbf{p}-\mathbf{k})_1 \partial_2) \psi(\mathbf{p})}{((\mathbf{p}-\mathbf{k})^2 + x^2)} d\mathbf{p},$$

which, added with the first term of (8), gives

$$\Delta E_2 = H \frac{2ieg^2}{(2\pi)^3} \int_{|\mathbf{k}| < P} d\mathbf{k} \frac{\bar{\psi}(\mathbf{p}) \tau_3 ((\mathbf{p}-\mathbf{k})_2 \partial_1 - (\mathbf{p}-\mathbf{k})_1 \partial_2) \psi(\mathbf{p})}{((\mathbf{p}-\mathbf{k})^2 + x^2)^2} d\mathbf{p}. \quad (9)$$

Quantities multiplying $(-H)$ in these expressions represent the change of magnetic moment. The second term of Eq. (8) is proportional to $\tau_3 \sigma_3$ and gives the decrease of anomalous magnetic moment explained in §2. The third term in (7) has arisen because of the asymmetrical distribution of the excluded meson cloud around the nucleon. As stated before, $\psi(\mathbf{p})$ of the extra odd nucleon differs from zero only when $|\mathbf{p}|=P$. In such a case integration with respect to \mathbf{k} is rather a simple task. The second term is readily integrated*. The change is negative for proton and positive for neutron and the absolute magnitude is given, in nuclear magnetons, by

* Use has been made of the formula

$$\int_{|\mathbf{k}| < P, |\mathbf{p}|=P} d\mathbf{k} f(\mathbf{p}-\mathbf{k}) = \frac{\pi}{P} \int_0^{2P} t^2 (2P-t) f(t) dt.$$

$$\begin{aligned}\delta\mu &= \frac{g^2}{4\pi} \frac{2}{\pi^2} \frac{M}{x^2} \int_{|k| < P, |p|=P} d\mathbf{k} \frac{(\mathbf{p}-\mathbf{k})^2}{((\mathbf{p}-\mathbf{k})^2 + x^2)^2} \\ &= \frac{g^2}{4\pi} \frac{2}{\pi} \frac{M}{P} \frac{1}{x^2} [2Pm(4) - m(5)]\end{aligned}\quad (10)$$

where

$$m(\nu) = \int_0^{\text{Min}(2P, k_0)} \frac{t^\nu}{(t^2 + x^2)^2} dt.$$

The upper limit is the smaller of $2P$ or the cut off momentum k_0 . (10) is to be compared with the magnetic moment of free nucleon, which is*

$$\mu_0 = \frac{g^2}{4\pi} \frac{8}{3\pi} \frac{M}{x^2} \int_0^{k_0} \frac{t^4}{(t^2 + x^2)^2} dt. \quad (11)$$

Ratios of (10) to (11) for various k_0 is tabulated in the following:

$k_0 =$	$2x$	$2.5x$	$3x$	∞
$\delta\mu/\mu_0$	0.85	0.53	0.44	0.54 (for p) 0.07 (for n)

The last column is obtained with μ_0 without cut off, relativistically calculated, including nucleon current contribution. We can thus see that the decrease of magnetic moment is about half of μ_0 . The above table is not very accurate because the contribution coming from the third term of Eq. (8) is considerably large for lower k_0 and reduces the '0.85' in the table to about 0.7. Also the expression (11) for μ_0 is not correct for k_0 as large as $3x$ because of the relativistic effect.

Correct expression for the third term of (8) is rather complicated if $k_0 < 2P$. Here we shall only give the expression for $k_0 > 2P$. Added with the second term, the change of the magnetic moment is given by

$$\Delta\mu = \left(\Delta\mu_1 \sigma_3 - \Delta\mu_2 \frac{\hat{p}_3(\boldsymbol{\sigma} \cdot \mathbf{p})}{p^2} \right) \tau_3$$

where

$$\Delta\mu_1 = \frac{g^2}{4\pi} \frac{1}{4\pi} \frac{M}{P} \frac{1}{x^2 p^2} [16P^3 n(2) - 8P^2 n(3) - n(5)] = \frac{g^2}{4\pi} 2.8, \quad (12)$$

$$\Delta\mu_2 = \frac{g^2}{4\pi} \frac{1}{4\pi} \frac{M}{P} \frac{1}{x^2 p^2} [8P^2 n(3) - 3n(5)] = \frac{g^2}{4\pi} 1.6, \quad (13)$$

$$n(\nu) = \int_0^{2P} \frac{t^\nu}{t^2 + x^2} dt.$$

* This expression was obtained with the same approximation which led (6) to (7).

Thus

$$\Delta\mu = 0.71\sigma_3 - 0.42p_3(\sigma p)/P^2 \quad (14)$$

where we have tentatively inserted 0.25 for $g^2/4\pi$ adjusted to fit the free nucleon moment when we take $k_0 = 3\lambda$. Although these figures are uncertain by a factor of ~ 2 . It is understood that change is $-\Delta\mu$ for proton and $+\Delta\mu$ for neutron.

E_2 (Eq. (9)) is independent of spin but proportional to the orbital angular momentum of the nucleon. The origin of this term can be explained as follows: A proton with angular momentum (around the z -axis) m makes a transition to neutron state with angular momentum m' and emits a π^+ with angular momentum $m-m'$. If this neutron state is occupied, this transition is forbidden and magnetic moment is changed by $\propto (m-m')$. As m' is as frequently positive as negative, summation of $\text{const.}(m-m')$ over all neutron states in the nuclear core, is proportional to m , which is the content of (9).

Unlike the term which is proportional to spin, this term cannot be compared with the free nucleon moment, for spin independent field also contribute to this term. However, as we shall see in the succeeding section, we have means to determine this value phenomenologically—from the known expression of nuclear forces.

§ 4. Comments on the exclusion principle in the intermediate state: Connection with the exchange current

Feynman states in his paper⁹⁾ that in his perturbation method, one can ignore the exclusion principle in the intermediate state, because there always exists corresponding process which just makes up the effect. One can verify this statement by examining all possible cases in which exclusion principle comes into question. Then what is the meaning of our theory in this viewpoint?

In our viewpoint, a proton in a state φ is forbidden the transition $\varphi \rightarrow \psi + \pi \rightarrow \varphi$ if state ψ is occupied, and this changes the magnetic moment. If we dispense with the exclusion principle in the intermediate state, this transition is allowed and no change of moment occurs. There exist, however, other processes which contribute to the magnetic moment. Thus the proton in state φ makes a transition to ψ with the emission of a meson, followed by the transition of a nucleon in state ψ to φ by absorbing the meson: and conversely, the nucleon in state ψ goes to state φ accompanied by the emission of a meson, the latter being subsequently absorbed by the proton in φ , which goes to ψ . These processes would not occur if we had taken the exclusion principle into account. These processes give just the same amount of magnetic moment as our method does. The magnetic moment arisen from the exchange of mesons between nucleons are usually called as exchange magnetic moment. Therefore the change of magnetic moment explained in previous sections is equally well interpretable as the effect of the exchange magnetic moment. The advantage of our method lies in the

simplicity attained by reducing to the single particle problem, of calculations as well as of physical meanings involved, and the possibility of comparison with the free nucleon moment.

The comparison of two viewpoints often affords an interesting interpretation. Consider the change of the self-energy of a nucleon φ in a nucleus, due to the forbiddenness of some of the virtual states. If $\varphi \rightarrow \psi + \pi$ were forbidden, the change of the self-energy of φ will be (in scalar theory)

$$\begin{aligned} f^2 \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\mathbf{x} \varphi(\mathbf{x}) \exp(i\mathbf{k}\mathbf{x}) \psi(\mathbf{x}) \cdot \int d\mathbf{y} \bar{\psi}(\mathbf{y}) \exp(-i\mathbf{k}\mathbf{y}) \varphi(\mathbf{y}) / 2(\mathbf{k}^2 + \kappa^2) \\ = \frac{1}{2} \frac{f^2}{4\pi} \int d\mathbf{x} d\mathbf{y} \varphi(\mathbf{x}) \bar{\psi}(\mathbf{y}) \frac{\exp(-\kappa|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} \psi(\mathbf{x}) \varphi(\mathbf{y}). \end{aligned}$$

But this is nothing but (half) the exchange integral of Yukawa potential taken between φ and ψ . The same situation also holds for all types of forces, if they are transmitted by intermediary quantized fields. Thus one can bestrew an intuitive meaning to the exchange integral which has no classical analogue: The exchange integral of a potential between states φ and ψ is the change of the self-energy of φ , caused by the prohibition of process $\varphi \rightarrow \psi + \pi$ due to the presence of ψ plus the change of the self-energy of ψ by the presence of φ . This form of statement will be more accessible to those who are accustomed in the self-energy problem.

Once realized the equivalence of our calculation to the exchange magnetic moment, we can turn to the evaluation of spin independent term (9). It has been proved by Osborn and Foldy⁹⁾ that spin independent exchange moment (which is referred to as space exchange moment or longitudinal exchange moment) is completely determined by the nuclear force solely by the requirement of gauge invariance. Their result is, that if the nuclear potential is

$$\tau_i \tau_j V_{ij},$$

the space exchange magnetic moment is given by

$$\mathbf{M} = -\frac{1}{2} e V_{ij} [\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j]_3 [\mathbf{r}_i \times \mathbf{r}_j].$$

With this expression the space exchange moment of the nucleus can be calculated. Again we use Fermi gas and for potential assume Yukawa type of the form

$$V = \dots + \tau \tau (a_\tau + a_{\sigma\tau} \sigma\sigma) \frac{e^{-\kappa r}}{r}.$$

Terms of the form a_0 or $a_\sigma \sigma\sigma$ are irrelevant for us. The result is

$$\begin{aligned} \mu' &= -(a_\tau + 3a_{\sigma\tau}) \frac{1}{\pi} \frac{M}{P^3} [2P^2 n(1) - n(3)] l \cdot \tau_3 \\ &= -(a_\tau + 3a_{\sigma\tau}) 1.04 l \cdot \tau_3, \end{aligned}$$

where l means the angular momentum of the odd nucleon. At present we have no satisfactory form of nuclear potentials, valid for both two nucleon problem and for heavy nucleus. From the nuclear potentials¹⁰⁾ adjusted to fit the high energy nucleon-nucleon scattering, we obtain the order of magnitude for μ' to be

$$\mu' = \pm 4\mu'l \approx \pm 0.1l \quad \left(\begin{array}{l} \text{for } p \\ \text{for } n \end{array} \right) \quad (15)$$

§ 5. Magnetic moments of heavy nuclei

We are now in position to compare our results with experiments. Magnetic moment of odd nucleus can easily be calculated by the vector model. But the

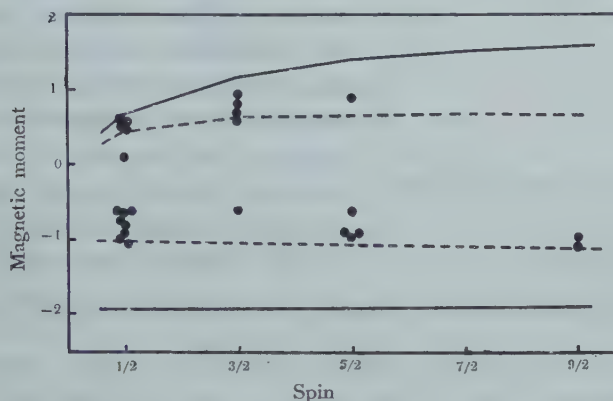


Fig. 1 Magnetic moments of odd neutron nuclei. Only heavy nuclei (mass number > 20) are plotted. Full lines represent the Schmidt lines. Dashed lines are obtained by our theory.

term of (14) can also be calculated. Magnetic moment of nucleus is given by

$$\mu = g_l l + g_s + g_a \frac{1}{2l+3}, \quad \text{for } I = l + \frac{1}{2},$$

$$\mu = g_l \frac{(l+1)(2l-1)}{2l+1} - g_s \frac{2l-1}{2l+1} - g_a \frac{3(2l-1)}{(2l+1)(2l+3)}, \quad \text{for } I = l - \frac{1}{2},$$

where

$$g_a = \pm 4\mu_2 \quad \left(\begin{array}{l} \text{for } p \\ \text{for } n \end{array} \right),$$

and $4\mu_2$ is given by (13) or (minus of) the numerical factor multiplying the second term of (14). In Figs. 1 and 2 the calculated moments are plotted. In plotting the curve we have used the values

gyromagnetic ratios g_s and g_l of spin and orbital angular momentum differ from usual ones because of the change of magnetic moments in a nucleus. Thus

$$\begin{aligned} g_l &= 1 + 4\mu', & \text{for } p, \\ &= -4\mu', & \text{for } n, \\ g_s &= 2.79 - 4\mu_1, & \text{for } p, \\ &= -1.91 + 4\mu_1, & \text{for } n, \end{aligned}$$

where $4\mu'$ is given by (15) and $4\mu_1$ is given by (12) or the first term of (14). The

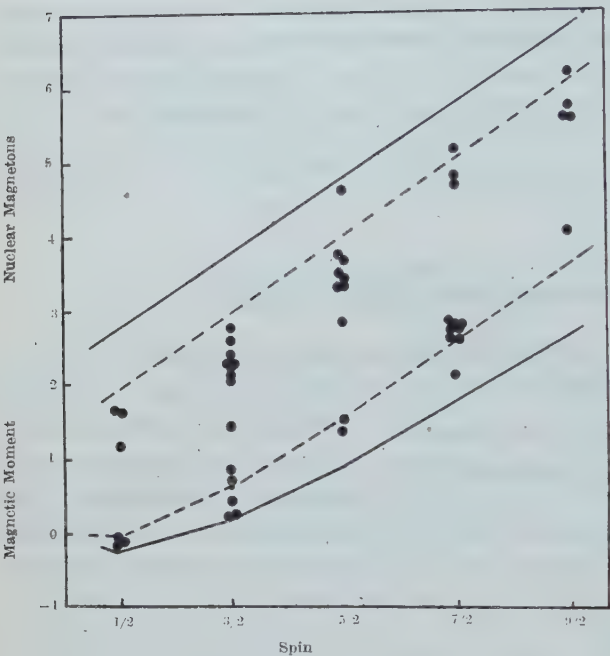


Fig. 2 Magnetic moments of odd proton nuclei.

$$\begin{aligned} \Delta\mu' &= 0.05, \\ \Delta\mu_1 &= 1, \end{aligned} \tag{16}$$

and $\Delta\mu_2 = 0.5$

to obtain the best fit with experiments, for the numerical factors in (15) or (14) are uncertain by a factor of ~ 2 .

The agreement of the present theory with the experiments seems satisfactory, in view of the crudeness of the assumptions adopted to simplify the calculation. Small deviations from the theoretical curves will be due to the rough calculation of $\Delta\mu$ or the lack of general validity of the extreme single par-

ticlę model. Also the deviation will be due to the coupling of states with $I=l\pm\frac{1}{2}$. Rainwater⁽¹¹⁾ pointed out that the large value of nuclear quadrupole moments can be attributed to the spheroidal shape of the nuclear core. In this asymmetrical field, the angular momentum of the extra odd nucleon is no longer an exact quantum number, and the magnetic moment of the nucleus deviates from the Schmidt lines. If so the deviation will be proportional to ϵ^2 , ϵ being the eccentricity of the nuclear core. The deviation cannot be proportional to ϵ , because negative as well as positive ϵ causes an inward deviation of moments. ϵ can be calculated from the known value of quadrupole moment. We have plotted in Fig. 3, magnetic moments μ of odd proton nuclei with $I=l+\frac{1}{2}=3/2$ versus ϵ^2 and found that in fact they lie co-linear. A rather unusual behaviour of Cu^{63} will be due to the experimental error. The extrapolated value of μ for $\epsilon^2 \rightarrow 0$ tends to the value predicted by theory.

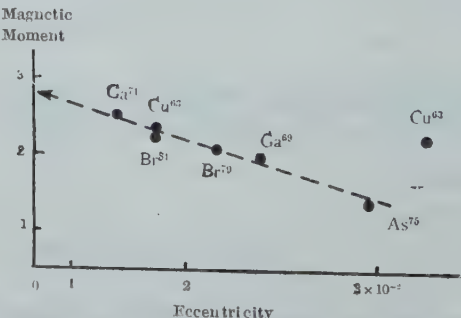


Fig. 3 Magnetic moments versus eccentricities of odd proton nuclei with $I=l+\frac{1}{2}=3/2$.

§ 6. Conclusion and discussions

The contribution of the virtual mesons to the magnetic moment of nucleus is calculated. A precise prediction of the moment is greatly hampered by the lack of a satisfactory meson theory which can give correct values of free nucleon magnetic moments. The problem of nucleon magnetic moments does not seem to be settled only by the second order perturbation calculation, but needs higher order corrections. However, such a way is prohibitively laborious. Taking a pseudoscalar theory and assuming that the correct solution will be given by cutting off the cumbersome high momentum mesons, the calculated effects are in accordance with experiments. One can conclude that a possible explanation for the deviations of nuclear magnetic moments from the Schmidt lines is given in this way. Conversely, we see that virtual mesons of momentum $\sim \pi$ must play a dominant part in the nucleon magnetic moment, and high momentum mesons with $2 \sim 3\pi$ is to be cut off. This conclusion will be useful in determining the distribution of charge and current densities of meson cloud around the nucleon.

Before concluding this work, some discussions are added.

We have used a sharp cut off (1 for $|\mathbf{k}| < k_0$ and 0 for $|\mathbf{k}| > k_0$) to calculate the magnetic moment and estimated $\delta\mu/\mu_0$ to be about 0.5. But this result is nearly independent of the choice of cut off factor. I have tried with the cut off factor $1/k^2$ and find $\delta\mu/\mu_0$ to be ~ 0.4 . Such cut off which saves the difficulty of nuclear forces is sufficient for our purpose.

It was shown that somewhat smaller value of $\Delta\mu'$ than predicted by theory (Eq. (15)) gives better fit with experiments. This may be taken as an indication of the invalidity of nuclear forces used to estimate (15). But it must be bear in mind that space exchange moment is not definitely determined in this way, for the orbital gyromagnetic ratio sensitively depends on special properties of the nuclear model. Thus the liquid drop model of Margenau and Wigner¹²⁾ gives $g_i \approx Z/A \approx 0.4$, in contrast to the single particle model which gives $g_i=1$ or 0. The true situation will probably correspond to a sort of compromise between these two extremities, rather near to the latter. So the space exchange moment would be larger than that given by (16). This tendency towards the liquid drop model is also inferred from Rainwater's discussion¹¹⁾ of nuclear quadrupole moments, who obtained too large value if single particle model were adopted, and thereby expected considerable "dilution" of the extreme single particle model.

The spin gyromagnetic ratio without meson effects is, on the contrary, independent of the model and is not obscured by the possible tendency towards the liquid drop model. However, this does not mean that (14) should just equal to (16). There is several reasons that this is not so. First of all, the problematical adoption of the current meson theory, as repeatedly stated. Secondly, the roughness of the Fermi gas model used to calculate the change of moments. True wave function of the nucleus is more tactfully arranged to increase the

overlapping of each two nucleons in order to gain more potential energy, yet not to increase the kinetic energy. The Fermi gas model is known to underestimate the potential energy by a factor of 2. It is possible that it also underestimates our change of moments equally. Thirdly, there must be other causes which change the magnetic moment inside a nucleus. We have good reason to suppose that meson-nucleon interaction as well as field equations of mesons differ from usual ones inside a nuclear matter. This is the mesonic analogue* of dielectrics in which, for instance, the Coulomb force is multiplied by $1/\epsilon$ if the dielectric constant is ϵ , and light velocity is reduced to $1/\sqrt{\epsilon}$. Similarly, meson-nucleon coupling constant or meson compton wave length will be diminished in a nucleus which naturally changes the anomalous magnetic moment of a nucleon.

The author wishes to express his appreciation to Professors T. Yamanouchi, S. Tomonaga and M. Kotani for their kind guidances and advices.

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* This problem is partly discussed by W. W. Wada (Phys. Rev. **81** (1951), 308) and by Y. Nambu, but results are not yet published.

On the Adiabatic Nuclear Potential, I

Kazuhiko NISHIJIMA

Department of Physics, Osaka City University

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The "nuclear force", inspite of its longest career, is one of the most complicated unsolved problems in the meson theory. We will investigate this problem on a non-relativistic basis. First, we analyze the special natures of the phenomena, "nuclear forces". Then making use of the foregoing analysis, we discuss the problem qualitatively, for instance, what kind of approximation we should employ. As the results of these discussions, we can conclude that no single approximation such as weak coupling or strong coupling will be capable of explaining the phenomena, and that the higher order calculations would not be able to help the situation out of discrepancies, if the adiabatic nuclear potential computed up to 4-th order could not fit the experimental data in the low energy region where the weak coupling theory is expected to hold. Second, based on the above considerations, we calculate the 4-th order adiabatic nuclear potential by the method of canonical transformations. In order to obtain a meaningful 4-th order adiabatic potential, it is necessary that the 2nd order one can be derived in the Schroedinger approximation without referring to the Pauli approximation. Interesting is the result that thus derived potential agrees with the one derived by the S -matrix method.

§ 1. Introduction

It was not later than three years after the birth of Yukawa theory when the serious difficulty concerning the singularity of the meson potential became emphasized, and so disencouraging was this difficulty, as compared with the Coulomb potential, for both theoretical and experimental reasons, that the efforts of early meson physicists were concentrated on this problem. Recently, however, the laboratory studies of mesons were rapidly advanced, and most meson physicists are interested in the mechanism of the production and capture of π -mesons leaving the problem of "nuclear forces" untouched. Still we have reasons to believe that this phenomenon, though implicitly related to the properties of mesons, will give us some informations about the correct method to be employed in the meson problem as will be shown later. Accordingly, we will study the problem in this paper.

Historically various methods were proposed in order to get rid of the singularity which will be stated simply in the following.

First, we must stress the mixed meson theory due to Møller and Rosenfeld.¹⁾⁻⁴⁾ It is interesting to remember the fact that later on this theory was

abstracted and developed to the mixed field theories. The cancellation of singularities in the mixed meson theory, however, is restricted only to the lowest order in the coupling constant, and moreover it is a general truth that such a theory breaks down in other phenomena.

Second, there appeared the method of "cut off" due to Bethe.^{5),6)} This method, contrary to other philosophies, leaves the difficulty as unknown, and treats the problem rather phenomenologically, which though imperfect theoretically, seems to be the orthodox in such a kind of problems.

Third, recently the following fact was emphasized by several authors that the relativistic effect, or synonymously the retardation effect, diminishes the order of singularity.⁷⁾⁻⁹⁾ But at the present stage, we know the strong interaction between nucleon and meson fields, and unless the higher singularities originated in the higher order calculations can be removed at the same time, we can no more regard this method as the final one than in the mixed meson theory.

Besides, a non-linear meson theory was proposed quite recently by Schiff.¹⁰⁾ This theory seems promising, but we know nothing about detailed results yet.

Thus we will confine ourselves to such extent that we can discuss with the present stage meson theory. As for the problem of singularity, it will not be solved unless the nature of the self field around a nucleon can fully be clarified, and we shall follow Bethe's standpoint for the time being.

We first discuss the method of approximation, i. e. which one of the weak, intermediate⁽¹¹⁾ and strong⁽¹²⁾⁻⁽¹⁵⁾ couplings will be the best.

As for the problem of nuclear forces, no single approximation will do, i. e. we must prepare every variety of approximation for every variety of condition. First we will examine if the weak coupling approximation is correct in the low energy region. Were it correct the main behaviour will be determined by the second plus fourth order potential. (§ 2)

The next problem is the computation of the 4-th order potential in the weak coupling approximation. As the method of computation, we make use of the canonical transformations. And in order to clarify what transformations we must use, we review the derivation of the 2nd order potential. (§ 3)

Based on the method of derivation of the 2nd order potential, further transformations for the 4-th order potential are studied. (§ 4)

By the above studied transformations, 4-th order potential is discussed. (§ 5)

And then the physical interpretation of the potential is given. (§ 6)

Finally we compute the 4-th order potential in individual cases. (§ 7)

§ 2 Qualitative discussions¹⁶⁾

(1) We first consider the reason why the concept of "potential" is necessary. It is almost certain that the mutual interaction between nucleons can be explained only on the basis of the meson theory, so that if we want to solve

the problem of a nucleon system, it is necessary to take account of the meson field. It turns out to be a quite complicated many body problem which, of course, we cannot solve at the present stage.

Therefore some convenient method to eliminate the meson field is desired. It will perhaps be impossible to perform such a procedure rigorously, but to some extent of approximation it will be possible.

From the standpoint of field theory, it corresponds to an approximate separation of nucleon and meson fields by some suitable method such as the canonical transformation.

In this connection, the nuclear potential plays an important role as the implicit representative of the eliminated meson field, and the mathematical treatment becomes much simpler than in the original form. Moreover it is established experimentally that the concept of nuclear potential is useful in describing the nucleon-nucleon interaction.

Next it is known that the Born approximation is meaningless in the low energy nucleon-nucleon scattering.¹⁷⁾ Theoretically, calculations of the covariant S-matrix is desired which, however, inevitably reduces to lower order Born approximations. Thus the non-relativistic potential is more useful in the analysis of the low energy nuclear forces. This is the second reason.

(2) Second, we will consider the special natures of nuclear forces. Although the character of nuclear forces so seriously depends on the nature of the meson field, only virtual mesons are concerned in these phenomena. While in other mesonic phenomena such as production or capture of π -mesons, we see π -mesons appear not virtually but really at least in one of the initial and final states. Thus these phenomena must involve high energy states in their processes in contrast to the nuclear force. What is essentially important is the fact that the nuclear force is the only low energy mesonic phenomenon.

(3) Third, we are concerned with the force range. Now, we define the essential $2n$ -th order nuclear force as the phenomenon in which n mesons are interchanged simultaneously between two nucleons.

Other types of nuclear forces are regarded as radiative corrections to the essential nuclear force. Then the radiative corrections in low energies give only the renormalizations.¹⁸⁾ (See Appendix.)

For this reason, we may investigate only the essential nuclear forces provided that we confine ourselves to the low energy phenomena. Our following discussions on the force range are due to Wick's idea.¹⁹⁾ Suppose a two nucleon system, then the uncertainty of energy of the system due to $2n$ -th order essential nuclear force is given by

$$\Delta E \gtrsim n\mu c^2 \quad (1)$$

where μ is the meson rest mass, and c the light velocity.

So the time of flight of mesons, Δt is limited by the following uncertainty

relation :

$$\Delta E \Delta t \sim \hbar. \quad (2)$$

Combining (1) and (2), we see

$$\Delta t \sim /n\mu c^2.$$

Thus the distance of flight of mesons, R is seen to be

$$R \lesssim c\Delta t \sim (1/n) (\hbar/\mu c) = (1/n) \lambda^{-1}. \quad (3)$$

The range of the $2n$ -th order essential nuclear force is $1/n$ times of that of the 2nd order. The above consideration is made by expansion in numbers of mesons but not referring to the perturbation method.

The point nucleon model employed in the above discussion, however, is not correct since a nucleon has its spread with a radius of about λ , the nucleon Compton wave length, due to its Zitterbewegung.

For $n \sim 6$, we see that $R \sim \lambda$, and the above consideration does not hold. The validity of the discussion will be at most to $n \sim 3$. The more mesons two nucleons interchange, the more uncertain the positions of nucleons will be due to their recoil.

In this way, the point nucleon model cannot be applied within about one third of the force range, and we need to use the velocity dependent potential in place of the ordinary static potential. We have estimated the validity region of the static potential to be outside of about one third of the force range (Cf. Bethe), however, it might be half the force range if other effects are taken into account.

Indeed, we must consider the dynamical effects due to meson clouds around nucleons besides the kinematical effects discussed above. This problem will be discussed later.

What we must notice next is the type of coupling between nucleon and meson fields. For instance, the discussions cannot be applied to the queer interaction γ_5 , i. e. Ps (ps) .

Here the capital letter Ps denotes the type of the meson field, and the small letter ps in parentheses the type of coupling. The existence of the coupling γ_5 makes the transitions from positive (or negative) energy states to negative (or positive) energy states easier than those from positive (or negative) to positive (or negative), and the former discussion breaks down. Thus the inequality (1) turns out to be

$$\Delta E \gtrsim 2Mc^2, \quad (M: \text{nucleon rest mass}) \quad (1')$$

and (3) to be

$$R \sim \hbar/2Mc. \quad (3')$$

As easily be seen from the above result, the meson clouds shrink together and the nucleon anomalous magnetic moment cannot be fit to the experiment.

To conclude, the former discussion is valid only when the 2nd order nuclear potential can be derived in the Schroedinger approximation without referring to the Pauli approximation.

Therefore we confine ourselves only to such couplings from now on.

(4) Fourth, we discuss what kind of approximation we should employ. This is a rather general problem, and we pick up only the low energy problem. The characteristic feature of the low energy phenomena is the applicability of the non-relativistic treatment together with the static (or adiabatic) approximation.

Now consider which approximation will be better, the weak coupling or the strong coupling. For this purpose, we first employ the weak coupling. Then the higher the order of approximation proceeds, the higher the order of singularity and shorter the force range will be.

Therefore, for a large separation of nucleons lower orders will be dominant because of shrinkage of the force range in higher orders. For an intermediary separation, lower orders turn out to be inferior since they are of low singularities and much higher orders will also be inferior because of their short force range. Finally for a suitably small separation higher orders will be dominant by their high singularities and the spread of nucleons, i. e. the criterion in terms of force range breaks down in this region.

Thus it is clear that the perturbation does never converge within some small separation, which cannot be determined in the present stage meson theory but is of great importance. We shall call this separation "the critical range" and denote it by r_c . The critical range is similar to the cut off radius in the phenomenological theory, and corresponds to the convergence radius of the perturbation calculation.

This is the dynamical limitation for the validity of static potential, strongly depending on the type of coupling contrary to the kinematical limitation discussed before.

As a whole, both limitations together determine the critical range.

From the above discussion, we can conclude that the approximation to be employed depends on the separation of nucleons, and that the weak coupling method cannot be applied within the critical range.

Although we do not know how to determine the critical range, we shall illustrate it by a simple model.

Suppose that the $2n$ -th order nuclear potential is given by

$$V_{2n} = (-g^2)^n \frac{F}{x^n} \left(\frac{e^{-x}}{x^{2n}} \right)^n,$$

where g is the coupling constant, F a constant with the dimension of energy. Then the whole potential is given by

$$V = \sum_{n=1}^{\infty} V_{2n} = \frac{F}{\kappa r} \sum_{n=1}^{\infty} \left(-\frac{g^2 e^{-\kappa r}}{\kappa^2 r^2} \right)^n.$$

In this case, the condition of convergence is readily seen to be

$$g^2 e^{-\kappa r} / \kappa^2 r^2 < 1.$$

This inequality can be transformed into the condition for r :

$$r > r_c.$$

Thus we get the value of the critical range. In general, the larger the coupling constant g , the larger the critical range r_c , i. e. the narrower the domain of perturbation method. Inside the critical range, it is clear that other methods than the weak coupling should be employed. But in this simple example, we will use the method of analytic continuation. From the outside solution by the perturbation method, we get the following potential inside the critical range:

$$\begin{aligned} V &= \frac{F}{\kappa r} \left(-\frac{g^2 e^{-\kappa r}}{\kappa^2 r^2} \right) \left/ 1 + \frac{g^2 e^{-\kappa r}}{\kappa^2 r^2} \right. \\ &= -\frac{F}{\kappa r} \cdot \frac{g^2 e^{-\kappa r}}{\kappa^2 r^2 + g^2 e^{-\kappa r}}. \end{aligned}$$

In this case, the singularity at the origin is only r^{-1} in contrast to the prediction due to perturbation method.

In actual problems, we cannot use the method of analytic continuation since higher orders are unknown, but this situation will be similar.

We know, in this way, that the weak coupling theory cannot explain the phenomena singly, so we shall examine the strong coupling theory next. Well, let us study the behaviour of the nuclear forces at comparatively large separation near the force range by the conventional strong coupling theory. Since large momentum transfer won't occur in this large separation, the adiabatic approximation employed in this theory will give fairly good informations. According to Serber and Dancoff,²⁰⁾ we know that with

$$\kappa a \lesssim 0.1, \quad (a: \text{the radius of a nucleon}) \quad (4)$$

no value of the coupling constant gives spin dependent forces large enough to agree with experience.

In general, the strong coupling theory gives, for large enough separation, forces between two nucleons of the same type as those obtained from perturbation theory, but at closer approach the forces become ordinary.

Thus the behaviour of the potential predicted from the strong coupling theory does not give the correct information near the force range, while the nature at closer distance cannot be trusted since the adiabatic approximation is not valid in

this region.

In order to investigate the property at small distances, we must study the relativistic strong coupling theory which to our regret is not known yet.

We conclude from the above discussions that inside the critical range the weak coupling theory is not valid and outside the strong coupling theory is not valid. The knowledge concerning the interior region is furnished only by the relativistic strong coupling theory, for which we must abandon the hope at the present stage and our subject is limited to the outer region. For these reasons we will test the validity of the weak coupling theory in the outer region. The treatment in the intermediate coupling theory is left open in this paper, though it is quite promising. We shall investigate how the adiabatic nuclear potential will be, provided that the weak coupling theory is valid.

If the perturbation treatment is allowed, we may suppose that

$$0.5 \gtrsim \kappa r_0 \gtrsim 0.3. \quad (5)$$

Of course, the smaller the value of κr_0 , the better the perturbation method. We assume for the moment $\kappa r_0 \sim 0.3$.

We employ the potential on the weak coupling theory in the outer region, and assume suitable cut off in the interior region. Notice here that the solution does not seriously depend on the mode of "cut off" as has been shown by Bethe. Otherwise the cut off procedure will lose its meaning.

The most difficult question in the perturbation treatment is to which order we must perform the calculation. We compute up to 4-th order in this paper. If it does not show good agreement with experiments, then 6-th order calculation will be required. On the other hand, if it shows good agreement with experiments, then we fear the 6-th order calculation would destroy the agreement.

Fortunately we have reasons to believe that the 4-th order computation will be the decisive one for right or wrong in the low energy nuclear forces which we discuss next.

(5) We compute the nuclear potential up to 4-th order which is different from the 4-th order computation of the S -matrix, since the S -matrix calculated by this potential involves the repetition of the 2nd plus 4-th order potential. The calculation of the 4-th order S -matrix is meaningless, for we are interested in the low energy nuclear force, and the Born approximation is no more valid in this region.

From the relation between the force range and the order of nuclear force stated before, the range of 4-th order nuclear force is half that of 2nd order, i. e. $(2\kappa)^{-1}$, and 6-th order $(3\kappa)^{-1}$.

Let us call the shell bounded by two spheres with the radii κ^{-1} and $(2\kappa)^{-1}$, the region A, the shell with the radii $(2\kappa)^{-1}$ and $(3\kappa)^{-1} \sim r_0$, the region B, and inside the sphere with the radius $(3\kappa)^{-1} \sim r_0$, the region C.

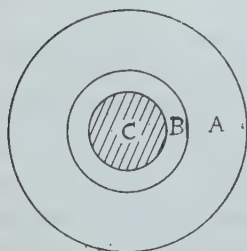


Fig. 1

Then the forces acting in each region are :

in A : 2nd order nuclear forces,

in B : 2nd+4-th order nuclear forces,

in C : higher orders, velocity dependent forces and forces due to the strong coupling theory, and possibly of heavy mesons.

Our standpoint is to regard the region C as the cut off region. The experimental evidences show that the nuclear force in the region C is not so singular as has been predicted from the weak coupling theory. Moreover the volume of the region C is small compared with the whole volume of nuclear force, i. e.

$$\frac{\text{volume}(C)}{\text{volume}(A+B+C)} = \frac{1}{27}. \quad (6)$$

Thus we see that the behaviour of the nuclear force in the region C or the mode of cut off will not seriously affect the results of the analysis, and the adiabatic nuclear potential computed up to 4-th order will give us a good criterion about the validity of the weak coupling meson theory of nuclear forces.

As will be shown later, the 4-th order potential in the case of Ps (pv) is large compared with the 2nd order one, and is comparable even near the force range.

If this largeness is due to some inevitable reason, such as differentiations, the higher orders will diverge even in the neighbourhood of the force range, i. e., $xr_0 \sim 1$, but if it is due to some accidental reason, then higher orders will be small and the perturbation method converges. At any rate, it is the question whether the 4-th order potential will fit to the experiments or not. For some reason the intermediate coupling theory seems to be best, and both the weak and strong coupling theories will be ruled out. And it seems to us that the validity of the weak coupling theory depends mainly on the type of coupling and only slightly on the value of the coupling constant.

§ 3. Second order potentials

As for the 2nd order potentials, we need not repeat calculations since they are well known, but we review them to get some instructive informations for the derivation of the 4-th order potentials.

The typical method to compute the 2nd order potentials are (1) the perturbation method which has developed into Feynman's *S*-matrix theory, and (2) the method of canonical transformations originated by Møller and Rosenfeld and developed into Tomonaga-Schwinger theory.

The former is simpler and convenient to analyze scattering problems, and Nambu computed the 4-th order potential on this method.²¹⁾ But the concept of

potential is too non-relativistic by nature to compute by the relativistic S -matrix theory, and the potential has the same transformation property with energy as a part of the Hamiltonian, quite different from the invariant S -matrix.

Moreover, the essential problem is how to separate the 4-th order potential from the repetition of the 2nd order potential.

For these reasons, we are inclined to choose the latter method and in fact we do so. In the method of canonical transformations, the original Hamiltonian is transformed into a more convenient form of other Hamiltonian without changing its transformation property and the separation of the 4-th order potential is automatically done.

We start from the Tomonaga-Schwinger equation for meson-nucleon system :

$$i \frac{\delta \Psi[\sigma]}{\delta \sigma(X)} = (H_1(X) + H_2(X)) \Psi[\sigma], \quad (7)$$

where H_1 and H_2 are Hamiltonian densities of the 1st and 2nd orders in the coupling constant, especially H_2 is added by the requirement of the integrability condition. We employ the unit $\hbar=c=1$ from now on.

In order to derive the 2nd order potential, we make use of the following customary transformation :

$$\Psi[\sigma] = \exp\left(-i \int^\sigma H_1(X) dX\right) \Psi_1[\sigma]. \quad (8)$$

For convenience in the non-relativistic approximation, we employed the following notations :

$$dX = dx dy dz dt, \quad dx = dx dy dz.$$

By the transformation (8), the equation (7) is transformed into

$$i \frac{\delta \Psi_1[\sigma]}{\delta \sigma(X)} = \left(H_2(X) - \frac{i}{2} [H_1(X), \int^\sigma H_1(X') dX'] \right) \Psi_1[\sigma]. \quad (9)$$

The nuclear force is obtained by taking the two nucleon, no meson part from the above Hamiltonian, i. e.

$$V_2(X) = \left\langle H_2(X) - \frac{i}{2} [H_1(X), \int^\sigma H_1(X') dX'] \right\rangle_{2,0} \quad (10)$$

For instance, in the case of neutral spinless meson theory

$$H_1 = f W \phi + \frac{g}{x} M_\mu \frac{\partial \phi}{\partial x_\mu}, \quad H_2 = \frac{1}{2} \left(\frac{g}{x} \right)^2 (M_\mu n_\mu)^2, \quad (11)$$

where ϕ is the wave function of the meson field, and W, M_μ are the bilinear forms of nucleon wave functions. n_μ is the unit normal of the space-like surface

σ at a point X .

Inserting (11) into (10), we face the following type of integrals:

$$\int^{\sigma} W(X) \mathcal{A}(X-X') W(X') dX',$$

where \mathcal{A} is the \mathcal{A} -function of the meson field defined by

$$\mathcal{A}(X) = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{k_0} e^{i\mathbf{k}\mathbf{x} \sin k_0 t},$$

and

$$\mathcal{A}^{(1)}(X) = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{k_0} e^{i\mathbf{k}\mathbf{x} \cos k_0 t},$$

with $k_0 = \sqrt{\mathbf{k}^2 + \pi^2}$. Gothic letters refer to three dimensional vectors throughout this paper.

The integrations are readily performed by means of the following formulae:

$$\int^{\sigma(X)} \mathcal{A}(X-X') \cdot F(X') dX' = \frac{F(X)}{x^2 - \square}, \quad (13a)$$

$$\int^{\sigma(X)} \frac{\partial}{\partial x_{\mu}} \mathcal{A}(X-X') \cdot F(X') dX' = \frac{\partial_{\mu} F(X)}{x^2 - \square}, \quad (13b)$$

$$\int^{\sigma(X)} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} \mathcal{A}(X-X') \cdot F(X') dX' = \frac{\partial_{\mu} \partial_{\nu} F(X)}{x^2 - \square} - F(X) \cdot n_{\mu} n_{\nu}. \quad (13c)$$

Applying these formulae to the case of (11), we have

$$\begin{aligned} V_2 = & -\frac{f^2}{4} \left\{ W, \frac{W}{x^2 - \square} \right\} + \frac{fg}{4x} \left(\left\{ W, \frac{\partial_{\mu} M_{\mu}}{x^2 - \square} \right\} - \left\{ \frac{\partial_{\mu} W}{x^2 - \square}, M_{\mu} \right\} \right) \\ & + \frac{1}{4} \left(\frac{g}{x} \right)^2 \left\{ M_{\mu}, \frac{\partial_{\mu} \partial_{\nu} M_{\nu}}{x^2 - \square} \right\}. \end{aligned}$$

It must be noticed that the normal dependent term appearing in the right hand side of (13c) just cancels H_2 . Terms like $\partial_{\mu} M_{\mu}$ can be simplified by means of Dirac equation.

In order to change V_2 into non-relativistic form, we have only to employ the Schroedinger or Pauli approximation, and to substitute like

$$x^2 - \square \rightarrow x^2 - \mathcal{A}, \quad (14)$$

which is the definition of the adiabatic approximation.

What we must notice here is the situation that the nucleons take only positive energy states all over the process in the 2nd order nuclear force, which no

more holds in the 4-th order. For instance, cases in which nucleons take negative energy states arise in the 4-th order as has been suggested in § 2, and indicated in fig. 2. The coupling γ_5 is an example. Thus the substitution (14)

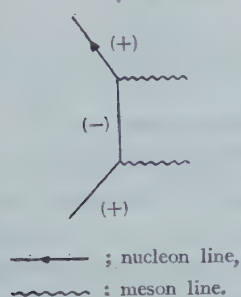


Fig. 2.

must carefully be performed, and we confine ourselves only to the cases in which the contributions due to the change of sign of energy in the virtual states can be neglected, i. e. the cases in which the 2nd order potential can be derived in the Schrodinger approximation. The adiabatic approximation holds only when the nucleons move so slowly compared with meson velocities that nucleons can be regarded as rest. Of course, this approximation can be applied only in the low energy regions. And the direct interaction H_2 required

by the mathematical condition has no physical meaning as has been shown in this section, so that we omit this term in the following calculations.

§4. Method of canonical transformations^{22)–26)}

Based on the qualitative discussions developed above, we shall compute the 4-th order nuclear potential. We first take up Bloch-Nordsieck transformation. We start from the Tomonaga-Schwinger equation also in this section :

$$i \frac{\partial \Psi[\sigma]}{\partial \sigma(X)} = H(X) \Psi[\sigma]. \quad (15)$$

We understand that the direct interaction H_2 is dropped already, and apply the first Bloch-Nordsieck transformation (8) to the equation (15), i. e.

$$\Psi[\sigma] = U_1[\sigma] \Psi_1[\sigma], \quad U_1[\sigma] = \exp\left(-i \int^\sigma H(X) dX\right), \quad (16)$$

then the transformed Hamiltonian is given up to 4-th order by

$$\begin{aligned} H'(X) = & -\frac{i}{2} [H(X), \int^\sigma H(X') dX'] - \frac{1}{3} [[H(X), \int^\sigma H(X') dX'], \int^\sigma H(X'') dX''] \\ & + \frac{i}{8} [[[H(X), \int^\sigma H(X') dX'], \int^\sigma H(X'') dX''], \int^\sigma H(X''') dX'''] \\ \equiv & H_2'(X) + H_3'(X) + H_4'(X), \end{aligned} \quad (17)$$

where H_2' , H_3' and H_4' are terms of 2nd, 3rd and 4th orders in the coupling constant, and H_3' does not contribute to the 4-th order nuclear force.

H_2' contains the following processes :

(1) nuclear forces, (2) self-energy, (3) Compton scattering and (4) double

emission or absorption of mesons. (Cf fig. 3)

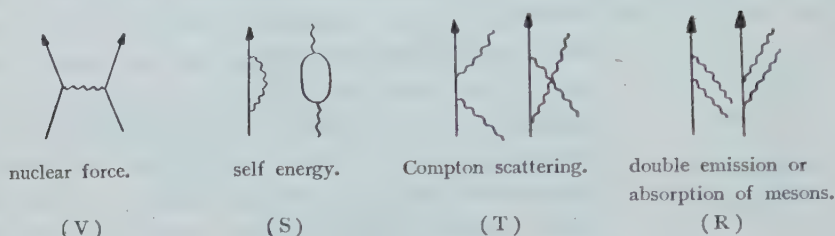


Fig. 3

Now we drop all the divergent terms, for our calculations are non-relativistic and they give only renormalizations in low energies.

So we drop the term (S) in H_2^I , moreover the diagram (T) does not contribute to the 4-th order nuclear force since we take into account only no meson states in both initial and final states. Important is the diagram (R), for its (2,0) part vanishes in this original form, whereas the iteration survives and contributes to the 4-th order potential, i. e. (R) is virtual in the 2nd order and turns out to be real in the 4-th order. Therefore we must eliminate (R) which gives meson clouds by the second Bloch-Nordsieck transformation. Notice that (R) contains the following real process if relativistically treated:

$$N + N' \rightleftharpoons \pi + \pi,$$

(N : nucleon, N' : anti-nucleon, π : meson)

and this process must be left from the second transformation, but since our consideration is restricted to non-relativistic approximation, we can ignore this process. As for Bloch-Nordsieck transformations in a relativistic treatment, they are discussed in detail by Takeda.²⁷⁾

Now we apply the second Bloch-Nordsieck transformation:

$$\Psi[\sigma] = U_2[\sigma] \Psi_2[\sigma], \quad U_2[\sigma] = \exp\left(-i \int^\sigma R(X) dX\right), \quad (18)$$

i. e.

$$\Psi[\sigma] = U_1[\sigma] U_2[\sigma] \Psi_2[\sigma]. \quad (19)$$

Thus the transformed Hamiltonian becomes up to 4-th order as

$$\begin{aligned} H''(X) = & V_2(X) + T(X) - i[V_2(X), \int^\sigma R(X') dX'] - i[T(X), \int^\sigma R(X') dX'] \\ & - \frac{i}{2} [R(X), \int^\sigma R(X') dX'] + H_3^I(X) + H_4^I(X), \end{aligned} \quad (20)$$

where R , T and V are terms in H_2^I corresponding to diagrams (R), (T) and

(V) respectively. Taking up (2, 0) part from (20), we obtain

$$\langle H''(X) \rangle_{2,0} = V_2(X) - \frac{i}{2} [R(X), \int^\sigma R(X') dX']_{2,0} + \langle H_4^I(X) \rangle_{2,0}. \quad (21)$$

The first term gives the 2nd order nuclear force, and the second term together with the third term give the 4-th order potential.

The advantage of the method of canonical transformations consists in the automatic separation of the 4-th order potential. And also the non-relativistic approximation in the calculation of the potential which is a purely non-relativistic concept has much benefit. The first is concerned with ambiguity of the potential,^{(28),(29)} i. e. although the 2nd order static potential has its definite meaning, the non-static part can be dropped by a suitable contact transformation which modifies the form of the static part of the 4-th order nuclear force.

This ambiguity does not occur in our calculation, for the indefinite terms vanish in the adiabatic approximation. Moreover we can regard the problem as a pure two-body problem.

Next we will write down the diagrams of the 4-th order nuclear forces:

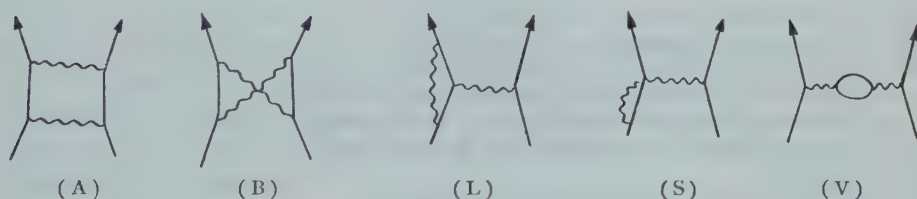


Fig. 4

We calculate only terms corresponding to the irreducible diagrams (A) and (B), for other diagrams give only renormalizations in low energies. For instance, the contribution from (V) vanishes in the adiabatic approximation in which nucleons are regarded as infinitely heavy, and (L), (S) give renormalizations of the coupling constant and the nucleon rest mass. (Cf. Appendix.) To pick up terms corresponding to the diagrams (A) and (B), we use the suffix 2N. Then the 4-th order adiabatic potential is found to be

$$V_4(X) = V_a(X) + V_b(X), \quad (22)$$

where

$$V_a(X) = -\frac{i}{2} [R(X), \int^\sigma R(X') dX']_{2N}, \quad (23a)$$

$$V_b(X) = \langle H_4^I(X) \rangle_{2N}. \quad (23b)$$

(to be continued)

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On the Quadrupole Moments of Light Nuclei

Hisashi HORIE and Shirô YOSHIDA

Department of Physics, University of Tokyo

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The values of quadrupole moments of light nuclei, which are calculated from the usual shell model, show unsatisfactory agreements with experimental values, except for nuclei shell \pm one. Consistent interpretation of quadrupole moments of light nuclei having more than one nucleons outside the closed shell, with magnetic moments can be obtained by the configuration interactions with higher nucleon configurations.

§ 1. Introduction

The spins and magnetic moments of atomic nuclei were successfully explained by the nuclear shell models except a few discrepancies¹⁾. Moreover, the deviations of magnetic moments of heavy nuclei from Schmidt limits were explained in various plausible manners²⁾. On the other hand, the relations between the values of quadrupole moments and fillings of nucleons into the shells were shown experimentally³⁾, and the large values of quadrupole moments of Lu^{175} etc. were explained by the introduction of ellipsoidal potential⁴⁾. The experimental values of quadrupole moments of some light nuclei also, have been obtained by the progress of microwave spectroscopy. We shall discuss the quadrupole moments as well as magnetic moments of light nuclei for which experimental values are available. We shall use the models of light nuclei such that nucleons (or nucleon holes) are moving around the spherically symmetric core, since the introduction of asymmetric potential seems rather unnatural for light nuclei. In the next section, we shall show that reasonable values of quadrupole moments of odd-odd nuclei cannot be obtained by the usual theory of single-body approximation and introduction of configuration interaction is necessary to give consistent values of magnetic and quadrupole moments for these nuclei. In § 3, we shall deal with Li^7 , and in the last section, some discussions will be given.

§ 2. Odd-odd nuclei

By the $j\text{-}j$ coupling shell model, Feenberg calculated the magnetic moments of odd-odd nuclei and obtained fairly good agreement with experimental values⁵⁾. Using the same wave functions, we computed the quadrupole moments of Li^6 , B^{10} and N^{14} for which their values have been measured experimentally⁶⁾. The results are shown in Table I in contrast with the experimental values. The

estimation of $\langle r^2 \rangle_{Av}$ was made roughly by $\langle r^2 \rangle_A = 2 \times A^{2/3} \times 10^{-26} \text{cm}^2$, where A is mass number⁷⁾. Although the agreements of calculated magnetic moments with experimental values are fairly good, the values of quadrupole moments of Li^6 and N^{14} show disagreement definitely.

In order to overcome this difficulty, we have considered the configuration interaction with higher configurations from the standpoint of j - j coupling shell model. Here, we are not concerning with the dynamical problems such as spin-orbit interaction, but restrict ourselves to investigate merely the problem how the states of different configuration have to be mixed to give consistent interpretation of quadrupole moments and magnetic moments.

Table I. Magnetic and quadrupole moments of odd-odd nuclei

Nuclei	Spin	Experimental* Magnetic moment	Quadrupole moment (10^{-26}cm^2)	Configu- ration	Calculated Magnetic** moment	Quadrupole moment (10^{-26}cm^2)
${}^3\text{Li}^6$	1	0.82189 ± 4	$ \leq 9 \times 10^{-2} $	$p_{3/2}^2$	0.63	1.06
${}^5\text{B}^{10}$	3	1.8004 ± 7	6 ± 4	$p_{3/2}^{-2}$	1.88	2.7
${}^7\text{N}^{14}$	1	0.40365 ± 3	2	$p_{1/2}^{-2}$	0.37	0

* J. E. Mack, Rev. Mod. Phys. **22** (1950), 64.

** E. Feenberg, Phys. Rev. **76** (1949), 1275.

(i) ${}^3\text{Li}^6$ As is shown in Table I, the calculated value of quadrupole moment is about ten times larger than observed value. Moreover, the agreement of calculated value of magnetic moment with experimental value is not very good. The configurations which can interact with the lowest $p_{3/2}^2$ are $p_{3/2} p_{1/2}$ and $p_{3/2} f_{5/2}$, since the parity must be same (even) and the spin unity. However, we can neglect the configuration interaction with $p_{3/2} f_{5/2}$ in comparison with $p_{3/2} p_{1/2}$, because the interval between $p_{1/2}$ and $f_{5/2}$ is expected to be much larger than the interval between $p_{3/2}$ and $p_{1/2}$. Furthermore, there is $p_{1/2}^2$ -configuration which does not interact with $p_{3/2}^2$ directly but does so through $p_{3/2} p_{1/2}$. However, we shall ignore this configuration assuming that the bulk of ground state consists of $p_{3/2}^2$ -configuration and its interaction with $p_{1/2}^2$ is the second order effect.

Table II. Matrix elements of magnetic moment operator

	A	B	C
A	0.63		
B	0.20	0.70	
C	0	0.13	0.37

Table III. Matrix elements of quadrupole moment operator

	A	B	C
A	4		
B	$-\sqrt{5/2}$	$-5/2$	
C	0	5	0

Tables II and III give the matrix elements of magnetic moment and quadrupole moment operators, respectively, for the systems with total angular momenta

unity involving two unlike p -nucleons. In these table, A, B and C denote $p_{3/2}^2$, $p_{3/2} p_{1/2}$ and $p_{1/2}^2$ -configurations, respectively. $\langle r^2 \rangle_{Av}$ is assumed to be the same for $p_{3/2}$ and $p_{1/2}$, for the sake of simplicity.

Ψ , the wave function of the ground state of Li^6 , can be written, according to the above-mentioned reasons by making use of normalized wave functions Ψ_A and Ψ_B , as follows:

$$\Psi = a \Psi_A + b \Psi_B,$$

where

$$a^2 + b^2 = 1.$$

Putting $a/b = \xi$, corresponding magnetic moment M and quadrupole moment Q are obtained from Tables II and III as

$$M = (0.63 + 0.40\xi + 0.70\xi^2)/(1 + \xi^2),$$

$$Q = (0.16 - 0.127\xi - 0.10\xi^2)\langle r^2 \rangle_{Av}/(1 + \xi^2),$$

where, $\langle r^2 \rangle_{Av}$ is estimated roughly as the order of magnitude $2 \times 6^{2/3} = 6.6 \times 10^{-26} \text{cm}^2$. Fig. 1 shows variation of the values of these M and Q with ξ ($|\xi| \leq 1$) and $1/\xi$ ($|\xi| \geq 1$). The calculated values of magnetic and quadrupole moments for $p_{3/2}^2$ -configuration as given in Table I, correspond to $\xi=0$, and the values for $p_{3/2} p_{1/2}$ correspond to $1/\xi=0$. Since the exchange magnetic moment vanishes for a self-conjugate nucleus as Li^6 , if we choose the value of ξ so that the computed magnetic moment agrees with experimental one, then the computed quadrupole moment is given by the value shown by an arrow. It might be fairly good agreement, remembering that the above estimate of $\langle r^2 \rangle_{Av}$ is rather overestimate. Then

$$\xi = 0.61,$$

and the probabilities of $p_{3/2}^2$ -and $p_{3/2} p_{1/2}$ -configurations are, respectively

$$a^2 \sim 0.73, \quad b^2 \sim 0.27.$$

(ii) ${}^6\text{B}^{10}$ For this nucleus, Table I gives almost satisfactory results. If we wish to obtain better agreement between computed and experimental values by the configuration interaction, we are forced to consider much higher levels such as $f_{7/2}$ in order to produce spin 3 and possess the same parity together with $p_{3/2}$. Thus, it seems natural that, for B^{10} , single configuration $p_{3/2}^2$ gives satisfactory

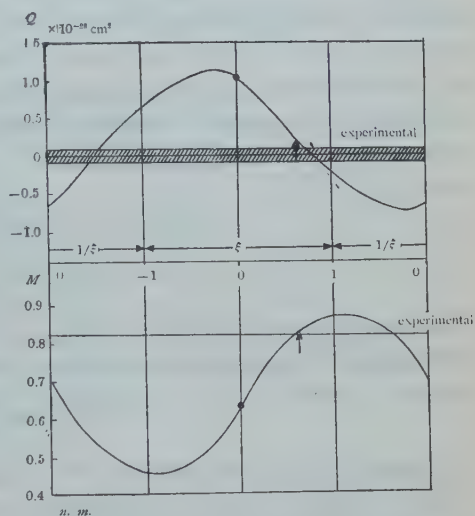


Fig. 1. Quadrupole moment and magnetic moment of Li^6 with the configuration interaction.

agreement with experiments.

(iii) ${}^7\text{N}^{14}$ Although the computed value of magnetic moment of N^{14} gives fairly good agreement with experimental value, the value of quadrupole moment calculated from the same $p_{1/2}^{-2}$ -configuration vanishes and it contradicts with experiments, as is seen from Table I. We assume that the ground state involves $p_{1/2}^{-1} p_{3/2}^{-1}$ in addition to $p_{1/2}^{-2}$ configuration, in the similar way as in the case of Li^6 . Let the wave function of ground state Ψ , and write as follows:

$$\Psi = c^* \Psi_{c^*} + b^* \Psi_{B^*},$$

where

$$c^{*2} + b^{*2} = 1,$$

and C^* and B^* denote the states with resultant angular momenta unity of $p_{1/2}^{-2}$ and $p_{1/2}^{-1} p_{3/2}^{-1}$ -configuration, respectively, and Ψ_{c^*} and Ψ_{B^*} are their normalized eigenfunctions. In this case also, Tables II and III can be applicable by considering that the signs of matrix elements of quadrupole moment must be reversed but those of magnetic moment remain unchanged according to hole theory of spectroscopy. Then, putting $b^*/c^* = \xi$, magnetic and quadrupole moments are given as

$$M = (0.37 + 0.26\xi + 0.70\xi^2)/(1 + \xi^2),$$

$$Q = (-0.4\xi + 0.1\xi^2)\langle r^2 \rangle_{Av}/(1 + \xi^2).$$

These are shown in Fig. 2. $\langle r^2 \rangle_{Av} \sim 2 \times 14^{2/3} = 11.6 \times 10^{-26} \text{cm}^2$, in a similar way as before. If we choose the value of ξ so as to make the value of magnetic moment coincide with experimental value, the points with arrows are obtained.

$\xi \sim -0.98$, and the probabilities of $p_{1/2}^{-2}$ - and $p_{1/2}^{-1} p_{3/2}^{-1}$ -configurations are, respectively,

$$c^{*2} \sim 0.51, \quad b^{*2} \sim 0.49.$$

The agreement of computed value of quadrupole moments with experiments are not so good but the estimate of $\langle r^2 \rangle_{Av}$ seems rather overestimate so that more refined value of $\langle r^2 \rangle_{Av}$ would likely to give better agreement between them. For instance, if we assume uniform density in the nuclear radius R ,

$$\langle r^2 \rangle_{Av} = 3/5 R^2,$$

then the values of quadrupole moment becomes $Q = 1.71 \times 10^{-26} \text{cm}^2$. Although there is such arbitrariness of the estimate of $\langle r^2 \rangle_{Av}$, our value of quadrupole moment gives right order.

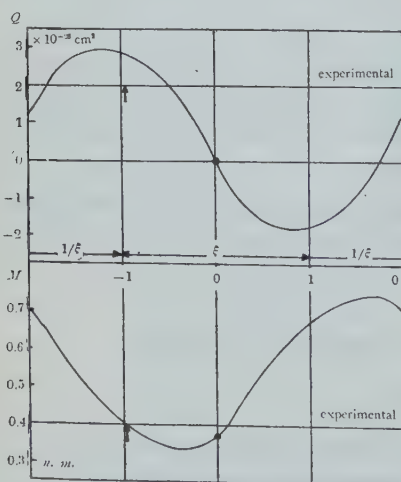


Fig. 2 Quadrupole moment and magnetic moment of N^{14} with the configuration interaction.

§ 3. Li^7

We shall study the magnetic and quadrupole moments of Li^7 , as an example of nuclei shell ± 3 , in this section. The magnetic moments of nuclei of this type have been investigated by Mizushima and Umezawa recently⁸⁾, and especially the properties of Li^7 were studied in detail by many authors⁹⁾. Here, we shall treat the ground state of Li^7 by the j - j' coupling shell model as was done in reference (8). The configuration of the ground state of this nucleus may be assigned as $p_{3/2}^3$ and, then, two distinct states with observed spin $3/2$ occur which correspond to total isotopic spin $T=1/2$ and $3/2$.*) The calculated magnetic and quadrupole moments for these two states are shown in Table IV with experimental values. It contains these moments of Cl^{35} also, which is a nucleus of the same type. $\langle r^2 \rangle_{Av}$ in the calculated values of quadrupole moments were estimated by $2 \times A^{2/3}$

 Table IV. Magnetic and quadrupole moments of Li^7 and Cl^{35}

Nuclei	Spin	Experimental*		Configu- ration	T	Calculated	
		Magnetic moment	Quadrupole moment (10^{-26}cm^2)			Magnetic** moment	Quadrupole moment (10^{-26}cm^2)
^7Li	$3/2$	3.25586	2	$p_{3/2}^3$	$1/2$	3.03	$-22/75 \langle r^2 \rangle_{Av} \sim -2.14$
		± 11	± 2		$3/2$	-0.01	$2/15 \langle r^2 \rangle_{Av} \sim 0.97$
^{35}Cl	$3/2$	0.82191	-7.95	$d_{3/2}^3$	$1/2$	-0.48	$-22/75 \langle r^2 \rangle_{Av} \sim -6.27$
		± 22	± 5		$3/2$	0.80	$2/15 \langle r^2 \rangle_{Av} \sim 2.86$

* J. E. Mack, Rev. Mod. Phys. **22** (1950), 64.

** M. Mizushima and M. Umezawa, Phys. Rev. **83** (1951), 463.

$\times 10^{-26}\text{cm}^2$, as before. From Table IV, we see at once that we cannot have any state from $p_{3/2}^3$ for which the magnetic and quadrupole moments agree with the experimental values of Li^7 simultaneously. Similar situation holds in $d_{3/2}$ -configuration for Cl^{35} . Therefore, we shall consider the configuration interactions with higher configurations for Li^7 as in the case of odd-odd nuclei.

The configurations which can interact with the lowest $p_{3/2}^3$, are $p_{3/2}^2 p_{1/2}$, $p_{3/2}^2 f_{7/2}$ and $p_{3/2}^2 f_{5/2}$, but we shall ignore the configurations containing $f_{7/2}$ and $f_{5/2}$ as in the case of Li^6 . The states of $p_{3/2}^2 p_{1/2}$ -configuration must be specified by parent terms $p_{3/2}^2(T_1 J_1)$ as well as resultants TJ . For instance, $p_{3/2}^2 p_{1/2}$ has two terms $T=1/2$, $J=3/2$ with parent terms $p_{3/2}^2(01)$ and $p_{3/2}^2(12)$. In addition to $p_{3/2}^2 p_{1/2}$, if we take into account $p_{1/2}^2 p_{3/2}$, the following eight states with spin $3/2$ will occur. The states signified by capital letters are states with $T=1/2$ and ones with small letters are $T=3/2$.

*) It might be convenient to consider total angular momentum J and total isotopic spin T as the analogues of total orbital angular momentum L and total spin S , respectively, in Russel-Saunders coupling of atomic spectroscopy, although J may take values of half integer.

Table V. The states of Li^7

	A	B	C	D	E	a	b	c
Configu- ration	$p_{3/2}^3$	$p_{3/2}^2 p_{1/2}$	$p_{3/2}^2 p_{1/2}$	$p_{1/2}^2 p_{3/2}$	$p_{1/2}^2 p_{3/2}$	$p_{3/2}^3$	$p_{3/2}^2 p_{1/2}$	$p_{1/2}^2 p_{3/2}$
T	1/2	1/2	1/2	1/2	1/2	3/2	3/2	3/2
J_1	—	2	1	1	0	—	2	0
T_1	—	1	0	0	1	—	1	1

The matrix elements of magnetic and quadrupole moments between these states are shown in Tables VI and VII.

Table VI. Matrix elements of magnetic moment for Li^7

M (n.m)	A	B	C	D	E	a	b	c
A	-0.293							
B	-0.053	0						
C	0	0.080	0.160					
D	0	-0.170	0.057	0				
E	0	0.042	-0.127	0	-0.267			
a	-0.239	0.119	0	0	0	0.133		
b	-0.038	0	-0.113	0.240	-0.060	0.085	0	
c	0	-0.060	0.179	0	-0.189	0	0.084	-0.133

Table VII. Matrix elements of quadrupole moments for Li^7

$Q/\langle r \rangle^2_{Av}$	A	B	C	D	E	a	b	c
A	-22/75							
B	-4/75	0						
C	0	2/25	4/25					
D	0	$-3\sqrt{2}/25$	$\sqrt{2}/25$	0				
E	0	$\sqrt{10}/75$	$-\sqrt{10}/25$	0	-4/15			
a	$-8\sqrt{5}/75$	$4\sqrt{5}/75$	0	0	0	2/15		
b	$-2\sqrt{2}/75$	0	$-2\sqrt{2}/25$	6/25	$-2\sqrt{5}/75$	$2\sqrt{10}/75$	0	
c	0	$-2\sqrt{5}/75$	$2\sqrt{5}/25$	0	$-2\sqrt{2}/15$	0	$2\sqrt{10}/75$	-2/15

In the first place, we shall investigate how the quadrupole moment and magnetic moment can be made to agree with experimental data only by the states which have configuration $p_{3/2}^3$ and $p_{3/2}^2 p_{1/2}$ and $T=1/2$ i.e., A, B, and C. The ground state of Li^7 is written as follows:

$$\Psi = a\Psi_A + b\Psi_B + c\Psi_C,$$

where

$$a^2 + b^2 + c^2 = 1.$$

We look for the values of a , b , and c which make Q maximum, and evaluate Q and M for it, then we get

$$Q = 1.32 \times 10^{-26} \text{cm}^2, \quad M = 1.11 \text{ n.m.}$$

This is not a good agreement with experimental data.

In the next place we construct the ground state by the state $T=1/2$ only. As an example we take for ground state

$$\Psi = \frac{1}{\sqrt{8}} [\Psi_A + \Psi_B - 2\Psi_C - \Psi_D + \Psi_E],$$

and then we get following values ;

$$Q = 1.20 \times 10^{-26} \text{cm}^2, \quad M = 2.34 \text{ n.m.}$$

Another example including various states is afforded by

$$\Psi = \frac{1}{3} [\Psi_A - 2\Psi_C - \Psi_D + \Psi_E - \Psi_a - \Psi_b],$$

$$Q = 1.47 \times 10^{-26} \text{cm}^2, \quad M = 1.56 \text{ n.m.}$$

We might obtain better agreement by trying various possible combinations, but we did not try such cumbersome calculations.

For Li^7 exchange magnetic moment has non-vanishing value, which we have also calculated. We take for exchange magnetic moment,

$$M_x = b \sum_{ij} \int (\psi, O_{ij} \psi) d\tau$$

as Spruch¹⁰⁾ did, where

$$O_{ij} = \frac{i}{\alpha_0^2 J_0} (\mathbf{r}_i \times \mathbf{r}_j)_z (\boldsymbol{\tau}_i \times \boldsymbol{\tau}_j)_z J_{ij} P_{ij},$$

$$b = \frac{J_0 \alpha_0^2 M}{\hbar^2}.$$

Here P_{ij} is the space exchange operator between i and j 'th particle, $J_{ij} P_{ij}$ is the space exchange interaction. \mathbf{r}_i is the distance of the nucleon i from center of gravity and α_0 and J_0 are range and strength of the space exchange interaction. If we take the same numerical values for J_0 , and α_0 and the same potential J_{ij} and radial wave function as Spruch, then the result for the state A are

$$M_x = -0.05 \text{ n.m.}$$

If we add this value to the one we obtained before, we have worse agreement. It is, however, small enough to be neglected,

§ 4. Conclusions

The above calculations reveal that the configuration interaction is necessary from the standpoint of j - j coupling shell model, in order to obtain consistent values of quadrupole and magnetic moments of light nuclei. This situation might

seem to introduce complexities into the shell structure theory of atomic nuclei, but it is quite natural for closer investigations of properties of atomic nuclei since it is evident that the shell model is not a complete picture of nuclei. However, neglects of the configuration interaction across the magic numbers seem justifiable by considering the analysis of quadrupole and magnetic moments of B^{10} in comparison with Li^6 and N^{14} . Thus, we may accept the shells and magic numbers as useful notions in nuclear structure, but the so-called sub-shells have not such definite meaning. Furthermore, we should notice that the charge multiplicity T is not a good quantum number for Li^7 and Cl^{35} as is seen from Table IV for the consistent interpretation of quadrupole with magnetic moment.

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On Solutions of New Field Equations of Einstein and Those of Schrödinger

Hyôitirô TAKENO, Mineo IKEDA and Shingo ABE

(Research Institute for Theoretical Physics, Hiroshima University;
Takehara-machi Hiroshima-ken)

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Recently Einstein has proposed a new unified theory and Schrödinger advanced it a step further. In this paper solutions of the field equations in these theories are searched for. Under the assumption of spherical symmetry, static solutions of their field equations are obtained, and it is shown that the exact solutions of Papapetrou are included in them. It is usually considered that affinity of the space-time is determined uniquely from the equation $g_{ij};_k=0$ for a given g_{ij} , but it is not necessarily the case. This circumstance is made clear and further some investigations are made with respect to the necessary and sufficient condition that affinity can be determined uniquely.

§ 1. Introduction

Einstein has proposed a new unified field theory as a direct and natural generalization of his general relativity.¹⁾ In the new theory two basic generalizations are introduced; that is, non-symmetric fundamental tensor g_{ij} and non-symmetric affinity I_{ij}^k . Advancing a step further, Schrödinger constructed a purely affine theory.²⁾ If one puts the 'cosmological constant' $\lambda=0$ in Schrödinger's field equations, then Einstein's ones are obtained at once. Papapetrou obtained some exact solutions of Schrödinger's equations and investigated their physical meanings.³⁾ On the other hand Straus discussed the existence of the meaningful solutions of Einstein's equations.⁴⁾

In this paper we shall obtain new spherically symmetric static solutions of these field equations and discuss the relations between these solutions and those of Papapetrou.

We shall adopt the notations used by Einstein. Then Schrödinger's field equations are given as follows:

$$[E_1] \quad g_{ik;l} \equiv g_{ik,l} - g_{sk} I_{il}^s - g_{is} I_{lk}^s = 0,$$

$$[E_2] \quad I_{ii}^i = 0,$$

$$[E_3] \quad R_{ik} + \lambda g_{ik} = 0,$$

$$[E_4] \quad (R_{ik,i} + \lambda g_{ik,i}) + (R_{\underset{\vee}{kl},i} + \lambda g_{\underset{\vee}{kl},i}) + (R_{\underset{\vee}{i},k} + \lambda g_{\underset{\vee}{i},k}) = 0,$$

where

$$R_{kl} = \Gamma_{kl,s}^s - \Gamma_{ks,l}^s - \Gamma_{ll}^s \Gamma_{ks}^l + \Gamma_{ts}^s \Gamma_{kl}^t. \quad (1.1)$$

To solve these equations, we shall first assume that the non-symmetric fundamental tensor g_{ij} is spherically symmetric; that is form-invariant under the group of 3-dimensional rotations. Further we consider that g_{ij} is of $(-, -, -, +)$ -type as used in general relativity. Such a tensor is known to be reducible to

$$(g_{ij}) = \begin{bmatrix} -A & 0 & 0 & f \\ 0 & -B & h \sin \theta & 0 \\ 0 & -h \sin \theta & -B \sin^2 \theta & 0 \\ -f & 0 & 0 & C \end{bmatrix} \quad (1.2)$$

and accordingly

$$g \equiv \det. g_{ij} = LM \sin^2 \theta, \quad (L \equiv f^2 - AC, \quad M \equiv B^2 + h^2), \quad (1.3)$$

by choosing a suitable coordinate system, where A, B, C, f and h are arbitrary functions of r and t satisfying the conditions $A, B, C > 0$, and $L \neq 0$.

In the following sections we shall solve $[E_1]$, $[E_2]$, $[E_3]$ and $[E_4]$ by using (1.2).

§ 2. Solutions of $[E_1]$

In general it is hard to solve $[E_1]$ in terms of Γ_{ij}^k for an arbitrarily given g_{ij} , but under the assumption (1.2) we can solve as follows.

If we substitute (1.2) into $[E_1]$, we have 64 algebraic linear equations with respect to Γ_{ij}^k , which can be reduced to five sets of simultaneous equations as follows:

4 equations, each one of which contains only one unknown quantity:

$$p_{44}^i \text{ (not summed),} \quad (2.1)$$

$$6 \text{ equations for 6 unknowns: } p_{11}^4, p_{14}^1, p_{14}^4, p_{44}^1; q_{14}^1, q_{14}^4, \quad (2.2)$$

$$6 \text{ equations for 6 unknowns: } p_{22}^3, p_{23}^2, p_{23}^3, p_{33}^2; q_{23}^2, q_{23}^3, \quad (2.3)$$

$$24 \text{ equations for 24 unknowns: } p_{11}^2, p_{11}^3, p_{12}^1, p_{12}^4, p_{13}^1, p_{13}^4, p_{14}^2, p_{14}^3, p_{24}^1, p_{24}^4, p_{34}^1, p_{34}^4, p_{44}^2, p_{44}^3, q_{12}^1, q_{12}^4, q_{13}^1, q_{13}^4, q_{14}^2, q_{14}^3, q_{24}^1, q_{24}^4, q_{34}^1, q_{34}^4, \quad (2.4)$$

$$24 \text{ equations for 24 unknowns: } p_{12}^2, p_{12}^3, p_{13}^2, p_{13}^3, p_{22}^1, p_{22}^4, p_{23}^1, p_{23}^4, p_{24}^2, p_{24}^3, p_{33}^1, p_{33}^4, p_{34}^2, p_{34}^3, q_{12}^2, q_{12}^3, q_{13}^2, q_{13}^3, q_{23}^1, q_{23}^4, q_{24}^2, q_{24}^3, q_{34}^2, q_{34}^3, \quad (2.5)$$

where we have put

$$\Gamma_{ij}^k = p_{ij}^k, \quad \Gamma_{ij}^k = q_{ij}^k \text{ and } (r, \theta, \varphi, t) \equiv (x^1, x^2, x^3, x^4). \quad (2.6)$$

From (2.1), (2.2) and (2.3), we have

$$p_{11}^1 = A'/2A, \quad p_{22}^2 = p_{33}^3 = 0, \quad p_{44}^4 = \dot{C}/2C, \quad (2.7)$$

$$\begin{aligned}
 p_{11}^4 &= \frac{1}{CL} \left\{ 2Af \left(\dot{f} - \frac{f\dot{C}}{2C} \right) - \frac{\dot{A}}{2} (f^2 + AC) \right\}, \\
 p_{14}^4 &= \frac{1}{L} \left\{ f \left(\dot{f} - \frac{f\dot{C}}{2C} \right) - \frac{\dot{A}C}{2} \right\}, \\
 p_{14}^4 &= \frac{1}{L} \left\{ f \left(f' - \frac{fA'}{2A} \right) - \frac{AC'}{2} \right\}, \\
 p_{41}^4 &= \frac{1}{AL} \left\{ 2Cf \left(f' - \frac{fA'}{2A} \right) - \frac{C'}{2} (f^2 + AC) \right\}, \\
 q_{14}^4 &= \frac{1}{L} \left\{ C \left(f' - \frac{fA'}{2A} \right) - \frac{fC'}{2} \right\}, \\
 q_{14}^4 &= -\frac{1}{L} \left\{ A \left(\dot{f} - \frac{f\dot{C}}{2C} \right) + \frac{f\dot{A}}{2} \right\},
 \end{aligned} \tag{2.8}$$

$$p_{23}^3 = \cot \theta, \quad p_{33}^3 = -\sin \theta \cos \theta, \quad p_{22}^3 = p_{23}^2 = q_{23}^2 = q_{23}^3 = 0, \tag{2.9}$$

respectively, where dashes and dots indicate the derivatives with respect to r and t . (2.4) are 24 linear homogeneous equations for 24 unknowns. By solving them, we can find that all of these unknowns must be zero.

Lastly by solving (2.5), the following two types of solutions are obtained:

Type I. When $(B^2 = h^2 \text{ and } f=0)$ does not hold good.

In this case 24 unknowns are determined uniquely as follows:

$$\begin{aligned}
 p_{22}^1 &= (-BP + hQ)/A, \quad p_{33}^1 = \sin^2 \theta p_{22}^1, \quad p_{22}^4 = (B\bar{P} - h\bar{Q})/C, \quad p_{33}^4 = \sin^2 \theta p_{22}^4, \\
 p_{12}^2 &= p_{13}^3 = P, \quad p_{24}^2 = p_{34}^3 = \bar{P}, \quad p_{23}^1 = p_{23}^4 = 0, \\
 p_{12}^3 &= -f\bar{Q}/C \sin \theta, \quad p_{13}^2 = -\sin^2 \theta p_{12}^3, \quad p_{24}^3 = -fQ/A \sin \theta, \quad p_{34}^2 = -\sin^2 \theta p_{24}^3, \\
 q_{12}^2 &= q_{13}^3 = f\bar{P}/C, \quad q_{24}^2 = q_{34}^3 = -fP/A, \\
 q_{12}^3 &= -Q/\sin \theta, \quad q_{13}^2 = -\sin^2 \theta q_{12}^3, \quad q_{24}^3 = \bar{Q}/\sin \theta, \quad q_{34}^2 = -\sin^2 \theta q_{24}^3, \\
 q_{23}^1 &= (BQ + hP) \sin \theta/A, \quad q_{23}^4 = -(B\bar{Q} + h\bar{P}) \sin \theta/C,
 \end{aligned} \tag{2.10}$$

where

$$P = \frac{BB' + hh'}{2(B^2 + h^2)}, \quad \bar{P} = \frac{B\dot{B} + h\dot{h}}{2(B^2 + h^2)}, \quad Q = \frac{B'h - Bh'}{2(B^2 + h^2)}, \quad \bar{Q} = \frac{B\dot{h} - B\dot{h}}{2(B^2 + h^2)}. \tag{2.11}$$

Type II. When $B^2 = h^2$ i.e. $B = \epsilon h$, ($\epsilon^2 = 1$), and $f=0$. (2.12)

In this case 24 unknowns are determined to within 4 arbitrary functions as follows:

$$\begin{aligned}
p_{22}^1 &= (2a - B')/2A, \quad p_{33}^1 = -(2a + B') \sin^2 \theta / 2A, \quad p_{22}^4 = -(2\bar{a} - \dot{B})/2C, \\
p_{33}^4 &= (2\bar{a} + \dot{B}) \sin^2 \theta / 2C, \\
p_{12}^2 &= (B' - a)/2B, \quad p_{13}^3 = (B' + a)/2B, \quad p_{24}^2 = (\dot{B} - \bar{a})/2B, \quad p_{34}^3 = (\dot{B} + \bar{a})/2B, \\
p_{12}^3 &= -A\beta/2B \sin^2 \theta, \quad p_{13}^2 = \sin^2 \theta \, p_{12}^3, \quad p_{24}^3 = C\bar{\beta}/2B \sin^2 \theta, \quad p_{34}^2 = \sin^2 \theta \, p_{24}^3, \\
p_{23}^1 &= \beta, \quad p_{23}^4 = \bar{\beta}, \\
p_{12}^2 &= \varepsilon A\beta/2B \sin \theta, \quad q_{13}^3 = -q_{12}^2, \quad q_{24}^2 = \varepsilon C\bar{\beta}/2B \sin \theta, \quad q_{34}^3 = -q_{24}^2, \\
q_{12}^3 &= -\varepsilon a/2B \sin \theta, \quad q_{13}^2 = \sin^2 \theta \, q_{12}^3, \quad q_{24}^3 = \varepsilon \bar{a}/2B \sin \theta, \quad q_{34}^2 = \sin^2 \theta \, q_{24}^3, \\
p_{23}^1 &= \varepsilon B' \sin \theta / 2A, \quad q_{23}^4 = -\varepsilon \dot{B} \sin \theta / 2C,
\end{aligned} \tag{2.13}$$

where a , \bar{a} , β and $\bar{\beta}$ are arbitrary functions of r , θ , φ and t . As is easily seen, in (2.10) if we tend h^2 and f to B^2 and 0 respectively, we have expressions of the form (2.13), in which a , \bar{a} , β , $\bar{\beta}$ are replaced by zero.

In this case where (2.12) holds good, (2.8) are reducible to the following simpler forms:

$$p_{11}^4 = \dot{A}/2C, \quad p_{14}^1 = \dot{A}/2A, \quad p_{14}^4 = C'/2C, \quad p_{44}^1 = \frac{C'}{2A}; \quad q_{14}^1 = q_{14}^4 = 0. \tag{2.14}$$

The fact that the solutions of (2.5) are not determined uniquely indicates that (2.12) is the condition that the determinant of the coefficients of the linear equations (2.5) vanish. In fact, some elementary but troublesome calculations show that this determinant of degree 24 is given by

$$A^2 C^2 (B^2 + h^2)^4 \{ AC(B^2 - h^2)^2 + 4f^2 B^2 h^2 \}^2 \sin^{14} \theta \tag{2.15}$$

to within constant factors, and this vanishes under the condition (2.12).

Thus we have determined completely the forms of Γ_{ij}^k for the general spherically symmetric g_{ij} .

It is generally considered that R_{ij} obtained from spherically symmetric g_{ij} is also spherically symmetric,⁵⁾ but we shall show in § 4 that it is not always the case. It seems to us that Einstein and Schrödinger developed their theories, and Straus and Papapetrou obtained their solutions, by excluding such a case.

§ 3. Equations $[E_2]$ for both types of Γ_{ij}^k

For Γ_{ij}^k of the type I, $[E_2]$ are reducible to

$$2q_{12}^2 + q_{14}^4 = 0, \quad 2q_{42}^2 + q_{41}^4 = 0. \tag{3.1}$$

By substituting (2.10) into (3.1) and integrating the results, we have

$$f^2 (B^2 + h^2) / L = \text{const.} \tag{3.2}$$

That is to say, $[\mathbf{E}_2]$ is equivalent to (3.2) in this case. And by making use of this condition, (2.8) reduces to the following simpler forms:

$$\begin{aligned} p_{11}^4 &= \frac{\dot{A}}{2C} + \frac{4}{C^2} f^2 \bar{P}, \quad p_{14}^4 = \frac{\dot{A}}{2A} + \frac{2f^2}{AC} \bar{P}, \quad p_{14}^4 = \frac{C'}{2C} + \frac{2f^2}{AC} P, \\ p_{44}^4 &= \frac{C'}{2A} + \frac{4}{A^2} f^2 P, \quad q_{14}^4 = \frac{2f}{A} P, \quad q_{14}^4 = -\frac{2f}{C} \bar{P}. \end{aligned} \quad (3.3)$$

For Γ_{ij}^k of the type II, as is easily seen, $[\mathbf{E}_2]$ is identically satisfied.

§ 4. Equations $[\mathbf{E}_3]$

For Γ_{ij}^k of the type I we obtain by (1.1),

$$\begin{aligned} R_{11} &= p_{11,4}^1 - 2p_{12,1}^2 - p_{14,1}^4 + p_{11}^1 (2p_{12}^2 + p_{14}^4) + p_{11}^4 (-p_{14}^1 + 2p_{24}^2 + p_{44}^1) - 2(p_{12}^2)^2 \\ &\quad - p_{12}^3 p_{13}^3 - (p_{14}^4)^2 + 2(q_{12}^2)^2 - 2q_{21}^3 q_{13}^3 + (q_{14}^4)^2 = R_{11}(r, t), \\ R_{44} &= p_{44,1}^1 - 2p_{24,1}^2 - p_{14,4}^4 + p_{44}^1 (2p_{24}^2 + p_{14}^4) + p_{44}^4 (-p_{14}^1 + 2p_{12}^2 + p_{11}^1) - 2(p_{24}^2)^2 \\ &\quad - 2p_{24}^3 p_{34}^3 - (p_{14}^4)^2 + 2(q_{24}^2)^2 - 2q_{24}^3 q_{43}^3 + (q_{41}^4)^2 = R_{44}(r, t), \\ R_{22} &= p_{22,1}^1 + p_{22,4}^4 + 1 + p_{22}^1 (p_{11}^1 + p_{14}^4) + p_{22}^4 (p_{14}^1 + p_{44}^1) - 2q_{12}^3 q_{23}^3 - 2q_{42}^3 q_{23}^4 = R_{22}(r, t), \\ R_{33} &= \sin^2 \theta R_{22}, \\ R_{14} &= p_{14,1}^1 - p_{11,4}^1 - 2p_{12,4}^2 + q_{14,1}^1 + q_{14,4}^4 + 2p_{12}^2 p_{14}^1 + 2p_{24}^2 p_{14}^4 + p_{14}^1 p_{14}^4 - p_{11}^1 p_{11}^4 \\ &\quad - 2p_{12}^2 p_{24}^2 - 2p_{12}^3 p_{34}^3 - 2q_{12}^2 q_{24}^2 - 2q_{14}^4 q_{14}^1 - 2q_{12}^2 p_{24}^2 - 2q_{12}^3 p_{34}^3 \\ &\quad - 2p_{12}^2 q_{24}^2 - 2p_{12}^3 q_{34}^3 + 2p_{12}^2 q_{14}^1 + 2p_{24}^2 q_{14}^4 = R_{14}(r, t), \\ R_{41} &= p_{14,4}^1 - p_{44,1}^1 - 2p_{24,1}^2 + q_{41,1}^1 + q_{41,4}^4 + 2p_{12}^2 p_{14}^1 + 2p_{24}^2 p_{14}^4 + p_{14}^1 p_{14}^4 - p_{11}^1 p_{11}^4 \\ &\quad - 2p_{12}^2 p_{24}^2 - 2p_{12}^3 p_{34}^3 - 2q_{21}^2 q_{42}^2 - 2q_{21}^3 q_{43}^3 - q_{14}^1 q_{14}^4 - 2q_{21}^2 p_{24}^2 - 2q_{21}^3 p_{34}^3 \\ &\quad - 2p_{21}^2 q_{42}^2 - 2p_{12}^3 q_{43}^3 + 2p_{12}^2 q_{41}^1 + 2p_{24}^2 q_{41}^4 = R_{41}(r, t), \\ R_{23} &= q_{23,1}^1 + q_{23,4}^4 + 2p_{33}^1 q_{12}^3 - 2p_{33}^4 q_{24}^3 + (p_{11}^1 + p_{14}^4) q_{23}^1 + (p_{14}^1 + p_{44}^1) q_{23}^4 \\ &= N(r, t) \sin \theta, \\ R_{32} &= -R_{23}, \quad \text{other } R_{ij} = 0. \end{aligned} \quad (4.1)$$

Evidently this R_{ij} is spherically symmetric and $[\mathbf{E}_3]$ are reducible to the four conditions:

$$R_{11} - \lambda A = 0, \quad R_{22} - \lambda B = 0, \quad R_{44} + \lambda C = 0, \quad R_{14} + R_{41} = 0. \quad (4.2)$$

For Γ_{ij}^k of the type II the following expressions hold good.

$$\begin{aligned} R_{11} &= p_{11,4}^4 - (p_{12}^2 + p_{13}^3 + p_{14}^4)_{,1} + (p_{12}^2 + p_{13}^3 + p_{14}^4) p_{11}^1 + (p_{42}^2 + p_{43}^3 + p_{44}^4) p_{11}^4 \\ &\quad - (p_{12}^2)^2 - (p_{13}^3)^2 - (p_{14}^4)^2 - p_{11}^4 p_{14}^1 - 2p_{12}^3 p_{13}^3 + 2(q_{12}^2)^2 + 2q_{12}^3 q_{13}^3, \end{aligned}$$

$$\begin{aligned}
R_{14} &= p_{14,1}^1 - (p_{11}^1 + p_{12}^2 + p_{13}^3)_{,4} + (p_{11}^1 + p_{12}^2 + p_{13}^3) p_{14}^4 + (p_{12}^2 + p_{13}^3) p_{14}^1 - p_{41}^1 p_{11}^4 \\
&\quad - p_{12}^2 p_{24}^4 - p_{13}^3 p_{34}^4 - p_{12}^2 p_{34}^2 - p_{13}^3 p_{24}^2 - q_{12}^2 q_{24}^2 - q_{13}^3 q_{34}^2 - q_{12}^2 q_{34}^2 - q_{13}^3 q_{24}^2 \\
&\quad - p_{12}^2 q_{24}^2 - p_{13}^3 q_{34}^2 - p_{12}^2 q_{34}^2 - p_{13}^3 q_{24}^2 - q_{12}^2 p_{24}^2 - q_{13}^3 p_{34}^2 - q_{12}^2 p_{34}^2 - q_{13}^3 p_{24}^2, \\
R_{22} &= p_{22,1}^1 + p_{22,4}^4 + (p_{11}^1 - p_{12}^2 + p_{13}^3 + p_{14}^4) p_{22}^1 + (p_{41}^1 - p_{42}^2 + p_{43}^3 + p_{44}^4) p_{22}^4 \\
&\quad - 2p_{12}^2 p_{23}^1 - 2p_{42}^2 p_{23}^4 - 2q_{12}^2 q_{23}^1 - 2q_{42}^2 q_{23}^4 + 1, \\
R_{12} &= [p_{12,2}^2 + p_{12,3}^3 + p_{23}^3 (p_{12}^2 - p_{13}^3)] + [q_{12,2}^2 + q_{12,3}^3 + p_{23}^3 (q_{12}^2 - q_{13}^3)], \\
R_{21} &= [\quad \quad \quad] - [\quad \quad \quad], \\
R_{13} &= [p_{13,2}^2 + p_{13,3}^3 - p_{33}^3 p_{12}^2] + [q_{13,2}^2 + q_{13,3}^3 - p_{33}^3 q_{12}^2], \\
R_{31} &= [\quad \quad \quad] - [\quad \quad \quad], \\
R_{23} &= [p_{23,1}^1 + p_{23,4}^4 - p_{22}^2 p_{13}^2 - p_{22}^4 p_{43}^2 - p_{33}^3 p_{21}^1 - p_{33}^3 p_{21}^4 + p_{23}^1 (p_{11}^1 + p_{14}^4) + p_{23}^4 (p_{14}^1 + p_{11}^4) \\
&\quad + q_{23}^1 (q_{12}^2 - q_{13}^3) + q_{23}^4 (q_{12}^2 - q_{13}^3)] + [q_{13,1}^1 + q_{13,4}^4 - p_{22}^2 q_{13}^2 - p_{22}^4 q_{43}^2 - p_{33}^3 q_{21}^1 - p_{33}^3 q_{21}^4 \\
&\quad + q_{23}^1 (p_{11}^1 + p_{14}^4) + q_{23}^4 (p_{14}^1 + p_{11}^4) + p_{23}^1 (q_{12}^2 - q_{13}^3) + p_{23}^4 (q_{12}^2 - q_{13}^3)], \\
R_{32} &= [\quad \quad \quad] - [\quad \quad \quad].
\end{aligned} \tag{4.3}$$

R_{44} , R_{41} , R_{42} , R_{24} , R_{43} and R_{34} are obtained from R_{11} , R_{14} , R_{12} , R_{21} , R_{13} and R_{31} respectively by interchanging the indices 1 and 4 throughout, and similarly $(R_{33} - \sin^2 \theta)$ is obtained from $(R_{22} - 1)$ by interchanging 2 and 3 throughout.

From the result above obtained it is clear that R_{ij} is not necessarily spherically symmetric, although g_{ij} is so.

By (4.3), $[\mathbf{E}_3]$ becomes

$$\begin{aligned}
R_{11} - \lambda A = 0, \quad R_{44} + \lambda C = 0, \quad R_{22} - \lambda B = 0, \quad R_{33} - \lambda B \sin^2 \theta = 0, \\
R_{23} + R_{32} = 0, \quad R_{14} + R_{41} = 0,
\end{aligned} \tag{4.4}$$

and

$$R_{12} = R_{21} = R_{13} = R_{31} = R_{24} = R_{42} = R_{34} = R_{43} = 0. \tag{4.5}$$

Further rewriting (4.5), we have

$$\begin{aligned}
a_{,2} + 2a \cot \theta + A\beta_{,3}/\sin^2 \theta = 0, \quad \bar{a}_{,2} + 2\bar{a} \cot \theta - C\bar{\beta}_{,3}/\sin^2 \theta = 0, \\
a_{,3} - A\beta_{,2} - A\beta \cot \theta = 0, \quad \bar{a}_{,3} + C\bar{\beta}_{,2} + C\bar{\beta} \cot \theta = 0.
\end{aligned} \tag{4.6}$$

§ 5. Equations $[\mathbf{E}_4]$

For Γ_{ij}^k of the type I, $[\mathbf{E}_4]$ are

$$R_{23,1} + \lambda g_{23,1} = 0, \quad R_{23,4} + \lambda g_{23,4} = 0, \tag{5.1}$$

and by integrating them we obtain

$$N(r, t) + \lambda h = \text{const.}, \tag{5.2}$$

where N is defined by (4.1).

For Γ_{ij}^k of the type II $[\mathbf{E}_4]$ becomes

$$R_{14,2}=R_{14,3}=0, \quad R_{23,1}+\lambda g_{23,1}=R_{23,4}+\lambda g_{23,4}=0. \quad (5.3)$$

Further by using (2.14) and (4.3) we have

$$\begin{aligned} & (A\bar{a}\beta + C\bar{a}\bar{\beta})/\sin\theta = \text{function of } r \text{ and } t, \\ & \left(\frac{B'}{2A}\right)' - \left(\frac{\dot{B}}{2C}\right)' + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{C'}{C}\right) - \frac{\dot{B}}{4C}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) + \frac{A\beta^2 - C\bar{\beta}^2}{B \sin^2 \theta} + \frac{a^2}{AB} - \frac{\bar{a}^2}{BC} + \lambda B \\ & \quad = \text{function of } \theta \text{ and } \varphi. \end{aligned} \quad (5.4)$$

§ 6. Approximately Minkowskian static solution for $\lambda=0$

In this section we shall deal with the case $\lambda=0$, that is, Einstein's field equations. First we assume that the space-time is static, of approximately Minkowskian type and $B=r^2$. Then Γ_{ij}^k must be of the type I. By the above assumptions we shall take

$$A=1+\bar{A}(r)m, \quad B=r^2, \quad C=1+\bar{C}(r)m, \quad h=\bar{h}(r)m, \quad f=\bar{f}(r)m, \quad (6.1)$$

where m is an infinitesimal parameter of the first order.

From (4.1),

$$R_{11}=(-\bar{C}''/2+\bar{A}'/r)m+\dots, \quad R_{22}=\{(\bar{A}'-\bar{C}')r/2+\bar{A}\}m+\dots, \quad (6.2)$$

$$R_{23}=-R_{32}=(-\bar{h}''/2+\bar{h}'/r)\sin\theta m+\dots, \quad R_{44}=(\bar{C}''/2+\bar{C}'/r)m+\dots,$$

where the terms not written are of the second and higher orders with respect to m . Substituting these expressions into (3.2), (4.2) and (5.2), we have an *approximate solution*:

$$A=1+k_1m/r, \quad C=1+(k_2-k_1/r)m, \quad (6.3)$$

$$f=k_3m/r^2, \quad h=(k_4r^2+k_5r^3+k_6)m,$$

where k_1, \dots, k_6 are arbitrary constants. By a suitable change of the scale of time t , k_2 in (6.3) may be replaced by zero.

§ 7. Exact static solution for Γ_{ij}^k of the type I

We shall put the following assumptions.

$[\mathbf{A}_1]$ A, C, f and h are all static in the coordinate system for which $B=r^2$. Since both g_{ij} and g_{ij}^* are tensors separately, it is obvious that we can take a coordinate system for which $B=r^2$ by a suitable transformation of r and $t^{(6)}$ (by which the character of the type I is conserved). Without this assumption $[\mathbf{A}_1]$ our field equations are too complicated to solve.

[A₂] $h=kr^2$ in the coordinate system for which $B=r^2$, where k is an arbitrary constant. But we must take $f \neq 0$ in the case of $k^2=1$, for we are dealing with the solutions of the type I. The results of the last section suggest us to take this assumption [A₂], without which we shall feel much difficulty in solving our field equations.

From the above assumptions the first three equations of (4.2) become

$$\begin{aligned} -\left(\frac{C'}{2C} + \frac{2f^2}{ACr} + \frac{2}{r}\right)' + \frac{A'}{2A}\left(\frac{C'}{2C} + \frac{2f^2}{ACr} + \frac{2}{r}\right) - \left(\frac{C'}{2C} + \frac{2f^2}{ACr}\right)^2 - \frac{2}{r^2} - \lambda A &= 0, \\ \left(\frac{C'}{2A} + \frac{4f^2}{A^2r}\right)' + \left(\frac{C'}{2A} + \frac{4f^2}{A^2r}\right)\left(\frac{A'}{2A} - \frac{C'}{2C} - \frac{2f^2}{ACr} + \frac{2}{r}\right) + \frac{6f^2}{A^2r^2} + \lambda C &= 0, \quad (7.1) \\ \frac{A'}{A} - \frac{C'}{C} - \frac{4f^2}{ACr} - \frac{2}{r} + \frac{2A}{r} - 2\lambda Ar &= 0, \end{aligned}$$

and the last becomes an identity. By putting the constant in the right-hand member of (3.2) to be $-k_1(k^2+1)$ and integrating (7.1), we obtain at once

$$A^{-1} = 1 - \frac{2m}{r} - \frac{\lambda}{3}r^2, \quad AC = a\left(1 + \frac{k_1}{r^4}\right), \quad f^2 = \frac{ak_1}{r^4}, \quad (7.2)$$

where a and m are integral constants. Conversely, it is obvious that these results satisfy the field equations. Then we have:

Theorem Under the assumptions [A₁] and [A₂], the general solutions of the type I of Schrödinger's field equations are given by

$$\begin{aligned} A^{-1} &= 1 - \frac{2m}{r} - \frac{\lambda}{3}r^2, \quad B = r^2, \quad C = a\left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)\left(1 + \frac{k_1}{r^4}\right), \\ h &= kr^2, \quad f^2 = ak_1/r^4, \end{aligned} \quad (7.3)$$

where k_1 , a and m are integral constants.

In the case of $a > 0$, we can take $a=1$ in (7.3) by a transformation of time-scale: $t' = \sqrt{a}t$.

By putting $k=0$ and $k_1=0$ in the above solution, we can obtain the solutions of the case I and II of Papapetrou respectively, and by putting $\lambda=0$, the solution of Einstein's field equations.

The non-vanishing components of $R_{ij} + \lambda g_{ij}$ which correspond to (7.3) are

$$R_{\sqrt{V}\sqrt{V}} + \lambda g_{\sqrt{V}\sqrt{V}} = \frac{g_{\sqrt{V}\sqrt{V}}}{r^2}, \quad R_{\sqrt{V}\sqrt{V}} + \lambda g_{\sqrt{V}\sqrt{V}} = g_{\sqrt{V}\sqrt{V}}\left(\frac{4m}{r^3} - \frac{\lambda}{3}\right). \quad (7.4)$$

§ 8. Exact static solution for Γ_{ij}^k of the type II

The properties of the type II are invariant under an arbitrary transformation of r and t . Therefore as in the last section we shall put the following assumption

tions.

[B₁] A and C are both static in the coordinate system for which $B=r^2$.

[B₂] α , $\bar{\alpha}$, β and $\bar{\beta}$ are functions of r and θ . This condition [B₂] is taken by analogy with the Christoffel symbols of the static spherically symmetric space-time in general relativity.

Under these assumptions the first five equations of (4.4) are reduced to

$$\begin{aligned} -\frac{C''}{2C} + \frac{C'}{4C} \left(\frac{A'}{A} + \frac{C'}{C} \right) + \frac{A'}{rA} - \lambda A &= 0, \\ \frac{C''}{2A} - \frac{C'}{4A} \left(\frac{A'}{A} + \frac{C'}{C} \right) + \frac{C'}{rA} + \lambda C &= 0, \\ \left(\frac{a-r}{A} \right)' + \frac{A\beta^2 - C\bar{\beta}^2}{r^2 \sin^2 \theta} + \frac{a-r}{A} \left(\frac{A'}{2A} + \frac{C'}{2C} + \frac{a}{r^2} \right) - \frac{\bar{a}^2}{r^2 C} + \frac{a}{rA} + 1 - \lambda r^2 &= 0, \\ -\left(\frac{a+r}{A} \right)' + \frac{A\beta^2 - C\bar{\beta}^2}{r^2 \sin^2 \theta} - \frac{a+r}{A} \left(\frac{A'}{2A} + \frac{C'}{2C} - \frac{a}{r^2} \right) - \frac{\bar{a}^2}{r^2 C} - \frac{a}{rA} + 1 - \lambda r^2 &= 0, \\ 2\beta' + \beta \left(\frac{A'}{A} + \frac{C'}{C} \right) &= 0, \end{aligned} \quad (8.1)$$

and the last becomes an identity. From (4.6), we have

$$a = u(r)/\sin^2 \theta, \quad \bar{a} = v(r)/\sin^2 \theta, \quad \beta = p(r)/\sin \theta, \quad \bar{\beta} = q(r)/\sin \theta, \quad (8.2)$$

where u , v , p and q are arbitrary functions of r . From these relations and (5.4) we can easily show:

Theorem Under the assumptions [B₁] and [B₂], the general solutions of the type II of Schrödinger's field equations are given by the following formulas to within a transformation of the time-scale, $t = a\bar{t}$:

$$\begin{aligned} A^{-1} = C = 1 - \frac{2m}{r} - \frac{\lambda}{3} r^2, \quad B = r^2, \quad h = \epsilon r^2, \quad f = 0, \\ a = k_2 A / \sin^2 \theta, \quad \bar{a} = -\epsilon k_2 / \sin^2 \theta, \quad \beta = k_3 / \sin \theta, \quad \bar{\beta} = \epsilon k_3 A / \sin \theta, \\ (\epsilon^2 = 1), \end{aligned} \quad (8.3)$$

where k_2 and k_3 are arbitrary constants.

In this case all components of $R_{ij} + \lambda g_{ij}$ vanish except

$$R_{23} + \lambda g_{23} = g_{23} / r^2 = \epsilon \sin \theta. \quad (8.4)$$

The expression obtained from (8.3) by putting $k_2 = k_3 = 0$ and (8.4) coincide with those obtained from (7.3) and (7.4) by putting $k \rightarrow \epsilon$, $k_1 \rightarrow 0$ and $a = 1$, respectively.

§ 9. Condition that $[E_1]$ can be solved uniquely

As is easily seen, if we put $g_{ij} = l_{ij}$ and $g_{ij} = f_{ij}$ (where $\det. l_{ij} \neq 0$), we have

$$g = \underline{g} + \underset{\vee}{g} + \underline{g} l^{ij} l^{jm} f_{ij} f_{lm} / 2, \quad (9.1)$$

where l^{ij} is the conjugate tensor of l_{ij} , and \underline{g} and $\underset{\vee}{g}$ are determinants of l_{ij} and f_{ij} respectively. Further let g^{ij} be the conjugate tensor of g_{ij} , that is,

$$g_{ij} g^{kj} = g_{ji} g^{jk} = \delta_i^k, \quad (9.2)$$

then there holds the relation

$$g g^{ab} = \underline{g} \left\{ l^{ab} + f^{ab} + \frac{\rho}{2} \epsilon^{ijab} f_{ij} + \frac{1}{4} \epsilon^{ijpa} \epsilon_{imp}^{\cdot b} f_{ij} f^{lm} \right\}, \quad (9.3)$$

between g^{ij} and l^{ij} , where both ϵ_{ijlm} and ϵ^{ijlm} are ϵ -tensors with respect to l_{ij} (not tensor density), $\rho = \epsilon^{ijlm} f_{ij} f_{lm} / 8$, i.e. $\rho^2 = \underset{\vee}{g}$, and the upper indices in the right-hand member have been obtained by using l^{ij} . (9.3) may be rewritten in the form:⁷⁾

$$g g^{ab} = \underline{g} \left\{ l^{ab} \left(1 + \frac{1}{2} f^{lm} f_{lm} \right) - f^{am} f_m^b + f^{ab} + \frac{1}{2} \epsilon^{abij} f_{ij} \right\}. \quad (9.4)$$

Next we shall consider about the condition that Γ_{ij}^k be determined uniquely from $[E_1]$. From the symmetric and skew-symmetric parts of $[E_1]$ with respect to i and k , we have

$$\begin{aligned} p_{ki}^i &= \{_{ki}^i \}_h - l^{ai} (f_{ka} q_{ia}^o + f_{ia} q_{ka}^o), \\ q_{ki}^i &= ({}^i_{ki})_f - l^{ai} (f_{ka} p_{ia}^o - f_{ia} p_{ka}^o), \end{aligned} \quad (9.5)$$

respectively, where $\{_{ki}^i \}_h$ are the Christoffel symbols with respect to l_{ij} and

$$({}^i_{ki})_f = \frac{1}{2} l^{ai} (f_{ka, i} - f_{ia, k} + f_{ki, a}) = -({}^i_{ik})_f. \quad (9.6)$$

Eliminating p or q from (9.5), we obtain

$$q_{is}^i N_{k'it}^{lsi} = ({}^i_{kh})_f - l^{ti} [f_{ks} \{_{ht}^i \}_h - f_{hs} \{_{ki}^i \}_h],$$

or

$$p_{is}^i \bar{N}_{k'it}^{lsi} = \{_{kh}^i \}_h - l^{ti} [f_{ks} ({}^o_{ht})_f + f_{hs} ({}^o_{ki})_f],$$

where

$$\begin{aligned} N_{k'it}^{lsi} &\equiv \partial_{[k}^i \partial_{h]}^s \partial_{t'}^i - l^{it'} (f_k^{[s} f_{ht'} - f_h^{[s} f_{kt'}]) + (\delta_k^{[t} f_k^{s]} - \delta_k^{[t} f_h^{s]}) f_{t'}^i, \\ \bar{N}_{k'it}^{lsi} &\equiv \delta_{(k}^i \delta_{h)}^s \partial_{t'}^i - l^{it'} (f_k^{[s} f_{ht'} + f_h^{[s} f_{kt'}]) + (\delta_h^{[t} f_k^{s]} + \delta_h^{[t} f_h^{s]}) f_{t'}^i. \end{aligned} \quad (9.7)$$

Accordingly the necessary and sufficient condition that Γ_{ij}^k be uniquely determined from $[E_1]$ is given by

(24-24) $\det. N^{1st}_{kht} \neq 0$ or (40-40) $\det. \bar{N}^{1st}_{kht} \neq 0$. (9.8)

It is not easy to write these determinants in simple ordinary tensor expressions.

But if we assume the space-time to be spherically symmetric, we have the following simple results:

Theorem *Let g_{ij} be spherically symmetric, a necessary and sufficient condition that Γ^k_{ij} be not uniquely determined from $[E_1]$ is given by*

$$g=2g_{\underline{2}}, \quad g_{\underline{v}}=0. \tag{9.9}$$

where $g \neq 0$, $g_{\underline{v}} \neq 0$ and g_{ij} is assumed to be of the type $(-,-,-,+)$. Further, these conditions can be rewritten in the following equivalent ones:

$$f^{im} f_{im}=2, \quad \epsilon^{ijkl} f_{ij} f_{lm}=0. \tag{9.10}$$

The proof is obvious from the above results.

Summary

We searched for the solutions of the new field equations proposed by Einstein and Schrödinger respectively. First of all we assumed that g_{ij} is spherically symmetric (§ 1), for which the general solutions of $[E_1]$ are determined (§ 2). Putting some further assumptions, we got a spherically symmetric static approximate solution:

$$(g_{ij}) = \begin{pmatrix} -1-k_1m/r, & 0, & 0, & k_3m/r^2 \\ 0, & -r^2, & (k_4r^2+k_5r^3+k_6)m \sin \theta, & 0 \\ 0, & -(k_4r^2+k_5r^3+k_6)m \sin \theta, & -r^2 \sin^2 \theta, & 0 \\ -k_3m/r^2, & 0, & 0, & 1+(k_2-k_1/r)m \end{pmatrix},$$

where m is an infinitesimal parameter, in the Einstein's case. Next we also obtained exact static solutions in Schrödinger's theory under some other additional assumptions (§§ 7 and 8). g_{ij} of these solutions are given by

$$(g_{ij}) = \begin{pmatrix} -1/\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right) & 0, & 0, & \epsilon \sqrt{ak_1}/r^2 \\ 0, & -r^2, & kr^2 \sin \theta, & 0 \\ 0, & -kr^2 \sin \theta, & -r^2 \sin^2 \theta, & 0 \\ -\epsilon \sqrt{ak_1}/r^2, & 0, & 0, & a\left(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\right)\left(1+\frac{k_1}{r^4}\right) \end{pmatrix}$$

Two kinds of exact solutions of Papapetrou are given by putting $k=0$ and $k_1=0$ respectively.

It is usually considered that equations $[E_1]$ are solved uniquely with respect to Γ^k_{ij} , but it is not necessarily the case. This circumstance was made clear in

this paper, and at the same time we investigated the necessary and sufficient condition that Γ_{ij}^k be determined uniquely.

The authors are very grateful to Prof. Y. Mimura for his valuable discussions.

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Note added in proof. After presenting this paper we received C. R. Acad. Sci. Paris, **231** (1950), where Tonnelat's paper concerning the general solution of $[\mathbf{E}_1]$ is appeared (p. 487). But her condition $\sqrt{g} \neq 0$ seems to be too stringent.

Production of Charged π -Meson by γ -Ray —Higher Order Corrections—

Ziro Koba, Tsuneyuki Kotani* and Shinzo Nakai

Department of Physics, Faculty of Science, Osaka University

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The effective cross section for the production of charged π -meson by γ -ray is calculated by the Lorentz-covariant perturbation method, including the effects of order $e^2 f^4 (e^2 g^4)$. The pseudoscalar meson field with both pseudoscalar and pseudovector coupling has been employed. Virtual mesons in intermediate states can be either charged or neutral.

The procedure of the renormalization of mass and mesic charge (f, g) is sufficient to determine unambiguously the finite fourth order matrix element except only one term to be interpreted as "electric charge of neutron," which we have struck out.

The general requirement of gauge invariance restricts the possible interaction of nucleon and meson with photon to only four types; consequently we have divided numerous Dyson diagrams into certain gauge invariant classes and then reduced them according to Dyson's general program, to obtain ultimately those four types.

In our second approximation the anomalous magnetic moment of the nucleon plays an essential part; consequently the lack of any correct analysis for the latter complicates our results considerably. Moreover, in our γ - π process the behaviour of the magnetic corrections, dependent on the incident energy of photon, are comparable with the static effects.

Three provisional measures have been taken with regard to the anomalous moments and the resulting values for the various combinations and couplings of meson fields are compared and discussed. Though the results depend upon the choice of these possibilities, it is shown, in general, that the fourth order contributions turn out for the energy region of the present experiments (incident X-ray being about 330 Mev.), not so large compared with the second order in the case of the symmetrical combination, while they are of the same order of magnitude with the second order in the case of the "charged plus pure neutral" combination.

Finally some qualitative aspects of the γ - π process are discussed, which might be valid independently of the order of approximation.

§ 1. Introduction and summary

Since the physicists at Berkeley made a successful experiment¹⁾⁻³⁾ of meson production by X-ray from the synchrotron, in 1949, this process aroused renewed interest among us⁴⁾⁻⁸⁾, because the artificial mesons, contrary to natural ones, provide us with fairly accurate and most suggestive informations. We have, for example, angular distribution, quantitative comparison of the yields of positive, negative and neutral mesons.⁹⁾⁻¹³⁾ Especially the use of a liquid hydrogen target¹¹⁾ has made it possible to set the experimental results directly against the theoretical predictions without being hindered by the complicated effects of nuclear binding. Further, the theoretical treatment is here less ambiguous than in the case of

* Now at Kobayashi Institute of Physical Research, Kokubunji, Tokyo.

nucleon-nucleon collision, since one of the colliding particles is the photon, the properties of which we know to some extent.

With regard to the reliability of the current meson theory most physicists agree: it is a good approximation within a certain energy region as long as the interaction of mesons with electromagnetic field is concerned;⁽¹⁾ it becomes dubious, however, when one has to do with mesons interacting with nucleons, and perhaps it would not be able to explain the experimental results quantitatively before some radical alteration is introduced into the theory.⁽¹⁵⁾

Thus the study of the electromagnetic properties of nucleon-meson system appears particularly interesting. Such study will include the problems concerning the anomalous magnetic moment* of nucleon, the meson creation by γ -ray, the capture of π -meson, the scattering of γ -ray by nucleon, and so on. The simplest of them, the a.m.m., which has not been yet explained satisfactorily,^{(16)–(23)} represents only static properties of the system, while our γ - π process depends essentially upon its dynamical behavior, so that we could expect to get a deeper insight into the nature of the nucleon-meson interaction by thoroughly investigating this photo-mesonic effect.

Various attacks^{(4)–(8), (24)–(32)} have been already made on this problem. We shall in the following try to evaluate the effective cross section for the process by the covariant perturbation method including higher order effects.

As is well known, the meson theory is characterized by its rather strong coupling with the nucleon field^{(24), (29)} and consequently the convergence of the perturbation series is quite questionable. Thus there is no authority for the expectation—analogueous to the quantum electrodynamics—that the first few terms of the series would yield the correct answer, though it might be still premature, on the other hand, to jump to the conclusion that here the perturbation method becomes entirely senseless. In any case one is not in general qualified to draw even rough conclusions before actually estimating—at least—the second approximation. In view of the provisional character of the current theory we can never be so optimistic as to anticipate that the higher order perturbation would immediately give us the predictions that agree well with experiments^{(10), (25)}; we suppose, however, that otherwise we could in general hardly find out virtues or defects of our present formalism and method. The recent success in the quantum electrodynamics, the renormalization method,^{(33), (34)} can be taken over into certain cases of the meson theory and enables us to carry out, —in principle, —such a higher order calculation.

The result of the fourth order perturbation for the pseudoscalar meson with the pseudoscalar coupling** is already included in Brueckner's excellent work on

* For short we shall write a.m.m. in place of anomalous magnetic moment throughout this paper.

** Hereafter the following abbreviations will be used: s.=scalar, p.s.=pseudoscalar, v.=vector, p.v.=pseudovector, p.s. (p.s.)=pseudoscalar meson with pseudoscalar coupling, p.s. (p.v.)=pseudoscalar meson with pseudovector coupling, and so on.

the $\gamma-\pi^\pm$ process.²⁴⁾ He relates his results, however, very briefly; and we hope it will be allowed for us to describe the methods and results of our calculation in more detail, because one could find in them some general aspects of higher order processes.

Our object, method and results. We have computed the effective cross section for the elementary process (γ -ray + nucleon \rightarrow nucleon + charged π -meson) to the order $e^2 f^4$.^{*} P.s. meson field, which is generally believed to be most promising^{**} and also has been rather successful in the second order computation for this transition,²⁴⁾ has been employed with both p.s. and p.v. coupling. Of course, virtual mesons in intermediate states can be either charged or neutral (in symmetrical formalism or "charged plus pure neutral" formalism).^{***}

The requirements of the over-all conservation of energy-momentum, and of the gauge invariance restrict the possible forms of the matrix element for the considered transition essentially to the following types:

- a) The photon is absorbed by the macroscopic electric current of the nucleon-meson system;
- b) The photon is absorbed by the internal spin electric current of the nucleon;
- c) The photon is absorbed by the mixed effect of spin and macroscopic currents.

The last type includes two independent terms, since the macroscopic current can be either the initial one or the final one. It is more convenient for our case, however, to take two suitable linear combinations of b) and c), instead of the two terms in c), because those correspond to the electromagnetic interaction of nucleons through their a.m.m. This statement is of general character and is valid independently of the order of approximation. Thus the estimation of higher order corrections consists in the more and more accurate computation of the coefficients to be multiplied into these four forms. These coefficients are of course functions of the energy of the incident photon and that of the emitted meson,

In carrying out the calculation we have followed the covariant formalism of Tomonaga-Schwinger-Feynman-Dyson and especially made use of Dyson's general scheme,³⁶⁾ reducing the higher order diagrams to those of the lower order, and, in view of the foregoing consideration, divided numerous diagrams into certain gauge-invariant classes.³⁷⁾ Such a procedure eliminates from the beginning those terms which turn out not gauge-invariant and are ultimately to be cancelled. Similarly the matrix elements concerned with p.v. coupling have been separated into terms that are equivalent to p.s. coupling and those that are not, and some

* The results of a similar calculation for the production of neutral meson by γ -ray carried out by S. Minami³²⁾ will be published shortly.

** i) The decay process of the neutral meson and ii) the π^+ -capture process by the deuterium are favourable to the meson having the zero spin. iii) The π^- -capture by deuterium excludes completely the scalar meson.³⁵⁾

*** In this paper we shall denote, for brevity's sake, these two cases by sym. and ch. + n. respectively.

of the latter terms have been combined suitably and reduced to simpler forms.

With regard to the applicability of Tomonaga-Schwinger's subtraction method, we have verified:

i) The renormalization procedure for the nucleon mass, the meson mass and the coupling constant of the meson with nucleon is sufficient for the evaluation of the observable effects of the fourth order γ - π transition.

ii) As an exception one correction term appears, which is not gauge-invariant and could be interpreted as "electric charge of neutron"²¹⁾ interacting with the photon. This term we have struck out according to the general argument concerning the ambiguities of the current field theory.³⁸⁾⁻⁴⁰⁾

As for the observable effects:

iii) The contribution of the fourth order quantities ($\sim e^2 f^4$ etc.) to the effective cross section is.....in the energy region of present experiments, incident X-ray being about 330 Mev.not much smaller, in general, than the second order ones ($\sim e^2 f^2$ etc.), because the coupling constant is to be chosen comparatively large: $f^2/(4\pi)^2$ (or $(2Mg/\mu)^2/(4\pi)^2$) ~ 1 .

iv) A characteristic feature of our second approximation is, as could have been expected, the intervention of the a.m.m.'s of the proton and of the neutron. Moreover, these a.m.m.'s include not only static effects but also non-static "corrections" dependent on the incident energy of the photon, and the latter are by no means small compared with the former. These circumstances prevent us from drawing a unique conclusion, because a straightforward evaluation of the fourth order cross section would be unsatisfactory, even if it should turn out rather small and appear to guarantee the convergence of the perturbation series, since the experimental a.m.m. cannot be explained field-theoretically (in the first approximation) by the p.s. (or any other single) meson. We could replace the static parts of the a.m.m. by their empirical values, but at present we have no reliable method to evaluate their non-static corrections.

v) We have therefore taken three provisional measures with regard to a.m.m.'s, namely (i) employed the empirical values for the static parts and neglected the non-static parts, (ii) employed calculated values for the static and non-static parts, or (iii) employed the empirical values for the static and calculated values for the non-static parts; and we have compared these results for a) sym. theory with p.s. coupling, b) sym. theory with p.v. coupling, c) ch.+n. theory with p.s. coupling, and d) ch.+n. theory with p.v. coupling. It has been found that the result is very sensitive to the choice from these twelve cases.

In the case a) the fourth-order corrections are not so large, and we have to choose $f^2/(4\pi)^2 \sim 2$ in order to explain the experiments. Besides, those corrections tend to increase the π^+ -production and to decrease the π^- -production. In the case a), i) one can most easily fit the calculated values to the empirical ones both for the absolute values and the negative-to-positive ratio of the cross-section. In the cases a), ii) and a), iii) the above-mentioned change in the cross-section ratio

is somewhat more conspicuous, owing to the non-static a.m.m. for iii) and incorrect static a.m.m. as well as non-static a.m.m. for ii). Thus in the latter two cases one cannot adjust the coupling constant to the observed cross-section ratio without departing from the empirical absolute value of the cross-sections. In the case b) the parts inequivalent to the p.s. coupling play an essential role, and, contrary to a), they reduce the π^+ -production and enlarge the π^- -production. Here, too, one cannot explain the absolute value and the ratio of the cross-section simultaneously. Only b), ii) becomes an exceptional case, due to the incorrect static a.m.m.'s; but this effect predominates only at large angles so that one would not obtain the required cross-section ratio at small or very large angles. In ch.+n. theory c) and d) the fourth order contributions turn out comparable with the second order, if one tries to explain the experimental data in this approximation. One can hardly justify oneself, therefore, in putting an end to the perturbation series at this order.

We are not qualified, thus, for comparing our fourth order results immediately with experiments. All we can say is that neither the first nor the second approximation can be relied upon quantitatively. All the same we could suppose that some of the qualitative features drawn from our calculation would be valid more or less independently of the order of approximation. Such arguments refer to the following points: i) the character of the dynamical corrections for the nucleon a.m.m. in the radiation field, ii) devices to supplement the values of the static a.m.m.'s, derived from the p.s. meson field in the first approximation, iii) comparison of various meson fields, and iv) the roles and significance of the four fundamental components of the matrix element for our γ - π transition.

§ 2. General scheme and second order matrix elements (*ef*)

In the covariant perturbation scheme the effective cross section for any transition process can be obtained by estimating the corresponding element of the S -matrix, which is, according to Dyson, expressed by

$$S \equiv U(\infty, -\infty) = \sum_{n=1}^{\infty} (-i)^n / n! \cdot \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_n \cdot P(H(x_1), H(x_2) \dots H(x_n)). \quad (2.1)$$

Throughout this paper we employ the natural unit $\hbar=c=1$; a thick letter, e.g. \boldsymbol{x} , means a four-vector, while a letter with an arrow, e.g. \vec{x} , represents a usual three-dimensional vector; $d\boldsymbol{x}$ stands for a four-dimensional volume element $d\boldsymbol{x} = dx_1 dx_2 dx_3 dx_0$.

$H(\boldsymbol{x})$ is the interaction Hamiltonian density function in the interaction representation and, since we are dealing with three kinds of interacting fields: nucleonic field $(\psi, \bar{\psi})$, electromagnetic field (A) , and pseudoscalar meson field,

charged (ϕ, ϕ^*) as well as neutral* (ϕ^0) , it is given by the following expression.**

$$H = H^{mn} + H^{ne} + H^{me} + H^{nme} + H_{\text{mass}} + H_{\text{coupl.}}, \quad (2.2)$$

with

$$\left. \begin{aligned} H^{mn} &= R^* \phi + R \phi^* + S_\mu \partial \phi^* / \partial x_\mu + S_\mu^* \partial \phi / \partial x_\mu + R^0 \phi^0 + S_\mu^0 \partial \phi^0 / \partial x_\mu, \\ R &= i f \bar{\psi} \gamma_5 \tau_{NP} \psi, \quad S_\mu = (i g / \mu) \bar{\psi} \gamma_5 \gamma_\mu \tau_{NP} \psi, \\ R^0 &= i f^0 \bar{\psi} \gamma_5 \tau_3 \psi, \quad S_\mu^0 = (i g^0 / \mu) \bar{\psi} \gamma_5 \gamma_\mu \tau_3 \psi, \end{aligned} \right\} \quad (2.3)$$

$$H^{ne} = -j_\mu A_\mu, \quad j_\mu = i e \bar{\psi} \gamma_\mu \tau_P \psi, \quad (2.4)$$

$$H^{me} = i e A_\mu (\phi^* \cdot \partial \phi / \partial x_\mu - \partial \phi^* / \partial x_\mu \cdot \phi), \quad (2.5)$$

$$H^{nme} = i e A_\mu (S_\mu \phi^* - S_\mu^* \phi). \quad (2.6)$$

The last two terms in (2.2), H_{mass} and $H_{\text{coupl.}}$ are so-called counter terms, which stand for the renormalization procedure of mass and coupling constant respectively. H_{mass} has the form; $-\delta M \bar{\psi} \psi - \delta M^0 \bar{\psi} \psi - \mu \delta \mu \phi^* \phi$, where δM (δM^0) and $\delta \mu$ are the corrections of the nucleon mass and the meson mass (self-energies) respectively.*** $H_{\text{coupl.}}$ is given if one replaces in R , R^0 , S_μ and S_μ^0 the parameters f , f^0 , g and g^0 by $-\delta f$, $-\delta f^0$, $-\delta g$ and $-\delta g^0$ respectively. The values of these corrections will be given later.

In H^{me} there is also a term quadratic in A , but we have omitted it since it has nothing to do with the process considered. Apart from this term the above expression for H is not perfect as a Hamiltonian and must be supplemented by certain terms which depend on the normal of the space-like surface, in order that the integrability condition may be satisfied. Nevertheless the above H is sufficient for the evaluation of the S -matrix, because one can suitably modify the definition of the P -symbol and then proceed as if there were no such normal-dependent terms.⁴¹⁾⁻⁴³⁾

First we recapitulate the results of the second order perturbation because our later arguments will be frequently referred to this first approximation.

For the process: $\gamma + P \rightarrow N + \pi^+$, we have the two diagrams in fig. (1) and fig. (2), from which we obtain**** the matrix element

$${}^{\text{II}}U^+ = (1 + g') ({}^{\text{II}}U_c^+ + {}^{\text{II}}U_m^+), \quad (2.8)$$

* Raised suffix 0 will always indicate a quantity which is referred to the neutral meson field.

** In R^0 and S^0 the τ_3 's are to be replaced by 1, if one deals with a pure neutral field.

*** δM^0 is the self-energy of nucleon due to neutral meson field and turns out slightly different from δM . (See § 4 [V].)

**** It is convenient to expand the wave functions following Fukuda and Takeda⁴²⁾. Then the meson wave function in the momentum space $\phi(q)$ becomes equal to unity, but we shall write it explicitly for the sake of clearness.

where*

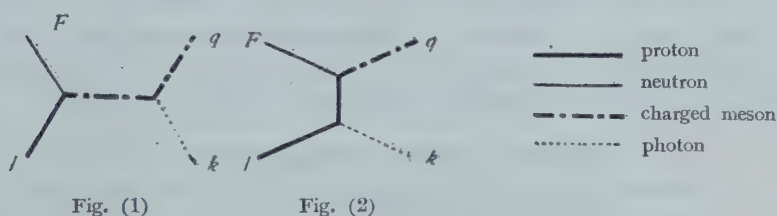
$$g' = 2Mg/\mu f, \quad (2.8)$$

$${}^{II}U_c^+ = C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}[iI_\mu/(\mathbf{k}\mathbf{I}) - iq_\mu/(\mathbf{k}\mathbf{Q})]\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(\mathbf{Q}), \quad (2.9)$$

$${}^{II}U_m^+ = C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}[\sigma_{\mu\nu}k_\nu/2(\mathbf{k}\mathbf{I})]\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(\mathbf{Q}), \quad (2.10)$$

with $C = -ef/4(2\pi)^2$, $\sigma_{\mu\nu} = (1/2i)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$,

and $(\mathbf{k}\mathbf{I}) = \vec{k}\vec{I} - k_0I_0$, etc.



Here the notation of Brueckner is followed: \mathbf{I} and \mathbf{F} are the initial and the final four-momentum of the nucleon respectively; \mathbf{k} represents the four-momentum of the incident photon, while \mathbf{Q} stands for that of the emitted charged π -meson.

It appears that ${}^{II}U_c^+$ exhibits characteristic features of the interaction of the photon with the convection electric current of the nucleon-meson system, and that ${}^{II}U_m^+$ represents something like the interaction of the photon with the "normal" magnetic moment of the proton. In the following this type of interaction will be provisionally called the "magnetic" interaction. (Cf. the discussions in § 3.) To verify the gauge-invariance of ${}^{II}U_c^+$ and ${}^{II}U_m^+$, one has only to substitute $ik_\mu A(\mathbf{k})$ for $A_\mu(\mathbf{k})$ in them and find that they do vanish.

For the process: $\gamma + N \rightarrow P + \pi^-$, the matrix element ${}^{II}U^-$ can be calculated in a similar way and we find it is connected with the above given ${}^{II}U^+$ by the relation

$${}^{II}U^- = \{ -(\mathbf{k}\mathbf{I})/(\mathbf{k}\mathbf{F}) \} {}^{II}U^+. \quad (2.11)$$

§ 3. Gauge invariance

As mentioned in § 1 one of the colliding particles in our γ - π process is the photon, which fact restricts the possible forms of interaction considerably because of the requirement of gauge invariance together with other general conditions, and so makes our theoretical treatment far less complicated than in other cases.

* It is convenient to decompose the matrix element into several parts. Each of these constituents will be called a "component."

The matrix element for the $\gamma-\pi$ process is a function of four energy-momenta \mathbf{I} , \mathbf{F} , \mathbf{k} and \mathbf{q} , one of which, however, can be eliminated by means of the over-all energy-momentum conservation $\mathbf{I}+\mathbf{k}=\mathbf{F}+\mathbf{q}$. We express \mathbf{q} , for example, in terms of \mathbf{I} , \mathbf{F} and \mathbf{k} . Then considering that any gauge invariant expression includes the vector potential A_μ necessarily in the form of field strength ($k_\nu A_\mu - k_\mu A_\nu$), the matrix element takes the form:

$$C\bar{\psi}(\mathbf{F})P_{\mu\nu}(\mathbf{k}, \mathbf{I}, \mathbf{F}, \gamma) \phi(\mathbf{I})A_\mu(\mathbf{k})k_\nu\phi^*(\mathbf{q}), \quad (3.1)$$

where the function $P_{\mu\nu}$ is antisymmetric with regard to the indices μ and ν .

First we shall investigate how $P_{\mu\nu}$ depends on γ -matrices. It has certainly a γ_5 as a factor since the emission of a pseudoscalar meson is involved, and is in general a polynomial in γ 's, containing arbitrary numbers of scalar factors $(\gamma\mathbf{I})$, $(\gamma\mathbf{F})$, and $(\gamma\mathbf{k})$. However, repeated application of the relations:

$$\left. \begin{aligned} (\gamma\mathbf{a})(\gamma\mathbf{b}) &= 2(\mathbf{ab}) - (\gamma\mathbf{b})(\gamma\mathbf{a}), \\ \gamma_\mu(\gamma\mathbf{a}) &= 2a_\mu - (\gamma\mathbf{a})\gamma_\mu, \\ \text{and } \gamma_5(\gamma\mathbf{a}) &= -(\gamma\mathbf{a})\gamma_5, \end{aligned} \right\} \quad (3.2)$$

allows us to transfer all $(\gamma\mathbf{F})$'s to the left and all $(\gamma\mathbf{I})$'s to the right. Then we notice that to the left of the expression stands $\bar{\psi}(\mathbf{F})$ and to the right $\phi(\mathbf{I})$, so that each of the above $(\gamma\mathbf{F})$'s and $(\gamma\mathbf{I})$'s can be replaced by iM according to the Dirac equation. Further, if the remaining expression for $P_{\mu\nu}k_\nu$ includes two or more $(\gamma\mathbf{k})$'s, it must vanish or practically vanish by the help of (3.2)* and the Lorentz condition $k_\mu A_\mu = 0$. Thus we conclude that γ 's in $P_{\mu\nu}k_\nu$ must be a linear combination of the following four forms:

$$\gamma_5, \gamma_5(\gamma\mathbf{k}), \gamma_5\gamma_\mu, \gamma_5\gamma_\mu(\gamma\mathbf{k}).$$

Taking this fact into consideration, we can determine the four fundamental antisymmetric expressions for $P_{\mu\nu}$;**

$$\text{i) } \gamma_5(I_\mu F_\nu - I_\nu F_\mu), \quad (3.3)$$

$$\text{ii) } \gamma_5(\gamma_\mu I_\nu - \gamma_\nu I_\mu), \quad (3.4)$$

$$\text{iii) } \gamma_5(\gamma_\mu F_\nu - \gamma_\nu F_\mu), \quad (3.5)$$

$$\text{iv) } \gamma_5(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \quad (3.6)$$

The six-vector (3.3) is proportional to the four-dimensional rotating current aroused by the change of the energy-momentum \mathbf{I} to \mathbf{F} , while (3.6) represents the rotating four-current due to the inner trembling motion of the fermion. As

* $(\gamma\mathbf{k})(\gamma\mathbf{k}) = k^2 = 0$, $\gamma_\mu(\gamma\mathbf{k}) = 2k_\mu - (\gamma\mathbf{k})\gamma_\mu$.

** We have discarded the combination $\gamma_5(X_\mu k_\nu - X_\nu k_\mu)$ since $k_\mu A_\mu = 0$ and $k_\nu k_\nu = 0$.

for (3.4) and (3.5) they stand for the mixed effects of the macroscopic and inner motions.

Any gauge-invariant expression for $P_{\mu\nu}$ can be deduced from these four by transforming, multiplying by arbitrary scalar functions of suitable dimension, independent of γ , and linearly combining them.* In (3.3), for example one can replace \mathbf{I} or \mathbf{F} by \mathbf{q} making use of the energy-momentum conservation and obtains

$$\gamma_5(F_\mu q_\nu - q_\mu F_\nu) \quad (3.3)'$$

$$\text{or} \quad \gamma_5(q_\mu I_\nu - I_\mu q_\nu), \quad (3.3)''$$

(3.3)' is convenient in the treatment of $\gamma-\pi^-$ process and (3.3)'' in the $\gamma-\pi^+$ process, since \mathbf{q} and \mathbf{F} or \mathbf{I} and \mathbf{q} are respectively proportional to the macroscopic current vectors which really carry the electric charge. It is to be noticed that the above considerations are of a quite general character. We have postulated only i) Lorentz invariance, ii) gauge invariance, iii) energy-momentum conservation, iv) p.s. field for the meson, and v) Dirac equation for the nucleon.** Thus our conclusion is independent of the method and order of approximation or the type of interaction. Our analysis can be, of course, applied *mutatis mutandis* to other processes involving a photon or photons.

The main parts (2.9) and (2.10) of the second order matrix element are (3.3)'' and (3.6) respectively, while that of (2.11) is given by (3.3)' and (3.6). In the next section we shall find that the main part of the fourth order matrix element is derived from (3.6) and (3.4) or (3.5). That is to say, in the evaluation of the fourth-order matrix element we shall not make use of (3.4) and (3.5) directly, but combine (3.4), for example, with (3.6) as follows:

$$\gamma_5(\gamma_\mu I_\nu - \gamma_\nu I_\mu) + M\gamma_5\sigma_{\mu\nu} = -(\mathbf{Ik})\gamma_5 S_{\mathbf{F}}(\mathbf{v})\sigma_{\mu\nu}, \quad (3.7)$$

* The four gauge invariant expressions obtained by interchanging the roles of the electric and the magnetic fields, namely $C\bar{\psi}(\mathbf{F})P'_{\mu\nu}\psi(\mathbf{I})\epsilon(\mu\nu\rho\sigma)A_\rho(\mathbf{k})k_\sigma\phi^*(\mathbf{q})$, with $P'_{\mu\nu}$ not containing γ_5 , which correspond to the diagrams involving closed loops of nucleon line and which will appear in the sixth or higher order, might at first seem independent of these fundamental four. In reality, however, the former can be expressed as certain linear combinations of the latter, though such transformation might not be always advantageous, as follows:

$$\begin{aligned} \epsilon(\mu\nu\lambda\rho)\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho &= 4!\gamma_5 \\ \epsilon(\mu\nu\lambda\rho)A_\mu\gamma_\nu\gamma_\lambda\gamma_\rho &= 3!(\boldsymbol{\tau A})\gamma_5 \\ \epsilon(\mu\nu\lambda\rho)A_\mu B_\nu\gamma_\lambda\gamma_\rho &= [(\boldsymbol{\tau B}), (\boldsymbol{\tau A})]\gamma_5 \\ \epsilon(\mu\nu\lambda\rho)A_\mu B_\nu C_\lambda\gamma_\rho &= (1/4)\{(\boldsymbol{\tau C}), [(\boldsymbol{\tau B}), (\boldsymbol{\tau A})]\}\gamma_5 \\ \epsilon(\mu\nu\lambda\rho)A_\mu B_\nu C_\lambda D_\rho &= (1/8)[(\boldsymbol{\tau D}), \{(\boldsymbol{\tau C}), [(\boldsymbol{\tau B}), (\boldsymbol{\tau A})]\}]\gamma_5 \end{aligned}$$

where

$$[X, Y] = XY - YX, \{X, Y\} = XY + YX.$$

We express our gratitude to Mr. Y. Watanabé and Mr. N. Mugibayashi for their valuable discussions and advices on this point.

** The last two assumptions can be further generalized.

$$\text{where} \quad S_F(\mathbf{v}) = (i\gamma\mathbf{v} - M) / (\mathbf{v}^2 + M^2) \quad (3.8)$$

$$\text{and} \quad \mathbf{v} = \mathbf{I} + \mathbf{k}. \quad (3.9)$$

In terms of the component matrix elements for the $\gamma - \pi^+$ process, the relation (3.7) is expressed as

$${}^{(1)}U_v^+ = {}^{(1)}U_m^+ - {}^{II}U_m^+, \quad (3.10)$$

where

$${}^{(1)}U_v^+ = C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}[(\gamma_\mu I_\nu - I_\mu\gamma_\nu)/2M(\mathbf{Ik})]\psi(\mathbf{I})A_\mu(\mathbf{k})k_\nu\phi^*(\mathbf{q}), \quad (3.11)$$

$${}^{(1)}U_m^+ = -C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}S_F(\mathbf{v})[\sigma_{\mu\nu}\tau_P/2M]\psi(\mathbf{I})A_\mu(\mathbf{k})k_\nu\phi^*(\mathbf{q}), \quad (3.12)$$

and ${}^{II}U_m^+$ is defined by (2.10). ${}^{(1)}U_m^+$ is easily seen just to represent a process in which the nucleon absorbs the photon with a Pauli-type interaction, going over into an intermediate state, and then emits the real meson. Replacing here the intermediate propagation function $S_F(\mathbf{v})/2M$ by the corresponding function for a scalar wave $1/2(\mathbf{Ik}) = 1/(\mathbf{v}^2 + M^2)$, one can formally obtain ${}^{II}U_m^+$, which have been referred to at the end of § 2 as the "magnetic" interaction. This would imply that the difference ${}^{(1)}U_v^+$ of these ${}^{(1)}U_m^+$ and ${}^{II}U_m^+$ stands for the effect of the nucleon spin in the intermediate state, and hence we may provisionally call ${}^{(1)}U_v^+$ "virtual spin effect."* In the same way as (3.7) one obtains from (3.5) another relation;

$${}^{(2)}U_v^+ = {}^{(2)}U_m^+ - {}^{II}U_m^+[-(\mathbf{Ik})/(\mathbf{Fk})], \quad (3.13)$$

where

$${}^{(2)}U_v^+ = C\bar{\psi}(\mathbf{F})[(\gamma_\mu F_\nu - \gamma_\nu F_\mu)/2M(\mathbf{Fk})]i\gamma_5\tau_{NP}\psi(\mathbf{I})A_\mu(\mathbf{k})k_\nu\phi^*(\mathbf{q}), \quad (3.14)$$

$${}^{(2)}U_m^+ = -C\bar{\psi}(\mathbf{F})[\sigma_{\mu\nu}\tau_N/2M]S_F(\mathbf{w})i\gamma_5\tau_{NP}\psi(\mathbf{I})A_\mu(\mathbf{k})k_\nu\phi^*(\mathbf{q}) \quad (3.15)$$

and

$$\mathbf{w} = \mathbf{F} - \mathbf{k}. \quad (3.16)$$

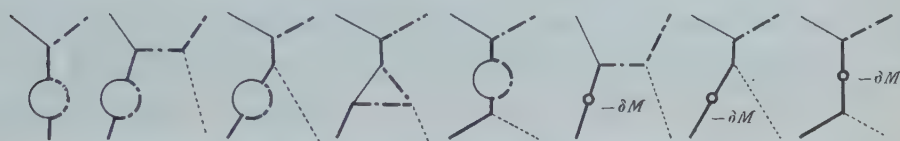
As four independent component matrix elements, it now appears more convenient to select (2.9), (2.10), (3.12) and (3.15) instead of those corresponding to (3.3)~(3.6).

The above considerations strongly suggest that one should carry out every step of calculation in such a way as to exhibit the gauge invariance of the component matrix elements most clearly. The gauge invariance of the S -matrix has been proved by Feynman⁽⁴⁶⁾ for the case of interacting electronic and electromagnetic field. One can easily extend his proof to the most general case including any kind of charged particles,⁽³⁷⁾ and this result we shall apply to our $\gamma - \pi$ process.

* We shall discuss the role of U_v for the $\gamma - \pi$ process in § 7 again.

The conclusion for such general case can be stated as follows: One may take off from the Dyson diagram that photon line with regard to which one will perform a gauge transformation, and thus one may obtain several "skeleton" diagrams. To each skeleton one may attach the photon line to every possible place, that is to say, to every charged line and to every vertex of charged lines, if the type of interaction allows such a diagram (e.g. p.v. coupling). In this way one derives one class of diagrams from each skeleton diagram, and this class as a whole yields a gauge-invariant expression.

As an example we take the case of $\gamma\text{-}\pi^+$ process, considering only charged p.s. mesons with p.s. coupling* in intermediate states. From the given skeleton (see the figure) one constructs four diagrams, as shown in the figs. (3.1), (3.2),



skeleton Fig. (3.1) Fig. (3.2) Fig. (4.1) Fig. (4.2) Fig. (3.1s) Fig. (3.2s) Fig. (4.2s)

(4.1) and (4.2),** which together form a gauge-invariant class. Consequently the expression which includes all the contributions from these four diagrams is certainly gauge-invariant.

This procedure of classifying diagrams not only makes the gauge-invariance of the results obvious, but also eliminates automatically the non-gauge-invariant contributions from individual diagrams and so lightens the burden of computation considerably. It turns out especially useful when one has to do with divergent integrals, because it indicates how to deal with it.

A slight modification of the above method is necessary when one is to carry out the mass-renormalization. In the above example one has then to add three more diagrams that come from the counter-self-energy term. (See figs. (3.1s), (3.2s) and (4.2s).) In these diagrams the small circle denoted by $(-\delta M)$ represents the action of the proton self-energy.

These three terms together form a gauge-invariant class, since they are constructed from a common skeleton. So the seven diagrams, the original four and the added three, are of course gauge-invariant in all. But these seven can be decomposed into two gauge-invariant classes in a different way from the original one, and to this new classification corresponds the physical interpretation: renormalization term and reactive term. Symbolically one may write,

* As for the p. v. coupling see § 5.

** Throughout this paper the figures are specified by two numbers. The first number indicates the (gauge-invariant) class and the second the individual diagrams in this class. The reason why we have nevertheless distinguished fig. (3) and fig. (4) as two classes will be explained shortly.

$$\begin{aligned}
 & (3 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 4 \cdot 2) + (3 \cdot 1_s + 3 \cdot 2_s + 4 \cdot 2_s) \\
 & = (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 1_s + 3 \cdot 2_s) + (4 \cdot 1 + 4 \cdot 2 + 4 \cdot 2_s). \quad (3 \cdot 17)
 \end{aligned}$$

We denote $(3 \cdot 1 + 3 \cdot 2 + 3 \cdot 1_s + 3 \cdot 2_s)$ by ${}^{IV}U_3^+$ and this renormalization term becomes, after some computation,

$${}^{IV}U_3^+ = (-aZ/4\pi) {}^{II}U^+, \quad (3 \cdot 18)$$

with

$$\left. \begin{aligned}
 Z &= (1/2i\pi^2) \int (d\mathbf{t}) \int_0^1 x \, dx \cdot (\mathbf{t}^2 + 2M^2x) / (\mathbf{t}^2 + L_0^2)^3 - 1/4, \\
 L_0^2 &= M^2x^2 + \mu^2(1-x) \quad \text{and} \quad a = f^2/4\pi,
 \end{aligned} \right\} \quad (3 \cdot 19)$$

which is evidently gauge invariant. The gauge invariance of the term $(4 \cdot 1 + 4 \cdot 2 + 4 \cdot 2_s)$ is manifest since it is the difference of two such expressions; also the direct proof is easy.

§ 4. Fourth order matrix element: pseudoscalar coupling

Now we shall proceed to the evaluation of the fourth order matrix element. In the first place we investigate the process $\gamma + P \rightarrow N + \pi^+$ taking into account only p.s. coupling.* The case of p.v. coupling will be analyzed in § 5 and the process $\gamma + N \rightarrow P + \pi^-$ will be referred to in § 6.

According to the method described in the preceding section we divide the Dyson diagrams for the fourth order $\gamma - \pi$ process into six sets, which we shall denote by [I], [II], ..., [VI]. The set [I] contains those which will yield unobservable reaction effects, that is to say, the renormalization terms. The sets [II], [III] and [IV] represent observable reactive effects due to virtual charged meson, and the set [V] those due to virtual neutral meson. The set [VI] contains both observable and unobservable effects due to virtual neutral meson.

[I] Renormalization

In the preceding section we have shown that contributions from the four diagrams figs. (3.1), (3.1s), (3.2) and (3.2s) are reduced to a simple form (3.18) which can be eliminated by renormalizing the coupling constant f to $(f + \delta f)$. Fig. (3.3) shows the diagram where a virtual charged meson is attached to the external neutron line, the exact analogue to fig. (3.1) in which a virtual charged meson is attached to the external proton line. The four diagrams, fig. (3.3), and those which correspond to figs. (3.1s), (3.2) and (3.2s), together make up a gauge invariant class and this class gives, like (3.10), a contribution which renormalizes the coupling constant f , the correction δf being $(-aZ/4\pi)$. The same holds also

* In § 4 and § 5 we treat only the process of π^+ -meson creation and we write instead of U^+ simply U since there will be no confusion.

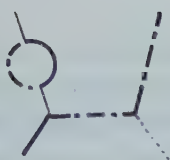


Fig. (3.3)

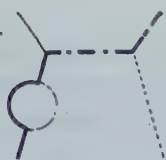


Fig. (3.4)

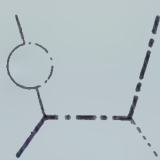


Fig. (3.5)

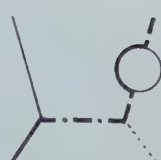


Fig. (3.6)

— · — · — neutral meson

when neutral mesons appear in intermediate states: two classes of diagrams modifying external proton and neutron lines respectively, the two representatives of which are given in figs. (3.4) and (3.5), yield renormalization effect $(-a^0 Z^0/4\pi)$ for each.* $(a^0 = f^0/4\pi)$. Besides there is a class whose member is illustrated by fig. (3.6). This gives the renormalization $(-aY/4\pi)$ due to the modification of the external meson line by virtual nucleon field.

The above mentioned five classes give rise to the following result, which amounts to simply renormalizing the coupling constant.**

$${}^{\text{II}}U + {}^{\text{IV}}U = \{1 - 2(a/4\pi)Z - 2(a^0/4\pi)Z^0 - (a/4\pi)Y\} {}^{\text{II}}U, \quad (4.1)$$

where Z is given by (3.19) and

$$Z^0 = Z + 1/4, \quad a^0 = f^0/4\pi, \quad (4.2)$$

$$Y = (2/i\pi^2) \int (d\mathbf{t}) \int_0^1 x dx (t^2 + 2M^2) / (t^2 + L_{m0}^2)^3, \quad (4.3)$$

$$L_{m0}^2 = M^2 - \mu^2 x(1-x).$$

[II] Anomalous magnetic moment of the proton (due to charged meson)

The diagram of fig. (4.1) is obtained by modifying the vertex γ_μ of the second order diagram fig. (2) by virtual emission and reabsorption of charged meson; the diagram of fig. (4.2) is similarly obtained by modifying the internal nucleon line of fig. (2). The component matrix element corresponding to the fig. (4.1) will be denoted by $U_{4.1}$ and that corresponding to fig. (4.2) and (4.2s) by $U_{4.2}$. They are expressed as follows.***

$$U_{4.1} = C \bar{\psi}(\mathbf{F}) i\gamma_5 \tau_{NP} S_F(\mathbf{v}) l_\mu^m(\mathbf{v}, \mathbf{I}) \tau_P \psi(\mathbf{I}) A_\mu \phi^* \quad (4.4)$$

and

$$U_{4.2} = C \bar{\psi}(\mathbf{F}) i\gamma_5 \tau_{NP} S_F'(\mathbf{v}) \gamma_\mu \tau_P \psi(\mathbf{I}) A_\mu \phi^*, \quad (4.5)$$

* Z^0 differs from Z by a finite term $1/4$. This somewhat strange result is forced by the general rule of selecting integration variables, in order to maintain gauge invariance. Cf. the set [V] of this section.

** Not all the renormalization terms are included in [II]. One will find in [VI] such an effect too.

*** Hereafter we shall write instead of $A_\mu(\mathbf{k})$ and $\phi^*(\mathbf{q})$ simply A_μ and ϕ^* .

where \mathbf{v} is given by (3.9).

Here $\Gamma_\mu^m(\mathbf{v}, \mathbf{I})$ denotes the modification of the γ_μ by virtual meson current and is related both to the four-momentum \mathbf{I} of the external nucleon line and to \mathbf{v} of the internal nucleon line.* On the other hand the propagation function of the internal nucleon $S_F(\mathbf{v})$ is in (4.5) altered to $S_F'(\mathbf{v})$.** After some calculations*** we obtain

$$\Gamma_\mu^m(\mathbf{v}, \mathbf{I}) = (a/2\pi) Z \gamma_\mu + (a/2\pi) \Sigma(\mathbf{v}) \gamma_\mu - (a/2\pi) \eta_I (\sigma_{\mu\nu} k_\nu / 2M), \quad (4.6)$$

$$S_F'(\mathbf{v}) = -(a/2\pi) Z S_F(\mathbf{v}) - (a/2\pi) S_F(\mathbf{v}) \Sigma(\mathbf{v}), \quad (4.7)$$

$$\begin{aligned} \Sigma(\mathbf{v}) = & \frac{1}{2i\pi^2} \int (d\mathbf{t}) \int_0^1 dx \int_0^1 dy \left[\frac{3(1-y)(L_0^2 - L_v^2)(x\mathbf{t}^2 + 2M^2x^3)}{y[\mathbf{t}^2 + L_v^2]^4} \right. \\ & \left. + \frac{(i\gamma\mathbf{v} + M)2Mx^2(1-x)}{[\mathbf{t}^2 + L_v^2]^3} \right], \end{aligned} \quad (4.8)$$

$$L_v^2 - L_0^2 = (\mathbf{v}^2 + M^2)x(1-x)y, \quad (4.9)$$

where L_0^2 is given in (3.19) and

$$\eta(\mathbf{I}) = \eta_I(a) + \eta_{II}(a) [(i\gamma\mathbf{v} + M)/M], \quad a = -2(\mathbf{Ik})/M^2. \quad (4.10)$$

η_I, η_{II} are functions of the energies of the incident photon and the emitted meson; their explicit forms will be given at the end of this section. η_I corresponds to the a.m.m. of proton in nuclear magneton unit. We shall later discuss the physical meaning of these terms together with similar terms appearing in the set [IV], and give here only the expression of the component matrix element;

$$\begin{aligned} U_4 = & U_{4,1} + U_{4,2} \\ = & C \bar{\psi}(\mathbf{F}) i\gamma_5 \tau_{NP} S_F(\mathbf{v}) [-(a/2\pi) \eta(\mathbf{I}) \cdot \sigma_{\mu\nu} k_\nu / 2M] \tau_P \psi(\mathbf{I}) A_\mu \phi^*. \end{aligned} \quad (4.11)$$

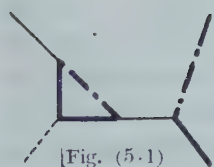
[III] Anomalous magnetic moment of neutron

The two diagrams represented by figs. (5.1) and (5.2) make up a gauge invariant class, since in this case the virtual meson and the virtual proton form a closed loop of electric charge, to which the photon line can be attached in only two ways.

* Our Γ_μ differs from what Dyson²⁶⁾ has given in his equation (38) (p. 1743) $\Gamma_\mu = \gamma_\mu + A_\mu$; ours is identical with Dyson's A_μ . Throughout this paper S_F', A_F', Σ , and Π are also somewhat different from those defined by Dyson.

** In $S_F'(\mathbf{v})$ is contained only reactive effect, since the renormalization effect has been subtracted by adding fig. (4.2s).

*** The four-momentum of the neutron is taken as the integration variable in computing Z . See the appendix of the reference²⁷⁾ for the integration formulas.



[Fig. (5.1)]



Fig. (5.2)

As there is no direct interaction between neutron and photon these figures cannot be reduced to the second order diagrams; but one can formally express the fourth order component matrix-elements as if they were the modification of the vertex γ_μ of neutron-photon interaction. There are two kinds of such modification as shown in the figures. One of them (fig. (5.1)), due to proton current, is denoted by Γ_μ^n , and the other (fig. (5.2)), due to meson current, by Γ_μ^m . The corresponding component matrix elements are given by

$$U_{5.1} = C \bar{\psi}(\mathbf{F}) \Gamma_\mu^n(\mathbf{F}, \mathbf{w}) \tau_N S_F(\mathbf{w}) i \gamma_5 \tau_{NP} \psi(\mathbf{I}) A_\mu \phi^*, \quad (4.12)$$

$$U_{5.2} = C \bar{\psi}(\mathbf{F}) \Gamma_\mu^m(\mathbf{F}, \mathbf{w}) \tau_N S_F(\mathbf{w}) i \gamma_5 \tau_{NP} \psi(\mathbf{I}) A_\mu \phi^*, \quad (4.13)$$

with $\mathbf{w} = \mathbf{F} - \mathbf{k} = \mathbf{I} - \mathbf{q}$. Γ_μ 's turn out, after some calculations

$$\Gamma_\mu^n(\mathbf{F}, \mathbf{w}) = (a/2\pi) Z^0 \gamma_\mu + (a/2\pi) \gamma_\mu \Sigma(\mathbf{w}) + (a/2\pi) (\sigma_{\mu\nu} k_\nu / 2M) \xi(\mathbf{F}), \quad (4.14)$$

$$\Gamma_\mu^m(\mathbf{F}, \mathbf{w}) = - (a/2\pi) Z \gamma_\mu - (a/2\pi) \gamma_\mu \Sigma(\mathbf{w}) + (a/2\pi) (\sigma_{\mu\nu} k_\nu / 2M) \eta(\mathbf{F}), \quad (4.15)$$

$$\xi(\mathbf{F}) = \xi_I(-b) + \xi_{II}(-b) [(i\gamma\mathbf{w} + M)/M],$$

$$\eta(\mathbf{F}) = \eta_I(-b) + \eta_{II}(-b) [(i\gamma\mathbf{w} + M)/M], \quad b = -2(\mathbf{F}\mathbf{k})/M^2. \quad (4.16)$$

The explicit definitions of ξ_I etc. will be given at the end of this section. They are of course functions of the energies of the incident photon and the emitted meson.

$$\begin{aligned} U_{5.1} + U_{5.2} = & - (a/2\pi) [\xi_I(-b) + \eta_I(-b)]^{(2)} U_m \\ & + (a/2\pi) (a/2) [\xi_{II}(-b) + \eta_{II}(-b)]^{II} U_m \\ & + (a/2\pi) (Z_0 - Z) C \psi(\mathbf{F}) \gamma_\mu \tau_N S_F(\mathbf{w}) i \gamma_5 \tau_{NP} \psi(\mathbf{I}) A_\mu \phi^*. \end{aligned} \quad (4.17)$$

Some comments on the right-hand side of (4.17) may not be superfluous. It is to be noticed that the coefficient of $^{(2)}U_m$ which has been stated in (3.15) includes, besides the "static" value of the a.m.m. of neutron first worked out by Case,¹⁷⁾ a correction term dependent on the frequency of the electromagnetic field, which could thus be called a "non-static" correction.*

The second line has the same form with the "magnetic" interaction, (2.10), only multiplied by a certain numerical factor. Thus it represents a simple correction to the component of the second order element.

* As for this name cf. § 7 (I).

The third line does not vanish, because we have fixed a scheme of calculation, according to which Z^0 is not identical with Z . In fact $Z^0 - Z = 1/4$ by (4.2). This fact gives rise to a trouble, since that term should be interpreted as "electric charge of the neutron"²¹⁾ and can never be renormalized. Also it is not gauge-invariant. Here one has to do with the so-called ambiguity of the current field theory which appears when the Dyson diagram contains a closed loop. By the help of the general discussion given by Mugibayashi and some of us²⁷⁾ with regard to such ambiguities, we strike out this problematic term.*

[IV] Mesonic polarization

The figs. (6.1) and (6.2) stand for the component matrix elements obtained from fig. (1) by modifying respectively the internal meson line ($\Delta_F(u)$) or the interaction of photon with meson current (q_μ). It is obvious from the requirement of gauge invariance, that these two elements cancel one another.** This can also be verified by a direct calculation. One has namely, with $u = q - k = I - F$,

$$\begin{aligned} U_{6.1} &= C \bar{\psi}(F) i \gamma_5 \tau_{NP} \psi(I) \Delta'_F(u) 2i q_\mu A_\mu \phi^*, \\ U_{6.2} &= C \bar{\psi}(F) i \gamma_5 \tau_{NP} \psi(I) \Delta_F(u) 2i Q_\mu A_\mu \phi^*. \end{aligned} \quad (4.18)$$

The propagation function of the virtual meson $\Delta_F(u)$ is defined by $\Delta_F(u) = 1/(u^2 + \mu^2)$. Δ'_F and Q_μ can be easily obtained; in the former the effect of the meson mass renormalization has been taken into account.

$$\Delta'_F(u) = -(a/2\pi) Y \Delta_F(u) + (a/2\pi) \Pi(u) \Delta_F(u), \quad (4.19)$$

$$Q_\mu = (a/2\pi) Y q_\mu - (a/2\pi) \Pi(u) q_\mu, \quad (4.20)$$

where Y is given in (4.3) and

$$\Pi(u) = \frac{1}{2i\pi^2} \int (d\mathbf{t}) \int_0^1 dx \int_0^1 dy \frac{3(1-y)(L_m^2 - L_{m0}^2)(x\mathbf{t}^2 + 2xM^2)}{[\mathbf{t}^2 + L_{m0}^2 + (L_m^2 - L_{m0}^2)y]^4}, \quad (4.21)$$

* This is the only case that one is bothered by the so-called ambiguities in calculating the fourth order matrix element for the $\gamma - \pi^\pm$ process.

** The reasoning is as follows. According to the general rule of making up a gauge-invariant class, one has to add to figs. (6.1) and (6.2) two more diagrams, which, however, are reduced to the renormalization of the meson mass. Thus the two figs. (6.1) and (6.2) must form by themselves a gauge-invariant subclass. Now from the diagrams one can easily infer that these give rise to a correction to the interaction of the meson with the photon. Since our p.s. meson has no magnetic moment of its own, this correction must take the form of convection current of the meson. But we have ascertained in § 3 that the interaction of the meson convection current with the photon can be gauge-invariant only when combined with the current of the proton. Thus we conclude that if figs. (6.1) and (6.2) should yield a non-vanishing result, it would violate the gauge invariance.

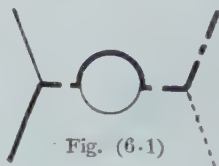


Fig. (6.1)

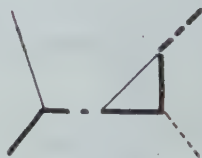


Fig. (6.2)

with

$$L_m^2 - L_{m0}^2 = (u^2 + \mu^2)x(1-x). \quad (4.22)$$

Thus we have indeed

$$U_{6,1} + U_{6,2} = 0. \quad (4.23)$$

[V] *Anomalous magnetic moment of the proton (due to neutral meson)*

One can modify the second order diagram (fig. (2)) by adding the emission and reabsorption of a neutral meson to the vertex (γ_μ) or to the internal nucleon line ($S_F(v)$). Thus one finds the figs. (7.1) and (7.2), which form of course a gauge-invariant subclass.* The corresponding component matrix elements are

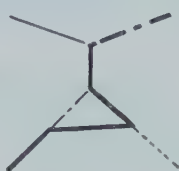


Fig. (7.1)

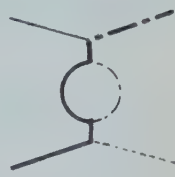


Fig. (7.2)

$$U_{7,1} = C\bar{\psi}(F)i\gamma_5\tau_{NP}S_F(v)\Gamma_\mu^{n,0}(v, I)\tau_P\psi(I)A_\mu\phi^*, \quad (4.24)$$

$$U_{7,2} = C\bar{\psi}(F)i\gamma_5\tau_{NP}S_F'^0(v)\gamma_\mu\tau_P\psi(I)A_\mu\phi^*,$$

where**

$$\Gamma_\mu^{n,0}(v, I) = (u^0/2\pi)Z^0\gamma_\mu + (u^0/2\pi)\Sigma(v)\gamma_\mu + (u^0/2\pi)\hat{\xi}(I)\sigma_{\mu\nu}k_\nu/2M, \quad (4.25)$$

$$S_F'^0(v) = -(u^0/2\pi)Z^0S_F(v) - (u^0/2\pi)S_F(v)\Sigma(v), \quad (4.26)$$

with $\Sigma(v)$ given by (4.8) and

$$\hat{\xi}(I) = \hat{\xi}_I(a) + \hat{\xi}_{II}(a)[(i\gamma v + M)/M]; \quad (4.27)$$

$\hat{\xi}_I(a)$ and $\hat{\xi}_{II}(a)$ are given explicitly at the end of this section.

Thus we obtain

$$\begin{aligned} U_7 &= U_{7,1} + U_{7,2} \\ &= C\bar{\psi}(F)i\gamma_5\tau_{NP}S_F(v)[(u^0/2\pi)\hat{\xi}(v)\sigma_{\mu\nu}k_\nu/2M]\tau_P\psi(I)A_\mu\phi^*. \end{aligned} \quad (4.28)$$

The total effect of the set [III] and the set [V], which gives the a.m.m. through the charged and the neutral meson field becomes

* Cf. the first few lines of the second footnote on the foregoing page.

** In computing $S_F'^0(v)$ we have chosen the four-momentum of the meson as the integration variable, while in the case of $S_F'(v)$, (4.7) we have chosen the four-momentum of the nucleon as much. This difference is required from the general rule of gauge-invariant computation that the integration variable should be the four momentum of the virtual neutral line. Cf. the set [I] of this section.

$$U_4 + U_7 = [(a/2\pi)\eta_I(a) - (a^0/2\pi)\xi_I(a)]^{(1)}U_m - (a/2)[(a/2\pi)\eta_{II}(a) - (a^0/2\pi)\xi_{II}(a)]^{II}U_m, \quad (4.29)$$

where $^{(1)}U_m$ is given by (3.12) representing the anomalous magnetic interaction of the proton. The coefficient of this term is reduced to the value of Case when $k \rightarrow 0$, as was the case with the first line of (4.17). The second line of (4.29) is, similarly to the second line of (4.17), a simple correction to the component matrix element of the second order.

[VI] Modification of meson emission



Fig. (8.1)

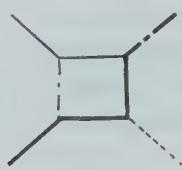


Fig. (8.2)

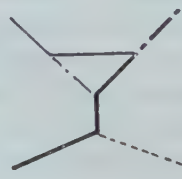


Fig. (8.3)

The skeleton of this class is characterized by the virtual emission and re-absorption of a neutral meson modifying the vertex γ_5 . This class includes three diagrams, figs. (8.1), (8.2) and (8.3), the second of which represents a new type, not being directly reducible. The corresponding component matrix elements are

$$\left. \begin{aligned} U_{8.1} &= C \bar{\psi}(\mathbf{F}) i \Gamma_5(\mathbf{F}, \mathbf{I}) \tau_{NP} \psi(\mathbf{I}) \Delta_F(u) 2i q_\mu A_\mu \phi^*, \\ U_{8.2} &= C \bar{\psi}(\mathbf{F}) A_\mu \psi(\mathbf{I}) A_\mu \phi^*, \\ U_{8.3} &= C \bar{\psi}(\mathbf{F}) i \Gamma_5(\mathbf{F}, \mathbf{v}) \tau_{NP} S_F(\mathbf{v}) \gamma_\mu \tau_P \psi(\mathbf{I}) A_\mu \phi^*. \end{aligned} \right\} \quad (4.30)$$

A_μ and Γ_5 in the above expressions are defined as follows;

$$A_\mu = (-if^{02})/(2\pi)^4 \cdot \int (d\mathbf{t}) i\gamma_5 \tau_3 S_F(\mathbf{I} + \mathbf{t}) i\gamma_5 \tau_{NP} S_F(\mathbf{v} + \mathbf{t}) \gamma_\mu \tau_P S_F(\mathbf{I} + \mathbf{t}) i\gamma_5 \tau_3 \Delta_F(\mathbf{t}), \quad (4.31)$$

$$\Gamma_5(\mathbf{F}, \mathbf{v}) = (a^0/2\pi) X \gamma_5 + (a^0/2\pi)^{II} \Gamma_5(\mathbf{F}, \mathbf{v}), \quad (4.32)$$

where

$$X = (1/i\pi^2) \int (d\mathbf{t}) \int_0^1 x dx \int_0^1 dy [t^2 + \mu^2(1-x)]/[t^2 + \Delta_0^2]^3 - 1/4, \quad (4.33)$$

$$\begin{aligned} {}^{II}\Gamma_5(\mathbf{F}, \mathbf{v}) &= 1/(i\pi^2) \int (d\mathbf{t}) \int_0^1 x dx \int_0^1 dy \\ &\times \gamma_5 \left[\int_0^1 d\varepsilon \frac{3(\Delta_0^2 - \Delta^2)(t^2 + \mu^2(1-x))}{[t^2 + \Delta_0^2 + (\Delta^2 - \Delta_0^2)\varepsilon]^4} + \frac{Mx(i\mathbf{r}\mathbf{v} + M)}{(t^2 + \Delta^2)^3} \right] \end{aligned} \quad (4.34)$$

with

$$\begin{aligned} \Delta_0^2 &= M^2 x^2 + \mu^2 (1-x) + \mu^2 x^2 y (y-1), \\ \Delta^2 - \Delta_0^2 &= (\nu^2 + M^2) (1-x) xy + [(\mathbf{F} - \nu)^2 + \mu^2] x^2 y (1-y), \\ \mathbf{v} &= \mathbf{I} + \mathbf{k} = \mathbf{F} + \mathbf{q}. \end{aligned} \quad (4.35)$$

$\Gamma_5(\mathbf{F}, \mathbf{I})$ can be derived from $\Gamma_5(\mathbf{F}, \nu)$ by replacing ν by \mathbf{I} and noticing the relation $\mathbf{I}^2 + M^2 = 0$.

We sum up these three component matrix elements and transform them into the following expression.*

$$\begin{aligned} U_{s,1} + U_{s,2} + U_{s,3} &= -(\alpha^0/2\pi) \lambda_I(a) [2(\mathbf{qk})/M^2]^{(1)} U_c - (\alpha^0/2\pi) \lambda_{II}(a) [(\mathbf{Ik})/M^2]^{(1)} U_m \\ &\quad - (\alpha^0/2\pi) \lambda_{III}(a) [2(\mathbf{Ik})/M^2]^{(1)} U_m + (\alpha^0/2\pi) \lambda_{IV}(a) [2(\mathbf{Fk})/M^2]^{(2)} U_m \\ &\quad + (\alpha^0/2\pi) X^{(1)} U. \end{aligned} \quad (4.36)$$

The coefficients λ 's are given below. (X is defined in (4.33)). The first two terms on the right-hand side of (4.36) represent respectively corrections to the convection current interaction and to the "magnetic" interaction of the second order. The third and the fourth terms are "non-static" corrections to the a.m.m.'s of the nucleon; and the last term, being the total second order matrix element multiplied by a constant numerical factor, can be eliminated by the renormalization of the coupling constant.

In foregoing we have enumerated all the possible component matrix elements of fourth order for the process $\gamma + P \rightarrow \pi^+ + N$ and reduced them to five types as those appearing in (4.36). As for the process of π^- -creation a similar arrangement can be carried out; the result will be mentioned in § 6. That no other types appears even when one takes the p.v. coupling into consideration, will be shown in the next section.

In concluding this section we give a list of the definitions of the functions used above.**

$$\eta_I(a) = \int_0^1 dx \int_0^1 dy x^2 (1-x) / L_v, \quad \eta_{II}(a) = \int_0^1 dx \int_0^1 dy x (1-x)^2 y / L_v, \quad (4.10)$$

* It is expected from the consideration in § 3 that such a transformation is always possible.

** It is seen that the integrands for $\xi(a)$, $\eta(a)$ and $\lambda(a)$ have singularities where the denominators vanish. This fact corresponds to the situation that in the diagrams [II], [V] and [VI], to which these functions are related, the energy-momentum of the system can be conserved in an intermediate stage. In these cases, one has to integrate round the singularities on the complex plane, when one carries out the parameter-integration. The exact prescription how to select the path results from the definitions of Δ_P (and S_P), by scrutinizing the methods of integration with and without parameters. Then one gets certain imaginary parts, besides the real parts which follow by taking the principal values for the integrals $\xi(a)$, $\eta(a)$ and $\lambda(a)$. Such imaginary parts, however, contribute nothing in our $e^2 f^4$ approximation (see (6.12)), [and are omitted also in (§ 7. I), though this is inexact.]

$$\xi_I(a) = \int_0^1 dx \int_0^1 dy x^3/l_v, \quad \xi_{II}(a) = \int_0^1 dx \int_0^1 dy x^2(1-x)y/l_v, \quad (4.27)$$

$$l_v = L_v^2/M^2 = x^2 + \varepsilon(1-x) - ax(1-x)y, \quad (3.19)(4.9)$$

$$a = -2(\mathbf{Ik})/M^2, \quad \varepsilon = \mu^2/M^2. \quad (4.10)$$

$\xi(-b)$'s and $\eta(-b)$'s are obtained from the corresponding $\xi(a)$'s and $\eta(a)$'s by changing the denominator l_v of the integrand to l_v . This amounts to replacing \mathbf{I} by $-\mathbf{F}$.

$$l_v = L_v^2/M^2 = x^2 + \varepsilon(1-x) + bx(1-x)y, \quad b = -2(\mathbf{Fk})/M^2, \quad (4.16)$$

$$\lambda_I(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^3 y(1-y) \left[\left(\frac{1}{l_{ad}} - \frac{1}{l_a} \right) + \frac{\varepsilon(1-x)}{2} \left(\frac{1}{l_{ad}^2} - \frac{1}{l_a^2} \right) + \frac{a}{2} \frac{xyz}{l_{ad}^2} \right],$$

$$\lambda_{II}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[(1-x)x^2 y \left(-\frac{2}{l_a} + \frac{\varepsilon(1-x)}{l_a^2} \right) - \frac{\varepsilon x^2 y(1-x)}{l_{ad}^2} + 2x^4 y \right],$$

$$\lambda_{III}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^4 y^2}{l_{ad}^2}, \quad \lambda_{IV}(a) = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{x^4 y(1-y)}{l_{ad}^2}, \quad (4.36)$$

$$l_a = x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - a(1-x)xy, \quad$$

$$l_{ad} = x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) - a(1-x)xy(1-z) + dx^2 y(1-y)z, \quad$$

$$l_d = x^2 + \varepsilon(1-x) + \varepsilon x^2 y(y-1) + dx^2 y(1-y)z, \quad d = -2(\mathbf{qk})/M^2.$$

§ 5. Fourth order matrix element: pseudovector coupling

As is well known, the p.v. coupling is derived from the p.s. coupling if one substitutes in the latter $\gamma_5 \gamma_\mu \partial/\partial x_\mu$ for γ_5 . When the electromagnetic interaction is taken into account one has further to introduce a term of direct coupling of nucleon, meson and photon, denoted previously by H^{nms} (2.6); otherwise the gauge-invariance would be violated.

In our method of diagram classification, as stated in § 3, this means: One may take any gauge-invariant class for the p.s. coupling, replace the vertex of each diagram γ_5 by $\gamma_5 \gamma_\mu \partial/\partial x_\mu$ and moreover add a new diagram containing a quadrifurcate vertex (with two nucleon lines, one meson line and one photon line) to this class and thus one obtains the corresponding gauge-invariant class for the p.v. coupling.

This class of p.v. coupling we divide into two parts; the one is what will yield results equivalent to the p.s. coupling and the other is the so-called inequivalent

part. The calculation can be for the most part performed in the momentum representation, but a scrupulous care must be taken when one has to do with a closed loop of charged lines that yields a divergent integral.*

We shall not give here the details of the calculation, but, illustrate our method taking a class of diagrams corresponding to [III] of § 4, which is just the ticklish case mentioned above. As is seen from the figure, our diagram has three vertices of nucleon-meson coupling. We can choose, for example, two of them as p.v.-coupled and the remaining one as p.s.-coupled. The former we denote by a double circle and the latter by a single one. Thus we can draw three diagrams; figs. (9.1), (9.2) and (9.3), which form a gauge-invariant class.

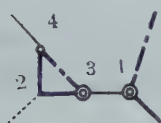


Fig. (9.1)

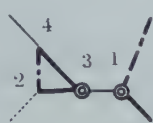


Fig. (9.2)



Fig. (9.1.1)



Fig. (9.1.2)

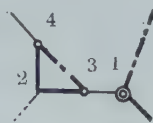


Fig. (9.1.3)



Fig. (9.1.4)

The vertices (3-2-4) in figs. (9.1) and (9.2) form a closed loop of charged lines and may yield an incorrect result if calculated in momentum representation.³⁷⁾ In fact the same diagram for the p.s. coupling has already given rise to an "electric charge of the neutron" (§ 4). So we had better proceed for the present in the coordinate representation.

For the component matrix element corresponding to the fig. (9.1) we have**

$$\begin{aligned}
 U_{9.1} = & - (e g / \mu) \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 \bar{\psi}(\mathbf{x}_1) [- (i g f / \mu) i \gamma_5 \tau_{NP} S_F(\mathbf{x}_4 - \mathbf{x}_2) \\
 & \times \tau_P \gamma_\mu S_F(\mathbf{x}_2 - \mathbf{x}_3) i \gamma_5 \tau_{PN} \gamma_\nu \cdot \partial A_F(\mathbf{x}_3 - \mathbf{x}_4) / \partial x_{3,\nu}] S_F(\mathbf{x}_3 - \mathbf{x}_1) \\
 & \times i \gamma_5 \tau_{NP} \gamma_\lambda \psi(\mathbf{x}_1) A_\mu(\mathbf{x}_2) \cdot \partial \phi^* / \partial x_{1,\lambda} .
 \end{aligned} \quad (5.1)$$

We integrate this expression by part with respect to \mathbf{x}_3 and making use of the relation

$$(\gamma_\nu \cdot \partial / \partial x_\nu + M) S_F(\mathbf{x}) = i \partial(\mathbf{x}) , \quad (5.2)$$

decompose it into three parts,**

* It will not be necessary to explain the other classes, not containing any closed loop. They can be separated into equivalent and inequivalent parts without difficulty. One has only to take care, for example, not to put the expression

$$\int (d\mathbf{t}) \bar{\psi}(\mathbf{F}) i \gamma_5 (i \gamma \mathbf{F} - M) S_F(\mathbf{F} + \mathbf{t}) i \gamma_5 S_F(\mathbf{F}) \dots \dots ,$$

equal to zero, because it contains the self-energy term. (See Karplus and Kroll.³⁷⁾)

** $S_F(\mathbf{x}) = (-i) / (2\pi)^4 \cdot \int (d\mathbf{W}) S_F(\mathbf{W}) \exp(i\mathbf{W}\mathbf{x})$.

*** We put the surface integral equal to zero, according to the general discussion given by Mugibayashi and others.

$$U_{9,1} = U_{9,1,1} + U_{9,1,2} + U_{9,1,3},$$

of which the first one, $U_{9,1,1}$ just cancels $U_{9,3}$ and the other two are, when expressed in the momentum representation, as follows:

$$U_{9,1,2} = (g/\mu f) C \bar{\psi}(\mathbf{F}) [(g/\mu f) \Gamma_{\mu}^n(\mathbf{F}, \mathbf{w}) \tau_N] i \gamma_5 \tau_{NP} (-i \gamma \mathbf{q}) \psi(\mathbf{I}) A_{\mu}(\mathbf{k}) \phi^*(\mathbf{q}), \quad (5.3)$$

$$U_{9,1,3} = (g/\mu f) C \bar{\psi}(\mathbf{F}) \times [(2Mg/\mu f) \Gamma_{\mu}^n(\mathbf{F}, \mathbf{w}) \tau_N] S_F(\mathbf{w}) i \gamma_5 \tau_{NP} (-i \gamma \mathbf{q}) \psi(\mathbf{I}) A_{\mu}(\mathbf{k}) \phi^*(\mathbf{q}) \quad (5.4)$$

with Γ_{μ}^n given by (4.14) and $\mathbf{w} = \mathbf{I} - \mathbf{q} = \mathbf{F} - \mathbf{k}$.

Further we divide $U_{9,1,3}$ into "equivalent" and "inequivalent" parts with respect to the vertex 1. This time we can make our way in the momentum space, since we are not concerned with any closed loop. To do this we substitute the relation

$$S_F(\mathbf{w}) i \gamma_5 \tau_{NP} (-i \gamma \mathbf{q}) \psi(\mathbf{I}) = \{2MS_F(\mathbf{w}) + 1\} i \gamma_5 \tau_{NP} \psi(\mathbf{I}), \quad (5.5)$$

into (5.4) and get

$$U_{9,1,3} = (g')^2 U_{5,1} + U_{9,1,4}, \quad (5.6)$$

where the first term on the right-hand side is equivalent to the p.s. coupling given by (4.12) and the second term

$$U_{9,1,4} = (g/\mu f) C \bar{\psi}(\mathbf{F}) [g' \Gamma_{\mu}^n(\mathbf{F}, \mathbf{w}) \tau_N] i \gamma_5 \tau_{NP} \psi(\mathbf{I}) A_{\mu}(\mathbf{k}) \phi^*(\mathbf{q}) \quad (5.7)$$

together with $U_{9,1,2}$ gives the inequivalent effect arising from the diagram (9.1).

The above mentioned reduction of the component matrix element with p.v. coupling may be performed with the help of diagrams as follows. Instead of a diagram with the double circle representative of p.v. coupling one may draw three diagrams, two of which are derived from the original figure by erasing the one or the other nucleon segment neighbouring the vertex in question, and the remaining one of which is obtained by replacing the double circle by a single one (p.s. coupling). Repeated application of this rule ultimately separates the essentially inequivalent effect.

* Now the inequivalent terms $U_{9,1,2}$ and $U_{9,1,4}$ from $U_{9,1}$ are of course not closed with regard to a gauge transformation. Those terms which are to be combined with them are derived from the diagram (9.2). We apply namely a similar reduction to the component matrix element of the diagram (9.2). After some calculation in the coordinate space - this is necessary because of the presence of the closed loop - we get expressions corresponding to the figs. (9.2.1), (9.2.2) and (9.2.3). Further application of the reduction procedure to the vertex 1 of (9.2.3) gives rise to an "equivalent" component matrix element (represented by

the diagram similar to (9.2.3), only with a single circle instead of the double), and an "inequivalent" element, (9.2.4). The component matrix element for the fig. (9.2.1) turns out zero. Thus (9.2) yields, besides an "equivalent" term which need not be written here explicitly, the following two "inequivalent" terms.

$$U_{9.2.2} = (g/\mu f) C \bar{\psi}(\mathbf{F}) [(g/\mu f) \Gamma_{\mu}^m(\mathbf{F}, \mathbf{w}) \tau_N] i \gamma_5 \tau_{NP} (-i \gamma \mathbf{q}) \phi(\mathbf{I}) A_{\mu}(\mathbf{k}) \phi^*(\mathbf{q}), \quad (5.8)$$

$$U_{9.2.4} = (g/\mu f) C \bar{\psi}(\mathbf{F}) [(2Mg/\mu f) \Gamma_{\mu}^m(\mathbf{F}, \mathbf{w}) \tau_N] i \gamma_5 \tau_{NP} \phi(\mathbf{I}) A_{\mu}(\mathbf{k}) \phi^*(\mathbf{q}), \quad (5.9)$$

with Γ_{μ}^m given by (4.6). It is obvious from the structure of these diagrams that $U_{9.2.2}$ with $U_{9.1.2}$ and $U_{9.2.4}$ with $U_{9.1.4}$ make up respectively gauge-invariant subclasses of inequivalent effects.

Thus the total "inequivalent" contribution from the diagrams (9.1), (9.2) and (9.3) becomes, after some calculation,

$$\begin{aligned} U_9 = & (g')^2 \frac{a}{2\pi} \left\{ \frac{\xi_I(-b) + \eta_I(-b)}{4} - \frac{\xi_{II}(-b) + \eta_{II}(-b)}{2} \right\} \frac{-2(\mathbf{Fk})}{M^2} {}^{(2)}U_m \\ & - (g')^2 \frac{a}{2\pi} \left\{ \frac{\xi_I(-b) + 2\xi_{II}(-b) + \eta_I(-b) + 2\eta_{II}(-b)}{4} \right. \\ & \left. + \frac{\xi_{II}(-b) + \eta_{II}(-b)}{8} \cdot \frac{-2(\mathbf{Fk})}{M^2} \right\} \frac{-2(\mathbf{Ik})}{M^2} {}^{(1)}U_m. \end{aligned} \quad (5.10)$$

It is seen that the "inequivalent" effect can be reduced to the known types of interaction: The first term is the anomalous magnetic interaction of the neutron, and the second term stands for a correction to the second order "magnetic" interaction. Thus the "inequivalence" consists in the fact that the coefficients to be multiplied into these interaction prototypes are not simply proportional to those from p.s. coupling.



Fig. (9.2.1) Fig. (9.2.2) Fig. (9.2.3) Fig. (9.2.4) Fig. (10) Fig. (11)

In the above example two of the vertices were assumed to be p.v.-coupled. The case of all three vertices taking p.v. coupling can be treated in an analogous way. Only one must be careful in treating the divergent integrals; the problematic term, which bears the form of the electric charge of the neutron, should be dropped, as has been done in § 4. Besides this term anything obscure does not appear.

The Dyson diagrams with p.v. coupling, which do not contain closed loops such as figs. (9.1) and so on, can be treated quite easily in the momentum representation. They give rise to the same prototypes of interaction with § 4, but of course with different and in general not proportional coefficients.

When one treats the Dyson diagrams containing virtual neutral meson with p.v. coupling, one can utilize a particular method analogous to the gauge-invariant classification. One may namely divide the diagrams into several classes, for each of which the equivalence theorem holds separately if it is valid at all. Such an "equivalent class" can be constructed from a "skeleton," which contains one less vertex of nucleon-meson coupling and so one meson-line left unconnected, by attaching that unfixed meson-line to every permissible place of the skeleton. The diagrams corresponding to figs. (10) and (11), for example, belong to the same "equivalent class," though they are members of different "gauge-invariant" classes. When one picks up the "inequivalent" terms from each diagram of a certain class, every two of them have an identical form, either with every the opposite or the same sign.

In our $\gamma-\pi^\pm$ process almost every class containing virtual neutral meson gives such pairs with the opposite sign, so that the inequivalent effect vanishes. Only a few classes afford pairs with the equal sign and this inequivalent contribution can be reduced, after some transformations, to only one prototypes: the anomalous magnetic interaction of the proton.

The renormalization of the coupling constant due to the inequivalent terms is given by

$$\begin{aligned} {}^{\text{II}}U + {}^{\text{IV}}U = & \left[1 + (1+g')^2 g' \frac{a}{2\pi} Z' - g'^3 \frac{a}{2\pi} Z'' \right. \\ & \left. + g'(1+g')(g'-3) \frac{a}{4\pi} Z'^0 - g'^3 \frac{a}{4\pi} Z'' \right] {}^{\text{II}}U, \end{aligned} \quad (5.11)$$

$$Z' = (1/4i\pi^2) \left[\int (\mathbf{dt}) \int_0^1 dx \cdot x / (\mathbf{t}^2 + L_0^2)^2 + i\pi^2/4 \right], \quad (5.12)$$

$$Z'^0 = (1/4i\pi^2) \left[\int (\mathbf{dt}) \int_0^1 d\mathbf{r} \cdot x / (\mathbf{t}^2 + L_0^2)^2 - i\pi^2/4 \right], \quad (5.13)$$

$$Z'' = (1/4i\pi^2 M^2) \int (\mathbf{dt}) \cdot 1 / (\mathbf{t}^2 + \mu^2), \quad (5.14)$$

with L_0^2 given by (3.19) and $g' = 2Mg/\mu f = g'^0 = 2Mg^0/\mu f^0$. The fourth and the fifth terms on the right-hand side come from the virtual neutral meson. We have here assumed the sym. field. If we take the ch.+n. field, the result is somewhat different, namely,

$${}^{\text{II}}U + {}^{\text{IV}}U = \left[1 + (1 + g')^2 g' \frac{a}{2\pi} Z' - g'^3 \frac{a}{2\pi} Z'' + (1 + g'^0) (2g' + 3g'^0 + g'g'^0) \frac{a^0}{2\pi} Z'^0 - (g'^0)^2 \frac{a^0}{2\pi} Z'' \right] {}^{\text{II}}U. \quad (5.15)$$

§ 6. Results

We shall summarize in this section the results derived from the calculations of §§ 4-5. Besides the renormalization terms, which have no direct physical significance, the fourth order matrix element for the $\gamma - \pi^+$ process can be classified into those components arising from i) the convection current interaction ${}^{\text{II}}U_c^+$, ii) the "magnetic" interaction of the proton ${}^{\text{II}}U_m^+$, iii) the a.m.m. interaction of the proton ${}^{(1)}U_m^+$ and iv) the a.m.m. interaction of the neutron ${}^{(2)}U_m^+$ for both p.s. coupling (f) and p.v. coupling (g).

As for the process $\gamma + P \rightarrow \pi^+ + N$, we have

$${}^{\text{IV}}U^+ = (1 + g') (G_c^+ {}^{\text{II}}U_c^+ + G_m^+ {}^{\text{II}}U_m^+ + {}^{(1)}G^+ {}^{(1)}U_m^+ + {}^{(2)}G_m^+ {}^{(2)}U_m^+), \quad (6.1)$$

where

$$G_c^+(k_0, q_0) = \pm (1 + g'^0)^2 (a^0/2\pi) \lambda_{\text{I}}(a) d, \quad (6.2)$$

$$\begin{aligned} G_m^+(k_0, q_0) = & (1 + g')^2 (a/4\pi) [\xi_{\text{II}}(-b) + \eta_{\text{II}}(-b) - \eta_{\text{II}}(a)] a \\ & + (1 + g'^0)^2 (a^0/4\pi) [\xi_{\text{II}}(a) \pm \lambda_{\text{II}}(a)] a \\ & - (2 + g') g' (a^0/4\pi) [\xi_{\text{I}}(-b)/2 + \xi_{\text{II}}(-b) + \eta_{\text{I}}(-b)/2 + \eta_{\text{II}}(-b) \\ & - \eta_{\text{I}}(a)/2 - \eta_{\text{II}}(a)] a \\ & + \left\{ (g')^2 (a/4\pi) \xi_{\text{II}}(a) \cdot a^2/4 \right. \\ & \left. (1 + g'^0) [g'^0 + g'/(1 + g')] (a^0/4\pi) [\xi_{\text{I}}(a) + \xi_{\text{II}}(a)] a, \right. \end{aligned} \quad (6.3)$$

$$\begin{aligned} {}^{(1)}G_m^+(k_0, q_0) = & (1 + g')^2 (a/4\pi) 2\eta_{\text{I}}(a) \\ & - (1 + g'^0)^2 (a^0/2\pi) [\xi_{\text{I}}(a) \mp \lambda_{\text{III}}(a) a] \\ & - g' (a/4\pi) [\eta_{\text{II}}(a) - g' \eta_{\text{I}}(a)/2] a \\ & - \left\{ g'^2 (a/4\pi) \xi_{\text{I}}(a) \cdot a/2 \right. \\ & \left. (1 + g'^0) [g'^0 + g'/(1 + g')] (a^0/2\pi) [\xi_{\text{I}}(a) + \xi_{\text{II}}(a)] a, \right. \end{aligned} \quad (6.4)$$

$$\begin{aligned} {}^{(2)}G_m^+(k_0, q_0) = & - (1 + g')^2 (a/4\pi) [2\eta_{\text{I}}(-b) \\ & + 2\xi_{\text{I}}(-b)] \mp (1 + g'^0)^2 (a^0/2\pi) \lambda_{\text{IV}}(a) b \\ & - g' (a/4\pi) [\xi_{\text{II}}(-b) + \eta_{\text{II}}(-b) - (g'/2) (\xi_{\text{I}}(-b) + \eta_{\text{I}}(-b))] b. \end{aligned} \quad (6.5)$$

a, b, d, ξ, η and λ have been defined at the end of §4. The upper sign corresponds to the sym. case, ($2a''=a=f^2/4\pi$), $g''=g'=(2Mg/\mu f)$, while the lower sign stands for the ch.+n. case (in general $2a''\neq a$). It goes without saying that the terms with $(1+g')^2$ represent the effects of the p.s. coupling and certain parts of the p.v. coupling which are equivalent to the former. The remaining terms are "inequivalent" contributions from the p.v. coupling.

The result for the process of negative meson creation $\gamma+N\rightarrow\pi^-+P$ is expressed in a similar form

$${}^{IV}U^- = (1+g')\{G_c^- {}^{II}U_c^- + G_m^- {}^{II}U_m^- + ({}^{(1)}G_m^- {}^{(1)}U_m^- + ({}^{(2)}G_m^- {}^{(2)}U_m^-)\}, \quad (6.6)$$

if one defines

$${}^{II}U_c^- = \{-(\mathbf{Ik})/(\mathbf{Fk})\} {}^{II}U_c^+, \quad {}^{II}U_m^- = \{-(\mathbf{Ik})/(\mathbf{Fk})\} {}^{II}U_m^+, \quad (6.7)$$

$${}^{(1)}U^- = -C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{PN}S_F(\mathbf{v})[\sigma_{\mu\nu}k_\nu/2M]\tau_N\psi(\mathbf{I})A_\mu\phi^*, \quad (6.8)$$

$${}^{(2)}U^- = -C\bar{\psi}(\mathbf{F})[\sigma_{\mu\nu}k_\nu/2M]\tau_P S_F(\mathbf{w})i\gamma_5\tau_{PN}\psi(\mathbf{I})A_\mu\phi^*. \quad (6.9)$$

The numerical factor $G^-(k_0, q_0)$ is derived from $G^+(k_0, q_0)$ by the substitution $a\rightarrow(-b)$ and $b\rightarrow(-a)$. Further, to obtain ${}^{(1)}G_m^-(k_0, q_0)$ one has to put $\lambda_{III}(-b)$ instead of $\lambda_{IV}(a)$ in ${}^{(2)}G_m^+(k_0, q_0)$ and to obtain ${}^{(2)}G_m^-(k_0, q_0)$ put $\lambda_{IV}(-b)$ instead of $\lambda_{III}(a)$ in ${}^{(1)}G_m^+(k_0, q_0)$.

From the above matrix element one can deduce the fourth order cross section ($\sim e^2 f^4$) by multiplying it into the conjugate complex of ${}^{II}U$, doubling the result and summing or averaging over the directions of spin and polarization.

For example, one has for the case of π^+ -production*

$$d\sigma^+/d(\cos\theta) = (1+g')^2(f^2/4\pi)\sigma_0(M/k_0)\sum_{i=1}^4\mathfrak{G}_i^+\chi_i^+, \quad (6.10)$$

where

$$\sigma_0 = (e^2/4\pi)(\pi/2)(1/M^2) = 5.07 \times 10^{-30} \text{ cm}^2, \quad (6.11)$$

$$\mathfrak{G}_1^+ = G_c^+ + G_c^{+*}, \quad \mathfrak{G}_2^+ = (G_c^+ + G_c^{+*} + G_m^+ + G_m^{+*})/2, \quad (6.12)$$

$$\mathfrak{G}_3^+ = G_m^+ + G_m^{+*}, \quad \mathfrak{G}_4^+ = ({}^{(1)}G_m^+ + ({}^{(1)}G_m^{+*} + ({}^{(2)}G_m^+ + ({}^{(2)}G_m^{+*}),$$

and

$$\chi_1^+ = \rho\{q^2 \sin^2\theta \cdot M(k_0 - q_0)/k_0^2(q_0 - q \cos\theta)^2\} > 0, \quad (6.13)$$

$$\chi_2^+ = -\rho\{\mu^2 q^2 \sin^2\theta/2k_0^2(q_0 - q \cos\theta)^2\} - \chi_1^+ < 0, \quad (6.14)$$

$$\chi_3^+ = \rho\{(q_0/k_0) - (\mu^2/2Mk_0)\} > 0, \quad (6.15)$$

$$\chi_4^+ = -\rho(q_0 - q \cos\theta)/2M < 0, \quad (6.16)$$

* In order to obtain the energy spectrum of the emitted meson one may utilize the relation $d(\cos\theta) = dq_0/k_0\rho$.

with

$$\rho = \frac{2q^3}{q_0(2Mk_0 + \mu^2) - 2\mu^2(M + k_0)}, \quad (6.17)$$

θ = the angle between the emitted meson and the incident photon in the laboratory system. χ_1^+ , χ_2^+ and χ_3^+ come respectively from the square of ${}^{\text{II}}U_e^+$, the product of ${}^{\text{II}}U_e^+$ with ${}^{\text{II}}U_m^+$ and the square of ${}^{\text{II}}U_m^+$. χ_4^+ is derived from the product of ${}^{\text{II}}U_m^+$ with ${}^{(1)}U_m^+$ and ${}^{(2)}U_m^+$. The product of ${}^{\text{II}}U_e^+$ with ${}^{(1)}U_m^+$ or ${}^{(2)}U_m^+$ vanishes.

To get a rough idea of these relative magnitude of these $\mathfrak{G}_i\chi_i$'s, a graph is given in fig. (12) for the case of $k_0=2\mu$ in the laboratory system. ($M=1837\text{ m}$, $\mu=276\text{ m}$).

The lowest order cross-section ($\sim e^2 f^2$) can of course be derived from (6.10) by putting $\mathfrak{G}_1=\mathfrak{G}_2=\mathfrak{G}_3=1$, $\mathfrak{G}_4=0$ and agrees with the value hitherto obtained.^{4) 24)} To find the angular distribution in this order one may superpose the three curves χ_1^+ , χ_2^+ and χ_3^+ in the fig. (12); it is seen that the contribution from the circulating current (χ_3^+) is predominant as is already well known.

Now we shall study how the fourth order contributions modify the above result, and for this purpose we shall compare χ_4^+ , which did not appear in the lowest order, with the main term χ_3^+

in the latter. The area under the curves χ_4^+ is approximately one-eighth of that under χ_3^+ , but we must of course take into account their coefficient \mathfrak{G}_3^+ , \mathfrak{G}_4^+ .

Table (I) Numerical values of \mathfrak{G}^+ 's (6.12). ($k_0=2\mu$, $M=6.6\mu$) $A\sim B$ means that the value varies from A at 0° to B at 180° monotonously.

		Pseudoscalar coupling		Pseudovector coupling	
		$f^0=f/\sqrt{2}$, $g'=0$, $a=4\pi$.		$g^0=g/\sqrt{2}$, $f=0$, $g'^2a=4\pi$.	
		a) Sym.	c) Ch.+n.	b) Sym.	d) Ch.+n.
\mathfrak{G}_1^+		0.00	0.00	0.00	0.00
\mathfrak{G}_2^+		0.02~0.05	0.34~0.37	-0.14~-0.15	0.40~0.39
\mathfrak{G}_3^+		0.04~0.10	0.68~0.74	-0.28~-0.30	0.80~0.78
\mathfrak{G}_4^+	(i)	-0.24	-0.24	-0.24	-0.24
	(ii)	-1.90~-1.96	-2.30~-2.48	-1.58~-1.76	-2.58~-2.88
	(iii)	0.70~0.64	0.30~0.12	1.02~0.84	0.03~-0.28

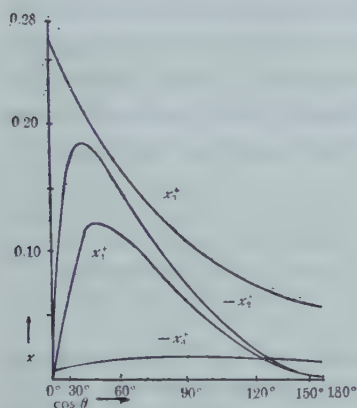


Fig. (12) Angular distribution of the emitted meson for $M=6.6\mu$ and $k_0=2\mu$ (in the laboratory system).

$$\chi_1^+ : (6.13)$$

$$\chi_2^+ : (6.14)$$

$$\chi_3^+ : (6.15)$$

$$\chi_4^+ : (6.16)$$

x should be replaced by χ

For the present case of $k_0=2\mu$ in the laboratory system, we shall give in Table (I) these quantities for four types: a) p.s. coupling in sym. theory, b) p.v. coupling in sym. theory, c) p.s. coupling in ch.+n. theory, d) p.v. coupling in ch.+n. theory.

The appearance of the anomalous magnetic interaction in \mathcal{G}_4 complicates the matter remarkably. As mentioned before, these interactions are due to static anomalous magnetic moments of the latter. To our regret the values of the static parts, which are to be compared with the observed anomalous magnetic moments of the nucleon, do not agree at all with experiments.¹⁷⁾ (See Table (II).) In view of this discrepancy we have taken the following three measures and compared the results. (More detailed discussion on this point will be given in § 7.)

Table (II). Anomalous magnetic moment of nucleon. The static value $G(0)$ has been derived from $G(k_0, q_0)$ ((6.4) and (6.5)) by putting k_0 equal to zero. The non-static correction has been evaluated, putting $k_0=2\mu$, $M=6.6\mu$ and taking an average over the direction of the emitted meson, when it depends on the latter.

	Observed value	Static value		Non-static correction	
		Symmetrical	Charged	a) Sym. (P.S.)	b) Sym. (P.V.)
Proton	-1.79	$^{(1)}G_m^+(0)=0.22$	0.69	$^{(1)}G_m^+ - ^{(1)}G_m^+(0)=0.29$	0.25
Neutron	-1.91	$^{(2)}G_m^+(0)=-1.64$	-1.64	$^{(2)}G_m^+ - ^{(2)}G_m^+(0)=0.17$	0.34

i) One replaces $^{(1)}G_m^+(k_0, q_0)$, (6.4) and $^{(2)}G_m^+(k_0, q_0)$, (6.5) by the observed values of the a.m.m. 1.79 and -1.91 respectively. This implies the neglect of the non-static corrections, which depend on the photon energy k_0 . Since those observed values have opposite signs, the coefficient \mathcal{G}_4^+ , which is twice their algebraic sum, turns out rather small (-0.24) and so $\mathcal{G}_4^+\chi_3^+ \ll \chi_3^+$.

ii) One computes $^{(1)}G_m^+(k_0, q_0)$ and $^{(2)}G_m^+(k_0, q_0)$ without introducing any further assumption. Then the numerical value of \mathcal{G}_4 is considerably larger than in the preceding treatment, because the a.m.m. of the nucleon, derived field-theoretically in the first approximation, differs from the experimental values to a great extent, so that their algebraic sum is not necessarily small.

iii) One replaces the static part of the derived a.m.m.'s by the experimental values, while one employs the calculated non-static correction. Then one has a much larger value 1.79 for the static part of $^{(1)}G_m^+$ than the computed 0.22 $(1+g'^2) a/4\pi$, but a not so much larger value -1.91 for that of $^{(2)}G_m^+$ than the computed $-1.64(1+g'^2)a/4\pi$. This treatment goes, so to speak, between the preceding two. Compared with ii), the static part of the a.m.m. is replaced by the experimental value and compared with i), the non-static correction to \mathcal{G}_4 is more or less taken into account; but this treatment iii) is inconsistent, of course,

since we have replaced only the static part of the a.m.m. by the empirical value. If it is possible to explain the static moment whether in higher orders or through virtual fields of other types, we should expect that the non-static part would be considerably modified too.

All these features are illustrated in fig. (13), where $u=4\pi$ (or $g^{f^2}u=4\pi$) has been assumed.

We can also obtain the cross section of the negative meson production in a quite similar way; i.e., we replace $\mathcal{G}_i^+ \chi_i^+$ in (6.10) by $\mathcal{G}_i^- \chi_i^-$. It may be seen that $\chi_1^-/\chi_1^+ = \chi_2^-/\chi_2^+ = \chi_3^-/\chi_3^+ = \{-(\mathbf{Ik})/(\mathbf{Ek})\}^2$, and $\chi_1^-/\chi_1^+ = -(\mathbf{Ik})/(\mathbf{Ek})$. We summarize in Table (III) the results of π^+ - and π^- -meson production for various cases, to compare and contrast their distinctive characters. Here we have adjusted the coupling constant so as to make the calculated value approximately agree with the experimental one* obtained by Panofsky et al.¹¹⁾ for π^+ -meson production.**

a), i) The fourth-order*** contributions to the cross section are smaller than the second-order ones (i.e. one-sixth even if $f^2/(4\pi)^2 \sim 2$, so that the main features of the latter are conserved, and this conclusion is identical with what Aidzu-Fujimoto-Fukuda³⁰⁾ and independently Kaplon³¹⁾ have inferred from a phenomenological treatment of the a.m.m.'s. Further the fourth-order effects act so as to increase the cross section for π^+ -meson production a little, while they somewhat reduce the π^- -meson production.

a), ii) In this case it appears necessary to choose a rather large value $f^2/(4\pi)^2 \sim 1.4$ for the coupling constant because of the smallness of the second order cross section. We have besides a large value for \mathcal{G}_4 , which results from the unreasonable values

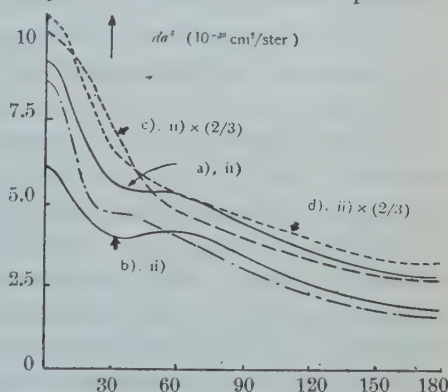


Fig. (13) Differential cross section. ($k_0=2\mu$, $M=6.6\mu$)

- a: Symmetrical theory with p.s. coupling.
 - b: Symmetrical theory with p.v. coupling.
 - c: "Charged plus neutral" theory with p.s. coupling.
 - d: "Charged plus neutral" theory with p.v. coupling.
- ($f^2/4\pi=4\pi$, $g^{f^2}f^2/4\pi=4\pi$)
 - - - - (2nd order)

* $d\sigma^+(90^\circ)/d\Omega \sim 8 \times 10^{-30} \text{ cm}^2/\text{ster.}$

** The coupling constant has been determined, for example in the case of p.s. coupling, as follows. The cross section of π^+ -meson production at 90° is given by

$$d\sigma^+(90^\circ)/d\Omega = (f^2/8\pi^2) \frac{\sigma_0(M/k_0)}{3} \left\{ \sum_{i=1}^3 \chi_i^+(90^\circ) + \sum_{i=1}^4 \mathcal{G}_i^+(90^\circ) \chi_i^+(90^\circ) \right\}.$$

Using the values $(M/k_0)=3.3$, $\sum_{i=1}^3 \chi_i^+(90^\circ) \sim 0.1$, (6.11) for σ_0 , and putting $f^2/(4\pi)^2 = x$,

$\sum_{i=1}^4 \mathcal{G}_i^+(90^\circ) \chi_i^+(90^\circ) \sim 0.1 \times \lambda x$, one obtains roughly $\lambda x^2 + x - 10 = 0$ and one can determine by inserting the calculated λ into the above equation.

*** From a certain point of view, the corrections in the cases i) and iii) may include not only the fourth order contributions but also those from still higher orders, since we have replaced the static magnetic moments by experimental values. We shall use the word "fourth order," though, for brevity's sake.

for a.m.m. of the nucleon. Thus the fourth order correction contributes a little too much in enlarging the cross section for the positive meson production on one hand, and in reducing that for the negative meson on the other. In this case, therefore, one cannot augment the cross section up to the experimental value without departing from the observed ratio of the positive mesons to the negative ones.

a) iii) This time, we have so modified the derived values of the static a.m.m. of the nucleon that the term $\mathcal{G}_4\chi_4$ may not play any significant role, and that, at the same time, the non-static corrections included in \mathcal{G}_4 may work so as to cancel the contributions of \mathcal{G}_3 (and \mathcal{G}_2) and that their balance may make the cross-section increase for π^+ -production and decrease for π^- -production on the other hand. These variations of cross-section in π^+ - and π^- -creation are for themselves slight for the value $f^2/(4\pi)^2 \sim 1.5$, but they are sensitively reflected upon the cross-section ratio, making it rather disagree with the experimental one.

b) The inequivalent part of p.v. coupling to p.s. coupling is larger than the equivalent one and \mathcal{G}_3 (as well as \mathcal{G}_2) changes its sign, contrary to the case a). Accordingly, the fourth order effects behave so as to decrease the cross section in the case of the positive meson production and to increase it in the case of the negative. In the cases b), i) and b), iii) one cannot, therefore, adjust the cross section of π^+ -meson production to the experimental value. Only the case b), ii) is exceptional owing to the large value of \mathcal{G}_4 . The ratio of the cross-sections for the production of negative and positive mesons is, however, quite sensitive to the value of the coupling constant, and one has to take the value of the coupling constant $a/4\pi$ (or $g'^2 a/4\pi$) below ~ 1.5 . Otherwise, one could not get a reasonable value for the cross-section ratio, which would agree with experiment. Table (III) shows that the fourth order contribution turns out in the case b), ii) considerably smaller than that from the second order. One cannot, however, take an optimistic view, since this result depends essentially upon the large value of \mathcal{G}_4^+ , which contradicts with the experiment of the static a.m.m. If one attributes this discrepancy to the effect of some other virtual fields, one must needs evaluate the fourth order $\gamma-\pi$ process through these fields, too. If one assumes, alternatively, that contributions from higher orders than the fourth fill up this gap, one cannot of course stop the work at this stage of approximation.

c) and d) The circumstances in ch.+n. theory are more or less different from the sym. case owing to the distinguished behaviour of $\tau_4(=1)$ from τ_3 when interchanged with τ_{NP} and τ_{PN} . As is seen from Table (II), \mathcal{G}^+ 's in these case are positive and not so small as in a) and b). As regards \mathcal{G}^- , the absolute value in the case c) is larger than in the case d), while those of \mathcal{G}_4^- 's (<0) both in c) and d) are of the same magnitude with those in b). Consequently the resulting fourth order contributions are somewhat less predominant in the case c). However that may be, they are comparable with the second order ones when $a/4\pi$ (or $g'^2 a/4\pi$) ~ 1 , so that it would be senseless to confront them with the experiment.

Table (III) The results of γ - π^\pm process including fourth order corrections.

		$\frac{a}{4\pi}$ or $\frac{g'^2 a}{4\pi}$	$\frac{d\sigma^-(90^\circ)}{d\sigma^+(90^\circ)}$	$\frac{d\sigma^+(90^\circ)}{d\Omega}, 10^{-30} \frac{\text{cm}^2}{\text{ster.}}$	The ratio of fourth order to second order (π^+)	$\lambda(\pi^+)$
2nd order		2.5	1.8	8.1		
a) p.s.	(i)	2.0	1.6	7.5	1/6	0.1
(p.s.)	(ii)	1.4	0.3	7.1	0.6	0.4
sym.	(iii)	1.5	0.8	8.1	0.2	-0.05
b) p.s.	(i)	—	—	—	—	—
(p.v.)	(ii)	1.5	1.1	5.5	1/7	0.1
sym.	(iii)	—	—	—	—	—
c) p.s.	(i)	1.3	0.7	7.3	0.8	0.6
(p.s.)	(ii)	1.1	0.1	7.3	1.1	1.0
ch.+n.	(iii)	1.4	0.2	8.6	0.9	0.5
d) p.s.	(i)	1.3	0.9	7.6	0.8	0.6
(p.v.)	(ii)	1.0	0.3	7.3	1.2	1.2
ch.+n.	(iii)	1.1	0.6	6.9	0.9	0.8

(As for λ , see the second footnote standing on p. 877.)

In closing this lengthy section we may again impress on our mind that the lack of a satisfactory theory for the a.m.m. has made our analysis rather dubious. In cynical words, we may have enumerated detailed values for various cases, not to present them as reliable predictions, but to convince one how unreliable they really are.

§ 7. Discussions

The result of a fourth-order calculation can be relied upon quantitatively only when it turns out negligible compared with the second order one. When it is found conspicuous, we can tell that the second order result is not to be trusted, but we can by no means insist upon the quantitative correctness of the fourth order result. In fact the latter may give an even worse value than the former. Nevertheless we can draw some general qualitative features from the higher order computations, which did not present themselves in the lowest order and which will be useful when we try to analyse the process in question by the help of other approximative methods. It appears, for example, that one should study at least two phenomena, the a.m.m. and the γ - π process, simultaneously. Thus we shall make in the following several qualitative remarks concerning the γ - π process.

(I) *The variation of the non-static part of the a.m.m. with the incident photon energy*

We have seen that our results depend essentially upon the non-static effects of the a.m.m. interaction. We shall, therefore, examine the characteristic behaviour of these dynamical "corrections," taking as an example those parts of nucleon a.m.m., which do not vanish in the static field, appearing in the coefficients $^{(1)}G_m^+$, (6.4) of $^{(1)}U_m^+$ and $^{(2)}G_m^+$, (6.5) of $^{(2)}U_m^+$.

Then, the a.m.m. of the proton and the neutron are given by the following $\mu_P(a)$ and $\mu_N(b)$ respectively :

$$\left. \begin{aligned} \mu_P(a) &= (a/2\pi)\eta_I(a) - (a^0/2\pi)\xi_I(a) , \\ \mu_N(b) &= -(a/2\pi) \{ \eta_I(-b) + \xi_I(-b) \} . \end{aligned} \right\} \quad (7.1)$$

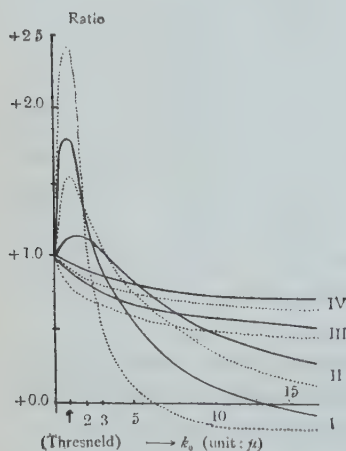


Fig. (14) The dependence of the ratio $\xi(-b)/\xi(0)$ and $\eta(-b)/\eta(0)$ on the photon energy. Solid lines; p.s. theory, dotted lines; s. theory I: $\eta(a)/\eta(0)$, II: $\xi(a)/\xi(0)$, III: $\eta(-b)/\eta(0)$, IV: $\xi(-b)/\xi(0)$. The argument b depends also on the angle (θ) between the incident photon and the emitted meson, and so those putting θ equal to 90° are shown.

As was explained in §4, the ξ 's represent the contributions from virtual nucleon current and the η 's those from virtual meson current. $\mu_P(0)$ and $\mu_N(0)$ are the static ones listed in Table (II).^{*} The dependence of these ξ and η on the incident photon energy is fig. (14), where the ratios of $\xi(-b)$ to $\xi(0)$ and of $\eta(-b)$ to $\eta(0)$ (solid lines) are given.^{**}

The following informations could be obtained from this figure.

(i) The non-static corrections of $\xi(a)$ and $\eta(a)$ have large values at the threshold energy for the $\gamma-\pi$ process, while those of $\xi(-b)$ and $\eta(-b)$ are not so conspicuous. The former functions have been derived from the figs. (4.1) and (7.1), which are characterized by the fact that the requirement of energy-momentum conservation can be fulfilled at an intermediate stage. Thus their behaviour near the threshold energy seems to indicate a violent distortion of the virtual meson and nucleon clouds, due to the circumstances that a real meson is about to be emitted in the inter-

* $\mu_P(0) = ^{(1)}G_m^+(0)$ and $\mu_N(0) = ^{(2)}G_m^+(0)$ in the static field and $a=2a^\circ$ for the sym. theory. Also, we find that $\xi_I(0)$ and $\eta_I(0)$ correspond to the B_1/x_0 and B_2/x_0 of Case¹⁷⁾ respectively; their numerical values will be listed in Table (IV).

** When the emitted meson line is replaced by the photon line, all diagrams cited in this section are translated into those for the Compton scattering by the nucleon, and so the curves in fig. (14) have been drawn from the origin which corresponds to the photon of energy zero. The variation of ξ and η for the $\gamma-\pi$ process begins, of course, at the threshold energy of the photon.

mediate stage, although such a real process has been subtracted.*

On the contrary, the latter functions have no such "resonance" effects,** since the $\gamma-\pi$ processes like the figs. (5.1) and (5.2), from which these functions have been derived, can not realize energy-momentum conservation in an intermediate state. The above considerations would apply to other ξ 's, η 's and λ 's as well.

(ii) The variation of the virtual meson current terms (η) is larger than that of virtual proton current term (ξ), probably showing the difference between the energy of virtual meson and that of virtual proton. Now one can picture, in a sense, that the p.s. meson cloud is closely bound to the nucleon and contains a considerable amount of energy, while the s. meson is rather loosely bound. Thus one could expect that the difference between the variations of ξ and η will be smaller in the p.s. theory than that in the s. theory. In fact the fig. (14) shows clearly such tendencies (dotted line).***

These arguments will hold also for the $\gamma-\pi^-$ process only with the substitution $a \rightarrow (-b)$ and $b \rightarrow (-a)$, which almost reverses the absolute values of the non-static corrections for the proton and neutron.

The name "non-static moment" may be justified by such complicated behaviour of the quantities mentioned above. If their dependence on k_0 could be expressed as polynomials, we should have called them "higher moments" of our dynamical system.

(II) How to supplement the static anomalous magnetic moments

It has been shown that the non-static a.m.m. interactions of the nucleons are, in general, not small in comparison with the static ones. Our computation, however, does not seem to have given their correct values. (Cf. Table (II).) It

* Cf. the footnote on p. 867.

** Such large variation would correspond to the resonance effects indicated by Sachs and Foldy⁽⁴⁸⁾. They have performed, in the non-relativistic approximation, an estimation on the Compton scattering by the nucleon. The relativistic calculation on the same problem has already been carried out by N. Mugibayashi et al. and their results will be published later in this journal. The results obtained by Sachs and Foldy do not indicate any appreciable resonance effect in the s. meson theory because the a.m.m.'s of the nucleon treated by their non-relativistic method are equal to zero, but the Lorentz-covariant perturbation method will yield for the s. meson the same resonance anomaly as in the p.s. theory though here arises a new complication owing to the fact that the calculated static anomalous moment has the opposite sign to the observed value. See the Table (IV).

*** Rosenbluth,⁽⁴⁹⁾ who has carried out a calculation on the "scattering of electrons by proton," has evaluated the variations of ξ and η for the process, in which virtual photon lines stand instead of our real ones, and real meson lines instead of our virtual ones. The variations of ξ and η estimated by him have no resonance effects, because evidently the energy-momentum conservation of the system does not take place in the intermediate state; consequently they show the same trend as the $\xi(b)$ and $\eta(-b)$ calculated here. The variation in the s. theory is larger than in the p.s. theory for the virtual meson current term (η) (fig. (3) in Rosenbluth's paper) and *vice versa* for the virtual proton current term (ξ), contrary to our case (fig. (4) in the same paper), but the modification of η is larger than that of similarly to our conclusion.

would be impossible for us to evaluate them reasonably, before we could give a convincing explanation of the static values of the a.m.m.

The second approximation ($\sim ef^4$) of the static a.m.m. of the nucleons in the p.s. theory have been found by Nakabayasi and Sato^{(50)*} to be quite large and to be able to remove considerably the discrepancies between the lowest order values (ef^2) and the observed ones. Moreover, the coupling constants required for this are nearly consistent with those for the $\gamma-\pi$ process, for example, $f^2/(4\pi)^2 \sim 2$. This fact suggests that higher approximations may be able to get rid of the present difficulties regarding the static moment. If this is true, we shall reach unambiguous values for the non-static parts, proceeding along this line. Without laborious computations, however, we cannot further advance in this way. What is worse, the convergence of the perturbation method is quite questionable. We shall, therefore, discuss this problem at another place.

Three measures have been taken in §6 to adjust the calculated values of a.m.m. in our approximation. A fourth possibility is to add to the Hamiltonian a Pauli type interaction,

$$(e/2)\bar{\psi}(\mu_P\tau_P + \mu_N\tau_N)(\sigma_{\mu\nu}/2M)\psi(\partial A_\mu/\partial x_\nu - \partial A_\nu/\partial x_\mu), \quad (7.2)$$

from the beginning and to carry out a straightforward calculation for the process to the order ef^3 . One can regard this Pauli-type interaction as an elementary one and treat it on completely equal footing with the usual coupling.** Evidently the static values of a.m.m. can be made to agree with experiment only if μ_P and μ_N are chosen properly. Alternatively this added interaction may be looked upon as standing for all static and non-static corrections, which we should have when all the higher order calculation above ef^3 would have been carried out without any Pauli term.*** This procedure has the advantage of introducing the non-static corrections as well as the static ones quite naturally, but it seems that the modification by them would not change the main features of our results, and so we shall take occasion to discuss this prescription when we shall treat the neutral

* They have calculated such corrections for p.s. meson with p.s. coupling in various theories: charged, sym. and "sym. plus pure neutral." From their results we can easily obtain, for the ch.+n. theory used in our paper, the following values, the meson mass being $273 m_e$:

$$\text{Proton moment: } \mu_P(0) = [0.35(a/2\pi) - 0.24(2a^\circ/2\pi)] + [0.25(a/2\pi)^2 - 0.13(2aa^\circ/(2\pi)^2) - 0.07(2a^\circ/(2\pi))^2],$$

$$\text{Neutron moment: } \mu_N(0) = -[0.35(a/2\pi) + 0.47(a/2\pi)] - [0.27(a/2\pi)^2 - 0.36(2aa^\circ/(2\pi)^2)].$$

If one puts a equal to $2a^\circ$ and adjusts $\mu_P(0)$ to the observed value (+1.79), the coupling constant (a) and neutron moment μ_N turn out: $a/2\pi = 5.33$ and $\mu_N = -1.65$. The p.s. (p.v.) will possibly give considerably different results.

** Such an opinion is entertained, for example, by S. Sakata.

*** This has been suggested by S. Tomonaga. We wish to express our sincere appreciation to Prof. S. Tomonaga for many discussions on this point.

π -meson production by γ -ray, in which the a.m.m. interaction plays an especially important role.*

(III) The other types of the meson field

In this paper, the p.s. meson theory has been selected for the reasons mentioned before. (Cf. § 1.) Our results have, however, revealed the important role of the a.m.m. interaction in the γ - π process, and since the p.s. theory is—at least in the first approximation—quite unsatisfactory regarding the static values of the a.m.m., it will be worth while to compare what would be expected from various types of the meson field.

The static a.m.m. of the proton ($\mu_P(0)$) and the neutron ($\mu_N(0)$) for the four types of the meson field, as have been worked out by many authors, are given by the formulas (7.1) with respective ξ 's and η 's listed in the Table (IV).

Table (IV) The static a.m.m. in the lowest order, calculated for various meson theories without derivative coupling.

Proton moment; $\mu_P(0) = (a/2\pi)\eta(0) - (a^0/2\pi)\xi(0)$,

Neutron moment; $\mu_N(0) = -(a/2\pi)[\eta(0) + \xi(0)]$.

To put $a=2a^0$ for the symmetrical, $a=0$ and $a^0 \neq 0$ for the pure neutral, $a \neq 0$ and $a^0=0$ for the charged and $a \neq 0$, $a^0 \neq 0$ (in general $a \neq 2a^0$) for the "ch.+n." theory. ξ 's and η 's differ from each other only in the numerators.

		$\int_0^1 dx \frac{\text{Numerator}}{x^2 + \epsilon(1-x)}, \quad \epsilon = \left(\frac{\mu}{M}\right)^2$	Numerical value for the meson of	
			mass 276 m	mass 1000 m
S. ^{17), 19), 21)**}	η_s	$(1-x)x(x-2)$	-2.02	-0.65
	ξ_s	$x^2(x-2)$	-1.16	-0.74
V. ^{16), 21), 22)**}	η_v	$2(1-x)x^2$	0.69	0.36
	ξ_v	$-2(1-x)x^2$	-0.69	-0.36
P.V. ^{21), 22)**}	$\eta_{p.v.}$	$2(1-x)(1-3x+x^2) + 4x^2(1-x)/\epsilon$	69.60	3.28
	$\xi_{p.v.}$	$2(1-x)(4-x)x + 4x^3/\epsilon$	92.79	8.06
P.S. ^{16)-21), 22)}	η_I	$(1-x)x^2$	0.35	0.18
	ξ_I	x^3	0.47	0.38

It can be inferred*** from the general consideration on the isotopic spin space that the contribution from the electromagnetic interaction of the virtual

* It is indicated by the Berkeley experiments¹⁰⁾ that the total cross section for the neutral meson production by the γ -ray is nearly equal to that for the charged meson production. The results of the second order perturbation for the γ - π process are very small owing to the cancellation of main terms, but the next order contributions, which depend essentially on the a.m.m. interaction, are large enough to give a reasonable value for the γ - π^0 cross section in comparison with the experiments.

** Misprints in the original works have been corrected.

*** We are indebted to Mr. H. Miyazawa for having pointed out this fact.

meson* to the a.m.m. of the proton is equal and opposite to that for neutron, irrespective of the order of approximation. This indicates a sufficient, but of course not necessary, condition to explain the experimental ratio of the a.m.m. of proton and neutron: the contribution from η should predominate. In the lowest order calculation shown in Table (IV) the absolute value of η larger than $|\xi|$ is given only by the scalar theory, and this may correspond to the circumstances that the meson cloud in the s. theory is, as it were, more widely spread than in other cases. The angular distribution of the emitted meson in the s. theory for the $\gamma-\pi^\pm$ process has, however, the form $\sin^2\theta$ in contradiction with the Berkeley experiments owing to this large spread of the meson cloud.²⁴⁾ Such a distribution results from χ_1 , which in its turn comes from the interaction of the photon with the convection meson current of the meson-nucleon system (${}^{II}U_c$), rather than χ_3 which is given by the magnetic interaction of the nucleon (${}^{II}U_m$). In other words, the experiments of the magnetic moments could be explained by such an extended meson cloud as in the s. theory, while the $\gamma-\pi$ experiments do not favour it.

Qualitative results of the v. meson theory in the lowest order perturbation do not agree either with experiments on the magnetic moment or with those on the angular distribution of the emitted meson. This theory gives namely an identical expression with opposite signs for $\hat{\xi}_v$ and η_v , so that the neutron moment vanishes. It predicts, on the other hand, a forward peak in the angular distribution, owing to the prevalent interaction of the photon with magnetic moment of the meson. (See the fig. (5) in Brueckner's paper.)

In order to examine the p.v. theory, we shall begin with a remark on the calculation of the first approximation (ef^2) for the a.m.m. in the meson theories of the spin one. If the meson wave function, ϕ_λ say, in the usual formalism is replaced by $\{\varphi_\lambda + (1/\mu)\partial B/\partial x_\lambda\}$ according to Stueckelberg and then a transformation is performed by a method similar to that used in § 5, to decompose the p.v. coupling of the B -field into a part equivalent to the coupling and an inequivalent part, it is found that the moments given by the equivalent part are analogous to that gained by the p.s. theory, only replacing the p.s. coupling constant (f) in the latter theory by a new constant $((2M/\mu)f_{p.v.} = (2/\sqrt{\epsilon})f_{p.v.})$ and turn out far larger than those given by the φ_λ -field, whereas the inequivalent part does not contribute to the moments in this order. In the v. theory the part equivalent to the s. coupling does not appear by the above transformation, as one could expect from the equivalence theorem for the s. theory.⁵¹⁾ The non-static a.m.m. obtained in the p.v. theory shows, owing to the above circumstances, much the same variation with the photon energy as that in the p.s. theory, while η_v and $|\hat{\xi}_v|$ are equal to η_I in the p.s. theory, whose variation is shown in the fig. (14). This effect could not, however, apparently reveal itself on account of the pre-dominant magnetic interaction of the meson with the photon in this theory.

* This is identical with our η in the first approximation. In a higher approximation it can include more than what is usually defined as meson current term.

It should seem that the p.v. theory could not entirely be excluded since in the first approximation it agrees to some extent with the present experiments of the $\gamma-\pi^\pm$ production and the π^- -capture,³⁶⁾ though the neutral meson is certainly spinless. But when we want to consider the next order corrections for the $\gamma-\pi^\pm$ process, we shall immediately come upon the same difficulties as in our paper, because this theory is disappointing in explaining the observed moments as was also the case with the p.s. theory.

The mixture theories of two or more meson fields can also be considered in order to overcome the present uncertainties in the corrections for the $\gamma-\pi$ process. To derive observed static a.m.m. in the second order calculation, there are several mixed theories possible:³²⁾ unsymmetrical spinless meson theory proposed by Hulthén which consists of the neutral s. field and the charged p.s. field, Schwinger's mixed type, Møller and Rosenfeld's mixed theory, and those which replace the p.s. field in the above theories by the p.v. field. We are of the opinion, though, that such mixture should be determined after having estimated higher order effects for individual fields. Mixed theories are very interesting in connection with both the V-particles observed in the recent experiments and the latest study of the nuclear scattering in the domain of relatively high energies. We shall have another chance when these theories are minutely considered.

(IV) The roles and significance of the four fundamental components

As has been deduced in § 3, the components of the matrix element for the $\gamma-\pi^+$ production can be limited, irrespective of the order of approximation, to the following four types:

$${}^{II}U_c^+ = C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}[iI_\mu/(\mathbf{Ik}) - iq_\mu/(qk)]\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(q), \quad (2.9)$$

$${}^{II}U_m^+ = C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}[\sigma_{\mu\nu}k_\nu/2(\mathbf{kI})]\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(q), \quad (2.10)$$

$${}^{(1)}U_m^+ = -C\bar{\psi}(\mathbf{F})i\gamma_5\tau_{NP}S_F(\mathbf{v})[\sigma_{\mu\nu}k_\nu\tau_P/2M]\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(q), \quad (3.12)$$

and

$${}^{(2)}U_m^+ = -C\bar{\psi}(\mathbf{F})[\sigma_{\mu\nu}k_\nu\tau_N/2M]S_F(\mathbf{w})i\gamma_5\tau_{NP}\psi(\mathbf{I})A_\mu(\mathbf{k})\phi^*(q). \quad (3.15)$$

The fourth order matrix element, for example, is expressed by

$${}^{IV}U^+ = (1+g')(G_c^+{}^{II}U_c^+ + G_m^+{}^{II}U_m^+ + {}^{(1)}G_m^+{}^{(1)}U_m^+ + {}^{(2)}G_m^+{}^{(2)}U_m^+). \quad (6.1)$$

The coefficients G 's, which depend on the incident energy of the photon and the angle of emission, are of course different functions according to the order of approximation. For the evaluation of the cross section we need, in general, besides the four χ 's (6.13-16), given in § 6, two expressions χ_5^+ and χ_6^+ , which are shown in Table (V), and graphically in fig. (15). χ_5^+ and χ_6^+ are derived from the square of ${}^{(1)}U_m^+$ or ${}^{(2)}U_m^+$ and the product of ${}^{(1)}U_m^+$ with ${}^{(2)}U_m^+$ respectively. χ_5^- and χ_6^- for the $\gamma-\pi^-$ production are equal to χ_6^+ and χ_5^+ individually.

Though we have used the above four types in order to simplify the results of the second order perturbation, to carry out the renormalization procedure conveniently, and to see easily the character of the angular distribution, there are other selections possible for the four independent gauge invariant components of the matrix element. For example, ${}^{\text{II}}U_m^+$ (2.10), being just the third term of (40) in Brueckner's paper²⁴⁾ (p. 648), is composed of the Pauli type interaction ${}^{(1)}U_m^+$ (3.12) and the "virtual spin effect" ${}^{(1)}U_v^+$ (3.11). (Cf. (3.10).) When one used ${}^{(1)}U_v^+$ instead of ${}^{\text{II}}U_m^+$, χ_3' and χ_4' appear in the expression (6.10), which are

Table (V). The various χ 's appearing in the expression (6.10) both for the s. and p.s. theories. (p.s.) and c.c. mean the same expression as is obtained by the p.s. theory and the complex conjugate respectively. R expresses $(-\mathbf{Ik})/(\mathbf{Fk})$.

The $\gamma-\pi^+$ production*				The $\gamma-\pi^-$ Production
The p.s. meson theory		The s. meson theory		Both the s. and p.s. theories
$\frac{\chi_1}{\rho} = \frac{q^2 \sin^2 \theta}{(\mathbf{qk})^2} M(F_0 - M) > 0$	$\mathfrak{G}_1 = G_c^* \cdot G_c$	$\frac{\chi_{1s}}{\rho} = \frac{q^2 \sin^2 \theta}{(\mathbf{qk})^2} 2M^2 + (\mathbf{p}, \mathbf{s}) \gg 0$	$\mathfrak{G}_{1s} = (\mathbf{p}, \mathbf{s})$	$\chi_1^- = R^2 \chi_1^+$
$\frac{\chi_2}{\rho} = \frac{q^2 \sin^2 \theta}{2(\mathbf{qk})} < 0$	$\mathfrak{G}_2 = \frac{1}{2} (G_m^* \cdot G_c + c.c.)$	$\frac{\chi_{2s}}{\rho} = (\mathbf{p}, \mathbf{s}) < 0$	$\mathfrak{G}_{2s} = (\mathbf{p}, \mathbf{s})$	$\chi_2^- = R^2 \chi_2^+$
$\frac{\chi_3}{\rho} = 1 - \frac{(\mathbf{qk})}{(\mathbf{Ik})} > 0$	$\mathfrak{G}_3 = G_m^* \cdot G_m$	$\frac{\chi_{3s}}{\rho} = (\mathbf{p}, \mathbf{s}) > 0$	$\mathfrak{G}_{3s} = (\mathbf{p}, \mathbf{s})$	$\chi_3^- = R^2 \chi_3^+$
$\frac{\chi_4}{\rho} = -\frac{(\mathbf{qk})}{2(\mathbf{Ik})} < 0$	$\mathfrak{G}_4 = G_m^* \cdot (1)G_m + G_m^* \cdot (2)G_m + c.c.$	$\frac{\chi_{4s}}{\rho} = 1 - (\mathbf{p}, \mathbf{s}) > 0$	$\mathfrak{G}_{4s} = G_m^* \cdot (1)G_m - G_m^* \cdot G_m^{(2)} + c.c.$	$\chi_4^- = R \chi_4^+$
$\frac{\chi_5}{\rho} = \frac{(k_0 - q_0)}{2M} > 0$	$\mathfrak{G}_5 = (1)G_m^* \cdot (1)G_m + (2)G_m^* \cdot (2)G_m$	$\frac{\chi_{5s}}{\rho} = 1 + (\mathbf{p}, \mathbf{s}) > 0$	$\mathfrak{G}_{5s} = (\mathbf{p}, \mathbf{s})$	$\chi_5^- = \chi_5^+$
$\frac{\chi_6}{\rho} = \frac{(\mathbf{qk})^2}{4(\mathbf{Fk})(\mathbf{Ik})} > 0$	$\mathfrak{G}_6 = (1)G_m^* \cdot (2)G_m + c.c.$	$\frac{\chi_{6s}}{\rho} = -1 - (\mathbf{p}, \mathbf{s}) < 0$	$\mathfrak{G}_{6s} = (\mathbf{p}, \mathbf{s})$	$\chi_6^- = \chi_6^+$

derived respectively from the square of ${}^{(1)}U_v^+$ and the product of ${}^{(1)}U_v^+$ with ${}^{(1)}U_m^+$, and are given by

$$\chi_3' = \rho \left[1 + \frac{k_0 - q_0}{2M} \right], \quad (7.3)$$

$$\chi_4' = -2\rho \left[\frac{k_0 - q \cos \theta}{2M} \right]. \quad (7.4)$$

* We have omitted the symbols (+) in this Table.

The relation, $\chi_3 = \chi_3' + \chi_4' + \chi_5$, is obtained easily. By the help of the other relation (3.10) between ${}^{II}U_m^+$ and ${}^{(2)}U_v^+$, one can reach the same result. The angular dependence of these functions is given explicitly in fig. (15). Further, χ_2 is derived rather from the product of ${}^{II}U_0^+$ with ${}^{(1)}U_v^+$ than from the product of ${}^{II}U_c^+$ with ${}^{II}U_m^+$, because the product of ${}^{II}U_c^+$ with ${}^{(1)}U_m^+$ vanishes. In other words, this χ_2 has an angular distribution like dipole radiation (fig. (12)), while χ_3' which represents the square of ${}^{(1)}U_v^+$ or χ_4' derived from the product of ${}^{(1)}U_v^+$ with ${}^{(1)}U_m^+$, assumes a character of the magnetic moment interaction. It is to be noticed that the contribution from ${}^{(1)}U_v^+$ is the largest one, as shown clearly in figs. (12) and (15), and this term corresponds to the relativistic effect, standing for both the spin and the recoil of the nucleon.

Since these χ 's are independent of the perturbation calculation, it would be of some use to introduce the coefficients (\mathcal{G} of G) of these χ 's parametrically and determine them numerically as the function of the energy (q_0) and the direction (θ) of the emitted meson (or the photon energy (k_0)) by confronting them with

the experiments.* It seems that the ratio of the emitted meson in direction 0° to 90° should be especially important by the reason of its sensitive dependence on the variants of meson theory in the energy region to the several hundreds Mev., and also in order to determine the coefficients of χ_4 , χ_5 and χ_6 which involve the non-static a.m.m. The contribution from χ_3 , principally χ_3' , to the total cross section is larger than others in the p.s. theory for the sufficiently high energy, provided that the coefficients are all of the same order, not only because the parts of the total cross section which come from $\chi_1 + \chi_2$ and χ_3 are approximately equal to $(\mu^2/M^2 k_0^2)$ and $(1/M k_0^0)$ severally, but also because those from χ_4 , χ_5 and χ_6 turn more rapidly into the smaller ones than the parts of χ_1 on account

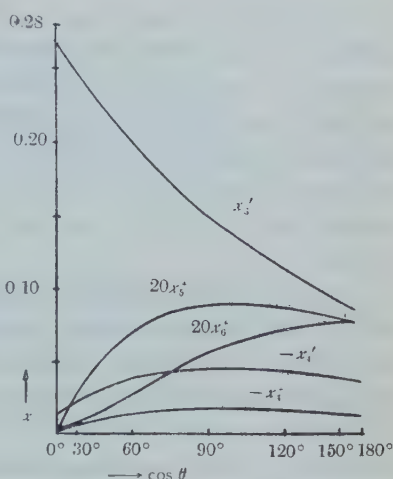


Fig. (15) Angular distribution of the emitted p.s. meson for $k_0 = 2\mu$ in the laboratory system. (C.f. Fig. (12).) χ_3' : (7.3), χ_4' : (7.4). Other χ 's are given in the Table (V).

x should be replaced by χ

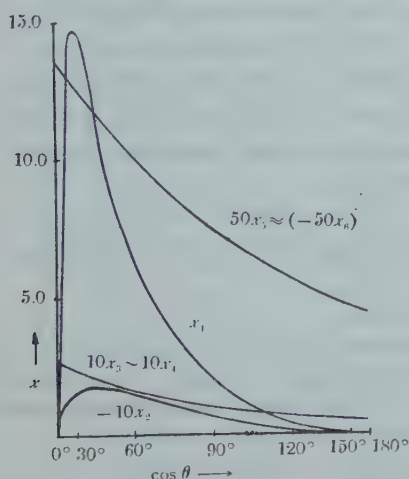


Fig. (16) Angular distribution of the emitted s. meson for $k_0 = 2\mu$ in the laboratory system. The definitions of χ 's are given in the Table (V).

x should be replaced by χ

* See the note added in proof, where an additional remark on this point is made.

of the numerator, e.g., $(q_0 - q \cos \theta)$ and $(k_0 - q_0)$. Such a determination of the coefficients may afford in some measure a check on the meson-nucleon interaction of the p.s. meson theory. The variations of six χ 's for the s. meson theory derived from the four components of the matrix element corresponding to those of the p.s. theory, $i\gamma_5$ in the latter being replaced by unity, are also shown fig. (16) and Table (V).^{*} In this case, χ_1 is larger than the other χ 's in the low energy region ($k_0 \lesssim M$), but χ_3 becomes the largest one in the sufficiently high energy.

Some of the qualitative aspects, which we have thus far examined, seem correct, but the quantitative studies have been quite unsatisfactory. It is in our case rather questionable whether a further advance along the line of weak coupling approximation would be duly rewarded. We are inclined, therefore, to conclude our lengthy report with the trite remark that an approximative method, which assumed a strong or intermediate coupling and which takes into account the relativistic effects must be established, in order to test the reliability of the current meson theory.

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* For the v. and p.v. meson theory, the matrix element corresponding to (3.1) in the spinless theory takes the form:

$$C\bar{\psi}(\mathbf{F})P_{\mu\nu\lambda}(\mathbf{k}, \mathbf{I}, \mathbf{F}, \boldsymbol{\gamma})\psi(\mathbf{I})A_{\mu}(\mathbf{k})\mathbf{k}_{\nu}\phi_{\lambda}^*(\mathbf{q}).$$

Using both the condition $q_{\lambda}\phi_{\lambda}^*=0$ and the over-all energy-momentum conservation $\mathbf{I}+\mathbf{k}=\mathbf{F}+\mathbf{q}$, the terms having the index λ in $P_{\mu\nu\lambda}$ are restricted to I_{λ} , F_{λ} and γ_{λ} . If one denotes, for short, these terms by ζ_{λ} , one obtains for $P_{\mu\nu\lambda}$ fifteen fundamental expressions antisymmetric with regard to the indices μ and ν , in the same way as described in § 3, as follows:

$$\gamma_5'(I_{\mu}F_{\nu}-I_{\nu}F_{\mu})\zeta_{\lambda}, \quad (7.6)$$

$$\gamma_5'(\gamma_{\mu}\gamma_{\nu}-\gamma_{\nu}\gamma_{\mu})\zeta_{\lambda}, \quad (7.9)$$

$$\gamma_5'(\gamma_{\mu}I_{\nu}-\gamma_{\nu}I_{\mu})\zeta_{\lambda}, \quad (7.7)$$

$$\gamma_5'(\zeta_{\mu}\delta_{\lambda\nu}-\zeta_{\nu}\delta_{\mu\lambda}), \quad (7.10)$$

$$\gamma_5'(\gamma_{\mu}F_{\nu}-\gamma_{\nu}F_{\mu})\zeta_{\lambda}, \quad (7.8)$$

where γ_5' means $i\gamma_5$, γ_5 or i according to the employed type and coupling of the meson field. Hence the χ 's from these expressions are very numerous compared with the six for the spinless meson and so will not be written down in this paper. We only remark that, as in the second order results, the parts of χ , which play relatively important roles, are derived from the interaction of the photon with the magnetic (or quadrupole) moment of the v. or p.v. meson.

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Note added in proof :

We have discussed the angular distribution of the emitted meson for scalar and pseudoscalar cases, which would be expected if one neglected the angular dependence of the coefficients of the χ 's. Such an assumption, however, could in general hardly be justified. This can be most clearly seen, when one considers the dual cases referred to in the footnote on page 857: If one interchanges the roles of the electric and the magnetic force of the incident γ -ray, our χ 's for p.s. meson go over into those for s. meson and vice versa. Thus our two groups of χ 's are linearly dependent on one another, according to the argument in the above-mentioned footnote. It is therefore impossible to distinguish between the two types of the spinless meson field by comparing the experimental angular distribution with our χ 's, except for the case of very weak coupling, where the perturbational picture may resemble reality. We express our cordial thanks to Mr. Y. Yamaguchi for his valuable advice and discussion on this point.

Letters to the Editor

S-Matrix and Nucleon Isobar

S. Kamefuchi, H. Nakai, and R. Kawabe

*Institute of Theoretical Physics,
Nagoya University*

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In order to obtain the energy eigen-value of closed system, the following two methods are usually employed, namely, (a) solving Schrödinger equation with suitable potential and (b) the analytic continuation method of S-matrix¹⁾. When the potential actually exists, both methods are shown to be equivalent. With the method (b), however, not only can we avoid the two step procedure of (a) (i.e. setting up the potential from the scattering matrix element and solving the Schrödinger equation), but also treat the case in which the potential in its ordinary sense does not exist, say, nucleon-meson system. In this note, we shall attempt to construct an approximate S-matrix, which may afford the binding energy in the same approximation as the method (a) if we start from the same scattering matrix element, and apply it to the problem of nucleon isobars.

For brevity, we consider the scattering problem of two-body system. (The extension to many body problem is straightforward.) The Schrödinger equation of the system is

$$H\Phi = E\Phi, \quad H = H_0 + H', \quad (1)$$

where H_0 is the free part of Hamiltonian in any field theory and H' interaction term describing the transition of two particles (obtained by a suitable canonical transformation). Now, *restricting the amplitudes of Φ to those of two body state only*, we expand Φ as follows,

$$\Phi = \sum_P \Psi(P) \varphi(P), \quad (2)$$

where $\varphi(P)$ is the eigenfunction of H_0 corresponding to the state of two particles with momenta $P_1 = P$ and $\Psi(P)$ the corresponding amplitude. Inserting (2) into (1), we get $\{E_P - E_0\} \Psi(P) = - \sum_q (P|H'|q) \Psi(q)$. (3)

The solution of (3), satisfying the boundary condition of the scattering problem, is given by

$$\Psi = \Psi_0 + \Psi_{sc}, \quad (4)$$

where Ψ_0 is the incident wave and

$$\Psi_{sc} = G\Psi, \quad (5)$$

$$(P|G|q) = -2\pi i \delta_+(E_P - E_0) (P|H'|q). \quad (6)$$

Then, (3) takes the following form

$$\Psi = \Psi_0 + G\Psi. \quad (3')$$

Therefore, the solution of (3') is reduced to

$$\Psi = \frac{1}{1-G} \Psi_0, \quad (7)$$

or

$$\Psi_{sc} = \frac{G}{1-G} \Psi_0. \quad (7')$$

Expanding (7') in power of G , the term proportional to G^n corresponds to the scattered wave in the n -th order Born approximation. Further, we define another operator \tilde{G} by

$$(P|\tilde{G}|q) = 2\pi i \delta_-(E_P - E_0) (P|H'|q). \quad (8)$$

(In the configuration space, the operators G , \tilde{G} correspond to Green functions $(-1/4\pi) \exp(\pm ik|x-x'|) V(x')/|x-x'|$, where $V(x)$ is a potential.) According to Heisenberg-Møller's definition¹⁾, S-matrix is given by

$$S = 1 + (G - \tilde{G}) \frac{1}{1-G}. \quad (9)$$

The scattered wave, the asymptotic amplitude of which satisfies Heitler's damping equation²⁾, is written as follows

$$\Psi_{se, H} = G \frac{1}{1 - \mathfrak{G}} \Psi_0, \quad (10)$$

where $\mathfrak{G} = (G - \tilde{G})/2$. The corresponding S-matrix³⁾ is therefore

$$S_H = \frac{1 - i\pi H'}{1 + i\pi H'}, \quad (11)$$

where $(\mathcal{U}|\mathbf{H}'|\mathcal{U}') = (\mathcal{U}|\mathbf{H}'|\mathcal{U}')\rho_{\mathcal{U}}$ is the sub-matrix in the same energy shell $E_{\mathcal{U}} = E_{\mathcal{U}'}$ and $\rho_{\mathcal{U}}$ the state density. The unitarity of S is evident from the properties of G , \tilde{G} .

The S-matrix thus obtained has the following property: If we take as \mathbf{H}' in (1) e.g., the collision matrix element of 2nd order in coupling constant (2nd order potential), it is easily found by expanding (7') in power of G , that our method corresponds to the approximation containing only the effect of interaction of 2nd order potential. In other words, it corresponds to the approximation neglecting the effect of higher order potential in Dyson's⁵⁾ expansion of S-matrix. Following the program of Heisenberg-Krammers,¹⁾ we can obtain the energy eigenvalue of closed system from the poles of these S-matrices.

Next, we shall apply our formula to nucleon-meson system (nucleon isobar). For simplicity, we take scalar meson theory with scalar coupling, and take an approximation neglecting the recoil of nucleon for the collision matrix element. Under this approximation,

the diagonalization of S becomes very easy. We cut off the high momenta, since, if not, (9) leads to divergent results. (This is unnecessary for (11), where only the diagonal element appears.) In this case, we have the following scattering processes:

scattering of $\pi^+(\pi^-)$ by proton (neutron) (A)

scattering of $\pi^-(\pi^+)$ by proton (neutron) (B)

(A) gives the charge isobar with total charge $2(-1)$, and (B) the one with $0(1)$. The collision matrix elements of (A) and (B) in the lowest approximation are given by

$$\left. \begin{aligned} \text{(A)} \quad (\mathbf{k}|\mathbf{H}'|\mathbf{k}') &= -g^2/2E_k^{1/2}E_{k'}^{3/2}, \\ \text{(B)} \quad (\mathbf{k}|\mathbf{H}'|\mathbf{k}') &= g^2/2E_k^{3/2}E_{k'}^{1/2}. \end{aligned} \right\} \quad (12)$$

Calculating G , \tilde{G} from (9), (11) and performing the diagonalizations, we can obtain the binding energy of the system from their zero points on negative imaginary axis of k . The results are shown in table (I), where, the isobar-states are allowed for the given binding energies (in the meson mass unit) and the indicated values of the coupling constant. * means no isobar-state for any value of them. k and μ represent the cut off factor and meson mass respectively.

When the number of particles, varies S-matrix becomes a piece-wise analytic function on the complex plane. In this case, remembering N. Hu's⁶⁾ remark, we can no longer make use of the method (b) for our present purpose. In view of this point we have fixed the number of particles just as usually

Bind. E.		0.01 μ	0.05 μ	0.01 μ	0.2 μ	0.3 μ	0.4 μ	0.5 μ	0.6 μ	0.7 μ	0.8 μ	0.9 μ
Coupl. const.												
$K=7\mu$	(A)	*	*	*	*	*	*	*	1.94	0.65	0.29	0.10
	(B)	0.53	0.58	0.66	0.82	1.26	3.68	15.0	*	*	*	*
$K=\mu$	(A)	*	*	10.3	2.06	1.10	0.72	0.50	0.36	0.24	0.15	0.07
	(B)	2.65	7.22	*	*	*	*	*	*	*	*	*
Damping theory	(A)	*	*	*	*	*	*	*	*	*	*	*
	(B)	7.14	3.05	2.07	1.33	0.98	0.75	0.58	0.44	0.31	0.20	0.10

done in method (a). Furthermore, we have neglected the recoil of nucleon to facilitate the diagonalization of S-matrix. These procedures can not always be regarded as a good approximation for the problem of meson field around nucleon, because it is expected that the reaction of eigen-field will play an important role in the vicinity of nucleon. It seems to be for this reason that the results of Table I show the considerable dependence on the cut off factor and the effect of damping. In this respect, we may say, S-matrix will not be so powerful method to determine the binding energy unless the exact S-matrix is known, because an approximation for S matrix will not always be the same order approximation for binding energy. It is very interesting, however, that without further assumptions we can expect the existence of nucleon isobar even under the weak coupling approximation in the present form of meson theory.

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V Particle as a Composite One

H. Suura

Department of Physics, Faculty of Science,
Hiroshima University

August 11, 1951

The possibility that the neutral V particle

with the mass $2240 m_e$, (m_e : electron mass), recently found by Butler et al.¹⁾, may be a composite particle composed of a proton and a negative τ meson is examined. This composite fermion model was originally applied to neutron* by the present author.²⁾ (This paper will be cited as B.) Following the method developed in B, we postulate a pair interaction between protons and τ mesons

$$H_1 = -\eta \int \bar{\psi}_P(x) \psi_P(x) \phi_\tau^*(x) \phi_\tau(x) d^3x, \quad (1)$$

and another pair interaction between protons and τ - π pairs

$$H_2 = -\zeta \int \bar{\psi}_P(x) \psi_P(x) \phi_\tau^*(x) \phi_\pi(x) d^3x + c.c. \quad (2)$$

H_1 is responsible for the formation of the bound system of a proton and a negative τ meson, ($P + \tau^-$), which represents a neutral V particle (or a neutron in B). In B, employing the conventional method of introducing a source function, after the usual procedure taken in meson pair theories³⁾, the contact interaction H_1 was smeared over a finite region and the resulting mesonic wave equation was solved. It was shown that for the existence of a bound mesonic state with energy E , η must be taken as

$$1/\eta = \frac{1}{4\pi} \frac{A}{1+W/A}, \quad (3)$$

where

$$W = \sqrt{\mu_\tau^2 - \left(\frac{E}{\hbar c}\right)^2}. \quad (4)$$

μ_τ is the reciprocal compton wave length for τ meson. A is a cutting momentum associated with the source function. In order that this bound state represents a V particle, E must be taken as

$$\begin{aligned} E &= m_\tau c^2 - m_P c^2 \approx (2240 - 1840) m_e c^2 \\ &= 400 m_e c^2. \end{aligned} \quad (5)$$

The second interaction H_2 gives rise to a process like

$$(P + \tau^-) \longrightarrow P' + \pi^-. \quad (6)$$

This was interpreted in B as a Yukawa process $N \rightarrow P + \pi^-$, with the value of ζ of the same order of magnitude as η . Here it represents the decay of the V^0 particle, $V \rightarrow P + \pi^-$, and now ζ must be taken far smaller in order to account for its rather long decay life-time of the order of 10^{-15} sec. According to B, by comparing the matrix element of H_2 for the process (6), that is, the annihilation of a bound τ^- meson and the creation of a free π^- meson, with that of a conventional Hamiltonian $g \int \bar{\psi}_P(x) \times \psi_V(x) \varphi_\pi(x) d^3x$, the 'equivalent' g is obtained as

$$\begin{aligned} \frac{g^2}{4\pi\hbar c} &= \frac{1}{(4\pi)^2} \frac{W\hbar c}{E} (A\zeta)^2 \\ &= \frac{W\hbar c}{E} \left(1 + \frac{W}{A}\right)^2 \left(\frac{\zeta}{\eta}\right)^2, \end{aligned} \quad (7)$$

where the relation (3) was used to deduce the last expression. On the other hand, according to Ishida and Ito⁴⁾, g must be taken as

$$\frac{g^2}{4\pi\hbar c} \sim 10^{-13}, \quad (8)$$

in order to obtain the right order of the life-time of the V particle. This implies, from (4), (5) and (7), that

$$(A\zeta)^2 \sim 10^{-11}. \quad (9)$$

It is interesting to note that this value can be obtained, though not necessarily, but quite reasonably, by considering the decay process (6) as the second order one resulting from the decay of the bound τ^- meson into π^- and π^0 , π^0 being absorbed in the second step by the proton, $P + \pi^0 \rightarrow P'$. Assuming the corresponding interaction Hamiltonians

$$H_3 = G' \frac{\mu_\tau}{\hbar c} \int \phi_\tau^*(x) \varphi_\pi(x) \varphi_{\pi^0}(x) d^3x + c.c. \quad (10)$$

and

$$H_4 = G \int \bar{\psi}_P(x) \psi_P(x) \varphi_{\pi^0}(x) d^3x, \quad (11)$$

with G' and G dimensionless, H_2 can be derived as the second order effective Hamiltonian, the coupling constant ζ being given by

$$\zeta = \frac{GG'}{\hbar c} \frac{\mu_\tau}{\mu_\pi^2}.$$

A may be taken from the consideration of virtual mesonic clouds as

$$A \sim \mu_\tau.$$

Therefore,

$$(A\zeta)^2 = \left(\frac{G^2}{\hbar c}\right) \left(\frac{G'^2}{\hbar c}\right) \left(\frac{\mu_\tau}{\mu_\pi}\right)^4.$$

With the values $\frac{G'^2}{\hbar c} \sim 10^{-12}$ and $\frac{G^2}{\hbar c} \sim 1$ taken, respectively, to account for the life-time of $\tau^- \rightarrow \pi^- + \pi^0$ decay and the nuclear force, this yields

$$(A\zeta)^2 \sim 10^{-10},$$

which is close to the required value (9). Strictly speaking, the second order Hamiltonian resulting from (10) and (11) in static approximation is not given by (2), but by

$$\begin{aligned} H_2' &= \frac{GG'}{\hbar c} \mu_\tau \iint \bar{\psi}_P(x) \psi_P(x) \frac{e^{-\mu_\pi|x-x'|}}{4\pi|x-x'|} \\ &\quad \times \phi_\tau^*(x') \phi_\pi(x') d^3x d^3x' \end{aligned} \quad (12)$$

which cannot be approximated to (2), since we are concerned in the bound state of τ^- meson, the wave function ϕ_τ having an extension of the order of $1/\mu_\tau$. Starting from (12), instead of (2), it is shown that the correct value of the equivalent g as given by (8) can be obtained. Alternatively, taking H_2 as the basic interaction, the decay can be considered as the second order one,

$$\tau^- \rightarrow P + P^- + \pi^-,$$

$$P + P^- \rightarrow \pi^0.$$

The production of V particles can take place via (6), through nuclear bremsung,

$$P \rightarrow (P + \tau^-) + \pi^+.$$

But the cross-section would be far too smaller

to account for the large production rate of V particles, with the value of the coupling constant ζ given by (9), or the equivalent g given by (8), which were determined from the life-time of V particles. However, the main contribution to V particle production is made by the interaction H_1 , which causes a process, by bremsung,

$$P \rightarrow (P + \tau^-) + \tau^+.$$

The equivalent coupling constant g' for this process is given by (7) with ζ replaced by η , thus

$$\frac{g'^2}{4\pi\hbar c} \sim 1,$$

which is the same order of magnitude as the coupling constant of π mesons with nucleons. Using this value of g' , the cross-section of V particle production can be calculated just in the same way as that of the production of π mesons. The resulting cross-section would be a little smaller than that of π meson production, since the energy denominator for the former process would be larger than for the latter, owing to the large masses of τ mesons and V particles.

In conclusion, the author wishes to express his cordial thanks to Prof. K. Sakuma for his kind interest in this work.

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π^0 -Meson Production by Gamma-Ray

S. Minami

Department of Physics, Osaka University

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In the process of $P + \gamma \rightarrow P + \pi^0$, it is well known that calculations of the lowest order perturbation theory do not agree with experimental facts. We have examined whether this fault might be relieved, as K. A. Brueckner et al. have predicted,¹⁾ by calculations of higher orders in symmetrical and neutral pseudoscalar theory.

Following the formulation adopted in the calculation of $\gamma - \pi^\pm$ production,²⁾ we have classified a number of Feynman-Dyson diagrams into some gauge-invariant classes and reduced the various terms of the matrix elements to four independent gauge-invariant fundamental expressions, that is to say,

$$U_e = \left\{ \frac{iF_p}{(FK)} - \frac{iI_p}{(IK)} \right\} \bar{\psi}(\mathbf{F}) i\gamma_5 \tau_P \psi(\mathbf{I}) \times A_p(\mathbf{K}),$$

$$U_m = \bar{\psi}(\mathbf{F}) i\gamma_5 \tau \frac{\sigma_{\rho\nu} K_\nu \tau_P}{M^2} \psi(\mathbf{I}) A_p(\mathbf{K}),$$

$$U_A^{(1)} = -\bar{\psi}(\mathbf{F}) i\gamma_5 \tau \frac{i\gamma(\mathbf{I} + \mathbf{K}) - M}{(\mathbf{I} + \mathbf{K})^2 + M^2} \frac{\sigma_{\rho\nu} K_\nu \tau_P}{2M} \times \psi(\mathbf{I}) A_p(\mathbf{K}), \quad (1)$$

and

$$U_A^{(2)} = -\bar{\psi}(\mathbf{F}) \frac{\sigma_{\rho\nu} K_\nu \tau_P i\gamma(\mathbf{F} - \mathbf{K}) - M}{2M} \frac{i\gamma(\mathbf{F} - \mathbf{K}) - M}{(\mathbf{F} - \mathbf{K})^2 + M^2} \times i\gamma_5 \tau \psi(\mathbf{I}) A_p(\mathbf{K}),$$

where \mathbf{I} , \mathbf{F} and \mathbf{K} denote the momenta of the initial and final nucleon and the incident photon respectively, and τ stands for τ_3 or τ_4 according to symmetrical or neutral theory. The S-matrix can be expanded according to the powers of f as follow:

$$efA_1 + ef^3A_3 + ef^5A_5 + \dots \quad (2)$$

The ef -term consists of two processes, in one of which the nucleon absorbs the incident

photon in the first step, and emits the π^0 meson in the second step and in the other the photon absorption and meson production occur in the reversed order. Owing to the fact that contributions from the main terms in these two processes cancel out, the ef^3 -terms overwhelm the ef -terms in magnitude, if the accepted value of the coupling constant is adopted. Consequently, we have to take into account not only $(ef) \times (ef^3)$, but also $(ef^3)^2$ -terms in the calculation of the cross section for the $\gamma-\pi^0$ process. This, in turn, necessitates the computation of the cross terms $(ef) \times (ef^5)$, which, however, has not still been done. In these circumstances it is impossible to say anything rigorous, but, for the sake of smallness of ef -terms, we may be allowed, for the present, to neglect the contribution from these cumbersome terms.

We here state the results only for the case of the pseudoscalar coupling in symmetrical theory which seems to be promising, and

the results for other cases will be reported shortly in this journal. For the sake of convenience in comparing the values of cross section up to e^2f^2 with those up to e^2f^6 , the following notations will be used :

- I..... e^2f^2 -terms only are taken into account⁽³⁾
- IIthe same as above, but with a Pauli type interaction^(4),5)
- III.....up to e^2f^6 , our computation.

The numerical values of the incident photon energy and the coupling constant which are used for the evaluation are $K_0=2\mu=276$ Mev and $f^2/4\pi=8\pi$ respectively. This value of $f^2/4\pi=8\pi$ is adopted because of the results in the work about charged meson production²⁾ and in order to conform with the experimental value of cross section at 90° , $3 \times 10^{-30}\text{cm}^2/\text{sterad.}$ ⁶⁾ In this case the differential cross sections in laboratory system are as follows:

		0°	30°	60°	90°	120°	150°	180°
$\sigma_0\left(10^{-30}\frac{\text{cm}^2}{\text{sterad.}}\right)$	I	0.016	0.033	0.15	0.34	0.46	0.51	0.52
	II	0.12	0.61	1.9	3.2	3.9	4.0	4.0
	III	12.4	10.9	7.6	4.4	2.4	1.4	1.0

		0°	30°	60°	90°	120°	150°	180°
$\sigma_0\left(10^{-30}\frac{\text{cm}^2}{\text{sterad.}}\right)$	III	8.1	6.4	3.6	1.9	1.1	0.76	0.69

Further, we have carried out the evaluation for $N+\gamma \rightarrow N+\pi^0$ with the same values of photon energy and coupling constant as for $P+\gamma \rightarrow P+\pi^0$, with the following results. A detailed paper will be published in near future.

In conclusion, we should like to express our deep gratitude to Prof. K. Husimi for his guidances, to Asst. Prof. Koba who has kindly given us his valuable suggestions with respect to the important points throughout this work and to Messrs. S. Nakai, T. Kotani and N. Mugibayashi for their valuable dis-

cussions. Moreover, we wish to express our gratitudes for the financial aid from the Yukawa Yomiuri Fellowship to us.

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On the Interaction of the Lattice Vibrations with the Conduction Electrons

T. Nishiyama

Department of Physics, Osaka University

August 31, 1951

G. Wentzel¹⁾ and W. Kohn and Vachaspati²⁾ made comments on Fröhlich's theory of superconductivity.³⁾ The latter confirmed that the shell construction in the momentum space is hardly possible if the interaction of the conduction electrons with the lattice phonons is so strong that Fröhlich's condition of superconductivity is satisfied. The former applied Tomonaga's theory of sound quanta⁴⁾ to derive a similar result pointing out that the frequency of the lattice vibration becomes imaginary so far as Fröhlich's condition is adopted. The objective of this report is two-fold: first we confirm the results of the abovementioned authors by an extension of Tomonaga's theory given in our paper⁵⁾ and next we examine Wentzel's suggestion that the motion of atom might become stable if the Coulomb interaction forces are taken into calculation. The result obtained is compared with that of the hydrodynamical treatment, the validity of which is computed from our point of view.

When a conduction electron with wave number k absorbs a lattice phonon with wave number w , its wave number becomes $k+w$. Such a process and the reciprocal process are represented by the following operator

$$M(w)[a^*(k+w)a(k)b(w) + a^*(k-w)a(k)b^*(w)], \quad (1)$$

in which $a^*(k)$ is the creation operator of electrons and $a(k)$ is the annihilation operator and $b^*(w)$ and $b(w)$ are those of lattice phonons respectively and $M(w)$ is the matrix element of the interaction energy which is expressed by

$$M(w)^2 = \frac{2}{3} \frac{F}{N} \hbar \omega_0(w) \zeta, \quad (2)$$

where F is a dimensionless constant of order unity and $\zeta = \hbar^2 k_0^2 / 2m$ and N is the total number of atoms. The interaction energy is written in the form

$$H_{INT} = \sum_w M(w) [\rho(w)b^*(w) + \rho(-w)b(w)] \quad (2)$$

$$= \frac{1}{2} \sum_w M(w) \left[\frac{2\omega_0(w)}{\hbar} \right]^{\frac{1}{2}} \times [\rho(w)q^*(w) + \rho^*(w)q(w)], \quad (3)$$

in which

$$\rho(w) = \rho^*(-w) = \sum_k a^*(k-w/2)a(k+w/2), \quad (4)$$

$$q(w) = q^*(-w) = \left[\frac{\hbar}{2\omega_0(w)} \right]^{\frac{1}{2}} [b^*(-w) + b(w)]. \quad (5)$$

Our approximation consists in regarding the density operator $\rho(w)$ as a sum of oscillators with various frequencies and with wave number w as follows:⁵⁾

$$\rho(w) = |w| \sum_s \left[\frac{L_s N_s}{m} \right]^{\frac{1}{2}} Q_s(w), \quad (6)$$

$V = L^3$: volume of vessel,

where

$$\dot{Q}_s(w) + \mathcal{Q}_s(w)^2 Q_s(w) = 0, \quad (7)$$

$$\mathcal{Q}_s(w)^2 = w^2 \frac{\hbar^2}{m^2} (k_0^2 - S^2), \quad (8)$$

$\hbar k_0$: the maximum momentum.

The hamiltonian of these oscillators is

$$H_E = \frac{1}{2} \sum \sum [P_s^*(w) P_s(w) + \mathcal{Q}_s(w)^2 Q_s^*(w) Q_s(w)], \quad (9)$$

in which

$$[Q_s(w), P_t(w')] = i\hbar \delta_{s,t} \delta_{w,w'}. \quad (10)$$

Therefore the total hamiltonian becomes

$$H_T = H_E + H_{INT} + H_L, \quad (11)$$

in which the hamiltonian of lattice phonons is

$$H_L = \frac{1}{2} \sum_w [p^*(w)p(w) + \omega_0(w)^2 q^*(w)q(w)], \quad (12)$$

Here we notice that

$$\omega_0(w) = |w|S, \quad \sigma_0 = mS/\hbar, \quad (\sigma_0/k_0)^2 \simeq 10^{-5},$$

S : sound speed.

Then one can easily find the proper frequency of lattice phonons:

$$\omega(w)^2 = \omega_0(w)^2 \left[1 - 2\nu F \left(1 + \frac{1}{2} \frac{\sigma_0}{k_0} \log \frac{k_0 - \sigma_0}{k_0 + \sigma_0} \right) \right],$$

in which ν is the number of conduction electrons per atom and the logarithmic term is a small negative quantity. If $2\nu F > 1 - O(\frac{\sigma_0^2}{k_0^2})$, the frequency becomes imaginary and the motion of atom becomes instable. G. Wentzel suggested that the motion might remain stable, if the electronic interaction of the electrons is taken into consideration. Here we calculate the effect of the Coulomb interaction energy in the scope of our approximation. One finds that Eq. (1) turns out

$$\omega(w)^2 = \omega_0(w)^2 = \omega_0(w)^2 [1 - 2\nu F / (1 + \lambda_w)], \quad (14)$$

in which

$$\lambda_w = 6\pi \left(\frac{n}{\zeta} \right) \left(\frac{e}{w} \right)^2, \quad n: \text{ number density of electrons,}$$

$$\omega(w)^2 < 0 \quad \text{if} \quad 2\nu F > 1 + \lambda_w. \quad (15)$$

This result is compared with that derived from the so-called acoustical approximation of the hydrodynamical treatment given in our previous paper⁵⁾ which has been proved to be compatible⁶⁾ with Landau's theory of quantum hydrodynamics. Some calculations show that

$$\omega(w)^2 = \omega_0(w)^2 [1 + \kappa(w)^2 - [1 - \kappa(w)^2]^2 + 4\gamma(w)^2]^{\frac{1}{2}} / 2,$$

$$\kappa(w) = \mathcal{Q}_0(w) / \omega_0(w),$$

$$\gamma(w)^2 = 4\nu F \zeta / 3m\omega_0(w)^2,$$

$$\omega(w)^2 < 0 \quad \text{if} \quad 2\nu > 3 + \lambda_w. \quad (16)$$

This is different from (15) by three. From our point of view the failure comes from the incorrect treatment about the curvature of the Fermi surface. If one sets $F=1$ and $\nu=2$, the squared frequency remains positive for

$$|w| \leq 2k_0 \text{ from (16), } |w| \leq \frac{2}{3}k_0 \text{ from (15).}$$

It is expected that the theory of Fröhlich is reconstructed introducing the Coulomb interaction energy.

The writer wishes to thank Prof. K. Husimi for his interest in this problem.

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On the Energy-Momentum Tensor of Bopp-Type Non-Local Fields

Y. Ôno

Department of Physics, Hokkaido University

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The symmetrical energy-momentum tensor is obtained for the Bopp-type non-local field, whose Lagrangian is of the type

$$L(x) = \int \varphi(x) \varepsilon(x-x') \varphi(x') dx',$$

where $\varphi(x)$ is a field variable of a certain

type, and $\varepsilon(x)$ the characteristic non-local function. The method is as follows: We expand $\varepsilon(x)$ in terms of the derivatives of $\delta(x)$, i.e.

$$\varepsilon(x) = \sum_p \lambda_p \square^p \delta(x),$$

and then derive the energy-momentum tensor for the Lagrangian

$$L(x) = \varphi(x) \square^p \varphi(x)$$

by the use of the theory of general relativity, that is, by varying the metrical tensor $g_{\mu\nu}$. Next, we carry out the summation with respect to p , which yields the form containing $\varepsilon(x)$ or its Fourier components $\tilde{\varepsilon}(k)$. In the spinor case, a slight modification is necessary, since $\varepsilon(x)$ is no longer a usual function but a matrix, and, moreover, the local coordinate vector h^μ must be used.

The results expressed in the Fourier components are:

a) For the extended electromagnetic field

$$\tilde{\Theta}_{\mu\nu}(k) = \frac{1}{16\pi(2\pi)^3} \int dk' \left[-\tilde{f}_{\alpha\beta}(k-k') \tilde{\varepsilon}(k') \right.$$

$$\tilde{f}_{\alpha\beta}(k') \delta_{\mu\nu} + 2\tilde{f}_{\nu\alpha}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\nu\alpha}(k')$$

$$+ 2\tilde{f}_{\nu\alpha}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\mu\alpha}(k')$$

$$2\left\{ \tilde{f}_{\alpha\beta}(k-k') \tilde{f}_{\alpha\beta}(k') \left(-(k-k')_\mu k'_\nu \right. \right.$$

$$+ \frac{1}{2} k_\tau k'_\tau \delta_{\mu\nu} \left. \right)$$

$$+ \tilde{f}_{\alpha\beta}(k-k') \tilde{f}_{\nu\beta}(k') k_\alpha (k-2k')_\mu$$

$$+ \tilde{f}_{\alpha\beta}(k-k') \tilde{f}_{\mu\beta}(k') k_\alpha (k-2k')_\nu \left. \right\}$$

$$\frac{\tilde{\varepsilon}(k-k') - \tilde{\varepsilon}(k')}{(k-k')^2 - k'^2};$$

b) For the spinor field

$$\tilde{\Theta}_{\mu\nu}(k) = \frac{1}{2} (\tilde{T}_{\mu\nu}(k) + \tilde{T}_{\nu\mu}(k)),$$

$$\tilde{T}_{\mu\nu}(k) = -\frac{1}{2} \frac{1}{(2\pi)^4} \int dk' \tilde{\psi}^+(k-k')$$

$$\left[\frac{\tilde{\varepsilon}(k'-k) + \tilde{\varepsilon}(k') \delta_{\mu\nu} + i(k-2k')_\nu \gamma_\mu}{i((k-k')_\lambda - 2(k-k')_\mu \delta_{\mu\lambda}) \gamma_\lambda - i k_\lambda' \gamma_\lambda} \right] \tilde{\psi}(k')$$

(with no summation for μ),

where the arrows in the denominator mean the following: if we multiply the fraction by the first term of denominator from the left, and by the second term from the right, and then subtract, we get the numerator.

These tensors are characterized by the following features:

- i) They are symmetrical in μ and ν .
- ii) They are gauge-invariant.
- iii) In the limiting case, they tend to the usual tensors.
- iv) They satisfy, due to the field equation, the equation of continuity. (Here, it is not necessary to introduce an extra inhomogeneous term, as has been done by Bopp, even when there is a source in the field.)

The expression proposed by Bopp¹⁾ for the case a) does not satisfy the conditions ii) and iii), and also i) for the field containing a source, while that proposed by Heisenberg²⁾ for the case b) does not in general satisfy the condition iv), notwithstanding his claim.

Thus our formulae might be considered as to express the most natural extension of the local fields.

Detailed account will be given in the later issue of this journal.

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The Quantum Mechanics of Assemblies of Interacting Particles

T. Matsubara

Department of Physics, Osaka University

September 11, 1951

Recently, H. S. Green¹⁾ has published in

J. Chem. Phys. a paper with the same title as given above. Some of the results derived in his theory happen to be very similar to those which the writer has obtained using a different method in the course of an investigation on liquid helium²⁾. Although the details of the writer's theory will appear shortly in this Journal, he would like to present here a brief account of some of the method used and the results obtained in the form convenient to compare with those of Green.

Let the Hamiltonian of a system of N identical particles be H , which may be divided into two parts, the total kinetic energies K and the total potential energies Φ . Then the density matrix for the system in thermal equilibrium will be given by

$$\rho = \exp(-\beta H) = \exp[-\beta(K + \Phi)], \quad \beta = \frac{1}{kT}. \quad (1)$$

ρ has to satisfy the following differential equation:

$$\frac{\partial \rho}{\partial \beta} = -H\rho = -(K + \Phi)\rho. \quad (2)$$

Assuming ρ in the form

$$\rho = \exp(-\beta K) \cdot (1 + \rho_1 + \rho_2 + \dots), \quad (3)$$

we can solve equation (2) by successive approximations similar to those used in the time-dependent perturbation method for the Schrödinger equation, and obtain

$$\begin{aligned} \rho_1(\beta) &= -\int_0^\beta V(t) dt, \\ \rho_2(\beta) &= -\int_0^\beta V(t) \rho_1(t) dt = -\int_0^\beta V(t) dt \int_0^t V(s) ds, \\ &\dots\dots\dots, \end{aligned} \quad (4)$$

where

$$V(t) = \exp(tK) \cdot \Phi \cdot \exp(-tK). \quad (5)$$

The Slater sum of the system is given by

$$n_N(x) = \sum \Psi_K^*(x) \rho \Psi_K(x), \quad (6)$$

where $\{\Psi_K(x)\}$ is an arbitrary set of orthogonal-normalized functions and x stands for

$(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$. Neglecting the effect of the quantum statistics, and taking for $\{\Psi_K(x)\}$ the set of plane waves, that is the eigenfunctions of K ,

$$\Psi_{\mathbf{p}_1 \dots \mathbf{p}_N}(x) = \frac{1}{V^{N/2}} \exp \left[\frac{i}{\hbar} (\mathbf{p}_1 \cdot \mathbf{r}_1 + \dots + \mathbf{p}_N \cdot \mathbf{r}_N) \right], \quad (7)$$

we can easily obtain from (3), (4), (5) and (6) the following results:

$$n_N(x) = \left(\frac{2\pi m}{\beta \hbar^2} \right)^{\frac{3N}{2}} \left\{ 1 - \beta \Phi_1(x) + \frac{1}{2!} \beta^2 \Phi_2(x) - \frac{1}{3!} \beta^3 \Phi_3(x) + \dots \right\}, \quad (8)$$

$$\begin{aligned} \frac{1}{k!} \Phi_K(x) &= \left(\frac{2\pi m}{\beta \hbar^2} \right)^{\frac{3KN}{2}} \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} \dots \\ &\quad \int_0^{z_{K-1}} dz_K \int_{(\beta K N)} \dots \\ &\quad \times \int \frac{\Phi(y_1) \Phi(y_2) \dots \Phi(y_K)}{\{(1-z_1)(z_1-z_2) \dots (z_{K-1}-z_K)z_K\}^{\frac{3KN}{2}}} \\ &\quad \times \exp \left[-\frac{2\pi^2 m}{\beta \hbar^2} \left\{ \frac{(x-y_1)^2}{1-z_1} + \frac{(y_1-y_2)^2}{z_1-z_2} + \dots \right. \right. \\ &\quad \left. \left. + \frac{(y_{K-1}-y_K)^2}{z_{K-1}-z_K} + \frac{(y_K-x)^2}{z_K} \right\} \right] d^N y_1 \dots d^N y_K. \end{aligned} \quad (9)$$

The formula (8) just corresponds to Green's formula (38) in his paper, up to the terms Φ_1 , except for a trivial difference in the normalization, as may readily be seen by noticing that our $\Phi_1(x)$ can be transformed by a change of integration variables into

$$\begin{aligned} \Phi_1(x) &= \left(\frac{2\pi m}{\beta \hbar^2} \right)^{\frac{3N}{2}} \int_0^1 dz \\ &\quad \int \Phi \left(x + \frac{1}{2} (1-z^2)^{1/2} y \right) \exp \left(-\frac{m y^2}{2\beta \hbar^2} \right) d^N y, \end{aligned} \quad (10)$$

which is just identical with Green's expression³⁾. Our treatment gives beyond this the general expression for the higher approximation terms, which are lacking in Green's paper.

The calculation of the formula (3) can

alternatively be done in the following way, and indeed in a more direct manner. Let H_ε be defined by

$$H_\varepsilon = K + \varepsilon \Phi \quad (11)$$

ε being an arbitrary parameter, and let $f(H_\varepsilon)$ be any function expressible as a power series of H_ε . Then we have

$$\frac{\partial f(H_\varepsilon)}{\partial \varepsilon} = \sum_{t=1}^{\infty} \left\{ \frac{\partial^t}{\partial H_\varepsilon^t} f(H_\varepsilon) \right\} \frac{1}{t!} \Phi^{(t-1)}(\varepsilon), \quad (12)$$

where $\frac{\partial}{\partial H_\varepsilon}$, $\frac{\partial^2}{\partial H_\varepsilon^2}$ etc. mean the formal differentiation with respect to H_ε treating H_ε as if it were an ordinary number, and $\Phi^{(t)}(\varepsilon)$'s are defined by

$$\begin{aligned} \Phi^{(t)}(\varepsilon) &= [\Phi^{(t-1)}(\varepsilon), H_\varepsilon] \\ &= \Phi^{(t-1)}(\varepsilon) \hat{H}_\varepsilon - H_\varepsilon \Phi^{(t-1)}(\varepsilon), \quad (13) \\ \Phi^{(0)} &= \frac{\partial H_\varepsilon}{\partial \varepsilon} = \Phi. \end{aligned}$$

Taking $\rho(\varepsilon) = \exp(-\beta H_\varepsilon)$ for $f(H_\varepsilon)$, and making use of (12) in a repeated manner, we find the following expansion for $\rho(\varepsilon)$:

$$\begin{aligned} \rho(\varepsilon) &= \rho(0) + \varepsilon \left(\frac{\partial \rho}{\partial \varepsilon} \right)_{\varepsilon=0} + \frac{\varepsilon^2}{2!} \left(\frac{\partial^2 \rho}{\partial \varepsilon^2} \right)_{\varepsilon=0} + \dots \\ &= \exp(-\beta K) \cdot \left\{ 1 + \varepsilon G_1(0) + \frac{\varepsilon^2}{2!} G_2(0) \right. \\ &\quad \left. + \frac{\varepsilon^3}{3!} G_3(0) + \dots \right\}. \quad (14) \end{aligned}$$

In this expression, $G_1(\varepsilon)$ is given by

$$G_1(\varepsilon) = \sum_{t=1}^{\infty} \frac{(-\beta)^t}{t!} \Phi^{(t-1)}(\varepsilon), \quad (15)$$

and $G_K(\varepsilon)$'s ($k \geq 2$) are determined from the recurrence formula

$$G_K(\varepsilon) = G_1(\varepsilon) G_{K-1}(\varepsilon) + \frac{\partial G_{K-1}(\varepsilon)}{\partial \varepsilon}. \quad (16)$$

It is not difficult to show that the expansion (14) for $\rho(1)$ (putting $\varepsilon=1$) is identical with the expansion (3). (This identity may most clearly be shown if one uses the representation in which K is diagonalized.)

Eq. (14) can be expressed in another way. If we put

$$\rho(\varepsilon) = \exp(-\beta K) \cdot \exp(-\beta \Phi^*), \quad (17)$$

then Φ^* is an operator which represents something of the nature of the effective potential energy of the system. It is expressible as a power series of:

$$-\beta \Phi^* = \varepsilon B_1 + \frac{1}{2!} \varepsilon^2 B_2 + \frac{1}{3!} \varepsilon^3 B_3 + \dots, \quad (18)$$

where B_K 's can be shown to be given by

$$\begin{aligned} B_1 &= G_1(0), \\ B_2 &= \left(\frac{\partial G_1(\varepsilon)}{\partial \varepsilon} \right)_{\varepsilon=0}, \\ B_3 &= \left(\frac{\partial^2 G_1(\varepsilon)}{\partial \varepsilon^2} \right)_{\varepsilon=0} - \frac{1}{2!} \left[\left(\frac{\partial G_1(\varepsilon)}{\partial \varepsilon} \right)_{\varepsilon=0}, G_1(0) \right], \\ B_4 &= \left(\frac{\partial^3 G_1(\varepsilon)}{\partial \varepsilon^3} \right)_{\varepsilon=0} - \left[\left(\frac{\partial^2 G_1(\varepsilon)}{\partial \varepsilon^2} \right)_{\varepsilon=0}, G_1(0) \right], \\ &\quad \text{etc.} \quad (19) \end{aligned}$$

The various formal expansions of the density matrix here obtained will converge for sufficiently small values of ε , so that they are valid in the whole range of temperature.

The details and the applications of our theory will be given in the forthcoming papers.

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- 2) T. Matsubara, Quantum-Statistical Theory of Liquid Helium, Prog. Theor. Phys. **6** (1951), 714
- 3) In his paper the argument of Φ appears as $\mathbf{x} + (1-z^2)^{1/2} \mathbf{y}$, instead of $\mathbf{x} + \frac{1}{2} (1-z^2)^{1/2} \mathbf{y}$. This would be a trivial misprint.

A Note on the Fermi's Theory of Meson Production

Y. Fujimoto and T. Tamura

Department of Physics, University of Tokyo,

September 12, 1951

In a recent observation of meson production in single nucleon-nucleon,¹⁾ it is shown,

that the Fermi's theory of meson production^{2),3)} is in good agreement with experiment at extreme high energies. That the theory can also explain the experiment at threshold energies, is shown by Fermi himself in his first paper.²⁾

The knowledge of the features of meson production at intermediate energy region, i.e., some or some ten Bev., is very important for the analysis of cosmic radiation, and as Fermi's theory agrees rather well with experiment at very low and high energies, it would be meaningful to investigate whether it can also explain the features of meson production at this energy region.

The most appropriate experimental data for this purpose is that of latitude effect of stars, given by Salant *et al.*⁴⁾. Since they given the mean number of charged mesons produced from light (C, O, N) and heavy (Ag, Br) elements by 1~8 Bev, and >8 Bev protons, we can compare the theory with experiment at these two energy regions, separately. We summarize their data in Table I.

We calculate the number of charged mesons produced in nucleon-nucleon collision by protons (1~5 Bev) and higher energy protons (>14 Bev) according to Eqs. (22) and (32) of A, respectively, and interpolate between these two energy regions.

Taking as the differential primary energy spectrum of Winckler *et al.*⁵⁾, we get mean number of charged mesons produced by 1~8 Bev protons. It amounts to 0.58, i.e. much smaller than the corresponding value in Table I. To diminish this difference, we first consider the effect of plural production. In the evaluation of secondary production, we assume one of the two nucleons, which take part in the initial production, goes forward, and the other goes backward, in the center of mass system, with same energy, so, in the laboratory system, they go forward with different energies. We further assume that they both collide with other different

nucleons, and produce mesons. In the same way we can calculate the number of mesons produced tertiary, and so on. If we include mesons produced as far as fourthly, the mean number amounts to 1.25. This would be the maximum value conceivable in light nuclei. This value is yet smaller than the experimental value 1.3 ± 0.3 . In heavier nuclei, seventhly or eighthly production could occur, but according to our calculation, the energies of nucleons reduce so small that the contribution becomes negligible after sixth production and is impossible to reach the value 1.5 in Table I.

We next calculate the number of mesons assuming the collision to be inelastic, but it is found, that the result becomes even smaller. This would moreover injure the angular distribution of mesons⁶⁾.

The expedient we finally take is to consider the effect of meson production by mesons. For the evaluation of this effect, we modify Fermi's Eqs. (22) and (32) in A, replacing the part of incident nucleon by an incident meson. It is found that the number of mesons produced increase much more rapidly with energy in the center of mass system than in the case of nucleon-nucleon collision. But the energy in center of mass system in meson-nucleon collision is smaller than the corresponding value in nucleon-nucleon collision, when the incident meson and nucleon have same energy in the laboratory system, so the contribution of meson production by mesons does not come to be so large.

As mentioned above, we obtained the number of mesons in the energy region 5~14 Bev by interpolation. This would perhaps be an underestimation. Therefore we may conclude that in this energy region, the Eq. (22) of A should be used rather than Eq. (32) of A.

In the above analysis we treated the number of thin tracks of table I to be all due to mesons. But as was shown by Camerini *et al.*⁷⁾, about 10% of them may be thought to be protons, so as the true number of

mesons, it would be better to take the value of table II. If we do so, then, though we use the above calculated interpolated value, it is possible to obtain the experimental value for 1~8 BeV protons, if in light nuclei meson production by nucleon-nucleon collision is considered up to secondary, and to this, secondary meson production by initially produced mesons is added, and in heavy nuclei both their contribution is considered up to tertiary. Similar consideration is sufficient to get agreement between theory and experiment for > 8 BeV protons.

Anyhow, as long as we believe in the validity of Fermi's theory, we can conclude that the meson production from nuclei by cosmic radiation is multiple and plural one.

Table I. Mean number of thin tracks

energy of protons	light nuclei	heavy nuclei
1 ~ 8 BeV	1.3 ± 0.3	1.5 ± 0.4
> 8 BeV	1.4 ± 0.4	2.8 ± 0.3

Table II. Mean number of mesons

energy of protons	light nuclei	heavy nuclei
1 ~ 8 BeV	1.17 ± 0.3	1.35 ± 0.4
> 8 BeV	1.26 ± 0.3	2.5 ± 0.3

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- 5) J. R. Winckler, T. Stix, K. Dwight and R. Sabin, Phys. Rev. **79** (1950), 556.
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Pri la Koncepto de la Libera Kampo

S. Hori

Department of Physics, Kanazawa University

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En la nuna kvantuma teorio de kampoj, la koncepto de la libera kampo ludas gravan rolon: Oni supozas ke por liberaj kampoj solvoj estas jam antaŭe donitaj laŭ spinoj, masoj kaj sargoj de korpuskloj kiujn la kampoj priskribas, kaj interagadoj estas aldone kaj adicie enkondukitaj inter ili.

Libera kampo, tamen, aktuale ne ekzistas en la naturo. Ni ne havas eĉ rimedon por certigi (eksperimente) ĉu la solvoj estas korektaj aŭ ne. Tial ni volas formeti la koncepton de la libera kampo forde la nuna teorio. Ni enkondukas jenan gvidantan principon: Libera kampo tiel nomata en la nuna teorio priskribas en fakto la korpusklon jam interaganta kun kampoj produktitaj de la korpusklo mem. Tiu ĉi gvidanta principo tamen ne estas facile aplikebla rekte al la teorio de due kvantumitaj kampoj. Ĝi estas aplikita, unue al la klasika teorio kaj ni havas modifitan teorion en kiu ne aperas klasika Coulomb'a memenergio, kaj due al la kvantuma teorio.

Bedaŭrinde tiu ĉi modifita kvantuma teorio estas malfacile due kvantumita, ĉar por apliki nian gvidantan principon, ni devis necese enkonduki distingeblecon inter fotonoj emanitaj de diferencaj ŝargaj korpuskloj. Ni tamen povas konjekti pri nia estonta teorio.

1. Nia teorio estos tia ke ŝarga korpusklo sola ekzistanta en la vakuo ne havas Coulomb'-an kampon ĉirkaŭ si, krom kiam oni alproksimigas alian ŝargan korpusklon por sondi aŭ mezuri la kampon.
2. En nia teorio ne aperas ne sole Coulomb'a memenergio sed ankaŭ kvantumteoria memenergio.
3. Ni devos eble enkonduki specon de Hamiltoniano kun memoro¹⁾ por eviti supre

menciitan distingbeleon.

Pli detala artikolo aperos en pli malfrua eldono de tiu ĉi gazeto.

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On the Fourth-Order Nuclear Potential

S. Machida, S. Ônuma

Department of Physics, University of Tokyo
and

M. Taketani, Tokyo

September 27, 1951

Recently, it has been pointed out¹⁾ that, if one considers the fourth-order term in addition to the second order one which has only been taken into account so far, the pseudoscalar meson potential with pseudo-vector coupling alters its feature considerably.

In view of the importance of this conclusion, we have made a investigation of the deuteron ground state and low energy neutron-proton scattering taking into account the fourth-order nuclear potential. The method of the shape-independent approximation has been used to analyse the low energy scattering and we have assumed the symmetrical theory throughout these calculations.

1) The singlet neutron-proton scattering

The calculated values of the singlet effective range, 1r , and cut-off radius, x_0 , are as follows:

a) zero cut-off

$f^2/4\pi\hbar c$	1r (in 10^{-13} cm)	x_0 (in unit of π -meson range)
0.075	1.925	0.5886
0.090	2.457	0.665

b) infinite repulsive square well inside (hard core)

$f^2/4\pi\hbar c$	1r	x_0
0.075	2.104	0.3285
0.090	2.585	0.384

Here the singlet scattering length 1a is adjusted to the experimental value 23.68×10^{-13} cm.²⁾ The experimental value of the singlet effective range is $-(2.7 \pm 0.5) \times 10^{-13}$ cm.²⁾

As can be seen from these results, a reasonable magnitude of the coupling constant $f^2/4\pi\hbar c$ is about $0.08 \sim 0.10$ for both ways of cutting-off considered above. However, in relation to the singlet hard core model suggested by Jastrow³⁾ and somewhat larger values of the singlet effective range, $f^2/4\pi\hbar c = 0.09 \sim 0.10$ and infinite repulsive cutting-off seems to be more favourable.

2) The deuteron ground state and triplet neutron-proton scattering

In this case, as the infinite repulsive cutting-off is unable to produce a sufficient binding energy, zero cut-off method is only adopted. The next table shows the values of the quadrupolemoment, Q , triplet effective range, 3r , and D -state probability, P_D .

$f^2/4\pi\hbar c$	Q (in 10^{-27} cm ²)	3r	P_D (%)	x_0
0.05	1.85	1.325	7.42	0.16
0.08	2.37	1.582	8.65	0.232
0.10	3.25	1.75	11.3	0.303

Experimental values for these quantities:

$$Q = 2.738 \times 10^{-27} \text{ cm}^2, ^4)$$

$$^3r = (1.704 \pm 0.030) \times 10^{-13} \text{ cm}. ^2)$$

It is generally accepted that the observed magnetic moment gives a measure of the D -state admixture, and the value 4%, calculated from the simple formula for the resultant magnetic moment, has been adopted so far. This, however, forms an unreliable restriction, not only because of the uncertain relativistic corrections but a dependence of the magnetic moment of one nucleon on the proximity of another. Actually, the latter effect has been confirmed by the experimental ratio of hyperfine structure in deuterium and hydrogen.⁵⁾ Taking these facts into consideration, it seems meaningless to adhere to the usually adopted value of 4%. The value

obtained here (9~10%) is not necessarily incompatible with present experimental data.

Straight cut-off procedure, using the value $f^2/4\pi\hbar c=0.08$, changes the cut-off radius x_0 from 0.232 to 0.33 altering the other quantities not seriously. If a deep square well is assumed inside,⁶⁾ the cut-off radius would increase and the quadrupole moment decrease a little.

According to these results, it is possible to conclude that the pseudoscalar meson potential with pseudovector coupling (inclusive of the second- and fourth-order terms) is compatible with the deuteron ground state and low energy neutron-proton scattering data, provided the proper choice of the coupling constant between π -meson and nucleon, namely the value 0.09~0.10, and cutting-off is made.

We have investigated, in addition, whether or not the above mentioned potential satisfies the saturation requirements,⁷⁾ and found that they are not satisfied in several respects; for example, spin saturated neutron cluster can be stable as far as reasonable cut-off radius for the potential is assumed. Even if we take into account the pseudoscalar coupling together, not all saturation requirements cannot be satisfied. So we must explain the saturation phenomena by some other effects, if we adopt the static potential deduced from the symmetrical pseudoscalar meson theory with both couplings including fourth-order effects.

We wish to express our thanks to Professors S. Nakamura and M. Sasaki for stimulating discussions throughout the course of this work. We are in addition grateful to Mr. K. Nishijima for sending us the manuscript of his paper before publication.

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Note on the Cascade Function for Heavy Elements

J. Nishimura and K. Ida

Departments of Physics,
Kobe University and Tokyo University

August 26, 1951

Recent calculations^{1),2)} of cascade functions using the accurate Bethe-Heitler cross-sections, have shown that there exist considerable differences between these functions and Snyder's³⁾ results based on the complete screening cross-sections.

In this paper, we should like to stress the point that in order to account the average numbers of the shower particles the Coulomb scattering effect is rather important especially in heavy elements. Efforts in this direction have already been made by Belenky⁴⁾, and we treat the same problem using the angular distribution functions⁵⁾ derived analytically in approximation B⁶⁾.

The approximations used for the calculation of this angular distribution are:

- (1) use of the complete screening cross-sections,
- (2) neglect of elongation of the path for the particles inclined from the shower axis, (see Fig. 1)
- (3) putting $\sin\theta \sim \theta$ (θ is the angle from the shower axis, see also Fig. 1).

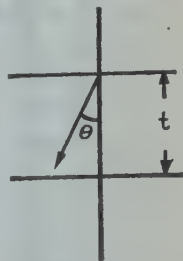


Fig. 1 Elongation of the path is given by

$$\left(\frac{1}{\cos\theta} - 1\right)t$$

The cross-sections for the low energy particles

are smaller than those of complete screening, but, fortunately the error due to these inaccurate cross-sections may be compensated by the error due to (2), because the lower the energy of the particles, the larger the inclination of the particles from the axis. Thus we have only to take into account the effect of (3) and the increase of ionization loss due to (2) to remove the error committed by our approximations. From the simple consideration it may be clear that the exact angular distribution function will be diminished with increasing θ as shown by dotted line in Fig. 2. It is, therefore, reasonable to assume that the shower particles which leave downward from the material are those lying between 0 and 1 radian from the shower axis in our distribution functions. The results thus obtained for considerably high incident energy particles are listed in Table 1, and compared with the experimental data obtained by Shapiro⁷⁾. (Fig. 3)

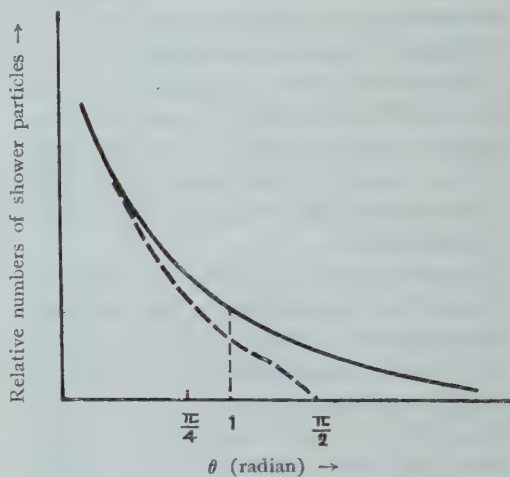


Fig. 2. Schematic representation of angular distribution function. The full line gives the angular distribution calculated using the approximations (1), (2), and (3). The exact distribution will be given by dotted line.

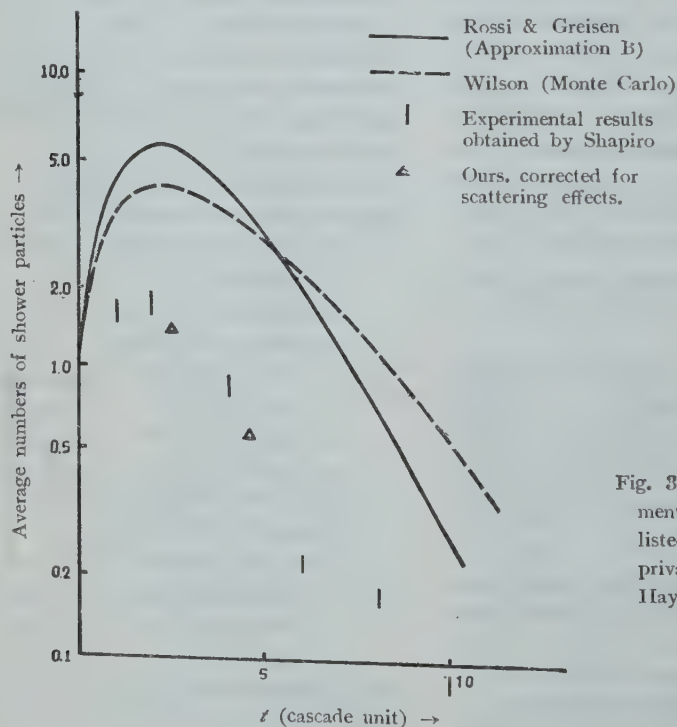


Fig. 3. Comparison with experimental results. The data here listed are mainly due to the private communication from Mr. Hayakawa.

The fair agreement between theoretical and experimental results seems to mean that our approximations are not so far from reality.

Table 1. Fractions of shower particles which leaves downward from the materials.
S: shower ages

S	1	1.4
Pb	0.25	0.18
Fe	0.55	0.47

These values are derived from the formula, $\frac{\int_0^1 \pi(\theta) 2\pi\theta d\theta}{\int_0^\infty \pi(\theta) 2\pi\theta d\theta}$ where $\pi(\theta)$ is the angular distribution function. In order to see the accuracy of these fractions, we have also estimated the formula $\frac{\int_0^{\pi/2} \pi(\theta) 2\pi\theta d\theta}{\int_0^\infty \pi(\theta) 2\pi\theta d\theta}$ which gives the upper limit of these fractions. These values are larger about 20% or less than those listed in this table. These values listed in this table may be considerably accurate.

Since these corrections for the average numbers of shower particles are considerably large, they may be of great importance for the interpretation of the cosmic-ray phenomena containing the cascade showers, i.e. burst, the transition effects of the cascade showers etc. For instance, the discrepancy between theoretical⁶⁾ and experimental results of burst under the thick absorber at sea level can be explained, if one takes account of this scattering effect. The theoretical size frequency curve is thus reduced by a factor about 3.5* and the discrepancy is Much diminished.

The authors wish to express their appreciations to messers. Hayakawa, Fujimoto and Ogawa for their valuable advices throughout this work.

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△ This is also true quantitatively for small shower ages.

* We take Fe ($s=1$) as a correction factor in Table 1, because the critical energy of the material Pb and Fe was determined experimentally by Lapp comparing with the data of the Fe alone.

Statistics of two-dimensional Ising Lattices of Chequered Types

T. Utiyama

Department of Physics, Osaka University

August 26, 1951

For the case of vanishing field, the problem of two-dimensional Ising lattices¹⁾ made up of "vertical" and "horizontal" bonds with two step periodicity can be treated unitarily and the results are exceedingly simple for lattices of "chequered type": we denote by this name, all the lattices which are obtained by replacing black squares of chess-board by one and the same "elementary figure" ($=E.F.$), the latter being assumed to be a rectangle with ν "vertical" cross-pieces. In the $E.F.$, the vertical, the upper and lower horizontal interactions are numbered, from left to right, by $\beta(=-1)$, $\beta_{2\mu+1}$; $\beta^{2\mu}$ and $\beta_{2\mu}$ ($\mu=0, 1, \dots, \nu$) respectively, where some β 's may be 0 or $+\infty$ (with $\beta=J/2kT$, $J/2$ being the interaction energy, $+$ or $-$, etc. See Fig. 1). For example, $\nu=0$: $\beta=+\infty$, $\beta_1=0$ (or $\beta_1=\beta$) represents the 4-(square-) lattice, $\beta_1=0$ or $+\infty$ corresponds respectively to the 6-(honeycomb-) or the 3-(triangular-) lattice; $\nu=1$: $\beta_1=+\infty$ or $\beta_3=0$, to the 3-6-(Kagome-) or the 4-8-lattice; $\nu=3$: $\beta_1=\beta_3=+\infty$, $\beta_2=\beta_7=0$, to the 3-12-lattice.

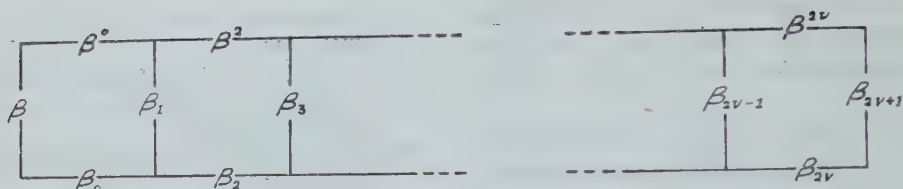


Fig. 1.

The partition function per atom λ of infinite lattice of chequered type is, in usual approximation, given by

$$2[2(\nu+1)-n_\infty](2\pi)^2 \ln(\lambda/\nu^2) = \int_0^{2\pi} \int_0^{2\pi} F.I(\omega_1, \pm\omega_2) d\omega_1 d\omega_2, \\ I(\omega_1, \omega_2) = [\{S_0^2 + (S^2)^2 + (S^1)^2 + S_3^2\} \\ - \cos(\omega_1 \mp \omega_2) - \{S_0^2 + (S^2)^2 - (S^1)^2 \\ - S_3^2 + 1\} \cos(\omega_1 \pm \omega_2) - 2\{S_3 S^1 \\ - S_0 S^2\} \cos \omega_1 - 2\{S_3 S^1 + S_0 S^2\} \cos \omega_2].$$

In this formula, all the quantities can be written down by mere inspection of the E.F.: n_∞ = total number of infinite bonds in the E.F. $F = F_0/F_\infty$,

$$F_0 = \prod_{\mu=0}^{\nu} sh 2\beta^{2\mu} sh 2\beta_{2\mu}, \quad F_\infty = \prod sh 2(\beta)_\infty,$$

where F_∞ is the product of $n_\infty sh 2(\beta)_\infty$'s which correspond to infinite interactions. The most important four parameters S_3, S_0, S^1, S^2 are determined by this rule (in the following prescription, β 's are treated as algebraic quantities regardless of their signs and magnitudes):

$$S_3 = \sum_{\mu=0}^{2\nu} [ch(\beta + \beta_1 + \dots + \beta_{2\nu-1} - \beta_{2\nu+1}) \times \\ \times ch(\beta^{0*} + \beta_0^*) ch(\beta^{2*} + \beta_2^*) ch(\beta^{4*} + \beta_4^*) \dots \times \\ \times ch(\beta^{2\nu*} + \beta_{2\nu}^*)],$$

where, we write first, the "leading factor" (L.F.): $ch(\beta + \dots + \beta_{2\nu-1} - \beta_{2\nu+1})$; second, all the $(2^{\nu}-2)$ $ch(\dots)$'s which are derived from the L.F. by alternation of signs before β 's, corresponding to the crosspieces, i.e. $\beta_{2\nu+1}(\rho=0, \dots, \nu-1)$; third, the $(\nu+1)$ "asterisk factors" of each term which are $ch(\beta^{2\mu*} + \beta_{2\mu}^*)$ or $sh(\beta^{2\mu*} + \beta_{2\mu}^*)$ according as the signs before $\beta_{2\mu-1}$ and $\beta_{2\mu+1}$ are equal

or opposite; at last we sum up all terms. S_0 is of the same structure with the L.F. $ch(\beta + \dots + \beta_{2\nu-1} + \beta_{2\nu+1})$. In the same manner we get S^1 and S^2 with the L.F.'s $sh(\beta + \beta_1 + \dots + \beta_{2\nu-1} + \beta_{2\nu+1})$ and $sh(\beta + \beta_1 + \dots + \beta_{2\nu-1} - \beta_{2\nu+1})$ respectively, but this time, the arguments of asterisk factors being $(\beta^{2\mu*} - \beta_{2\mu}^*)$'s. For practical calculations, it suffices to know only one of the two sets (S_3, S^1) and (S_0, S^2) ; the other can be obtained by simple substitution t having the properties: $t: \beta_{2\nu+1} \rightarrow -\beta_{2\nu+1}$, $\beta_{2\nu} \rightarrow -\beta_{2\nu}$ (i.e. $\beta_{2\nu}^* \rightarrow \beta_{2\nu}^{*-j}$, $j=i(\pi/2)(2l+1)$ l being an integer); $t[S_3 S^1] = S_0 S^2$, $-t[(S^1)^2 + S_3^2] = (S^2)^2 + S_0^2$, $t[|S_3^1|] = |S_0^2|$.

Example: $\nu=0$: $S^1 = sh(\beta + \beta_1) \times ch(\beta^{0*} - \beta_0^*)$, $S_3 = ch(\beta - \beta_1) \cdot sh(\beta^{0*} + \beta_0^*)$.

$$2S_3 S^1 = (1/2)(sh 2\beta + sh 2\beta_1) \times (sh 2\beta^{0*} + sh 2\beta_0^*),$$

$$2S_0 S^2 = t[2S_3 S^1] \\ = (1/2)(sh 2\beta - sh 2\beta_1) \times (sh 2\beta^{0*} - sh 2\beta_0^*).$$

$$(S^1)^2 + S_3^2 = (1/4)[(ch 2(\beta + \beta_1) - 1) \times (ch 2(\beta^{0*} - \beta_0^*) + 1) \\ + (ch 2(\beta - \beta_1) + 1) \times (ch 2(\beta^{0*} + \beta_0^*) - 1)],$$

$$(S^2)^2 + S_0^2 = -t[(S^1)^2 + S_3^2] \\ = (1/4)[(ch 2(\beta - \beta_1) - 1) \times (ch 2(\beta^{0*} - \beta_0^*) - 1) \times \\ + (ch 2(\beta + \beta_1) + 1) \times (ch 2(\beta^{0*} + \beta_0^*) + 1)].$$

Generally, the Curie point T_c is the root of the Eq. in T : $|S_3| = |S^1|$ or $|S_0| = |S^2|$. (When some J 's are $+\infty$, we equate the coefficients of the highest powers in $exp(+\infty)$; we can not enter into details

here, the situation being rather complicated for the case of mixed signs of J 's.) Even if all the J 's are equal and positive, T_c is not determined only by the number of nearest neighbours, the gd -relation proposed by Onsager being characteristic for the case $\nu = 0$.²⁾ By the way, the values of $\exp(2\beta)$ at T_c of the 4-8- and the 3-12-lattices with positive equal J 's are 4.01 and 5.07 respectively. On examining the structure of S 's, we can show the existence of the Curie point under fairly general conditions and when there is no infinite J , the symmetry properties for the reversal of signs of all J 's. (In the opposite case, the eventual disappearance of transition point is connected with the fact that, after the limiting process, $sh()$, $ch()$ become $\exp()$ and even their absolute values lose the sign symmetry in argument.)

In the Euclidian space where the $\gamma^{(+)}$'s are plotted against their argument ϕ and the absolute temperature T ($\gamma^{(+)}$: half of the

real parts of the eigenvalues of the *even* rotation, and $0 < \phi \leq 2\pi$), the value at T_1 of $\ln(\lambda)$ is, apart from analytic additive term and constant factor, equal to the area on the plane $T=T_1$ between the ϕ - T -plane and the $\gamma^{(+)}$ -surface. Roughly speaking, the Curie point can be characterized intuitively as a temperature where the $\gamma^{(+)}$ -surface has a cusp, resulting from the fact that a $\gamma_0^{(-)}$ of *odd* rotation crosses the ϕ - T -plane and that a $\gamma_0^{(+)}$ is equal to $|\gamma_0^{(-)}|$ at all temperatures. Using this image, we can understand the reason why the infinity appears first in the specific heat and the relation between the degeneracy of the maximum eigenvalue and the order disorder transition. Coming to the end, the author is very grateful to the *Yukawa* Foundation of Osaka University.

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On the Adiabatic Nuclear Potential, II

Kazuhiko NISHIJIMA

Department of Physics, Osaka City University

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§ 5. Derivation of the fourth order adiabatic potential

Now we will derive the adiabatic potential from the equation (22). For the sake of simplicity and experimental evidences, we restrict our subject to the spinless meson theories.

We express the wave function of the meson fields by the notation:

$$\phi_L, \quad (L = 1, 2, 3, 4)$$

where all ϕ 's are real, and the suffices 1 and 2 correspond to the charged fields, 3 and 4 to symmetrical neutral and pure neutral respectively. The corresponding isotopic spins are written as

$$\tau_L, \quad (L=1, 2, 3, 4)$$

especially $\tau_4=1$.

We regard the suffix L as a dummy index. Then the commutation relations for the meson fields are given by

$$[\phi_L(X), \phi_M(X')] = -i\delta_{LM}\Delta(X-X'). \quad (24)$$

We decompose the meson field into positive and negative frequency parts, since it is necessary for our calculation:

$$\phi_L(X) = \phi_L^+(X) + \phi_L^-(X), \quad (25)$$

where positive (or negative) frequency parts are annihilation (or creation) operators of the mesons. We obtain the following commutation relations:

$$[\phi_L^+(X), \phi_M^-(X')] = -i\delta_{LM}\Delta_+(X-X'), \quad (26)$$

other commutators = 0,

where

$$\Delta_+(X) = (1/2)(\Delta(X) + i\Delta^{(1)}(X)).$$

The interaction between nucleon and meson fields can be written in the following form:

$$H = \bar{\psi} O_L \psi \cdot \phi_L. \quad (27)$$

For instance, in the case of $Ps(pv)$, we have

$$O_L = i(g/\kappa)\gamma_5\gamma_\mu\tau_L \cdot D_\mu,$$

where D_μ is a differential operator applied only to the meson fields.

With these notations, the expressions for R and T are

$$\begin{aligned} R &= -\frac{i}{2} \int [\bar{\psi} O_L \psi, \bar{\psi}' O'_M \psi'] (\phi_L^- \phi_M^{-'} + \phi_L^+ \phi_M^{+'}) dX', \\ T &= -\frac{i}{2} \int [\bar{\psi} O_L \psi, \bar{\psi}' O'_M \psi'] (\phi_L^- \phi_M^{+'} + \phi_M^- \phi_L^+) dX'. \end{aligned} \quad (28)$$

Inserting the above expression for R into (23a), we see

$$\begin{aligned} V_a(X) &= -\frac{i}{2} [R(X), \int R(X') dX']_{2N} \\ &= \frac{i}{16} \int dX' \int dX'' \int^{\sigma(X'')} dX''' \{ [\phi O_L \psi, \bar{\psi}' O'_M \psi'], [\bar{\psi}'' O''_P \psi'', \bar{\psi}''' O'''_Q \psi'''] \} \\ &\quad \times [\phi_L^- \phi_M^{-'} + \phi_L^+ \phi_M^{+'}, \phi_P^{-''} \phi_Q^{-'''} + \phi_P^{+''} \phi_Q^{+'''}]_0 \\ &= \frac{1}{32} \int dX' \int dX'' \int^{\sigma(X'')} dX''' \{ [\bar{\psi} O_L \psi, \bar{\psi}' O'_M \psi'], [\bar{\psi}'' O''_P \psi'', \bar{\psi}''' O'''_Q \psi'''] \} \\ &\quad \times [\delta_{LP} \delta_{MQ} (\Delta(X-X'') \Delta^{(1)}(X'-X''') + \Delta^{(1)}(X-X'') \Delta(X'-X''')) \\ &\quad + \delta_{LQ} \delta_{MP} (\Delta(X-X''') \Delta^{(1)}(X'-X'') + \Delta^{(1)}(X-X''') \Delta(X'-X''))]. \end{aligned} \quad (29)$$

We integrate this equation in the adiabatic approximation.

As an example, we shall consider the case of $Ps(pv)$. The choice of pv -coupling is due to the following three reasons:

- (1) $Ps(ps)$ theory gives an ordinary force in the 4th order, and therefore does not fit the experiments.³⁰⁾
- (2) $Ps(pv)$ is more favourable than $Ps(ps)$ to explain other phenomena.^{31), 32)}
- (3) $Ps(pv)$ is preferred, for the 2nd order potential can be derived in the Schrodinger approximation, and thus the 4th order potential has a comparatively definite physical meaning.

Now since nucleons are almost unable to move in the adiabatic approximation, their freedoms are limited to spin orientations and charge states, and of course the freedom of changing the energy signs are prohibited. In the case of $Ps(pv)$, the nucleon part of the coupling is

$$i\bar{\psi}\gamma_5\gamma_\mu\psi = (\psi^*\sigma\psi, i\psi^*\rho_2\psi).$$

The spatial part contributes to the nuclear force in that it may change the spin orientation, while the temporal part can only change the energy sign which is considered to be prohibited in the present discussion.

The ratio of the order of magnitude of the spatial part to the temporal part,

taking into account of the meson wave functions, is given by

$$\left(\frac{v}{c}\right)^0 \cdot |q| : \left(\frac{v}{c}\right)^1 \cdot q_0 \sim 1 : \frac{x}{M}, \quad (\text{for low energy meson})$$

where v is the nucleon velocity and (q, q_0) the momentum energy four vector of the meson. Our adiabatic approximation is concerned only with terms of order $(x/M)^0$, therefore only the spatial part survives and we drop the temporal part from our calculation. Inserting the expression of \mathcal{A} -function (12) into (29), in virtue of the formula:

$$\mathcal{A}(X)\mathcal{A}^{(1)}(Y) + \mathcal{A}^{(1)}(X)\mathcal{A}(Y) = -\frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}\cdot\mathbf{r} + i\mathbf{l}\cdot\mathbf{y}} \sin(k_0 x_0 + l_0 y_0),$$

we have

$$\begin{aligned} V_a(X) = & \frac{1}{32} \int dX' \int dX'' \int dX''' \frac{g_L g_M g_P g_Q}{x^4} \\ & \times \{ [\phi^*(\sigma \nabla) \tau_L \phi, \phi^{*'}(\sigma \nabla') \tau_M \phi'], [\phi^{*''}(\sigma \nabla'') \tau_P \phi'', \phi^{*'''}(\sigma \nabla''') \tau_Q \phi'''] \} \\ & \times \frac{1}{(2\pi)^6} \left[\delta_{LP} \delta_{MQ} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') + i\mathbf{l}\cdot(\mathbf{x}'-\mathbf{x}''')} \sin(k_0(t-t'') + l_0(t'-t''')) \right. \\ & \left. + \delta_{LQ} \delta_{MP} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'') + i\mathbf{l}\cdot(\mathbf{x}-\mathbf{x}''')} \sin(k_0(t'-t'') + l_0(t-t''')) \right], \end{aligned} \quad (30)$$

where the operator ∇ in $(\sigma \nabla)$ should be understood to be applied only on the meson part. Well, we can regard the bilinear forms of the nucleon wave function to be time independent in the adiabatic approximation and so the time integration is performed referring only to the meson part.

The time integrations are

$$\begin{aligned} \int^t dt' \int^t dt'' \int^{t'''} dt''' \sin(k_0(t-t'') + l_0(t'-t''')) &= \frac{1}{l_0^2(l_0 + k_0)}, \\ \int^t dt' \int^t dt'' \int^{t'''} dt''' \sin(k_0(t'-t'') + l_0(t-t''')) &= \frac{1}{l_0 k_0(l_0 + k_0)}, \end{aligned}$$

where the following definition of the time integration is employed:

$$\int^t \cos k_0 t' dt' = \frac{1}{k_0} \sin k_0 t, \quad \int^t \sin k_0 t' dt' = -\frac{1}{k_0} \cos k_0 t.$$

Also in this approximation of stopping nucleons, the spatial coordinates referring to the same nucleon can be equated, i.e., $\mathbf{x} = \mathbf{x}'$, $\mathbf{x}'' = \mathbf{x}'''$, and this substitution is guaranteed by the occurrence of $\delta(\mathbf{x} - \mathbf{x}')$, and $\delta(\mathbf{x}'' - \mathbf{x}''')$ from the nucleon commutators, so that we may put $\mathbf{x} = \mathbf{x}' \equiv \mathbf{x}_1$, $\mathbf{x}'' = \mathbf{x}''' \equiv \mathbf{x}_2$ and $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{r}$. Then (30) is rewritten as

$$\begin{aligned}
V_a(X) = & \frac{g_L g_M g_P g_Q}{32(2\pi)^6 x^4} \int d\mathbf{x}' d\mathbf{x}'' d\mathbf{x}''' \\
& \times \{ [\psi^* \sigma_\pi \tau_L \psi, \psi^{*'} \sigma_\lambda \tau_M \psi'], [\psi^{*''} \sigma_\mu \tau_P \psi'', \psi^{*'''} \sigma_\nu \tau_Q \psi'''] \} \\
& \times \left(\partial_{LP} \delta_{MQ} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} (k_\pi k_\mu l_\lambda l_\nu) e^{i(k+l)\mathbf{r}} \frac{1}{l_0^2(l_0+k_0)} \right. \\
& \left. + \partial_{LQ} \delta_{MP} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} (k_\lambda k_\mu l_\pi l_\nu) e^{i(k+l)\mathbf{r}} \frac{1}{l_0 k_0(l_0+k_0)} \right). \quad (31)
\end{aligned}$$

In order to replace the form $(kkll)$ by the differential operator, we distinguish the \mathbf{r} 's appearing in the forms $k\mathbf{r}$ and $l\mathbf{r}$, i.e. we make the substitution

$$(k+l)\mathbf{r} \rightarrow k\mathbf{r} + l\mathbf{r}'.$$

Thus the problem of integrating (31) is reduced to the following integrals:

$$\begin{aligned}
G_1(\mathbf{r}, \mathbf{r}') &= \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'} \frac{1}{l_0^2(l_0+k_0)}, \\
G_2(\mathbf{r}, \mathbf{r}') &= \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'} \frac{1}{l_0 k_0(l_0+k_0)}. \quad (32)
\end{aligned}$$

We insert these functions into (31) and omit $d\mathbf{x}''$ to obtain the potential instead of the interaction Hamiltonian density.

$$\begin{aligned}
V_a(\mathbf{r}) = & \frac{g_L g_M g_P g_Q}{32x^4} \int d\mathbf{x}' d\mathbf{x}''' \\
& \times (\partial_{LP} \delta_{MQ} \{ [\psi^*(\sigma \nabla) \tau_L \psi, \psi^{*'}(\sigma' \nabla) \tau_M \psi'], [\psi^{*''}(\sigma \nabla) \tau_P \psi'', \psi^{*'''}(\sigma' \nabla) \tau_Q \psi'''] \} G_1(\mathbf{r}, \mathbf{r}') \\
& + \partial_{LQ} \delta_{MP} \{ [\psi^*(\sigma \nabla) \tau_L \psi, \psi^{*'}(\sigma \nabla) \tau_M \psi'], [\psi^{*''}(\sigma \nabla) \tau_P \psi'', \psi^{*'''}(\sigma' \nabla) \tau_Q \psi'''] \} G_2(\mathbf{r}, \mathbf{r}'))_{\mathbf{r}'=\mathbf{r}},
\end{aligned}$$

where ∇, ∇' are operated on the variables \mathbf{r} and \mathbf{r}' respectively and the substitution $\mathbf{r}'=\mathbf{r}$ must be done after differentiations. The occurrence of two distances \mathbf{r} and \mathbf{r}' corresponds to the interchange of two mesons. Transforming the above expression from quantized field operators into the configuration space representation, we have

$$\begin{aligned}
V_a = & \frac{1}{8} \delta_{LP} \delta_{MQ} \frac{g_L g_M g_P g_Q}{x^4} [(\sigma \nabla) \tau_L, (\sigma' \nabla) \tau_M]^{(1)} [(\sigma \nabla) \tau_P, (\sigma' \nabla) \tau_Q]^{(2)} \\
& \times (G_1(\mathbf{r}, \mathbf{r}') - G_2(\mathbf{r}, \mathbf{r}'))_{\mathbf{r}'=\mathbf{r}}, \quad (33)
\end{aligned}$$

or in the original expression (27),

$$V_a = \frac{1}{8} [O_L, O'_M]^{(1)} [O_L, O'_M]^{(2)} (G_1(\mathbf{r}, \mathbf{r}') - G_2(\mathbf{r}, \mathbf{r}'))_{\mathbf{r}'=\mathbf{r}}, \quad (34)$$

where the upper suffices refer to the number of nucleons.

As for the concrete expressions of the functions G_1 and G_2 , it will be discussed later in detail. The next task is the computation of V_b , which in a

completely analogous way to V_a is given by

$$\begin{aligned}
 V_b(X) &= \frac{i}{8} [[[H(X), \int^\sigma H(X') dX'] \int^\sigma H(X'') dX''] \int^\sigma H(X''') dX''']_{2N} \\
 &= \frac{3}{32} \frac{g_L g_M g_F g_Q}{x^4} \int^\sigma dX' \int^\sigma dX'' \int^\sigma dX''' \\
 &\quad \times \{ [\psi^* (\sigma \nabla) \tau_L \psi, \psi^{*'} (\sigma \nabla') \tau_M \psi'], [\psi^{*''} (\sigma \nabla'') \tau_F \psi'', \psi^{*'''} (\sigma \nabla''') \tau_Q \psi'''] \} \\
 &\quad \times \delta_{LP} \delta_{MQ} \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} e^{i\mathbf{k}(x-x'') + i\mathbf{l}(x'-x''')} \sin k_0(\ell-\ell'') \cdot \cos l_0(\ell'-\ell''').
 \end{aligned} \tag{35}$$

We define the following function G_3 :

$$G_3(r, r') = \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0^2} \cdot \frac{d\mathbf{l}}{l_0^3} e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'}. \tag{36}$$

Then V_b in the configuration space representation is given by

$$V_b = \frac{3}{8} \delta_{LP} \delta_{MQ} \frac{g_L g_M g_F g_Q}{x^4} [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(1)} [(\sigma \nabla) \tau_F, (\sigma \nabla') \tau_Q]^{(2)} G_3(r, r')_{r'=r}. \tag{37}$$

Corresponding to (34), we have

$$V_b = \frac{3}{8} [O_L, O'_M]^{(1)} [O_L, O'_M]^{(2)} G_3(r, r')_{r'=r}. \tag{38}$$

Now put

$$G(r, r') = G_1(r, r') - G_2(r, r') + 3G_3(r, r'), \tag{39}$$

then the whole 4th order potential is

$$V_4 = \frac{1}{8} \delta_{LP} \delta_{MQ} \frac{g_L g_M g_F g_Q}{x^4} [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(1)} [(\sigma \nabla) \tau_F, (\sigma \nabla') \tau_Q]^{(2)} G(r, r')_{r'=r}, \tag{40}$$

or generally

$$V_4 = \frac{1}{8} [O_L, O'_M]^{(1)} [O_L, O'_M]^{(2)} G(r, r')_{r'=r}. \tag{41}$$

This form of potential agrees with the results obtained by the S-matrix method.³³⁾

For the sake of completeness, we will write down the 2nd order potential

$$V_2 = \frac{1}{4\pi} \delta_{LM} \frac{g_L g_M}{x_2} (\sigma \nabla_r \tau_L)^{(1)} (\sigma \nabla_r \tau_M)^{(2)} \frac{e^{-\kappa r}}{r}, \quad P_S(pv)$$

or generally

$$V_2 = -\frac{1}{4\pi} O_L^{(1)} O_L^{(2)} \frac{e^{-\kappa r}}{r}.$$

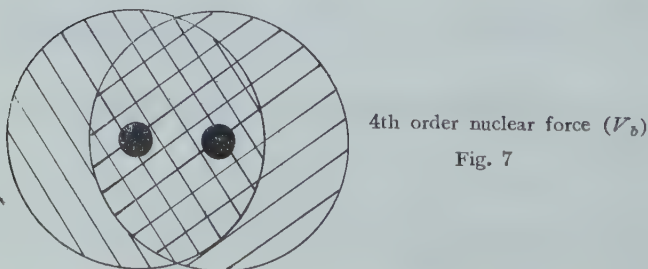
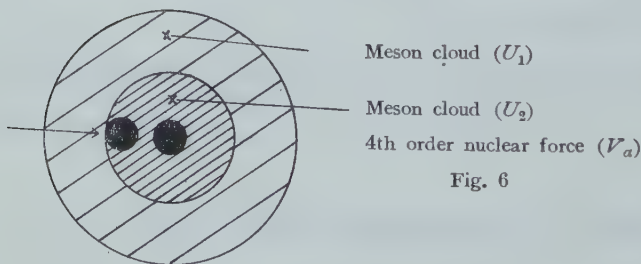
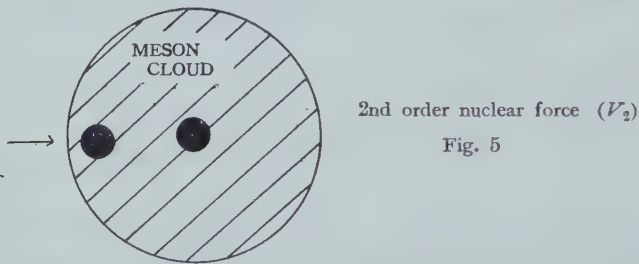
§ 6. Physical interpretation of nuclear forces

The method of canonical transformations employed in this paper provides means to analyze the mesonic self field around a nucleon so that we will investigate how the meson clouds contribute to the nuclear forces by an intuitive method. First the 2nd order potential is of the form :

$$V_2 \sim \bar{\psi} O \psi \cdot \frac{\bar{\phi} O \phi}{x^2 - \Delta}.$$

Now it comes into question how to interpret this potential. The bare nucleons have no direct interaction between themselves in the original form, but once they are clothed with meson clouds the overlapping of meson clouds induces forces between clothed nucleons.

As is seen from the above expression, the one nucleon is bare and the other clothed. Thus we interpret the 2nd order nuclear force as such a phenomenon that one bare nucleon rushes into the meson cloud of the other nucleon and intends to carry away the clothes, which is interrupted by the owner. (fig. 5)



Next, the 4th order potential V_a is derived after two canonical transformations, i.e. one nucleon is doubly clothed. The second clothes consist of mesons promenading always in couples around a nucleon. V_a is interpreted as such that one bare nucleon rushes into the second clothes of the other nucleon just as in the case of V_2 . (fig. 6.) Finally, the nuclear force V_b can be induced only by the first Bloch-Nordsieck transformation, nevertheless interchanges two mesons. Therefore it must occur by the collision of two singly clothed nucleons. (fig. 7.) These figures are essentially spatial in contrast to the temporal descriptions by Feynman diagrams. This difference corresponds to that between the canonical transformation method and the S -matrix method.

§ 7. Adiabatic nuclear potential

In this section we will push forward the discussions in previous sections and derive explicit forms of potentials. For this purpose, we have to calculate the integrals defining the function G .

A. Derivation of the function $G(r, r')$

First we calculate $G_1 - G_2$:

$$G_1 - G_2 = \frac{1}{(2\pi)^6} \int d\mathbf{k} d\mathbf{l} e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'} \frac{k_0 - l_0}{l_0^3 k_0^2 (l_0 + k_0)}. \quad (42)$$

We decompose the integrand into the following form:

$$\frac{k_0 - l_0}{l_0^3 k_0^2 (l_0 + k_0)} = \left(\frac{1}{l_0^3} - \frac{2}{l_0^2 k_0} + \frac{1}{l_0 k_0^2} \right) \frac{1}{k^2 - l^2},$$

and calculate the following integral which corresponds to the second term in the above decomposition, while the third is of the same type.

$$\begin{aligned} & \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k} d\mathbf{l}}{k_0 l_0^2} \cdot \frac{1}{l^2 - k^2} \cdot e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'} \\ &= \frac{(4\pi)^2}{(2\pi)^6} \int_0^\infty \frac{k dk}{2k \cdot k_0} \int_0^\infty \frac{l dl}{l_0^2} \cdot \frac{\sin kr}{r} \cdot \frac{\sin lr'}{r'} \cdot \left(\frac{1}{l-k} - \frac{1}{l+k} \right) \\ &= \frac{(2\pi)^2}{2(2\pi)^6 r r'} \int_{-\infty}^\infty \frac{dk}{k_0} \int_{-\infty}^\infty \frac{dl}{l_0^2} \cdot \sin kr \cdot \sin lr' \cdot \left(\frac{1}{l-k} - \frac{1}{l+k} \right). \end{aligned} \quad (43)$$

We first integrate with respect to l , by means of Cauchy method, i.e.

$$\int_{-\infty}^\infty \frac{dl}{l^2} \sin lr' \left(\frac{1}{l-k} - \frac{1}{l+k} \right) = 2\pi k \frac{\cos kr' - e^{-\pi r'}}{k^2 + x^2}.$$

Inserting this result into (43), we have

$$\frac{(2\pi)^3}{2(2\pi)^6 r r'} \left\{ \int_{-\infty}^\infty \frac{k dk}{k_0^3} \cos kr' \sin kr - e^{-\pi r'} \int_{-\infty}^\infty \frac{k dk}{k_0^3} \sin kr \right\}$$

$$= \frac{1}{2rr'} \{ F(r+r') + F(r-r') - 2e^{-\kappa r'} F(r) \}, \quad (44)$$

where

$$F(R) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{k dk}{k_0^3} \sin kR. \quad (45)$$

Multiplying (44) by 2 we have the second term, and exchanging the arguments r and r' between them we have the third term. The integral corresponding to the first term is

$$\begin{aligned} & \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k} d\mathbf{l}}{l_0^3} \cdot \frac{1}{k^2 - l^2} \cdot e^{i\mathbf{k}\mathbf{r} + i\mathbf{l}\mathbf{r}'} \\ &= \frac{1}{(2\pi)^4 rr'} \int_{-\infty}^\infty k dk \int_{-\infty}^\infty \frac{ldl}{l_0^3} \sin kr \cdot \sin lr' \frac{1}{k^2 - l^2}. \end{aligned} \quad (46)$$

First integrating with respect to k , we have

$$\int_{-\infty}^\infty dk \frac{k}{k^2 - l^2} \sin kr = \pi \cos lr, \quad (r > 0)$$

which gives when inserted into (46)

$$\begin{aligned} & \frac{\pi}{(2\pi)^4 rr'} \int_{-\infty}^\infty \frac{ldl}{l_0^3} \sin lr' \cos lr \\ &= \frac{1}{2rr'} \{ F(r+r') - F(r-r') \}. \end{aligned} \quad (47)$$

Thus we have the whole expression of $G_1 - G_2$:

$$\begin{aligned} G_1 - G_2 &= \frac{1}{2rr'} \{ F(r+r') - F(r-r') \} + \frac{1}{rr'} \{ F(r+r') + F(r-r') - 2e^{-\kappa r'} F(r) \} \\ &+ \frac{1}{2rr'} \{ F(r+r') - F(r-r') - 2e^{-\kappa r} F(r) - e^{-\kappa r} F(r') \}. \\ &= \frac{1}{rr'} \{ 2F(r+r') - 2e^{-\kappa r'} F(r') \}. \end{aligned}$$

The function G_3 is readily integrated by the separation of variables, and yields

$$G_3 = \frac{1}{rr'} e^{-\kappa r} F(r'). \quad (48)$$

Therefore we have

$$G = G_1 - G_2 + 3G_3 = \frac{2}{rr'} F(r+r') + \frac{2}{rr'} (e^{-\kappa r} F(r') - e^{-\kappa r'} F(r')). \quad (49)$$

The integral (45) can be expressed by the modified Bessel function,³⁴⁾ i.e.

$$F(R) = \frac{R}{(2\pi)^3} K_0(xR), \quad (50)$$

where the function K is connected with Hankel's function by the following relation :

$$K_\nu(x) = \frac{\pi i}{2} \exp\left(\frac{\nu\pi i}{2}\right) \cdot H_\nu^{(2)}(ix).$$

Since the expressions of the 4th order potential (40) and (41) are symmetric with respect to the differential operators ∇ and ∇' , terms which survive after the substitution $r=r'$ are only the symmetric part of (49).

So effectively

$$G \sim \frac{2}{rr'} F(r+r'). \quad (51)$$

This is the fundamental function of the 4th order potential,⁽³⁵⁾⁻⁽³⁷⁾ just as the Yukawa function e^{-x}/x in the 2nd order.

B. Symmetrical scalar theory

As the examples of the theory, we take up only the symmetrical meson theories, since the charge independence of nuclear forces⁽³⁸⁾ seems favourable from the analyses of the mirror nuclei⁽³⁹⁾ and the nucleon-nucleon scattering.⁽⁴⁰⁾⁽⁴¹⁾

We first illustrate by the symmetrical scalar theory.

In this case

$$O_L = f\tau_L,$$

and the 2nd order potential is given by

$$V_2 = -\frac{f^2}{4\pi} (\tau^{(1)} \tau^{(2)}) \frac{e^{-xr}}{r}.$$

The 4th order potential is written as

$$V_4 = \frac{1}{8} f^4 [\tau_L, \tau_M]^{(1)} [\tau_L, \tau_M]^{(2)} G(r, r). \quad (52)$$

Noting the relations :

$$[\tau_L, \tau_M]^{(1)} [\tau_L, \tau_M]^{(2)} = -8 (\tau^{(1)} \tau^{(2)}),$$

$$G(r, r) = \frac{2}{r^2} F(2r) = \frac{2}{(2\pi)^3 r} K_0(2xr),$$

we obtain the final result :

$$V_4 = -\left(\frac{f^2}{4\pi}\right)^2 (\tau^{(1)} \tau^{(2)}) \frac{4}{\pi} \cdot \frac{1}{r} K_0(2xr).$$

Thus the 2nd plus 4th order potential is equal to

$$V = -(\tau^{(1)}\tau^{(2)}) \left[\frac{f^2}{4\pi} \cdot \frac{e^{-\kappa r}}{r} + \left(\frac{f^2}{4\pi} \right)^2 \cdot \frac{4}{\pi} \cdot \frac{1}{r} K_0(2\kappa r) \right]. \quad (53)$$

The ratio of the 4th order to the 2nd at the force range $\kappa r = 1$ is given by

$$\left(\frac{V_4}{V_2} \right)_{\kappa r = 1} = \frac{f^2}{4\pi} \cdot \frac{4e}{\pi} K_0(2) = 0.39 \frac{f^2}{4\pi}.$$

This is a fairly small value, and we may suppose that the perturbation theory has its validity in the scalar meson theory. We see from (52) that the 4th order potential survived because of the charge correlations between mesons, i.e. the non-commutativity of isotopic spins.

So it is easily understood that the 4th order potential vanishes in the neutral scalar meson theory since no correlation due to spin or charge exists provided that nucleon recoils are neglected in the adiabatic approximation.

C. Symmetrical pseudoscalar theory⁽²⁾

In this case, we put $g_1 = g_2 = g_3 = g$, $g_4 = 0$, then the 4th order potential is given by

$$V_4 = \frac{1}{8} \left(\frac{g}{\kappa} \right)^4 [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(1)} [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(2)} G(r, r')_{r'=r}. \quad (54)$$

The commutator is calculated as follows:

$$\begin{aligned} & [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(1)} [(\sigma \nabla) \tau_L, (\sigma \nabla') \tau_M]^{(2)} G(r, r')_{r'=r} \\ &= -4[2(\nabla \nabla')^2 (\tau^{(1)} \tau^{(2)}) + 3(\sigma^{(1)}, \nabla, \nabla')(\sigma^{(2)}, \nabla, \nabla')] G(r, r')_{r'=r} \\ &= -\left[8(\tau^{(1)} \tau^{(2)}) \left(\frac{2}{r^2} \cdot \frac{\partial^2}{\partial r \partial r'} + \frac{\partial^4}{\partial r^2 \partial r'^2} \right) + 24(\sigma^{(1)} \sigma^{(2)}) \frac{1}{r^2} \cdot \frac{\partial^2}{\partial r \partial r'} \right. \\ &\quad \left. + 12 \left((\sigma^{(1)} \sigma^{(2)}) - \frac{(\sigma^{(1)} \mathbf{r})(\sigma^{(2)} \mathbf{r})}{r^2} \right) \frac{1}{r} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial r'} - \frac{2}{r} \right) \frac{\partial^2}{\partial r \partial r'} \right] G(r, r')_{r'=r}. \end{aligned}$$

In the above consideration the derivatives of the function K_0 arise which can be reduced to the linear combinations of K_ν 's by means of the following formulae:

$$K_{\nu+1} + K_{\nu-1} = -2K'_\nu,$$

$$K'_\nu = K_{-\nu}.$$

Performing the above differentiation, we get after tedious calculations

$$V_4 = \kappa \left(\frac{g^2}{4\pi} \right)^2 [(\tau^{(1)} \tau^{(2)}) U_\tau(\kappa r) + (\sigma^{(1)} \sigma^{(2)}) U_\sigma(\kappa r) + S_{12} U_T(\kappa r)], \quad (55)$$

where

$$S_{12} = \frac{3(\sigma^{(1)} \mathbf{r})(\sigma^{(2)} \mathbf{r})}{r^2} - (\sigma^{(1)} \sigma^{(2)}),$$

$$U_\tau = (8/\pi) (-A_1 + 2A_2 - 4A_3 + 2A_4) < 0,$$

$$U_o = -(\delta/\pi)(2A_2 - 2A_3 + A_4) > 0, \quad (56)$$

$$U_T = (4/\pi)(2A_2 - 5A_3 + 4A_4) < 0,$$

and

$$A_1 = \frac{K_0^{IV}(2x)}{x}, \quad A_2 = \frac{K_0'''(2x)}{x^2}, \quad A_3 = \frac{K_0''(2x)}{x^3}, \quad A_4 = \frac{K_0'(2x)}{x^4}.$$

These functions have singularities of x^{-5} , and numerically

$$U_o \sim |U_T| \sim (1/2)|U_T|. \quad (57)$$

Thus obtained 4th order potential in $Ps(pv)$ has a completely different form from the ordinary force in $Ps(ps)$. For the sake of completeness we write down the 2nd plus 4th order potential:

$$V = x \left(\frac{g^2}{4\pi} \right) (\tau^{(1)} \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \sigma^{(2)}) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) S_{12} \right] \frac{e^{-x}}{x} \\ + x \left(\frac{g^2}{4\pi} \right)^2 [(\tau^{(1)} \tau^{(2)}) U_T + (\sigma^{(1)} \sigma^{(2)}) U_o + S_{12} U_T], \quad (x = xr) \quad (58)$$

For the comparison with the 2nd order potential, we will give their rough numerical values at the force range $x = xr = 1$,

$$U_T(1) = -5.1, \quad U_o(1) = 2.7, \quad U_T(1) = -2.6, \quad e^{-x}/x(1) = 0.37.$$

As is seen from the above values, the 4th order potential has a large possibility of being comparable with the 2nd order one if suitably superposed, and this is really the case.

Thus the validity of the weak coupling theory becomes quite doubtful in this case. Now we will distinguish the potential (58) in various cases.

(a) *Singlet even state*

Substituting $(\sigma^{(1)} \sigma^{(2)}) = -3$, $(\tau^{(1)} \tau^{(2)}) = 1$ into (58), we have

$$V = -x\alpha Y + x\alpha^2 (U_T - 3U_o), \quad (59a)$$

where we have put

$$Y = \frac{e^{-x}}{x}, \quad \alpha = \frac{g^2}{4\pi}.$$

Since $U_T - 3U_o < 0$, both the 2nd and the 4th orders are attractive, and at the force range $x = 1$, the ratio of the 4th order to the 2nd order is

$$(V_4/V_2)_{x=1} \sim 3.5\alpha.$$

Even for $\alpha = 0.1$, this ratio has a large value 3.5 and we cannot neglect the effect of the 4th order potential.

(b) *Singlet odd state*

In this case, $(\sigma^{(1)} \sigma^{(2)}) = -3$, $(\tau^{(1)} \tau^{(2)}) = -3$ and

$$V = 3\kappa a Y - 3\kappa a^2 (U_\tau + U_\sigma); \quad (59b)$$

Since $U_\tau + U_\sigma < 0$, both the 2nd and the 4th orders are repulsive, and the ratio is

$$(V_4/V_2)_{x=1} \sim 7a.$$

This value is comparatively small.

(c) *Triplet even state*

$$(\sigma^{(1)}\sigma^{(2)}) = 1, \quad (\tau^{(1)}\tau^{(2)}) = -3, \quad A = \frac{1}{3} + \frac{1}{x} + \frac{1}{x^2},$$

$$V = -\kappa a (Y + 3S_{12}AY) + \kappa a^2 [-3U_\tau + U_\sigma + S_{12}U_T]. \quad (59c)$$

In this case, the 2nd order central force is attractive and the 4th order central force is repulsive, and the ratios are

$$\left| \frac{V_4}{V_2} \right|_{x=1} \sim 49a \text{ for central part, } \left| \frac{V_4}{V_2} \right|_{x=1} \sim a \text{ for tensor part.}$$

In this case too, the 4th order is very large.

(d) *Triplet odd state*

$$(\sigma^{(1)}\sigma^{(2)}) = 1, \quad (\tau^{(1)}\tau^{(2)}) = 1,$$

$$V = \kappa a \left(\frac{1}{3}Y + S_{12}AY \right) + \kappa a^2 (U_\tau + U_\sigma + S_{12}U_T). \quad (59d)$$

The 2nd order central force is repulsive and 4th order attractive. The ratios are

$$\left| \frac{V_4}{V_2} \right|_{x=1} \sim 21a \text{ for central part, } \left| \frac{V_4}{V_2} \right|_{x=1} \sim 3a \text{ for tensor part.}$$

In the 4th order potentials, U 's are superposed in the same sign in the even states and in the opposite sign in the odd states.

We will not touch here the comparison with experiments since it is a very tedious complicated problem, they will be discussed elsewhere.

§ 3. Concluding remarks

From the previous calculations we may conclude that the convergence of the perturbation theory in the scalar meson theory is fairly good, while it is quite doubtful whether the perturbation calculation is meaningful or not in the pseudo-scalar case. This conclusion does not seriously depend on the value of the coupling constant.

Although the 4th order potential becomes quite large as compared to the 2nd order one, it is generally believed that the perturbation method gives rather larger values than the true ones if it is not a good approximation. The 4th order is larger than with the 2nd order even at the neighbourhood of the force range. Thus, if the higher orders are also large, we have to suppose that

$$xr_c \gtrsim 1.$$

The value of r_c is insensitive to the choice of the coupling constant, and in this case we must abandon the hope of both weak and strong coupling theories and depend on the intermediate coupling theory.

The nuclear force can roughly be separated into static and non-static parts, and we discussed in this paper only the cases when the former is supposed to be larger than the latter, i.e. the cases in which we can calculate only on the Schroedinger approximation but not Pauli approximation. But as is seen in the calculation of the case $P_3(pv)$, the higher orders, consequently the non-static part, will give larger contributions than are predicted before.

Thus the physically meaningful potential will be only the velocity dependent one, and the representation of the phenomenological potential in the form of static nuclear potential will have a meaning only as a suitable average of the velocity dependent one.

Such a point of view will play an important rôle in the saturation problem. For the present phenomenological potential determined from the scattering data cannot satisfy the saturation requirements so long as it is expressed in the form of a static potential, while the velocity dependent potentials are capable of explaining the saturation properties. There the idea of the velocity dependent potential may be a key point together with the non-linear meson theory.

In connexion with this problem, the many body force must be analyzed, but it is of the higher order in the nucleon velocities and thus vanishes in the adiabatic approximation as is seen from the expression (41).

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Appendix

Renormalizations in the Adiabatic Approximation In the text, we picked up only terms corresponding to the diagrams (A) and (B). (Cf. fig. 4.) According to Watson and Lepore, other diagrams do not contribute to the 4th order nuclear force in the low energy region. But as we have employed a different type of coupling $P_3(pv)$, we have to prove this fact again. We shall prove that other diagrams give only renormalizations of the coupling constant and the rest mass of a nucleon in the adiabatic approximation. In this appendix, we illustrate it only by the diagram (L), but as for other diagrams we see completely the same situations.

Picking up terms corresponding to the diagram (L) from $H''(X)$ in (20),

$$\langle H''(X) \rangle_L = \frac{i}{8} [[[H(X), \int^0 H(X') dX'] \int^0 H(X'') dX''] \int^0 H(X''') dX''']_L$$

$$\begin{aligned}
&= \frac{1}{8} \frac{g_L g_M g_F g_Q}{x_4} \int^{\sigma} dX' \int^{\sigma} dX'' \int^{\sigma} dX''' \\
&\times [(3/2) \delta_{LQ} \delta_{MP} [[\bar{\psi} O_L \psi, \bar{\psi}' O'_M \psi'] \bar{\psi}'' O''_P \psi'''] \cdot \bar{\psi}''' O'''_Q \psi'''] \\
&\times \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} e^{i\mathbf{k}(x-x''')} \sin k_0(t-t''') \cdot \int \frac{d\mathbf{l}}{l_0} e^{i\mathbf{l}(x'-x'')} \cos l_0(t'-t'') \\
&+ (1/2) \delta_{LM} \delta_{PQ} [[\bar{\psi} O'_M \psi', \bar{\psi}'' O''_P \psi''] \bar{\psi}''' O'''_Q \psi'''] \cdot \bar{\psi} O_L \psi \\
&\times \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} e^{i\mathbf{k}(x-x')} \sin k_0(t-t') \cdot \int \frac{d\mathbf{l}}{l_0} e^{i\mathbf{l}(x''-x''')} \cos l_0(t''-t''').
\end{aligned}$$

Performing the time integration in the adiabatic approximation, we transform it into the configuration space representation.

For the symmetrical $Ps(pv)$ theory, we obtain

$$\begin{aligned}
V_L &= \frac{1}{2} \left(\frac{g}{x} \right)^4 \delta_{LQ} \delta_{MP} \frac{1}{(2\pi)^6} \int \frac{d\mathbf{k}}{k_0} \cdot \frac{d\mathbf{l}}{l_0} \cdot \frac{1}{k_0 l_0^2} e^{i\mathbf{k}\mathbf{r}} \\
&\times [[\sigma\mathbf{k} \cdot \tau_L, \sigma\mathbf{l} \cdot \tau_M] \sigma\mathbf{l} \cdot \tau_P]^{(1)} \cdot (\sigma\mathbf{k} \cdot \tau_Q)^{(2)} \\
&= \frac{1}{2} g^4 \cdot \frac{1}{(2\pi)^6} \int \frac{l^2 d\mathbf{l}}{l_0^3} \cdot \int \frac{d\mathbf{k}}{k_0^2} e^{i\mathbf{k}\mathbf{r}} \frac{(\sigma\mathbf{k} \cdot \tau_L)^{(1)} \cdot (\sigma\mathbf{k} \cdot \tau_L)^{(2)}}{x^4} \\
&= (\text{divergent constant}) \times V_2.
\end{aligned}$$

This gives just the renormalization of the coupling constant.

In a similar way, the diagram (S) gives the renormalization of the nucleon rest mass, but (V) vanishes from the beginning in this adiabatic approximation.

Thus we see that other diagrams than (A) and (B) give nothing but renormalizations in the low energy regions.

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On the Energy-Momentum Tensor of Bopp-Type Non-Local Field

Yôro Ôno

Department of Physics, Hokkaido University

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Expanding the characteristic function $\epsilon(x)$ in the Lagrangian in derivatives of $\delta(x)$ and applying the variation principle of general relativity, we obtained the symmetrical energy-momentum tensors for the Bopp-type non-local fields. They are free from the defects accompanying the expressions proposed by Bopp and Heisenberg and can be regarded as the most natural extension of the local fields.

§ 1. Introduction

In his series of papers Bopp considered the field whose Lagrangian is given by the form

$$L(x) = \int \varphi(x) \epsilon(x-x') \varphi(x') dx',$$

where $\varphi(x)$ is a field variable of certain type (scalar, vector or tensor), and $\epsilon(x)$ the characteristic non-local function.* Concerning the energy-momentum tensor of this field investigations have been performed by several authors.¹⁾ But the expressions proposed in concrete for each type of the field involve serious defects and cannot be deemed satisfactory.

For the extension of electromagnetic field, Bopp²⁾ proposed the following Fourier component of the tensor:

$$\begin{aligned} \tilde{T}_{\mu\nu}(k) = & \frac{1}{16\pi} \frac{1}{(2\pi)^4} \int dk' \left[\tilde{\varphi}_\mu(k-k') \tilde{\varphi}_\nu(k') \left\{ -2(k-k')^\alpha \tilde{\epsilon}(k-k') \delta_{\mu\nu} \right. \right. \\ & + (k-2k')_\mu (k-2k')_\nu \frac{(k-k')^\alpha \tilde{\epsilon}(k-k') - k'^\alpha \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} \left. \right\} \\ & \left. + 4\tilde{\varphi}_\mu(k-k') \tilde{\varphi}_\nu(k') k'^\alpha \tilde{\epsilon}(k') \right], \end{aligned} \quad (1.1)$$

where $\varphi_p(x)$ is the potential.** Owing to the field equation

* In this paper we adopted the notation of Bopp. $\varphi(x)$ and $\epsilon(x)$ represent the function of four space-time variables x_λ , and the integral $\int dx$ denotes the quadruple integral $\iiint dx_1 dx_2 dx_3 dx_0$.

** The Fourier components $\tilde{\varphi}_p(k)$ and $\tilde{\epsilon}(k)$ are defined by

$$\varphi_p(x) = \frac{1}{(2\pi)^4} \int dk \tilde{\varphi}_p(k) \exp i k_\lambda x_\lambda, \quad \epsilon(x) = -\frac{1}{(2\pi)^4} \int dk \tilde{\epsilon}(k) \exp i k_\lambda x_\lambda$$

and are the functions of four components k_λ . The integral $\int dk$ denotes the quadruple integral $\iiint dk_1 \times dk_2 dk_3 dk_0$. The suffix appearing twice in the same term indicates as usual that the summation must be performed with respect to the suffix. k^2 means $k_\lambda k_\lambda = k_\lambda^2$.

$$k^\nu \tilde{\epsilon}(k) \tilde{\varphi}_\nu(k) = \frac{4\pi}{c} \tilde{s}_\rho(k), \quad (1.2)$$

this tensor satisfies the required condition

$$ik_\nu \tilde{T}_{\mu\nu}(k) = -\frac{i}{c(2\pi)^4} \int dk' \{ k'_\mu \tilde{\varphi}_\nu(k') - k'_\nu \tilde{\varphi}_\mu(k') \} \tilde{s}_\nu(k'), \quad (1.3)$$

or in the x -space
$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = -\frac{1}{c} f_{\mu\nu} s_\nu. \quad (1.3')$$

But, in the first place, this tensor is not symmetrical in μ and ν , due to the last term of (1.1)—*extra inhomogeneous term*—, which seems, indeed, to have been added *ad hoc* to satisfy (1.3). In the second place, the tensor reduces, in the limit of local field, i.e. when $\epsilon(x) = \delta(x)$ or $\tilde{\epsilon}(k) = 1$ to

$$T_{\mu\nu}(x) = \frac{1}{8\pi} \varphi_\rho \square \varphi_\rho \delta_{\mu\nu} + \frac{1}{8\pi} \left(\frac{\partial \varphi_\rho}{\partial x_\mu} \frac{\partial \varphi_\rho}{\partial x_\nu} - \varphi_\rho \frac{\partial^2 \varphi_\rho}{\partial x_\mu \partial x_\nu} \right) - \varphi_\mu \square \varphi_\nu. \quad (1.4)$$

In other words, even when the extra inhomogeneous term is excluded, it does not tend to the usual Maxwell energy-momentum tensor. Finally, the expression (1.1), which is constructed from the potentials, is not gauge-invariant.

In our opinion, the tensor should be constructed only of the field quantities $f_{\alpha\beta}$, and the equation (1.3) should be deduced only from the field equation, which is also described by $f_{\alpha\beta}$; that is, the extra inhomogeneous term should be dispensed with.

For the spinor field, Heisenberg³⁾ gave (in our notation) for the free field

$$\tilde{T}_{\mu\nu}(k) = -\frac{1}{2} \frac{1}{(2\pi)^4} \int dk' \tilde{\psi}^+(k-k') i(k-2k')_\nu \gamma_\mu \frac{\tilde{\epsilon}(k-k') - \tilde{\epsilon}(k')}{i(k-k')_\lambda \gamma_\lambda - ik'_\lambda \gamma_\lambda} \tilde{\psi}(k'), \quad (1.5)$$

$$\tilde{\theta}_{\mu\nu}(k) = \frac{1}{2} (\tilde{T}_{\mu\nu}(k) + \tilde{T}_{\nu\mu}(k)). \quad (1.6)$$

When $\tilde{\epsilon}(k) = ik_\lambda \gamma_\lambda + x$, this reduces to the usual Dirac case. He claims that, owing to the field equation, the equation of continuity

$$ik_\nu \tilde{T}_{\mu\nu}(k) = ik_\nu \tilde{\theta}_{\mu\nu}(k) = 0 \quad (1.7)$$

is satisfied. It is to be noted, however, that in the spinor case $\tilde{\epsilon}(k)$ (and $\epsilon(x)$) contains in general Dirac matrices γ_λ , so that it does not always commute with γ 's. The difficulty arises first from the definition of the fraction

$$\frac{\tilde{\epsilon}(k-k') - \tilde{\epsilon}(k')}{i(k-k')_\lambda \gamma_\lambda - ik'_\lambda \gamma_\lambda},$$

which may be interpreted equally well as

$$-\frac{\tilde{\epsilon}(k-k') - \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} (i(k-k')_\lambda \gamma_\lambda + ik'_\lambda \gamma_\lambda)$$

or

$$-(i(k-k')_{\lambda}\gamma_{\lambda}+ik'_{\lambda}\gamma_{\lambda})\frac{\tilde{\epsilon}(k-k')-\tilde{\epsilon}(k')}{(k-k')^2-k'^2}$$

or the like, which take in general mutually different values. The second more serious difficulty is that for the very same reason (1.5) (and (1.6)) does *not* in fact satisfy the equation (1.7) (see section § 5). Heisenberg's claim is true only when $\tilde{\epsilon}(k)$ is not a matrix, i.e. $\tilde{\epsilon}(k)=f(k)\times$ unit matrix, or when $\tilde{\epsilon}(k)$ is linear in $ik_{\lambda}\gamma_{\lambda}$, including the Dirac case.

In this paper it is attempted to get the energy-momentum tensor, which is free from the above-mentioned defects.

§ 2. General method

Let the Lagrangian $L(x)$ be considered as a scalar in general relativity transformations, and denote

$$\mathfrak{L}=\int L(x)\sqrt{g}(x)dx; \quad (2.1)$$

where g is the determinant $|g_{\mu\nu}|$ of the metrical (gravitational) tensor $g_{\mu\nu}$. Then, as is well-known, $2/\sqrt{g}$ times the functional derivative of \mathfrak{L} with respect to $g^{\mu\nu}(x)$ is equal to the symmetrical energy-momentum tensor of the field

$$\theta_{\mu\nu}(x)=2/\sqrt{g}\frac{\delta\mathfrak{L}}{\delta g^{\mu\nu}},^* \quad (2.2)$$

and is essentially equal to the usually defined tensor at a point where the space-time is flat, i.e.

$$g_{\mu\nu}=\delta_{\mu\nu}, \quad \sqrt{g}=1.$$

When the Lagrangian is of the type

$$L(x)=\int\varphi_{\rho}(x)\epsilon(x-x')\varphi_{\rho}(x')dx' \quad (2.3)$$

(vector field, for example), it seems natural to take

$$\mathfrak{L}=\frac{1}{2}\iint(\varphi_{\rho}(x)\bar{\varphi}^{\rho}(x)+\varphi_{\rho}(x')\bar{\varphi}^{\rho}(x'))\epsilon\left(\int_{s(x)}^{s(x')}ds\right)\sqrt{g}(x)dx\sqrt{g}(x')dx'. \quad (2.4)$$

In this expression, $\int_{s(x)}^{s(x')}ds$ is the length of the geodesic line joining the space-time points x and x' , and can also be written in the form, with t as the parameter of the line:

* We select the sign of L such that $\theta_{44}(x)$ is equal to the minus of energy density W .

$$\int_{t(x)}^{t(x')} \sqrt{g_{\mu\nu}(x'')} \frac{dx''^\mu}{dt} \frac{dx''^\nu}{dt} dt.$$

$\varphi_p(x)$ is the covariant component of the field vector, and $\bar{\varphi}^p(x)$ is defined as the contravariant component of the vector, which undergoes parallel displacement from x' to x along the geodesic line. Explicitly this is given by

$$\bar{\varphi}^p(x) = \varphi^p(x') + \int_{t(x')}^{t(x)} \Gamma_{\lambda\varepsilon}^p \varphi^\lambda \frac{dx^\varepsilon}{dt} dt,$$

where $\Gamma_{\lambda\varepsilon}^p$ denotes the Christoffel three index symbol. The general invariancy of (2.4) is thus evident. We should therefore be able to obtain the tensor using (2.4), at least in principle. Nevertheless, it seems very difficult to calculate the functional derivative of \mathfrak{L} .

In this paper we adopted the procedure given in the paper of Pais and Uhlenbeck⁴⁾, expressing the non-local function $\varepsilon(x)$ in the form:

$$\varepsilon(x) = \sum_{p=0}^{\infty} \lambda_p \square^p \delta(x) \equiv F(\square) \delta(x), \quad (2.5)$$

where \square is the D'Alembertian, and λ_p a pure number*. The Lagrangian (2.3) becomes

$$L(x) = \sum_{p=0}^{\infty} \lambda_p \varphi_p(x) \square^p \varphi_p(x) \equiv \sum_{p=0}^{\infty} \lambda_p L^{(p)}(x). \quad (2.6)$$

We then apply to each term $L^{(p)}(x)$ the equation (2.2). The problem is thus separated into three parts: First, to find the expression $\square_{(ge)}$ in the general relativity corresponding to the D'Alembertian \square for each type of the field; second, to obtain the tensor $\theta_{\mu\nu}^{(p)}(x)$ for $L^{(p)}(x)$; third, to carry out the summation with respect to p and obtain the expression in terms of $\varepsilon(x)$ or $\tilde{\varepsilon}(k)$.

Using the equation (2.2), Chang⁵⁾ has already derived the general formula of the tensor when L contains any higher order derivatives of field variable. But his obtained expression is inconvenient for applying it to the actual cases, since he did not put any restriction on the form of L . In our case where L is of the definite form $L^{(p)}$, the tensor can be calculated in a simple manner, and the summation easily carried through.

§ 3. Scalar field

As is well-known, we have in this case

$$\square_{(ge)} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\rho} \left(\sqrt{g} g^{\rho\lambda} \frac{\partial}{\partial x^\lambda} \right). \quad (3.1)$$

Adopting the charged field φ^* , φ , we then put

* For the spinor case see section § 5.

$$\mathfrak{L}^{(p)} = \int \varphi^* \square_{(p)} \varphi \sqrt{g} dx. \quad (3.2)$$

In deriving the functional derivative, the following formulae are utilized:

$$\frac{\partial \sqrt{g}}{\partial g^{\mu\nu}} = -\frac{1}{2} \sqrt{g} g_{\mu\nu}, \quad \frac{\partial \left(\frac{1}{\sqrt{g}} \right)}{\partial g^{\mu\nu}} = \frac{1}{2} \frac{1}{\sqrt{g}} g_{\mu\nu}.$$

Here it is further to be noticed that at the point where the space-time is flat

$$g_{\mu\nu} = \delta_{\mu\nu}, \quad \sqrt{g} = 1$$

and the derivatives of $g_{\mu\nu}$ or $g^{\mu\nu}$ vanish.

Then we obtain

$$\begin{aligned} \theta_{\mu\nu}^{(p)}(x) = & -\varphi^* \square^p \varphi \delta_{\mu\nu} + \sum_{n=0}^{p-1} \{ (\square^n \varphi^* \square^{p-n} \varphi + \square^n \varphi_\alpha^* \square^{p-1-n} \varphi_\alpha) \delta_{\mu\nu} \\ & - \square^n \varphi_\mu^* \square^{p-1-n} \varphi_\nu - \square^n \varphi_\nu^* \square^{p-1-n} \varphi_\mu \}, \end{aligned} \quad (3.3)$$

where φ_α means $\partial\varphi/\partial x_\alpha$. This tensor has as its Fourier component

$$\begin{aligned} \tilde{\theta}_{\mu\nu}^{(p)}(k) = & \frac{1}{(2\pi)^4} \int dk' \tilde{\varphi}^*(k-k') \varphi(k') (-k'^2)^p \delta_{\mu\nu} \\ & + \{ -k_\alpha k'_\alpha \delta_{\mu\nu} + (k-k')_\mu k'_\nu + (k-k')_\nu k'_\mu \} \cdot \sum_{m=0}^{p-1} [-(k-k')^2]^m (-k'^2)^{p-1-m}. \end{aligned} \quad (3.4)$$

Making use of the relations

$$\sum_{m=0}^{p-1} A^m B^{p-1-m} = \frac{A^p - B^p}{A - B}, \quad (3.5)$$

$$\sum_{p=0}^{\infty} \lambda_p (-k^2)^p = F(-k^2) = \tilde{\epsilon}(k), \quad (3.6)$$

we get finally

$$\begin{aligned} \tilde{\theta}_{\mu\nu}(k) = & \frac{1}{(2\pi)^4} \int dk' \tilde{\varphi}^*(k-k') \tilde{\varphi}(k') \left[-\tilde{\epsilon}(k') \delta_{\mu\nu} + (k_\alpha k'_\alpha \delta_{\mu\nu} - (k-k')_\mu k'_\nu \right. \\ & \left. - (k-k')_\nu k'_\mu) \frac{\tilde{\epsilon}(k-k') - \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} \right]. \end{aligned} \quad (3.7)$$

Returning to the x -space, we get the formula

$$\begin{aligned} \theta_{\mu\nu}(x) = & -\int \varphi^*(x) \epsilon(x-x') \varphi(x') dx' \delta_{\mu\nu} + \iint dx' dx'' \varphi^*(x') \varphi(x'') \times \\ & \times \left\{ -\left(\frac{\partial}{\partial x'_\alpha} + \frac{\partial}{\partial x''_\alpha} \right) \frac{\partial}{\partial x''_\alpha} \delta_{\mu\nu} + \frac{\partial}{\partial x'_\mu} \frac{\partial}{\partial x''_\nu} + \frac{\partial}{\partial x'_\nu} \frac{\partial}{\partial x''_\mu} \right\} Z(x-x', x-x''), \end{aligned}$$

$$Z(x-x', x-x'') = \frac{1}{(2\pi)^8} \iint dk' dk'' \frac{\epsilon(k') - \epsilon(k'')}{k'^2 - k''^2} \exp ik'_\lambda (x-x')_\lambda \exp ik''_\lambda (x-x'')_\lambda. \quad (3.7')$$

In case $\epsilon(x) = (-\square + x^2)\delta(x)$ or $\tilde{\epsilon}(k) = k^2 + x^2$, the formula (3.7') reduces to that of the local scalar field:

$$\theta_{\mu\nu}(x) = -\left(\frac{\partial\varphi^*}{\partial x_\alpha} \frac{\partial\varphi}{\partial x_\alpha} + x^2\varphi^*\varphi\right)\delta_{\mu\nu} + \frac{\partial\varphi^*}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\nu} + \frac{\partial\varphi^*}{\partial x_\nu} \frac{\partial\varphi}{\partial x_\mu}.$$

This was not the case with Bopp's formula¹⁾.

For the neutral (real) field, (3.7) becomes

$$\begin{aligned} \tilde{\theta}_{\mu\nu}(k) = & \frac{1}{2} \frac{1}{(2\pi)^4} \int dk' \tilde{\varphi}(k-k') \tilde{\varphi}(k') \left[-\tilde{\epsilon}(k') \delta_{\mu\nu} \right. \\ & \left. + (k_\alpha k'_\alpha \delta_{\mu\nu} - 2(k-k')_\mu k'_\nu) \frac{\tilde{\epsilon}(k-k') - \tilde{\epsilon}(k')}{(k-k')^2 - k'^2} \right]. \end{aligned} \quad (3.7'')$$

In the case when the field satisfies the inhomogeneous equation

$$\int \epsilon(x-x') \varphi(x') dx' = \rho(x) \quad (3.8)$$

or

$$\tilde{\epsilon}(k) \tilde{\varphi}(k) = \tilde{\rho}(k), \quad (3.8')$$

the equation of continuity becomes

$$ik_\nu \tilde{\theta}_{\mu\nu}(k) = -\frac{i}{(2\pi)^4} \int dk' k'_\mu \tilde{\rho}(k-k') \tilde{\varphi}(k') \quad (3.9)$$

or

$$\frac{\partial\theta_{\mu\nu}}{\partial x_\nu} = -\rho \frac{\partial\varphi}{\partial x_\mu}, \quad (3.9')$$

which is just the same as the equation (46) of Bopp's paper¹⁾. Thus in our case it is not required to discriminate between $\theta_{\mu\nu}^{\text{inh}}$ and $\theta_{\mu\nu}^{\text{hom}}$, and the equation (3.9') has been derived solely from the field equation.

§ 4. Extended electromagnetic field

The Lagrangian in this case is given by

$$L(x) = \frac{1}{16\pi} \int f_{\alpha\beta}(x) \epsilon(x-x') f_{\alpha\beta}(x') dx' *, \quad (4.1)$$

where $f_{\alpha\beta}$ is an antisymmetric tensor satisfying the relation

* In this section we adopted the Heaviside unit in order to compare the result with that of Bopp.

$$\frac{\partial f_{\alpha\beta}}{\partial x_\tau} + \frac{\partial f_{\tau\alpha}}{\partial x_\beta} + \frac{\partial f_{\beta\tau}}{\partial x_\alpha} = 0. \quad (4.2)$$

In order to derive $\square_{(ge)}$ we first take the covariant derivative of $f_{\alpha\beta}$:

$$f_{\lambda,\alpha\beta} = \frac{\partial f_{\alpha\beta}}{\partial x^\lambda} - \Gamma_{\lambda\alpha}^\rho f_{\rho\beta} - \Gamma_{\lambda\beta}^\rho f_{\alpha\rho}, \quad (4.3)$$

where the Christoffel three index symbol $\Gamma_{\lambda\alpha}^\rho$ is given by

$$\Gamma_{\lambda\alpha}^\rho = g^{\rho\mu} \Gamma_{\mu,\lambda\alpha},$$

$$\Gamma_{\mu,\lambda\alpha} = \frac{1}{2} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\alpha}}{\partial x^\mu} \right).$$

Putting then

$$F_{\alpha\beta}^\tau \equiv g^{\tau\lambda} f_{\lambda,\alpha\beta}, \quad (4.4)$$

we can define

$$\begin{aligned} \square_{(ge)} &= F_{\tau,\alpha\beta}^\tau \\ &= \frac{\partial F_{\alpha\beta}^\tau}{\partial x^\tau} + \Gamma_{\rho\tau}^\tau F_{\alpha\beta}^\rho - \Gamma_{\tau\alpha}^\rho F_{\rho\beta}^\tau - \Gamma_{\tau\beta}^\rho F_{\alpha\rho}^\tau. \end{aligned} \quad (4.5)$$

The $\mathfrak{L}^{(p)}$ is given by

$$\mathfrak{L}^{(p)} = \frac{1}{16\pi} \int g^{\alpha\mu} g^{\beta\nu} f_{\mu\nu} \square_{(ge)}^p f_{\alpha\beta} \sqrt{g} dx. \quad (4.6)$$

Noting that the Christoffel symbol and its derivatives vanish on a flat space-time point, the similar but rather tedious procedure as in the foregoing section leads to

$$\begin{aligned} \theta_{\mu\nu}^{(p)}(x) &= -\frac{1}{16\pi} f_{\alpha\beta} \square_{(ge)}^p f_{\alpha\beta} \delta_{\mu\nu} + \frac{1}{8\pi} f_{\mu\alpha} \square_{(ge)}^p f_{\nu\beta} + \frac{1}{8\pi} f_{\nu\alpha} \square_{(ge)}^p f_{\mu\beta} \\ &+ \frac{1}{8\pi} \sum_{m=0}^{p-1} \left\{ -\frac{\partial \square_{(ge)}^m f_{\alpha\beta}}{\partial x_\mu} \frac{\partial \square_{(ge)}^{p-1-m} f_{\alpha\beta}}{\partial x_\nu} + \frac{1}{2} \frac{\partial}{\partial x_\tau} \left(\square_{(ge)}^m f_{\alpha\beta} \cdot \frac{\partial \square_{(ge)}^{p-1-m} f_{\alpha\beta}}{\partial x_\tau} \right) \delta_{\mu\nu} \right. \\ &+ \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \square_{(ge)}^m f_{\alpha\beta}}{\partial x_\nu} \square_{(ge)}^{p-1-m} f_{\mu\beta} + \frac{\partial \square_{(ge)}^m f_{\alpha\beta}}{\partial x_\mu} \square_{(ge)}^{p-1-m} f_{\nu\beta} \right) \\ &\left. - \frac{\partial}{\partial x_\alpha} \left(\square_{(ge)}^m f_{\alpha\beta} \frac{\partial \square_{(ge)}^{p-1-m} f_{\nu\beta}}{\partial x_\mu} + \square_{(ge)}^m f_{\alpha\beta} \frac{\partial \square_{(ge)}^{p-1-m} f_{\mu\beta}}{\partial x_\nu} \right) \right\}. \end{aligned} \quad (4.7)$$

Transforming to the Fourier component and carrying out the summation, we get

$$\begin{aligned} \tilde{\theta}_{\mu\nu}(k) &= \frac{1}{16\pi} \frac{1}{(2\pi)^4} \int d\tilde{k}' \left[-\tilde{f}_{\alpha\beta}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\alpha\beta}(k') \delta_{\mu\nu} \right. \\ &\quad \left. + 2\tilde{f}_{\mu\alpha}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\nu\alpha}(k') + 2\tilde{f}_{\nu\alpha}(k-k') \tilde{\varepsilon}(k') \tilde{f}_{\mu\alpha}(k') \right] \end{aligned}$$

$$\begin{aligned}
& +2\left\{\tilde{f}_{\alpha\beta}(k-k')\tilde{f}_{\alpha\beta}(k')(-(k-k')_{\mu}k'_{\nu}+\frac{1}{2}k_{\tau}k'_{\tau}\delta_{\mu\nu})\right. \\
& +\tilde{f}_{\alpha\beta}(k-k')\tilde{f}_{\nu\beta}(k')k_{\alpha}(k-2k')_{\mu} \\
& \left. +\tilde{f}_{\alpha\beta}(k-k')\tilde{f}_{\mu\beta}(k')k_{\alpha}(k-2k')_{\nu}\right\}\frac{\tilde{\epsilon}(k-k')-\tilde{\epsilon}(k')}{(k-k')^2-k'^2}. \quad (4.8)
\end{aligned}$$

In the limit of local field, i.e. $\tilde{\epsilon}(k)=1$, this reduces to the Maxwell energy-momentum tensor

$$\theta_{\mu\nu}(x)=\frac{1}{4\pi}f_{\mu\alpha}f_{\nu\alpha}-\frac{1}{16\pi}f_{\alpha\beta}f_{\alpha\beta}\delta_{\mu\nu}.$$

When the source of field $s_{\mu}(x)$ exists, the field equation is deduced from the variation of potential in the Lagrangian

$$L'(x)=L(x)-\frac{1}{c}\varphi_{\alpha}s_{\alpha}, \quad (4.9)$$

where $L(x)$ is given by (4.1). That is,

$$\int \epsilon(x-x')\frac{\partial f_{\alpha\beta}(x')}{\partial x'_{\beta}}dx'=\frac{4\pi}{c}s_{\alpha}(x) \quad (4.10)$$

or

$$ik_{\beta}\tilde{\epsilon}(k)\tilde{f}_{\alpha\beta}(k)=\frac{4\pi}{c}\tilde{s}_{\alpha}(k). \quad (4.10')$$

Using this and (4.2) or its Fourier component

$$k_{\tau}\tilde{f}_{\alpha\beta}(k)+k_{\beta}\tilde{f}_{\tau\alpha}(k)+k_{\alpha}\tilde{f}_{\beta\tau}(k)=0, \quad (4.2')$$

it is verified that

$$ik_{\nu}\tilde{\theta}_{\mu\nu}(k)=-\frac{1}{c(2\pi)^4}\int dk'\tilde{f}_{\mu\nu}(k-k')\tilde{s}_{\nu}(k') \quad (4.11)$$

or

$$\frac{\partial \theta_{\mu\nu}}{\partial x_{\nu}}=-\frac{1}{c}f_{\mu\nu}s_{\nu}. \quad (4.11')$$

Moreover, since (4.8) is constructed exclusively of the field quantities $f_{\alpha\beta}$, its gauge-invariance is self-evident.

§ 5. Spinor field

In this case the general method of § 2 must be slightly modified. Let the spinor field and its conjugate be denoted by $\psi(x)$ and $\psi^+(x)(=\psi^*(x)\gamma_4)$ respectively. The scalar which is bilinear in these quantities is then given by

$$\phi^+ \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^p \phi.$$

Thus the Lagrangian must be in general of the form

$$L(x) = \int \phi^+(x) \varepsilon(x-x') \phi(x') dx', \quad (5.1)$$

$$\varepsilon(x) = \sum_{p=0}^{\infty} \lambda_p \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^p \delta(x) \equiv F \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right) \delta(x) \quad (5.2)$$

or

$$\tilde{\varepsilon}(k) = F(ik_\lambda \gamma_\lambda). \quad (5.2)$$

Consequently, $\varepsilon(x)$ (and $\varepsilon(k)$) is in general a matrix, constructed of Dirac γ 's.

We have now to find the expression $\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)_{(g)}$ in the general relativity corresponding to $\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)$. This has been done by several authors⁽⁶⁾, whose result is

$$\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)_{(g)} = \gamma_p \left\{ h^\lambda_{(p)} \frac{\partial}{\partial x^\lambda} + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\lambda} \left(\sqrt{g} h^\lambda_{(p)} \right) \right\}. \quad (5.3)$$

$h^\lambda_{(p)}$ is the local coordinate vector, which is related to the metrical tensor by

$$h^\mu_{(p)} h^\nu_{(p)} = g^{\mu\nu}$$

and to its conjugate vector $h_{(p)\lambda}$ by

$$h^\mu_{(p)} h_{(p)\nu} = h^\rho_{(\mu)} h_{(\rho)\nu} = \delta_{\mu\nu}.$$

Incidentally,

$$h_{(\mu)} h_{(\rho)} = g_{\mu\rho},$$

$$g = |g_{\mu\nu}| = |h_{(\rho)}| = \frac{1}{|h^\lambda_{(p)}|},$$

$$\frac{\partial \sqrt{g}}{\partial h^\mu_{(p)}} = - \sqrt{g} h_{(\rho)\mu}.$$

At a point where the space-time is flat these vectors reduce to

$$h^\lambda_{(p)} = h_{(p)\lambda} = \delta_{p\lambda},$$

and their derivatives vanish.

Finally, to construct the energy-momentum tensor from

$$\mathfrak{L} = \int L \sqrt{g} dx,$$

we use instead of (2.2) the following relation :

$$\theta_{\mu\nu}(x) = \frac{1}{2} \frac{1}{\sqrt{g}} \left(\frac{\partial \mathfrak{L}}{\partial \dot{h}^\mu_{(\lambda)}} \dot{h}^\nu_{(\lambda)} + \frac{\partial \mathfrak{L}}{\partial \dot{h}^\nu_{(\lambda)}} \dot{h}^\mu_{(\lambda)} \right) \equiv \frac{1}{2} (T_{\mu\nu}(x) + T_{\nu\mu}(x)). \quad (5.4)$$

Thus, from the Lagrangian

$$L^{(p)}(x) = \phi^+ \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^p \phi, \quad (5.5)$$

we obtain

$$\begin{aligned} T_{\mu\nu}^{(p)}(x) = & -\frac{1}{2} \left\{ \phi^+ \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^p \phi + \phi^+ \left(-\frac{\partial}{\partial x_\lambda} \gamma_\lambda \right)^p \phi \right\} \delta_{\mu\nu} \\ & + \frac{1}{2} \sum_{m=0}^{p-1} \left\{ \phi^+ \left(-\frac{\partial}{\partial x_\lambda} \gamma_\lambda \right)^m \left(\gamma_\mu \frac{\partial}{\partial x_\nu} \right) \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^{p-1-m} \phi \right. \\ & \left. + \phi^+ \left(-\frac{\partial}{\partial x_\lambda} \gamma_\lambda \right)^m \left(-\frac{\partial}{\partial x_\nu} \gamma_\mu \right) \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} \right)^{p-1-m} \phi \right\}^*. \end{aligned} \quad (5.6)$$

The Fourier component is

$$\begin{aligned} \tilde{T}_{\mu\nu}^{(p)}(k) = & \frac{1}{2} \frac{1}{(2\pi)^4} \int d k' \tilde{\phi}^+(k-k') [-\{ (i k'_\lambda \gamma_\lambda)^p + (i(k'-k)_\lambda \gamma_\lambda)^p \} \delta_{\mu\nu} \\ & - i(k-2k')_\nu \sum_{m=0}^{p-1} (i(k'-k)_\lambda \gamma_\lambda)^m \gamma_\mu (i k'_\lambda \gamma_\lambda)^{p-1-m}] \tilde{\phi}(k'). \end{aligned} \quad (5.6')$$

Now, we put $A = a_\lambda \gamma_\lambda$, $B = b_\lambda \gamma_\lambda$ where a_λ , b_λ are ordinary numbers. Then,

$$\begin{aligned} & \sum_{m=0}^{p-1} A^m \gamma_\mu B^{p-1-m} \\ &= \gamma_\mu B^{p-1} + A \gamma_\mu B^{p-2} + \cdots + A^{p-2} \gamma_\mu B + A^{p-1} \gamma_\mu \\ &= \gamma_\mu (B^{p-1} + A' B^{p-2} + \cdots + A'^{p-2} B + A'^{p-1}) \\ &= \gamma_\mu \sum_{m=0}^{p-1} A'^m B^{p-1-m} \equiv \gamma_\mu C, \end{aligned} \quad (5.7)$$

where

$$A' = \gamma_\mu A \gamma_\mu = (-a_\lambda + 2a_\mu \delta_{\lambda\mu}) \gamma_\lambda^{**}. \quad (5.8)$$

Since A and B do not mutually commute, C cannot be expressed in the form

* The differentiation in $\left(-\frac{\partial}{\partial x_\lambda} \gamma_\lambda \right)^m$ or $\left(-\frac{\partial}{\partial x_\nu} \gamma_\mu \right)$ is to be interpreted as to operate on ϕ^+ .

** Throughout the following we must not take the summation with respect to the suffix μ , even when it appears twice in the same term.

$$\frac{A'^p - B^p}{A' - B}$$

as in the foregoing cases. But it holds that

$$A'C - CB = A'^p - B^p. \quad (5.9)$$

In this sense we write

$$C = \frac{A'^p - B^p}{\begin{matrix} A' & - & B \\ \rightarrow & & \leftarrow \end{matrix}} \quad (5.10)$$

which means: multiply C by A' from the left, and by B from the right, and subtract, then we get $A'^p - B^p$. In order to obtain the explicit value of C , we multiply C by A' from the left, and by B from the right, i.e.

$$A'^2 C - CB^2 = A'(A'^p - B^p) + (A'^p - B^p)B.$$

But since $A'^2 (=a_\lambda^2)$ and $B^2 (=b_\lambda^2)$ are no longer matrices, we can write

$$\begin{aligned} C &= \frac{A'(A'^p - B^p) + (A'^p - B^p)B}{A'^2 - B^2} \\ &= \frac{(A'^p - B^p) \begin{matrix} \rightarrow & \leftarrow \end{matrix}}{a_\lambda^2 - b_\lambda^2}. \end{aligned} \quad (5.11)$$

Applying these formulae to our case where $a_\lambda = i(k' - k)_\lambda$ and $b_\lambda = ik'_\lambda$, and carrying out the summation, we get for the term

$$\sum_{m=0}^{p-1} (i(k' - k)_\lambda \gamma_\lambda)^m \gamma_\lambda (ik'_\lambda \gamma_\mu)^{p-1-m}$$

the expression

$$\gamma_\mu \frac{F(\gamma_\mu i(k' - k)_\lambda \gamma_\lambda \gamma_\mu) - F(ik'_\lambda \gamma_\lambda)}{\gamma_\mu i(k' - k)_\lambda \gamma_\lambda \gamma_\mu - ik'_\lambda \gamma_\lambda}.$$

But, since we have, from (5.2') and (5.8), the relations

$$F(ik'_\lambda \gamma_\lambda) = \bar{\epsilon}(k'),$$

$$F(\gamma_\mu i(k' - k)_\lambda \gamma_\lambda \gamma_\mu) = \tilde{\epsilon}((k - k')_\lambda - 2(k - k')_\mu \delta_{\lambda\mu}),$$

we obtain after all

$$\begin{aligned} \tilde{T}_{\mu\nu}(k) &= \frac{1}{2} \frac{1}{(2\pi)^4} \int dk' \tilde{\varphi}^+(k - k') \left[-(\tilde{\epsilon}(k' - k) + \tilde{\epsilon}(k')) \delta_{\mu\nu} \right. \\ &\quad \left. - i(k - 2k')_\nu \gamma_\mu \frac{\tilde{\epsilon}((k - k')_\lambda - 2(k - k')_\mu \delta_{\lambda\mu}) - \tilde{\epsilon}(k')}{i((k - k')_\lambda - 2(k - k')_\mu \delta_{\mu\lambda}) \gamma_\lambda - ik'_\lambda \gamma_\lambda} \right] \tilde{\varphi}(k'). \end{aligned} \quad (5.12)$$

As the field equations for the free field are given by

$$\tilde{\varepsilon}(k)\tilde{\phi}(k)=0, \quad \tilde{\phi}^+(k)\tilde{\varepsilon}(-k)=0, \quad (5.13)$$

the $\delta_{\mu\nu}$ term in the integrand drops away. When $\tilde{\varepsilon}(k)=ik_\lambda\gamma_\lambda+\alpha$, (5.12) reduces to the Dirac case, except for the $\delta_{\mu\nu}$ term which is zero. It is to be noticed that in the free field the formula (5.12) coincides with that of Heisenberg (1.5), if the term $(k-k')_\lambda-2(k-k')_\mu\delta_{\lambda\mu}$ is replaced by $(k-k')_\lambda$. As will be demonstrated below, the term $-2(k-k')_\mu\delta_{\lambda\mu}$ cannot be dispensed with in order that the equation of continuity may be satisfied.

We first put

$$\tilde{\varepsilon}(k)=F(ik_\lambda\gamma_\lambda)\equiv f_1(-k^2)+f_2(-k^2)ik_\lambda\gamma_\lambda^*, \quad (5.14)$$

where

$$\left. \begin{aligned} f_1(-k^2) &= \sum_{n=0}^{\infty} \lambda_{2n} (-k^2)^n, \\ f_2(-k^2) &= \sum_{n=0}^{\infty} \lambda_{2n+1} (-k^2)^n, \end{aligned} \right\} \quad (5.15)$$

remembering that

$$(ik_\lambda\gamma_\lambda)^2 = -k^2.$$

Then owing to (5.10), (5.11) and the relation

$$(\gamma_\mu i(k'-k)_\lambda \gamma_\lambda \gamma_\mu)^{2n} = (i(k'-k)_\lambda \gamma_\lambda)^{2n} = (-(k'-k)^2)^n,$$

the integrand of the expression $ik_\nu \tilde{T}_{\mu\nu}(k)$ becomes

$$\begin{aligned} & \tilde{\phi}^+(k-k')\gamma_\mu [\{f_1(-(k'-k)^2) - f_2(-k'^2)\} \{\gamma_\mu i(k'-k)_\lambda \gamma_\lambda \gamma_\mu + ik'_\lambda \gamma_\lambda\} \\ & \quad + \{f_2(-(k'-k)^2) (\gamma_\mu i(k'-k)_\lambda \gamma_\lambda \gamma_\mu) \\ & \quad - f_2(-k'^2) ik'_\lambda \gamma_\lambda\} \underbrace{\{\gamma_\mu i(k'-k)_\lambda \gamma_\lambda \gamma_\mu + ik'_\lambda \gamma_\lambda\}}_{\leftarrow}] \tilde{\phi}(k') \\ &= \tilde{\phi}(k-k') [\{f_1(-(k'-k)^2) + f_2(-(k'-k)^2) i(k'-k)_\lambda \gamma_\lambda\} \times \\ & \quad \times \{i(k'-k)_\lambda \gamma_\lambda \gamma_\mu + \gamma_\mu ik'_\lambda \gamma_\lambda\} \\ & \quad - \{\gamma_\mu ik'_\lambda \gamma_\lambda + i(k'-k)_\lambda \gamma_\lambda \gamma_\mu\} \times \\ & \quad \times \{f_1(-k'^2) + f_2(-k'^2) ik'_\lambda \gamma_\lambda\}] \tilde{\phi}(k') \\ &= \tilde{\phi}(k-k') \tilde{\varepsilon}(k'-k) \{i(k'-k)_\lambda \gamma_\lambda \gamma_\mu + \gamma_\mu ik'_\lambda \gamma_\lambda\} \tilde{\phi}(k') \\ & \quad - \tilde{\phi}(k-k') \{\gamma_\mu ik'_\lambda \gamma_\lambda + i(k'-k)_\lambda \gamma_\lambda \gamma_\mu\} \tilde{\varepsilon}(k') \tilde{\phi}(k'), \end{aligned}$$

which vanishes according to the equation of motion (5.13). Thus

$$ik_\nu \tilde{T}_{\mu\nu}(k) = 0$$

* Precisely it should be written that $f_1(-k^2) \cdot 1$ instead of $f_1(-k^2)$, where 1 is unit matrix.

and also

$$ik_\nu \tilde{\theta}_{\mu\nu}(k) = 0 \quad (5.16)$$

has been proved.

It can be easily verified, on the other hand, that the proposition (1.5) by Heisenberg does not satisfy (5.16), when $f_2(-k^2) \neq 0$, that is, when $\tilde{\epsilon}(k)$ contains odd powers of $ik_\lambda \gamma_\lambda$, except in the case $\tilde{\epsilon}(k) = \lambda_0 + \lambda_1 ik_\lambda \gamma_\lambda$ including the Dirac case.

§ 6. Conclusion

It has been proved by Iskraut⁷⁾ that the symmetrical energy-momentum tensor cannot be uniquely determined when the Lagrangian contains the unlimited order of derivatives of field variables. It is true, also in our method some ambiguity might be introduced in the process \sum_p , but the formulae (3.7), (4.8) and (5.12) may be considered as a most natural extension of the usual fields, inasmuch as they are characterized by the following features, which do not all belong to the propositions of Bopp and Heisenberg:

- i) They are symmetric in μ and ν .
- ii) They are gauge-invariant.
- iii) In the limiting case they tend to the usual tensors.
- vi) They satisfy, due to the field equation, the equation of continuity, irrespective of the existence or non-existence of the source in the field. That is, the extra inhomogeneous term is not necessary to be introduced even when the source is present.

The present work was stimulated by the discussions with Mr. M. Sugawara, whom I wish to express my cordial thanks for the interest he has taken in this work.

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- 3) W. Heisenberg, *Naturforschung* **5a** (1950), 251.
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- 6) H. Weyl, *ZS. f. Phys.* **56** (1929), 330, V. Fock, *ZS. f. Phys.* **57** (1929), 261.
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Addendum

When we put in the expression (4.8) $\tilde{\varepsilon}(k) = 1 + k^2/x^2$ or $\varepsilon(x) = (1 - \square/x^2)\delta(x)$, which corresponds to the case of Bopp's first paper*, we obtain

$$\begin{aligned} \theta_{\mu\nu}(x) = & \frac{1}{4\pi} f_{\mu\alpha} f_{\nu\alpha} - \frac{1}{16\pi} f_{\alpha\beta} f_{\alpha\beta} \delta_{\mu\nu} \\ & - \frac{1}{x^2} \left[-\frac{1}{16\pi} f_{\alpha\beta} \square f_{\alpha\beta} \delta_{\mu\nu} + \frac{1}{8\pi} f_{\mu\alpha} \square f_{\nu\alpha} + \frac{1}{8\pi} f_{\nu\alpha} \square f_{\mu\alpha} \right. \\ & + \frac{1}{8\pi} \left\{ -\frac{\partial f_{\alpha\beta}}{\partial x_\mu} \frac{\partial f_{\alpha\beta}}{\partial x_\nu} + \frac{1}{2} \frac{\partial}{\partial x_\tau} \left(f_{\alpha\beta} \frac{\partial f_{\alpha\beta}}{\partial x_\tau} \right) \delta_{\mu\nu} \right. \\ & \left. \left. + \frac{\partial}{\partial x_\alpha} \left(\frac{\partial f_{\alpha\beta}}{\partial x_\nu} f_{\mu\beta} + \frac{\partial f_{\alpha\beta}}{\partial x_\mu} f_{\nu\beta} \right) - \frac{\partial}{\partial x_\alpha} \left(f_{\alpha\beta} \frac{\partial f_{\nu\beta}}{\partial x_\mu} + f_{\alpha\beta} \frac{\partial f_{\mu\beta}}{\partial x_\nu} \right) \right\} \right]. \quad (a) \end{aligned}$$

Bopp gave as the tensor for this case

$$\left. \begin{aligned} \theta'_{\mu\nu}(x) &= \theta_{\mu\nu}^M(x) - \theta_{\mu\nu}^V(x), \\ \theta_{\mu\nu}^M(x) &= \frac{1}{4\pi} F_{\mu\alpha} F_{\nu\alpha} - \frac{1}{16\pi} F_{\alpha\beta} F_{\alpha\beta} \delta_{\mu\nu}, \\ \theta_{\mu\nu}^V(x) &= \frac{1}{4\pi} U_{\mu\alpha} U_{\nu\alpha} + \frac{1}{4\pi} U_\mu U_\nu \\ &\quad - \frac{1}{16\pi} U_{\alpha\beta} U_{\alpha\beta} \delta_{\mu\nu} - \frac{1}{8\pi} U_\alpha U_\alpha \delta_{\mu\nu}, \end{aligned} \right\} \quad (b)$$

where

$$F_{\alpha\beta} = f_{\alpha\beta} + U_{\alpha\beta}, \quad U_{\alpha\beta} = -\frac{\square}{x^2} f_{\alpha\beta}, \quad U_\alpha = \frac{1}{x} \frac{\partial f_{\alpha\beta}}{\partial x_\beta}.$$

This also satisfies all the conditions mentioned in § 6. Sugawara** has already proved, using the relation (4.2), that (b) reduces after all to (a) except for the divergent term

$$\begin{aligned} \theta_{\mu\nu}(x) - \theta'_{\mu\nu}(x) = & \frac{1}{8\pi x^2} \frac{\partial}{\partial x_\alpha} \left\{ f_{\mu\alpha} \frac{\partial f_{\nu\beta}}{\partial x_\beta} + f_{\nu\alpha} \frac{\partial f_{\mu\beta}}{\partial x_\beta} + f_{\alpha\beta} \frac{\partial f_{\nu\beta}}{\partial x_\mu} + f_{\alpha\beta} \frac{\partial f_{\mu\beta}}{\partial x_\nu} \right. \\ & \left. + \left(f_{\beta\tau} \frac{\partial f_{\alpha\beta}}{\partial x_\tau} + f_{\alpha\tau} \frac{\partial f_{\beta\tau}}{\partial x_\beta} \right) \delta_{\mu\nu} \right\}. \quad (c) \end{aligned}$$

Nevertheless, the total energy and momentum can be easily verified to be the same for both cases:

$$\int \theta_{4\nu} dV = \int \theta'_{4\nu} dV. \quad (d)$$

* F. Bopp, Ann. d. Phys. **38** (1940), 345.

** M. Sugawara, unpublished.

Pressure Effect on the Second Sound Velocity in Liquid Helium II

Atsushi KONDOH, Sadao NAKAJIMA and MASAO SHIMIZU

Physical Institute, Nagoya University

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The pressure dependence of the second sound velocity in liquid helium II is explained by taking account of the phonon entropy on the basis of the models of Tisza and Landau. To calculate the velocity we make use of the generalized formula derived by Gorter and Usui, according to which, however, the second sound velocity at the absolute zero of temperature in Landau's model is $\sqrt{3/4}$ times the ordinary sound velocity. The discrepancy between this value and that predicted by Landau is due to the different assumptions made on the entropy and the relaxation time.

§ 1. Introduction

The phenomenon of the second sound wave is well known as a characteristic of liquid helium II and its behavior above 1°K has been successfully explained by the two fluid theory. According to the recent measurements the second sound velocity as a function of temperature exhibits successively a maximum and a minimum and increases sharply with decreasing temperature.¹⁾ It is also well known that Landau's theory²⁾ based on the roton and phonon model of liquid helium II explains this aspect qualitatively and that in its original form Tisza's theory³⁾ does not so. In our previous paper,⁴⁾ however, we have shown that the minimum of the second sound velocity can be explained by Tisza's model as well as by Landau's, if one takes into account the contribution of the phonon entropy as pointed out by Landau and Tisza and calculates the velocity by means of the formula of the second sound velocity introduced by Gorter-Kasteleijn-Mellink⁵⁾ and Usui⁶⁾ (Eq. 4).

On the other hand, Maurer and Herlin⁷⁾ have investigated the pressure dependence of the second sound velocity to examine the phonon effect on second sound and obtained the results that the minimum of the velocity goes to lower temperature with increasing pressure, as shown in Fig. 2.

As seen from the Debye theory of the vibrational specific heat, the entropy due to the phonon is inversely proportional to the cube of the first sound velocity, which increases with pressure in liquid helium II. Therefore, the shift of the minimum of the second sound velocity to lower temperature with increasing pressure is expected to be attributable to the diminishing of the phonon effect on second sound. From this standpoint we calculate numerically the second sound velocity as a function of temperature and pressure using the experimental data of the entropy and the first sound velocity, etc.

Now Atokins and Osborne⁸⁾ have measured the second sound velocity in the vicinity of the absolute zero of temperature and found that the velocity at 0°K is nearly equal to $c_1/\sqrt{3}$, which has been predicted by Landau, where c_1 is the first sound velocity in liquid helium II. However, if one uses the formula of the second sound velocity obtained by Gorter-Kasteleijn-Mellink⁵⁾ and Usui,⁶⁾ the velocity at 0°K in Landau's model is equal to the ordinary sound velocity divided by $4/\sqrt{3}$, instead of $\sqrt{3}$.

In the case of Tisza's model, as the temperature dependence of the normal fluid concentration of liquid helium II is given by an empirical formula only above 1°K, it must be extrapolated below 1°K to investigate the behavior of the second sound velocity in the vicinity of 0°K. If the normal fluid concentration at 0°K vanishes faster than the fourth order of temperature, the velocity at 0°K tends to infinity.

§ 2. The second sound velocity based on Tisza's model

In Tisza's theory,³⁾ the entropy per unit mass of liquid helium II at temperature T is given by the following expression:

$$s = s_0\xi + s_{ph.}, \quad (1)$$

where $\xi = \rho_n/\rho$ is the normal fluid concentration, s_0 is a constant independent of temperature and $s_{ph.}$ is the phonon entropy which is given by the well-known Debye formula as follows:

$$s_{ph.} = 34.64\pi \frac{k^4 T^3}{\rho h^3 c_1^3} = aT^3, \quad (2)$$

where k is Boltzmann constant, h is Planck constant, ρ is the total density of He II and c_1 is the first sound velocity. Eq. (1) is well known as Tisza's relation, if we neglect the second term. The meaning of this relation in the two fluid model has been previously examined by us.⁹⁾

The temperature dependence of the normal fluid concentration ξ in Eq. (1) would be obtainable from the measurements of the viscosity using the method of oscillating disc, or the measurements of the thermal conduction. Usually, however, ξ is expressed by an empirical formula

$$\xi = (T/T_\lambda)^r, \quad (3)$$

where T_λ is the transition temperature.

Among various quantities in the right hand sides of Eqs. (1), (2) and (3), the pressure dependence of ρ , T_λ and c_1 are determined directly by the use of experimental data.¹⁰⁾ s_0 is determined as a function of pressure from the values of the entropy at the transition temperatures. Then, using Eqs. (1), (2), (3) and the observed values of the entropy, r is determined at each temperature. Fig. 1 represents the results in which r is calculated by both data of Kapitza¹¹⁾

and Gorter-Kasteleijn-Mellink⁵⁾ under the vapour pressure. It contains also the points which are estimated from the measurements of the viscosity of Andronikashivilli¹²⁾ by means of Eq. (3). From Fig. 1 it will be concluded that the approximation of $\hat{\epsilon}$ as a constant power of T/T_λ such as Eq. (3) has no justification.

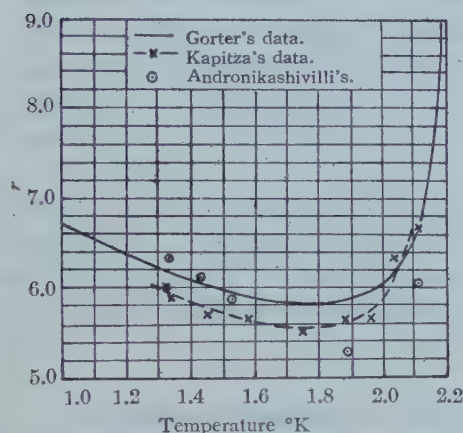


Fig. 1. The values of r in Eq. (3) calculated from the experimental data of entropy of Kapitza and Gorter-Kasteleijn-Mellink and the data of viscosity of Andronikashivilli.

Table I

Pressure atm.	T_λ °K	ρ g/cm	c_1^* m/sec	s_λ cal/g. deg	s_0 cal/g. deg	$a \times 10^3$ cal/g. deg
v. p.	2.19	0.145	230.0	0.405	0.386	1.85**
2.5	2.16	0.149	355.7	0.393	0.380	1.31
5.0	2.13	0.153	276.0	0.385	0.376	1.01
10.0	2.06	0.159	303.5	0.373	0.366	0.717
25.0	1.84	0.174	367.6	0.358	0.356	0.377

* c_1 at higher pressure is calculated from the value of compressibility.

** The value of a which has been taken up in our preceding paper⁴⁾ is underestimated and corresponds to the case of 10 atm.

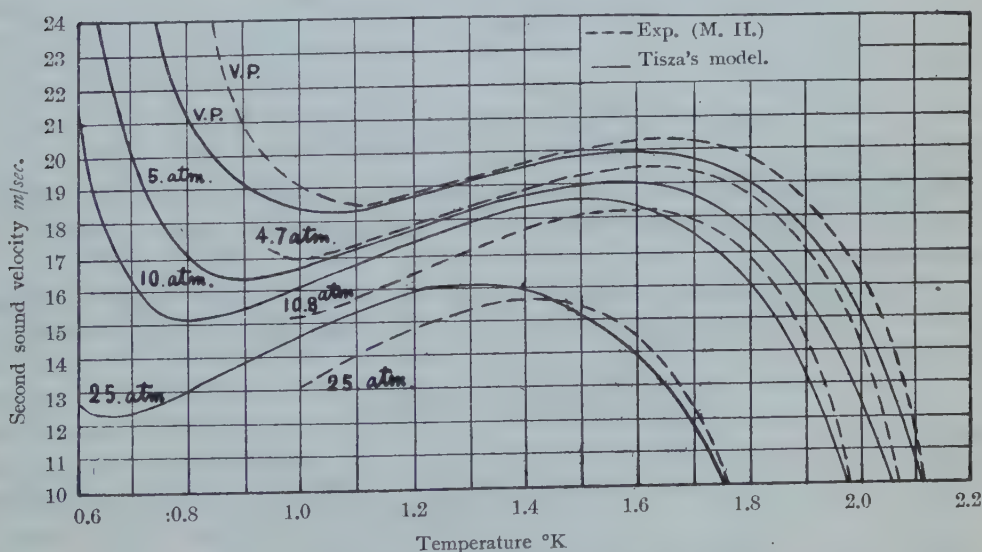


Fig. 2a. The second sound velocity based on Tisza's model.

Now we can compute the temperature and pressure dependence of the second sound velocity using the following formula which has been obtained by Gorter-

Kasteleijn-Mellink and Usui assuming that the normal- and superfluids of liquid helium II are always in local equilibrium :

$$c_2^2 = \xi(1-\xi) \left(\frac{\partial s}{\partial \xi} \right)_p \cdot \left(\frac{\partial T}{\partial \xi} \right)_p. \quad (4)$$

Fig. 2a represents the results which are calculated by means of the numerical values in Table I and the extrapolated values of r below 1°K in Fig. 1. For simplicity, it is assumed that r is not altered by pressure. This assumption may be allowed in view of the fact that the values of the entropy given by Eqs. (1), (2) and (3) coincide well with the experimental ones as long as the pressure is not so large.

As stated previously,⁴⁾ c_2 tends to infinity as temperature decreases to 0°K, if $r > 4$. This result seems to be contradictory with the results of Atkins and Osborne.⁸⁾ However, owing to the lack of the experimental results giving the temperature dependences of the normal fluid concentration and the entropy, we cannot regard this discrepancy as a defect of Tisza's theory.

§ 3. The second sound velocity based on Landau's model

Landau has considered the normal fluid as composed of the phonons and rotons and represented the normal fluid concentration and the entropy of liquid helium II respectively as follows:²⁾

$$\xi = \frac{4}{3} \frac{E_{ph}}{c_1^2} + \frac{\mu}{\rho} \left(\frac{\mu k}{2\pi \hbar^2} \right)^{3/2} \cdot T^{3/2} \cdot e^{-\Delta/kT}, \quad (5)$$

and

$$s = s_{ph} + \frac{\Delta}{\rho} \left(\frac{\mu k}{2\pi \hbar^2} \right)^{3/2} \cdot T^{1/2} \left(1 + \frac{5kT}{2\Delta} \right) \cdot e^{-\Delta/kT}, \quad (6)$$

where E_{ph} is the internal energy of the phonons, μ is the effective mass of the roton and Δ is the gap energy between the lowest energies of the phonon- and roton-spectra.

To eliminate ρ and μ in Eqs. (5) and (6), we normalize ξ and s so that $\xi=1$ and $s=s_\lambda$ at the transition temperature T_λ . Then Eqs. (5) and (6) can be rewritten as

$$\xi = a' x^4 + x^{3/2} (1 - a') \cdot e^{\beta(1-1/x)}, \quad (5')$$

and

$$s = a T_\lambda^3 x^3 + (s_\lambda - a T_\lambda^3) \cdot x^{1/2} \frac{2\beta + 5x}{2\beta + 5} \cdot e^{\beta(1-1/x)}, \quad (6')$$

where $x = T/T_\lambda$, $\beta = \Delta/kT_\lambda$ and $a' = a T_\lambda^4 / c_1^2$ with a in Eq. (2).

In these expressions $s_\lambda(p)$, $T_\lambda(p)$, $c_1(p)$ and $a(p)$ are determined experimentally as before. The energy gap Δ may be dependent of pressure, but for the sake of simplicity we fix it as $\beta = \Delta/kT_\lambda = 4$.

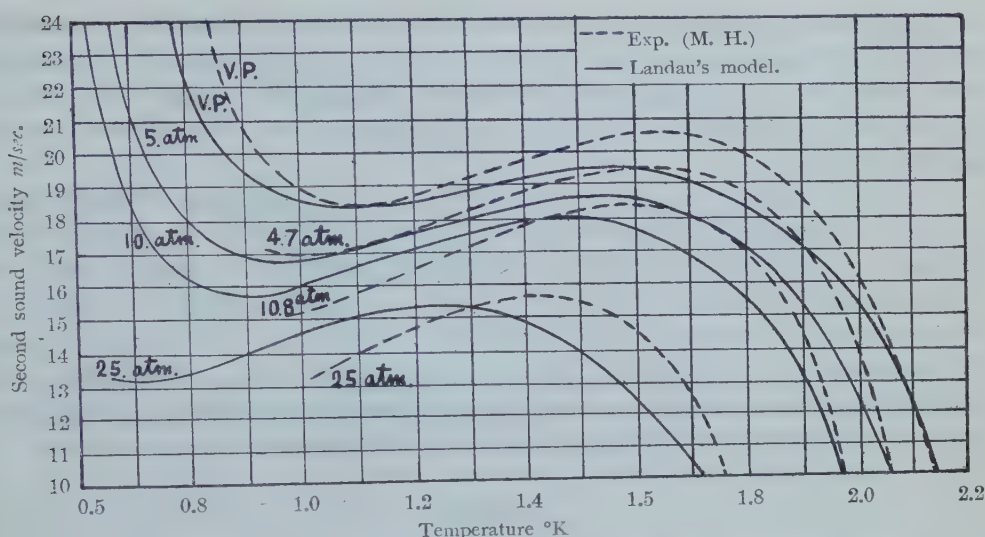


Fig. 2b. The second sound velocity based on Landau's model.

Using these expressions of ξ and s , Eqs. (5') and (6'), the temperature and pressure dependence of the second sound velocity will be obtained by Eq. (4). The results are shown in Fig. 2b. and Fig. 3. The value of the second sound velocity at 0°K is calculated from Eq. (4) as $\sqrt{3c_1/4}$, but since the extrapolated value of the ordinary sound velocity is used in our computation the computed values in Fig. 3 have not so serious significance.

The shift of the minimum of the velocity to lower temperature with increasing pressure is represented by this model as well as by Tisza's model. The flatness of the velocity below 0.5°K under vapour pressure is related with the fact that the contribution of the rotons becomes negligibly small below this temperature. The pressure dependence of the second sound velocity in this temperature region is expected, in Landau's model, to be identical with that of the ordinary sound velocity. This expectation, however, has not yet been experimentally assured.

§ 4. Conclusion

From the present analysis of the second sound velocity, we obtain the following conclusions. Firstly the shift of the minimum of the second sound

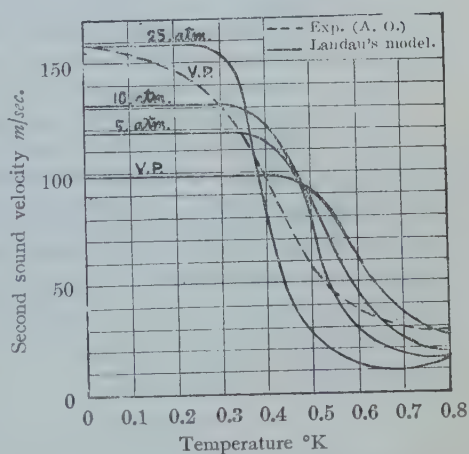


Fig. 3. The second sound velocity at the vicinity of 0°K.

velocity can be explained by Tisza's model as well as by Landau's, if one takes into account the phonon effect on the entropy of liquid helium II. Secondly a finite value of the second sound velocity at 0°K is due to the fact that the normal fluid concentration contains the term proportional to T^4 such as the phonon part in Eq. (5).

From the first result we may conclude that the eigenstates of liquid helium II at lower energy is represented by the quantized wave field of ordinary sound. As for the values of the second sound velocity at 0°K in Landau's model, it is to be noted that our value $\sqrt{3}c_1/4$ is based on Eq. (4), which is valid if the fluctuation of the normal fluid concentration is instantaneously followed by that of temperature. On the other hand, Landau's value $c_1/\sqrt{3}$ is obtained, if one assumes that the normal fluid concentration fluctuates independently of temperature and that the partial molal entropy of the superfluid vanishes.

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Electronic States of C_2 -Molecule, II— Effect of $2s$ -Shells —

Gentaro ARAKI and Wataro WATARI
Faculty of Engineering, Kyoto University

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Adiabatic potentials corresponding to 62 electronic states, arising from various configurations of the C_2 -molecule which consists of two carbon atoms with two $2p$ -electrons each, are calculated taking into account the effect of $2s$ -shells. The complete adiabatic Hamiltonian of eight electrons, but not a so-called interaction operator, is used, and direct, exchange, higher permutation, and overlapping integrals are all taken into account. The same effective charge x is assumed for $2s$, $2p\sigma$, and two $2p\pi$ orbitals, and the minimum electronic energy is obtained by varying x for each nuclear distance. It is found that $2s$ -shells repel $2p\sigma$ electrons and attract $2p\pi$ -electrons. Thus $(s^2\sigma^2, s^2\sigma^2)^1\Sigma_g^+$ state, which was found to be attractive and deepest when the effect of $2s$ -shells was neglected in the previous paper, becomes very higher and repulsive. The results of the present calculation are that the deepest state is $(s^2\sigma\pi, {}^2\sigma\pi)^1\Delta_g$ and that $(s^2\sigma\pi, {}^2\sigma\pi)^1\Sigma_g^+$ and $(s^2\sigma\pi, s^2\pi\pi)^1\Pi_u$ are next and third respectively.

Introduction

In the first part⁽¹⁾ (which will be referred to in what follows as I) of this paper we discussed the interaction between $2p$ -electrons of two carbon atoms in their ground states. The purpose was not only to separate the contribution of p -electrons to the binding energy of the C_2 -molecule from other effect but also to study the general character of the interaction between p -electrons in the complex molecule with double bonds due to p -electrons. It was found that the deepest attractive state was $(2p\sigma^2, 2p\sigma^2)^1\Sigma_g^+$ in contradiction to Pauling's theory⁽²⁾ which does not allow atoms with fully occupied orbitals to bind. The reasons for this result were that the binding energy is mainly supplied by the Coulomb energy and the effect of exchange and permutation energy is small owing to the large value of effective charge involved in the $2p$ -orbitals, and that the Coulomb energy is the larger the more σ -orbitals involved in the wave function. The same phenomenon was observed in the result of calculation by Bartlett⁽³⁾ on the single pp -bond. This shows that Pauling's theory can not literally be applied to any simple model for complex molecules. This was an important result of the study in the first part as well as Bartlett's result that the triplet state is deepest in the single pp -bond in contradiction to Pauling's fundamental idea.⁽²⁾

We shall consider the effect of $2s$ -shells in the present paper in order to see the general character of interactions of a closed s -shell with $p\sigma$ - and $p\pi$ -electrons as well as the quantitative contribution of these interactions to the binding energy

of the C_2 -molecule. For this purpose we shall examine 62 electronic states of the C_2 -molecule taking into account all integrals and varying the effective charge of orbitals for each nuclear distance.

The effect of inner shells on the binding of the Li_2 -molecule was studied by James.⁽⁴⁾ He found that there is a large repulsion between a s -electron and a closed s -shell. It may be imagined from his result that the closed shell repels the valence electron in general. This is in accord with the fact that there is a repulsion between two He atoms or He and H atoms.⁽⁵⁾ It is more natural, however, to consider that the nature of the interaction between a valence electron and a closed shell is different when the valence electron is different from a s -electron. We shall find in what follows that the latter is true. We shall see in fact that the closed $2s$ -shell repels σ -electrons whereas it attracts π -electrons. This is one of the important results in the present paper. Owing to this peculiar property of the interaction the rearrangement of the order of potential curves occurs when the $2s$ -shells are taken into account for the calculation of the electronic energy of the C_2 -molecule. The deepest state is replaced by $(2s^2 2p\sigma 2p\pi, 2s^2 2p\sigma 2p\pi)^1 A_g$ instead of $(2p\sigma^2, 2p\sigma^2)^1 \Sigma_g^+$.

In order to find the deepest electronic state of the C_2 -molecule we have further to take into account the effect of $1s$ -shells. This will be postponed in the future research. We may consider that the property of the interaction between $2p$ -electrons and $2s$ -shells is exaggerated in the present calculation because we use a more diffuse function as the $2s$ -orbital. If the effect of the $1s$ -shells is similar, the result of the present calculation may include the effect to a certain extent. If this consideration is adequate the present result can be considered as representing an approximate property of the electronic state of the C_2 -molecule. On this assumption the above mentioned result for the deepest state may be interpreted as corresponding to Penney's model⁽⁶⁾ of the C_2 -molecule in which the double bonds consist of a $\sigma\sigma$ -bond and a $\pi\pi$ -bond.

Reduction formulas and numerical values of integrals which appear in our calculation will be published in the third part.

§ 1. Method of Calculation

In order to examine the effect of $2s$ -shells on the electronic states of the C_2 -molecule we represent a carbon atom by a simplified model which consists of two $2s$ -electrons, two $2p$ -electrons, and a nucleus of atomic number 4. The C_2 -molecule is considered as consisting of two atoms with such a structure. The effect of $1s$ -shells is still omitted except for their effect of complete screening of nuclear charges. The adiabatic Hamiltonian of the molecule is thus given by

$$H = - \sum_{k=1}^8 \left(\frac{1}{2} A_k + \frac{4}{r_{ak}} + \frac{4}{r_{bk}} \right) + \sum_{k=j+1}^8 \sum_{j=1}^8 \frac{1}{r_{kj}} + \frac{16}{R} \quad (1.1)$$

where all quantities are measured in atomic unite, and notations are the same as

those in I. The following calculation will be carried out employing this complete Hamiltonian but not a so-called interaction operator. Therefore our result will not include any error which arises from employing an interaction operator.

$2s$ - and $2p\sigma$ -orbitals are denoted by s and σ respectively. Two $2p\pi$ -orbitals are distinguished by π and ϖ . We assume the same radial part for all orbitals for the sake of practical convenience. Thus they are given by

$$\left. \begin{aligned} s &= N_s r \exp(-\kappa r/2) & \sigma &= N_\sigma z \exp(-\kappa r/2) \\ \pi &= N_\pi (x + iy) \exp(-\kappa r/2) & \varpi &= N_\pi (x - iy) \exp(-\kappa r/2) \end{aligned} \right\} \quad (1.2)$$

where N_s , N_σ , and N_π are normalization constants, κ is an effective charge, x , y , and z are Cartesian coordinates, the origin of the coordinates is an atomic nucleus, and r is a distance of (x, y, z) from the origin. The better approximation can be obtained if we employ a different effective charge for the $2s$ -orbital. The above assumption makes the $2s$ -orbital too diffuse.⁽⁷⁾ Consequently the effect of $2s$ -shells is exaggerated. This may compensate for the error due to the neglect of $1s$ -shells to a certain extent, because the effect of $1s$ -shells may be similar to that of $2s$ -shells.

The $2s$ -orbital given by (1.2) is nodeless and its polynomial part does not contain a constant term. The reason for this assumption is mainly to reduce the labour of computation. The omitted part of the function is large at the immediate vicinity of a nucleus. We are interested in the region at the vicinity of the equilibrium distance between two carbon atoms. In this region the neglected part gives a small effect on the one hand, and the error of its neglect may be compensated to some extent by a large norm of the nodeless function on the other hand.

62 electronic states arise from the present configuration. They are the same as those in the case discussed in I. The multiplicity, parity, reflexion symmetry and axial quantum number of the states can be determined in quite the same way as in I. The wave functions of these states are represented by the similar expressions as those in I. Only the difference is that the wave functions contain more one-electron functions, s^+ and s^- , besides those in I.

The matrix elements of H between different configurations are still neglected whereas direct, exchange and higher permutation integrals are all taken into account. The numerical evaluation of these integrals is carried out by the aid of the tables given by Kotani, Amemiya and Simose.⁽⁸⁾ The secular equations are at most cubic, as is seen from Table II in I, p. 142. Four examples of formulas for energy eigenvalues or secular matrices will be shown at the end of the third section.

The eigenvalues depend on R (a nuclear distance) and κ (an effective charge). Therefore we denote them by $W_1(R, \kappa)$. We seek for the minimum value of $W_1(R, \kappa)$ by varying κ for a definite R where the variation is begun with $\kappa=4$. The value of κ for which $W_1(R, \kappa)$ takes its minimum value depends on R . We

denote it by $x_0(R)$. Some examples of $x_0(R)$ are shown in Fig. 5. That x_0 depends on R means that orbitals change as two carbon atoms approach each other. The better approximation would have been obtained if we had further let orbitals deform as two atoms approach, but it may be practically impossible.

We denote an adiabatic potential between two carbon atoms by $W(R)$. This function is given by $W_1(R, x_0(R))$ which means a minimum of electronic energy at a given R . The dissociation energy of the C_2 -molecule in a given state is equal to $W(\infty) - W(R_0)$ where $W(R_0)$ is a minimum of $W(R)$ with respect to R . For attractive states $W_1(R, 4) - W(R)$ for $R=2 \sim 3$ au is smaller, in general, than $W_1(\infty, 4) - W(\infty)$, therefore the dissociation energy for $x=4$ is larger than its value calculated by the variational method. The absolute energy is of course lower in the variational case.

§ 2. Outline of the Result

In order to see the general property of the interaction between $2p$ -electrons and $2s$ -shells we first compute $W(R)$ for Σ -states assuming $x=4$. By comparing the resulting potentials with those obtained in I we find that $2s$ -shells repel

$p\sigma$ -electrons whereas they attract $p\pi$ -electrons. Such phenomena are found in II and Δ states too. The typical examples are shown in Fig. 1, where $W(R) - W(\infty)$ is plotted.

The $(\sigma^2, \sigma^2)^1\Sigma_g^+$ state was attractive and deepest when the $2s$ -shells were neglected. The corresponding $(s^2\sigma^2, s^2\sigma^2)^1\Sigma_g^+$ state now becomes very higher and repulsive when the $2s$ -shells are taken into account. The $(\sigma\pi, \sigma\pi)^3\Sigma_g^-$ state was attractive and had a shallow minimum whereas the corresponding $(s^2\sigma\pi, s^2\sigma\pi)^3\Sigma_g^-$ state is repulsive. The $(\pi\pi, \pi\pi)^1\Sigma_g^+$ state has only a very shallow minimum whereas the minimum of the corresponding state becomes deeper owing to the effect of the $2s$ -shells. We see in Fig. 1 that the repulsion between $p\sigma$ -electrons and $2s$ -shells is large and effective over all range of R whereas the attraction between $p\pi$ -electrons and $2s$ -shells is large only in the narrower region of R .

Next, in order to determine the relative order of the potentials we calculate

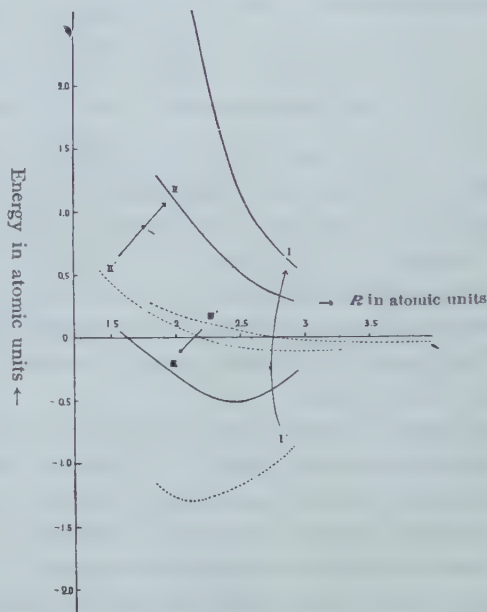


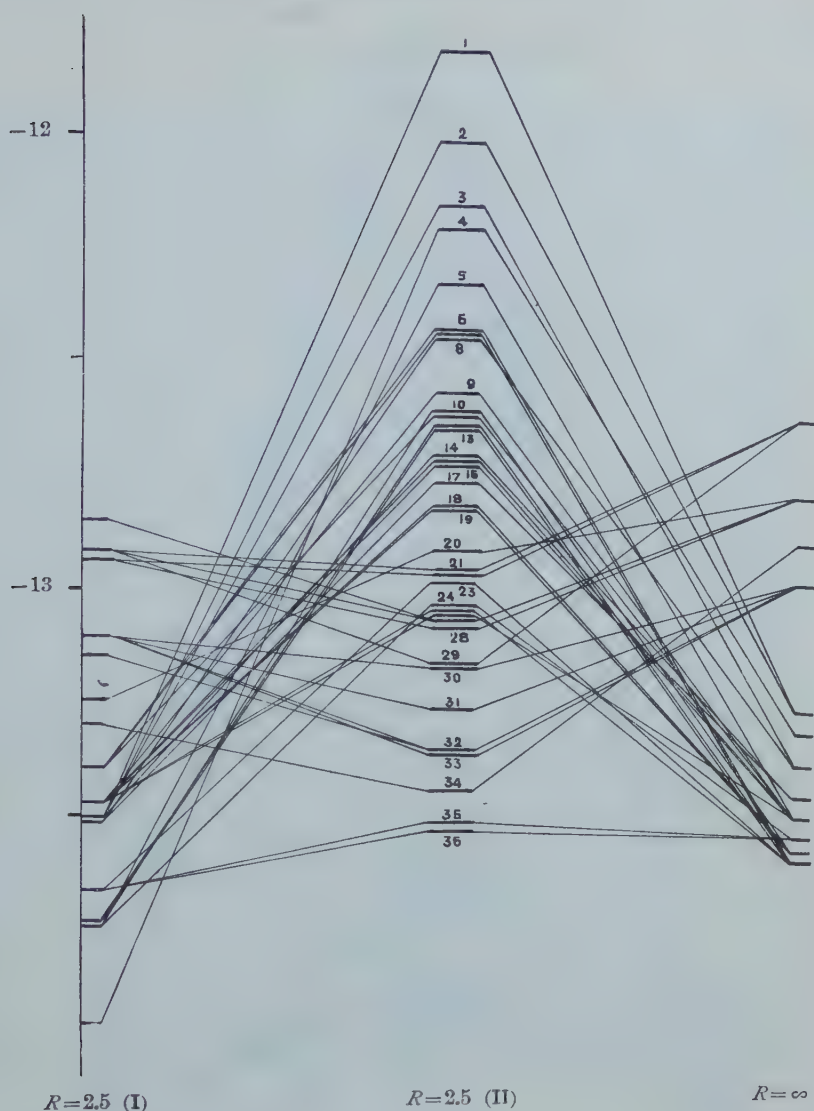
Fig. 1. Effect of $2s$ -shells on adiabatic potentials
Dotted curves: $2s$ -shells are neglected.

Full curves: $2s$ -shells are taken into account.

I': $(\sigma^2, \sigma^2)^1\Sigma_g^+$, I: $(s^2\sigma^2, s^2\sigma^2)^1\Sigma_g^+$

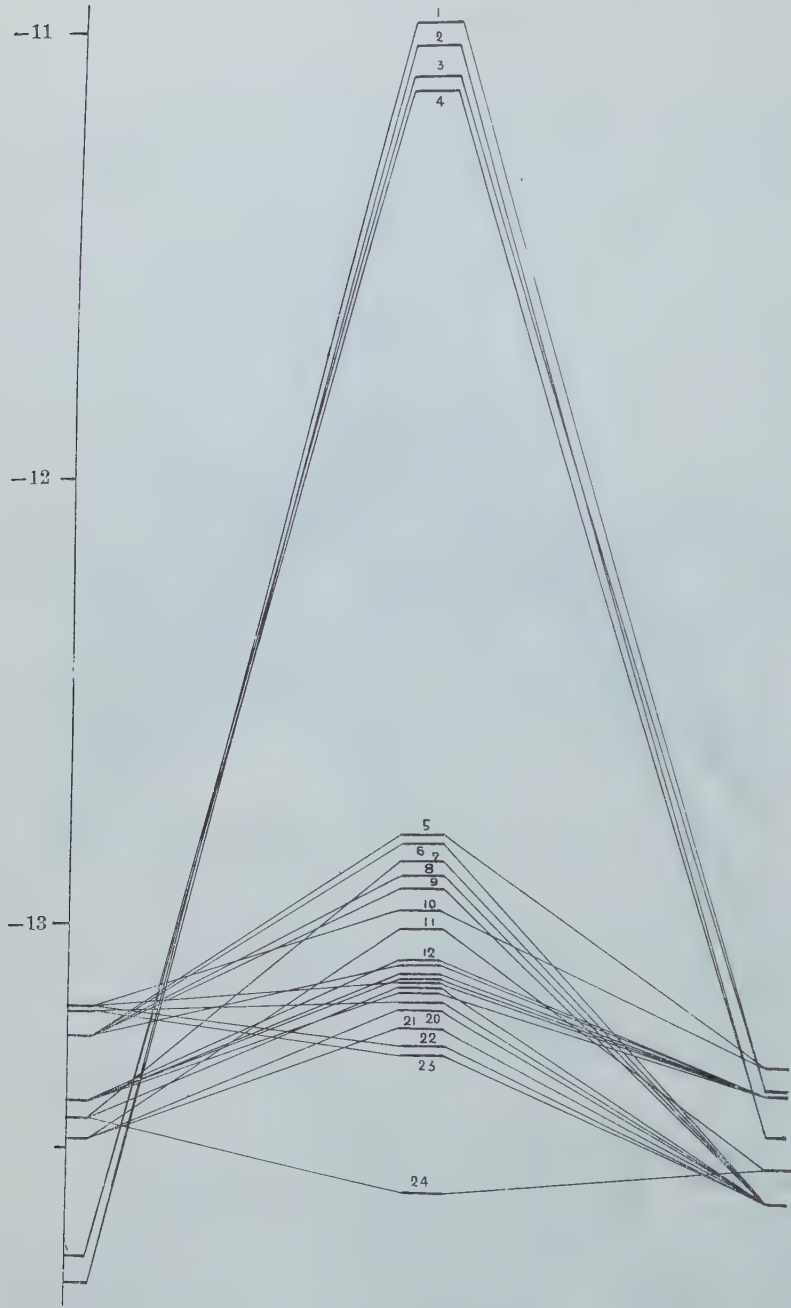
II': $(\sigma\pi, \sigma\pi)^3\Sigma_g^-$, II: $(s^2\sigma\pi, s^2\sigma\pi)^3\Sigma_g^-$

III': $(\pi\pi, \pi\pi)^1\Sigma_g^+$, III: $(s^2\pi\pi, s^2\pi\pi)^1\Sigma_g^+$

Fig. 2. Σ , Δ , and Γ -states

The left column: Electronic energy at $R=2.5$ au in I
 The middle column: Electronic energy at $R=2.5$ au in the present calculation
 The right column: Electronic energy at $R=\infty$ in the present calculation
 Corresponding states are joined

1) $(s^2\sigma^2, s^2\pi\bar{\sigma})^1\Sigma_u^+$	14) $(s^2\sigma^2, s^2\pi\bar{\sigma})^3\Sigma_u^-$	27) $\left\{ \begin{array}{l} (s^2\pi^2, s^2\pi)^1\Gamma_g \\ (s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})^3\Sigma_g^- \end{array} \right.$
2) $(s^2\sigma^2, s^2\pi\bar{\sigma})^1\Sigma_g^+$	15) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^1\Sigma_g^+$	28) $(s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})\Sigma_g^+$
3) $(s^2\sigma^2, s^2\pi^2)^1\Delta_u$	16) $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$	29) $(s^2\pi^2, s^2\pi\bar{\sigma})^3\Delta_g$
4) $(s^2\sigma^2, s^2\sigma^2)^1\Sigma_g^+$	17) $(s^2\sigma^2, s^2\pi\bar{\sigma})^3\Sigma_g^-$	30) $(s^2\pi^2, s^2\pi\bar{\sigma})^1\Delta_u$
5) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_u^-$	18) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^5\Sigma_g^+$	31) $(s^2\pi^2, s^2\pi\bar{\sigma})\Delta_u$
6) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_u^+$	19) $(s^2\sigma\pi, s^2\sigma\pi)^5\Delta_g$	32) $\left\{ \begin{array}{l} (s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})^3\Sigma_u^- \\ (s^2\pi^2, s^2\sigma\bar{\sigma})^1\Sigma_u^- \end{array} \right.$
7) $(s^2\sigma\pi, s^2\sigma\pi)^3\Delta_u$	20) $(s^2\pi^2, s^2\sigma\bar{\sigma})^1\Sigma_g^+$	33) $(s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})^3\Sigma_u^+$
8) $(s^2\sigma^2, s^2\pi^2)^1\Delta_g$	21) $(s^2\pi^2, s^2\pi\bar{\sigma})\Delta_g$	34) $(s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})^1\Sigma_g^+$
9) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_g^+$	22) $(s^2\pi\bar{\sigma}, s^2\pi\bar{\sigma})^1\Sigma_g^+$	35) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^1\Sigma_g^+$
10) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^5\Sigma_u^-$	23) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_g^-$	36) $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$
11) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})\Sigma_u^-$	24) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^1\Sigma_u^-$	
12) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_u^+$	25) $(s^2\sigma\pi, s^2\sigma\bar{\sigma})^3\Sigma_g^-$	
13) $(s^2\sigma\pi, s^2\sigma\pi)^3\Delta_u$	26) $(s^2\sigma\pi, s^2\sigma\pi)^3\Delta_g$	



$R=2.5$ (I)

$R=2.5$ (II)

$R=\infty$

Fig. 3. Π and Φ -States

- | | | |
|--|---|---|
| 1) $(s^2\sigma^2, s^2\sigma\pi)^1\Pi_u$ | 9) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^5\Pi_u$ | 17) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_u$ |
| 2) $(s^2\sigma^2, s^2\sigma\pi)^3\Pi_u$ | 10) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^1\Pi_g$ | 18) $(s^2\pi^2, s^2\sigma\bar{\sigma})^3\Pi_g$ |
| 3) $(s^2\sigma^2, s^2\sigma\pi)^1\Pi_g$ | 11) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^1\Pi_g$ | 19) $(s^2\pi^2, s^2\sigma\pi)^1\Phi_g$ |
| 4) $(s^2\sigma^2, s^2\sigma\pi)^3\Pi_g$ | 12) $(s^2\pi^2, s^2\sigma\bar{\sigma})^1\Pi_u$ | 20) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_g$ |
| 5) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^1\Pi_u$ | 13) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_u$ | 21) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_g$ |
| 6) $(s^2\pi^2, s^2\sigma\pi)^3\Phi_u$ | 14) $(s^2\pi^2, s^2\sigma\bar{\sigma})^3\Pi_u$ | 22) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^5\Pi_g$ |
| 7) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_u$ | 15) $(s^2\pi^2, s^2\sigma\bar{\sigma})^1\Pi_g$ | 23) $(s^2\pi^2, s^2\sigma\pi)^3\Phi_g$ |
| 8) $(s^2\pi^2, s^2\sigma\pi)^1\Phi_u$ | 16) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^3\Pi_g$ | 24) $(s^2\sigma\pi, s^2\pi\bar{\sigma})^1\Pi_u$ |

the energy minima at $R=2.5$ au and $R=\infty$ by varying α . The result is shown in Fig. 2 and Fig. 3. Levels shown in the middle column of the figures are the energy minima at $R=2.5$ au, those at $R=\infty$ are shown on the right, and the levels obtained in I are shown on the left. The absolute values of the results in I and in the present calculation can not directly be compared in these figures. Two cases are joined so as to agree at $R=\infty$ for $(\sigma^2, \sigma^2)^1\Sigma_g^+$. Differences between these adiabatic potentials are roughly computed for other values of R , and it is ascertained that the order determined at $R=2.5$ au does not change in the region of smaller R and in the vicinity of larger R within each of Fig. 2 and Fig. 3. We find that the deepest competing states are $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$, $(s^2\sigma\pi, s^2\pi\sigma)^1\Pi_u$ and $(s^2\sigma\pi, s^2\sigma\sigma)^1\Sigma_g^+$.

We calculate $W(R)$ more carefully at other points for these three states. The result is shown in Fig. 4 and Fig. 5. The final conclusion is that $^1\Delta_g$ is deepest. Its dissociation energy is 0.75 au (20 eV) and its equilibrium distance is 1.9 au (1.0 Å). Mulliken⁽⁹⁾ discussed electronic levels of the C_2 -molecule on the basis of empirical data. According to him the order of the low energy electronic levels is given by Table I, where energy is measured from the deepest level, R_0 is an equilibrium nuclear distance, electron configurations mean those in his molecular orbital theory, the orbitals $x\sigma_g$, $y\sigma_u$, and $z\sigma_g$ are linear combinations of $2s$ and $2p\sigma$, and $v\pi_g$ and $w\pi_u$ are those of $2p\pi$. His lowest three states $^1\Delta_g$, $^1\Pi_u$, and $z\sigma_g^2y\sigma_u^2w\pi_u^2x\sigma_g^2(^1\Sigma_g^+)$ correspond to three states in Fig. 4. His $z\sigma_g^2y\sigma_u^2w\pi_u^4(^1\Sigma_g^+)$ corresponds to the $(s^2\pi\sigma, s^2\pi\sigma)^1\Sigma_g^+$ state (No. 34) in Fig. 2. These results of Mulliken are in qualitative agreement with our result. The $z\sigma_g^2y\sigma_u^2w\pi_u^3x\sigma_g(^3\Pi_u)$ state is one of the lowest states in Mulliken's result whereas the corresponding $(s^2\sigma\pi, s^2\pi\sigma)^3\Pi_u$ state (No. 17) is repulsive in our result as is seen from Fig. 3.

To completely discuss the ground state of the C_2 -molecule the following effects should further be taken into account: (i) the effect of the configurations in the first and second columns of Table II in I, p. 142, (ii) the effect of $1s$ -shells,

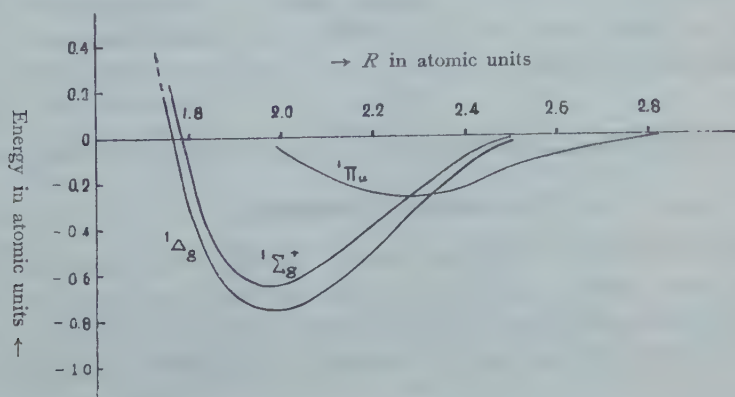


Fig. 4. Adiabatic potentials for the lowest three states
 $(s^2\sigma\pi, s^2\pi\sigma)^1\Pi_u$ $(s^2\sigma\pi, s^2\sigma\sigma)^1\Sigma_g^+$ $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$

(iii) the deformation of orbitals, (iv) the effect of ionic configurations, and (v) the effect of other atomic configurations. It may be possible that the second effect makes $(s^2\sigma\pi, s^2\pi\sigma)^1\Pi_u$ or $^3\Pi_u$ deepest because these contain less σ -orbital, where the latter has been considered as the final state of the Swan bands.⁽¹⁰⁾ Heitler and Poeschel⁽¹¹⁾ considered that last effect made $^3\Pi_u$ deepest.* Our result for the dissociation energy is too large to compare with experiment (the dissociation energy for $^3\Pi_u$ is estimated at the order of 5.5 eV from the Swan bands.)⁽¹⁰⁾ This discrepancy may be due to the neglect of the repulsion by 1s-shells.

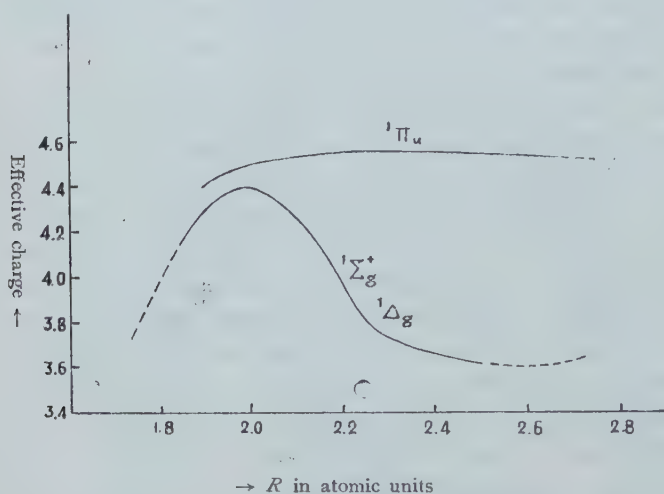


Fig. 5. Effective charges as functions of R
 $(s^2\sigma\pi, s^2\pi\sigma)^1\Pi_u$ $(s^2\sigma\pi, s^2\sigma\sigma)^1\Sigma_g^+$ $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$

Table 1.

States	Energy in eV	R_0 (obs.) in Å	Electron configurations
$3, ^1\Pi_g$	3	1.266 1.257	$2\sigma_g^2 y\sigma_u^2 w\pi_u^3 x\sigma_g^2$
$3, ^1\Sigma_u^+$	3	1.240	$2\sigma_g^2 y\sigma_u^2 w\pi_u^4 x\sigma_g$
$^1\Delta_g$ $^3\Sigma_g^-$ $^1\Sigma_g^+$	0	—	$2\sigma_g^2 y\sigma_u^2 w\pi_u^2 x\sigma_g^2$
$3, ^1\Pi_u$	0	1.313 1.318	$2\sigma_g^2 y\sigma_u^2 w\pi_u^3 x\sigma_g$
$^1\Sigma_g^+$	0	1.243	$2\sigma_g^2 y\sigma_u^2 w\pi_u^4$

§ 3. Energy Formulas

The general formulas for electronic energy are very complicated and lengthy. To write down here their complete expressions is therefore given up, but some examples for the states, which are important in the present study, will be illustrated in what follows, where the adiabatic electronic energy will be denoted by W .

* Erratum. I, P. 152, the second line from the bottom, for 2s-electrons read the 5S state of the C-atom.

For a non-degenerate state (see Table II in I, p. 142), W is given by

$$W = (\psi, H\psi) / (\psi, \psi) \quad (3.1)$$

where ψ denotes a wave function of the state and the parenthesis means a Hermitian inner product. For a doubly degenerate state W is given by

$$W = (B \pm \sqrt{B^2 - 4AC}) / (2A) \quad (3.2)$$

where

$$\left. \begin{aligned} A &= A_{11}A_{22} - A_{12}^2 & C &= H_{11}H_{22} - H_{12}^2 \\ B &= A_{11}H_{22} + A_{22}H_{11} - 2A_{12}H_{12} \end{aligned} \right\} \quad (3.3)$$

$$H_{kj} = (\psi_k, H\psi_j), \quad A_{kj} = (\psi_k, \psi_j) \quad k=1, 2 \quad (3.4)$$

and ψ_1 and ψ_2 are two linearly independent wave functions of the state.

The $(s^2\sigma\pi, s^2\pi\varpi)^3\Pi_g$ and ${}^3\Pi_u$ state are trebly degenerate. The secular equations for these states have following form:

$$\begin{vmatrix} H_{11} - A_{11}W & H_{12} - A_{12}W & H_{13} \\ H_{12} - A_{12}W & H_{22} - A_{22}W & H_{23} \\ H_{13} & H_{23} & H_{33} - A_{33}W \end{vmatrix} = 0 \quad (3.5)$$

Our numerical calculation shows that H_{13} and H_{23} are small. For example, in case of ${}^3\Pi_g$ ($x=4$, $R=2.5$ au) we have

$$\left. \begin{aligned} H_{11} &= -9.4868 \text{ au}, & H_{22} &= -8.4384 \text{ au}, & H_{33} &= -7.0732 \text{ au} \\ H_{12} &= 0.3884 \text{ au}, & H_{13} &= -0.0444 \text{ au}, & H_{23} &= 0.0239 \text{ au} \\ A_{11} &= A_{22} = 0.7229, & A_{33} &= 0.7509, & A_{12} &= -0.0280 \end{aligned} \right\} \quad (3.6)$$

In order to obtain approximate roots of the equation we first neglect H_{13} and H_{23} . Then the problem reduces to those in the case of non-degenerate and doubly degenerate states. The approximate roots show that these states are not competing with the deepest state. Therefore we give up to obtain more precise values of the roots.

In what follows, examples of W or H_{kj} and A_{kj} are given for the states $(s^2\sigma^2, s^2\sigma^2)^1\Sigma_g^+$, $(s^2\sigma\pi, s^2\sigma\varpi)^1\Sigma_g^+$, $(s^2\sigma\pi, s^2\pi\varpi)^1\Pi_u$, and $(s^2\sigma\pi, s^2\sigma\pi)^1\Delta_g$. The first state was deepest in I and becomes higher and repulsive in the present case, the last state is deepest in the present case, and the middle two are competing with the deepest.

Wave functions are denoted by abbreviated symbols. For example, $s_a s_a \sigma_a \pi_a s_b s_b \sigma_b \varpi_b$ stands for $s_a(1)s_a(2)\sigma_a(3)\pi_a(4)s_b(5)s_b(6)\sigma_b(7)\varpi_b(8)$, and $\pi_a(4)$ means a π -orbital which belongs to the a -atom and contains coordinates of the fourth electron as its argument. A Hermitian inner product of two functions φ and ψ is denoted by (φ, ψ) . Integrals given by $(\varphi\psi\chi\mu\xi\eta\zeta\omega, H\varphi'\psi'\chi'\mu'\xi'\eta'\zeta'\omega')$ are

divided into four classes which are distinguished by Q , J_k , N_k ($k=1, 2, 3, \dots$), and \mathcal{O} . Q is a direct integral, J_k is an integral which does not vanish when II becomes unity, N_k is an integral which becomes including a factor of a vanishing overlapping integral when II becomes unity, and integrals of having two such factors are gathered together in \mathcal{O} .

The overlapping integrals of two orbitals are denoted by

$$\left. \begin{aligned} S_0 &= (\sigma_a, \sigma_b) & S_1 &= (\pi_a, \pi_b) = (\varpi_a, \varpi_b) \\ S &= (s_a, s_b) & \delta &= (s_a, \sigma_b) \end{aligned} \right\} \quad (3.7)$$

The notations of Kotani, Amemiya and Simose⁽⁸⁾ are used:

$$\left. \begin{aligned} I_{\varphi\psi} &= (\varphi_a, r_a^{-2}\psi_b) \\ I_{\varphi\psi} &= (\varphi_a, r_b^{-1}\psi_b) & K_{\varphi\psi} &= (\varphi_a, r_b^{-1}\psi_a) \end{aligned} \right\} \quad (3.8)$$

$$\left. \begin{aligned} C_{\varphi\psi\eta} &= (\varphi_a\psi_b, r_{12}^{-1}\hat{\zeta}_b\eta_a) & D_{\varphi\psi\eta} &= (\varphi_a\psi_b, r_{12}^{-1}\hat{\zeta}_a\eta_b) \\ F_{\varphi\psi\eta} &= (\varphi_a\psi_a, r_{12}^{-1}\hat{\zeta}_a\eta_a) & L_{\varphi\psi\eta} &= (\varphi_a\psi_a, r_{12}^{-1}\hat{\zeta}_a\eta_b) \end{aligned} \right\} \quad (3.9)$$

where $I_{\varphi\psi}$ and $F_{\varphi\psi\eta}$ were not defined by Kotani, Amemiya and Simose. $F_{\varphi\psi\varphi\psi}$ and $F_{\varphi\psi\psi\varphi}$ are atomic integrals which can be represented by Slater's F_k and G_k .⁽¹²⁾

$$(i) \quad (s^2\sigma^2, s^2\sigma^2) \sum_g^+$$

$$W = (Q + \mathcal{J} + \mathcal{H} + \mathcal{O}) / \mathcal{S}^2 \quad (3.10)$$

where \mathcal{S} is the norm of the wave function of the state

$$\mathcal{S} = (1 - \delta^2)^2 - S_0^2(1 - S)^2 - S(S - 2S_0\delta^2) \quad (3.11)$$

$$Q = (s_a s_a \sigma_a \sigma_a s_b s_b \sigma_b \sigma_b, H s_a s_a \sigma_a \sigma_a s_b s_b \sigma_b \sigma_b) \quad (3.12)$$

$$\begin{aligned} \mathcal{J} &= 8J_1 + 6J_2 + 4(J_3 + J_4 + J_5 + J_6 + J_7 + J_8) + 2(J_9 + J_{10} + J_{11} + J_{12} + J_{13} + J_{14}) \\ &\quad - 4(J_{15} + J_{16} + J_{17} + J_{18} + J_{19} + J_{20}) - 2(J_{21} + J_{22} + J_{23} + J_{24} + J_{25}) \end{aligned} \quad (3.13)$$

$$\begin{aligned} \mathcal{H} &= 8(N_1 + N_2 + N_3 + N_4 + N_5 + N_6) - 16(N_7 + N_8) \\ &\quad - 8(N_9 + N_{10} + N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}) \end{aligned} \quad (3.14)$$

$$\mathcal{O} = 4D_{s\sigma s\sigma} \{ \mathcal{S}(\delta^2 - S_0 S) - 2\delta^2(S_0^2 - S^2) \} - 4F_{s\sigma s\sigma} \{ \mathcal{S} - 2\delta^2(S_0^2 - S^2) \} \quad (3.15)$$

In order to reduce the explanation of the notations as simple as possible the abbreviation such as

$$(s_a s_a \sigma_a \sigma_a s_b s_b \sigma_b \sigma_b, H s_b s_b s_b \sigma_a s_a s_b \sigma_a s_a) = (\sigma_b s_b s_b \sigma_a s_a s_b \sigma_a s_a) \quad (3.16)$$

is used in the case (i), where the operand of II is only shown in the abbreviated formula and H and the function on its left is omitted because it is the same throughout the case (i). In this way J_k and N_k are given by

$$J_1 = (\sigma_b s_b s_b \sigma_a s_a s_b \sigma_a s_a) \quad J_2 = (\sigma_b s_b s_b \sigma_b \sigma_a s_a s_a \sigma_a) \quad J_3 = (s_a \sigma_b s_b \sigma_a \sigma_a s_b \sigma_b s_a)$$

$$\begin{aligned}
J_4 &= (\sigma_b s_b \sigma_a \sigma_a s_b s_a s_a s_b) & J_5 &= (s_a \sigma_b \sigma_b \sigma_a s_b s_b \sigma_a s_a) & J_6 &= (s_a s_b \sigma_b \sigma_a s_b s_a \sigma_a \sigma_b) \\
J_7 &= (s_b s_b \sigma_b \sigma_a \sigma_a s_a s_a \sigma_b) & J_8 &= (s_b s_a \sigma_b \sigma_b \sigma_a s_b s_a \sigma_a) & J_9 &= (s_a s_a \sigma_b \sigma_b \sigma_a s_b s_b \sigma_a \sigma_b) \\
J_{10} &= (\sigma_b \sigma_b \sigma_a \sigma_a s_b s_b s_a s_a) & J_{11} &= (s_b s_b \sigma_a \sigma_a s_a s_a \sigma_b \sigma_b) & J_{12} &= (s_a s_a \sigma_b \sigma_b s_b s_b \sigma_a \sigma_a) \\
J_{13} &= (\sigma_b \sigma_b s_b s_b \sigma_a \sigma_a s_a s_a) & J_{14} &= (s_b s_b \sigma_b \sigma_b s_a s_a \sigma_a \sigma_a) & J_{15} &= (\sigma_b s_a \sigma_a \sigma_a s_b s_b s_a s_b) \\
J_{16} &= (s_b s_a \sigma_b \sigma_a \sigma_a s_b s_a \sigma_b) & J_{17} &= (\sigma_b \sigma_b s_b \sigma_a \sigma_a s_b s_a s_a) & J_{18} &= (s_b \sigma_b \sigma_b \sigma_a s_a s_b \sigma_a s_a) \\
J_{19} &= (s_b \sigma_b \sigma_b s_b \sigma_a \sigma_a s_a s_a) & J_{20} &= (s_b s_b \sigma_b \sigma_b \sigma_a s_a s_a \sigma_a) & J_{21} &= (s_b s_a \sigma_a \sigma_a s_a s_b \sigma_b \sigma_b) \\
J_{22} &= (\sigma_b s_b s_b \sigma_a \sigma_a s_a s_a \sigma_b) & J_{23} &= (s_b s_a s_b \sigma_b \sigma_a s_b s_a \sigma_a) & J_{24} &= (s_b s_a \sigma_b \sigma_b s_b s_a \sigma_a \sigma_a) \\
J_{25} &= (\sigma_b s_b s_b \sigma_a s_a s_a \sigma_a \sigma_b) & & & &
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
N_1 &= (\sigma_a s_a \sigma_b \sigma_a s_a s_b \sigma_b \sigma_b) & N_2 &= (\sigma_a s_a \sigma_b \sigma_a s_b s_b s_a \sigma_b) & N_3 &= (\sigma_b \sigma_a s_b s_b \sigma_a s_a s_a \sigma_b) \\
N_4 &= (\sigma_b \sigma_a s_b \sigma_b \sigma_a s_b s_a s_a) & N_5 &= (s_b \sigma_a \sigma_b \sigma_b s_a s_a \sigma_a \sigma_b) & N_6 &= (s_b \sigma_a \sigma_b \sigma_b s_a s_b s_a s_a) \\
N_7 &= (\sigma_a \sigma_b s_b \sigma_a s_a s_b \sigma_b s_a) & N_8 &= (\sigma_a s_a s_b s_b s_a \sigma_a \sigma_b \sigma_b) & N_9 &= (\sigma_a \sigma_b \sigma_b \sigma_a s_b s_b s_a s_a) \\
N_{10} &= (\sigma_a s_a \sigma_b s_b \sigma_a s_a s_a \sigma_b) & N_{11} &= (\sigma_a s_b s_b \sigma_a s_a s_a \sigma_b \sigma_b) & N_{12} &= (\sigma_a \sigma_b \sigma_b \sigma_a s_b s_a s_a \sigma_b) \\
N_{13} &= (\sigma_a s_a s_b \sigma_b s_b s_b \sigma_a \sigma_a) & N_{14} &= (\sigma_a s_a \sigma_b \sigma_b s_b s_b s_a \sigma_a) & N_{15} &= (\sigma_b \sigma_a s_b s_b s_a s_a \sigma_a \sigma_b) \\
N_{16} &= (\sigma_b \sigma_a s_b \sigma_b s_b s_b \sigma_a s_a) & & & &
\end{aligned} \tag{3.18}$$

$$\text{(ii)} \quad (s^2 \sigma \pi, s^2 \sigma \varpi)^1 \Sigma_g^+ \quad \begin{aligned} \mathcal{A}_{11} &= (1 + S_1^2) (\mathcal{D}_1^2 + \mathcal{J}_1^2) & \mathcal{A}_{22} &= \mathcal{A}_{11} - 3\mathcal{J}_1^2/2 & \mathcal{A}_{12} &= -\mathcal{A}_{11}/2 \end{aligned} \tag{3.19}$$

$$\left. \begin{aligned} H_{11} &= Q + 2(\mathcal{J}_1 + \mathcal{J}_2) + 4(\mathcal{N}_1 + \mathcal{N}_2) + \mathcal{A}_1 \\ H_{22} &= Q + 2\mathcal{J}_1 - \mathcal{J}_2 + 4\mathcal{N}_1 - 2\mathcal{N}_2 + \mathcal{A}_2 & H_{12} &= -(H_{11}/2) + \mathcal{A} \end{aligned} \right\} \tag{3.20}$$

where

$$\mathcal{D}_1 = 1 - \delta^2 - S^2 \quad \mathcal{J}_1 = S_0(1 - S^2) - S\delta^2 \tag{3.21}$$

$$Q = (s_a s_a \sigma_a \pi_a s_b s_b \varpi_b, H s_a s_a \sigma_a \pi_a s_b s_b \varpi_b) \tag{3.22}$$

$$\left. \begin{aligned} \mathcal{J}_1 &= J_1 + J_2 + (J_5 + J_6 + J_7 + J_8 + J_9)/2 - (J_{16} + J_{17} + J_{18} + J_{19}) \\ \mathcal{J}_2 &= J_3 + J_4 + (J_{10} + J_{11} + J_{12} + J_{13} + J_{14} + J_{15})/2 - (J_{20} + J_{21} + J_{22} + J_{23}) \end{aligned} \right\} \tag{3.23}$$

$$\mathcal{N}_1 = N_1 + N_2 - (N_5 + N_6 + N_7 + N_8), \quad \mathcal{N}_2 = N_3 + N_4 - (N_5 + N_6 + N_9 + N_{10}) \tag{3.24}$$

and, using notations \mathcal{J} given by (3.11) and

$$\mathcal{D}_2 = 1 - \delta^2 - S_0 S \tag{3.25}$$

\mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A} are given by

$$\begin{aligned}
\mathcal{A}_1 &= -C_{s\pi\pi s} \{ (2 - \delta^2) \mathcal{D}_1 + 2S_0(1 - S^2) \mathcal{J}_1 \} + C_{ss\pi\varpi} \{ S_1 \mathcal{J}_1 (2SS_0 + \delta^2) + 2SS_1 \mathcal{D}_1 \\
&\quad + \delta^2 - 2S^2 \delta \} + 2C_{s\sigma\pi\varpi} (\mathcal{D}_1 + S \mathcal{J}_1) S_1 \delta + 2C_{s\pi\pi\sigma} (S \mathcal{D}_1 + \mathcal{J}_1) \delta - C_{\sigma\pi\pi\sigma} (1 - S^2) \mathcal{D}_1 \\
&\quad - C_{\sigma\sigma\pi\varpi} (1 - S^2) S_1 \mathcal{J}_1 + C_{\pi\varpi\pi\pi} (\mathcal{D}_1 + \mathcal{J}_1^2) + D_{\sigma\sigma s s} \{ 2S(1 + S_1^2) \mathcal{J}_1
\end{aligned}$$

$$\begin{aligned}
& -\delta^2(2+2S^2+S_1^2+4S^2S_1^2)\} + D_{\pi\theta\pi}\{2S\mathcal{D}_1+(2SS_0+\delta^2)\phi_1\}S_1 \\
& + 2D_{\pi\theta\pi}SS_1\delta\phi_1 - D_{\pi\theta\pi}(1-S^2)S_1\phi_1 + D_{\pi\theta\pi}\{\mathcal{D}_1+(1-S^2)S_0^2\} \\
& - 2F_{\theta\theta\pi}\{(2+S_1^2)\mathcal{D}_1+(1-S^2)\phi_1-\delta^2\} - F_{\theta\pi\pi}(1-S^2)\mathcal{D}_1 \\
& + 2L_{\pi\pi\pi}\{2S\mathcal{D}_1+2S\phi_1(1-S^2)+\delta^2\phi_1\} - 2L_{\pi\pi\pi}\{(2-\delta^2)\mathcal{D}_1+2S\phi_1\}S_1 \\
& + 2(L_{\pi\theta\pi}+L_{\pi\theta\pi})(S\mathcal{D}_1+\phi_1)S_1\delta - 2L_{\pi\theta\pi}(\mathcal{D}_1-S\phi_1)\delta - 2L_{\theta\pi\pi}S\phi_1\delta \\
& - 2L_{\theta\pi\pi}(1-S^2)\phi_1 - 2L_{\pi\theta\pi}(1-S^2)S_1\mathcal{D}_1 \quad (3.26) \text{ (a)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_2 = & -C_{\pi\pi\pi}\{(1-\delta^2)\mathcal{D}_1+\phi_1-2\delta^2\mathcal{D}_2\} + C_{\pi\pi\pi}\{2S\delta^2\mathcal{D}_2-S\phi_1+(SS_0+\delta^2)\phi_1\}S_1 \\
& + 2C_{\pi\theta\pi}\{(1-S^2)\mathcal{D}_2-S\phi_1\}S_1\delta + 2C_{\pi\theta\pi}(S\mathcal{D}_1+\phi_1)\delta \\
& - \{C_{\theta\pi\pi}\mathcal{D}_1-2C_{\theta\pi\pi}S_1\phi_1\}(1-S^2) - C_{\theta\pi\pi}\{(1-S^2)\phi_1-\phi_1^2\}/2 \\
& + D_{\theta\theta\pi}\{SS_1^2\phi_1+(1-S^2)(1-S_1^2)(\delta^2-SS_0)+\delta^2-2S^2\delta\} \\
& - D_{\pi\theta\pi}\{S\phi_1-SS_0\phi_1+2(S_0-S)\delta^2\}S_1 - 2D_{\pi\theta\pi}(\mathcal{D}_1-S\phi_1)S_1\delta \\
& - D_{\pi\theta\pi}(1-S^2)S_1\phi_1 + D_{\pi\theta\pi}\{\mathcal{D}_1^2+(1-S^2)\phi_1\}/2 \\
& - F_{\theta\theta\pi}\{2+\phi_1+(1-S^2)(1+S_1^2)-\delta^2(1+2S^2-2SS_0)\} \\
& + 2F_{\theta\pi\pi}(1-S^2)\mathcal{D}_1 + 2L_{\pi\pi\pi}\{S\mathcal{D}_1-S\delta^2\mathcal{D}_2+S\phi_1+(S_0-S)\delta^2\} \\
& - 2L_{\pi\pi\pi}\{S_1\delta^2\mathcal{D}_2-\phi_1+S_0\phi_1-(1+S_0)S^2S_1\delta^2\}S_1 \\
& - 2L_{\pi\theta\pi}\{\phi_1+(1-S^2)(S_0-S)\}S_1\delta + 2L_{\pi\theta\pi}(S\mathcal{D}_1+\phi_1)S_1\delta \\
& - 2L_{\pi\theta\pi}(\mathcal{D}_1-S\phi_1)\delta + 2L_{\theta\pi\pi}\{\mathcal{D}_1+(1-S^2)\mathcal{D}_2\}\delta - 2L_{\theta\pi\pi}(1-S^2)\phi_1 \\
& - 2L_{\pi\theta\pi}(1-S^2)S_1\mathcal{D}_1 \quad (3.26) \text{ (b)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A} = & -(3/2)C_{\pi\pi\pi}\delta^2 - C_{\pi\pi\pi}\{(\mathcal{D}_2-\delta)SS_1+S^2+S_0S_1\delta+(\phi_1-\delta)(S_1\delta/2)\}\delta \\
& + C_{\pi\theta\pi}\{S^2\mathcal{D}_2+(S_0-S)S+2S\phi_1\}S_1\delta \\
& - C_{\pi\theta\pi}\{2S\mathcal{D}_1+\phi_1+(1-S^2)(S_0-S)\}\delta \\
& + (3/2)C_{\theta\pi\pi}(1-S^2)\mathcal{D}_1 - (3/2)C_{\theta\pi\pi}(1-S^2)S_1\phi_1 - D_{\theta\theta\pi}(S^2+S_1^2/2)\delta^2 \\
& + (3/2)D_{\pi\theta\pi}S_1\phi_1\delta^2 + D_{\pi\theta\pi}\{\mathcal{D}_1+(1-S^2)\mathcal{D}_2-2S\phi_1\}S_1\delta \\
& + (3/2)D_{\pi\theta\pi}(1-S^2)S_1\phi_1 + F_{\theta\theta\pi}\mathcal{D}_1\delta^2 - (3/2)F_{\theta\pi\pi}(1-S^2)\mathcal{D}_1 \\
& - L_{\pi\pi\pi}(1-S_1)\{\mathcal{D}_2+(S-S_0)S\}\delta^2 \\
& + (L_{\pi\theta\pi}-L_{\pi\theta\pi})\{S\mathcal{D}_1+2\phi_1+(1-S^2)(S_0-S)\}S_1\delta \\
& - L_{\pi\theta\pi}\{(1-S^2)\mathcal{D}_2+S\phi_1\}\delta - L_{\theta\pi\pi}\{\mathcal{D}_1+(1-S^2)\mathcal{D}_2+S\phi_1\}\delta \quad (3.26) \text{ (c)}
\end{aligned}$$

In order to explain J_k and N_k the abbreviation such as

$$(s_a s_b s_c \pi_a s_b s_b \pi_b, \text{H } s_a s_b s_b \pi_a s_a s_b \pi_b) = (s_a s_b s_b \pi_a s_a s_b \pi_b) \quad (3.27)$$

is used in the case (ii), where the meaning of the symbol is similar as is in the preceding case.

$$\begin{aligned}
J_1 &= (s_a s_b s_b \pi_a \sigma_a s_b \sigma_b \varpi_b) & J_2 &= (\sigma_b s_a s_b \pi_b s_a s_b \sigma_a \varpi_a) & J_3 &= (\sigma_b s_a s_b \pi_a s_a s_b \sigma_a \varpi_b) \\
J_4 &= (s_a s_b s_b \pi_b \sigma_a s_b \varpi_a) & J_5 &= (s_a \sigma_b s_b \pi_a \sigma_b s_a \varpi_b) & J_6 &= (s_b s_b \sigma_a \pi_a s_a s_b \varpi_b) \\
J_7 &= (s_a s_a \sigma_b \pi_b s_b s_b \sigma_a \varpi_a) & J_8 &= (s_b s_b \sigma_b \pi_b s_a \sigma_a \varpi_a) & J_9 &= (\sigma_b s_b s_b \pi_b s_a \sigma_a \varpi_a) \\
J_{10} &= (s_a s_a \sigma_b \pi_a s_b s_b \sigma_a \varpi_b) & J_{11} &= (s_b s_b \sigma_b \pi_a s_a s_a \sigma_a \varpi_b) & J_{12} &= (\sigma_b s_b s_b \pi_a \sigma_a s_a s_a \varpi_b) \\
J_{13} &= (s_a s_a \sigma_a \pi_b s_b s_b \sigma_b \varpi_a) & J_{14} &= (s_b s_b \sigma_a \pi_b s_a s_a \sigma_b \varpi_a) & J_{15} &= (s_a \sigma_b s_b \pi_b \sigma_a s_b s_a \varpi_a) \\
J_{16} &= (s_a \sigma_b \sigma_a \pi_a s_b s_b s_a \varpi_b) & J_{17} &= (s_a s_b \sigma_a \pi_a s_b s_a \sigma_b \varpi_b) & J_{18} &= (s_b s_a \sigma_b \pi_b s_a s_b \sigma_a \varpi_a) \\
J_{19} &= (\sigma_b s_b s_b \pi_b s_a s_a \sigma_a \varpi_a) & J_{20} &= (s_b s_a \sigma_b \pi_a s_a s_b \sigma_a \varpi_b) & J_{21} &= (\sigma_b s_b s_b \pi_a s_a s_a \sigma_a \varpi_b) \\
J_{22} &= (s_b s_a \sigma_a \pi_b s_a s_b \sigma_b \varpi_a) & J_{23} &= (s_a s_a s_b \pi_b \sigma_a s_b \sigma_b \varpi_a) & & (3.28)
\end{aligned}$$

$$\begin{aligned}
N_1 &= (s_a s_b \sigma_a \pi_a \sigma_b s_a \varpi_b) & N_2 &= (s_b s_a \sigma_b \pi_b s_a \sigma_a s_b \varpi_a) & N_3 &= (\sigma_a \sigma_b \sigma_b \pi_a s_b s_a s_a \varpi_b) \\
N_4 &= (\sigma_a s_a s_b \pi_b s_a s_b \sigma_b \varpi_a) & N_5 &= (\sigma_a \sigma_b s_b \pi_a s_a s_b s_a \varpi_b) & N_6 &= (\sigma_b s_a s_b \pi_b s_a \sigma_a s_b \varpi_a) \\
N_7 &= (\sigma_a s_b s_b \pi_a s_a s_a \sigma_b \varpi_b) & N_8 &= (\sigma_b s_a s_a \pi_b s_b s_b \sigma_a \varpi_a) & N_9 &= (\sigma_a s_a \sigma_b \pi_a s_b s_b s_a \varpi_b) \\
N_{10} &= (\sigma_a s_b s_b \pi_b s_a s_a \sigma_b \varpi_a) & & & & (3.29)
\end{aligned}$$

The integrals Q , J_k and N_k can be expressed in terms of those defined by (3.7), (3.8), and (3.9). Since their complete explanation is too lengthy we here give some typical examples:

$$\begin{aligned}
Q &= (2x^2/3) + (16/R) - 8(x + K_{\sigma\sigma} + K_{\pi\pi}) - 16K_{ss} + 4(D_{ssss} + D_{ss\sigma\sigma} + D_{\pi\pi\sigma\pi}) \\
&\quad + 2D_{\sigma\pi\sigma\pi} + D_{\sigma\sigma\sigma\sigma} + D_{\pi\pi\pi\pi} + 10F_{ssss} + 2F_{\sigma\pi\sigma\pi} & (3.30)
\end{aligned}$$

$$\begin{aligned}
J_{16} &= \{ (x^2/4) + (16/R) - 6x - 12K_{ss} - 4K_{\sigma\sigma} - 8K_{\pi\pi} \} \delta^2 - 2xJ_{\sigma\sigma}\delta \\
&\quad + (2D_{ssss} + 2D_{ss\sigma\sigma} + 3D_{ss\pi\pi} + D_{\pi\pi\pi\pi} + D_{\sigma\pi\sigma\pi} + 5F_{ssss} + F_{\sigma\pi\sigma\pi}) \delta^2 & (3.31) (a)
\end{aligned}$$

$$\begin{aligned}
J_6 &= \{ (16/R) - 4x - 8K_{\sigma\sigma} - 8K_{\pi\pi} \} S^4 + 4(xJ_{ss} - 8J_{ss} - I_{ss}) S^3 + 6C_{ssss} S^2 \\
&\quad + (D_{\sigma\sigma\sigma\sigma} + D_{\pi\pi\pi\pi} + 2D_{\sigma\pi\sigma\pi} + 2F_{\sigma\pi\sigma\pi}) S^4 + 8(L_{ssss} + L_{\pi\pi\pi\pi}) S^3 & (3.31) (b)
\end{aligned}$$

$$\begin{aligned}
J_{11} &= \{ -(x^2/2) + (16/R) - 2x - 8K_{\pi\pi} + D_{\pi\pi\pi\pi} \} S^4 S_0^2 + 4(xJ_{ss} - 8J_{ss} - I_{ss} + 2L_{\pi\pi\pi\pi}) S^3 S_0^2 \\
&\quad + 2(xJ_{\sigma\sigma} - 8J_{\sigma\sigma} + 2L_{\pi\pi\sigma\pi}) S^4 S_0 + 6C_{ssss} S^2 S_0^2 + 8C_{ss\sigma\sigma} S^3 S_0 + C_{\sigma\sigma\sigma\sigma} S^4 & (3.31) (c)
\end{aligned}$$

$$\begin{aligned}
J_8 &= \{ -x^2 + (16/R) \} S^4 S_0^2 S_1^2 + 4(xJ_{ss} - 8J_{ss} - I_{ss}) S^3 S_0^2 S_1^2 + 2(xJ_{\sigma\sigma} - 8J_{\sigma\sigma}) S^4 S_0 S_1^2 \\
&\quad + 2(xJ_{\pi\pi} - 8J_{\pi\pi}) S^4 S_0^2 S_1 + (6C_{ssss} S^2 S_0^2 + 8C_{ss\sigma\sigma} S^3 S_0 + C_{\sigma\sigma\sigma\sigma} S^4) S_1^2 \\
&\quad + 8C_{ss\pi\pi} S^3 S_0^2 S_1 + 4C_{\sigma\pi\sigma\pi} S^4 S_0 S_1 + C_{\pi\pi\pi\pi} S^4 S_0^2 & (3.31) (d)
\end{aligned}$$

$$N_1 = (4K_{ss} + D_{ssss} + D_{\sigma\sigma\sigma\sigma} + D_{\pi\pi\pi\pi}) S\delta - L_{ssss}\delta + L_{\sigma\sigma\sigma\sigma}S \quad (3.32) (a)$$

$$N_3 = (-4K_{ss} + D_{ssss} + D_{ss\sigma\pi}) S^2 S_0\delta + 2L_{ssss} S S_0\delta + (L_{ss\sigma\sigma}\delta + L_{ssss} S_0) S^2 \quad (3.32) (b)$$

$$N_{10} = (4K_{ss} - D_{ss\sigma\sigma}) S^3 S_1^2\delta + (L_{ss\sigma\sigma} S - 3L_{ssss}\delta) S^2 S_1^2 - 2L_{ss\sigma\pi} S^3 S_1\delta \quad (3.32) (c)$$

In these examples J_{16} , J_6 , J_{11} , and J_8 are of the second, fourth, sixth, and eighth degrees in overlapping integrals respectively, and N_1 , N_3 , and N_{10} are of the

second, fourth, and sixth degrees respectively.

$$(iii) \quad (s^2\sigma\pi, s^2\pi\varpi)^1\Pi_u$$

$$\begin{aligned} A_{11} &= (1+S_1^2)(1-S^2)(\mathcal{D}_1-S_1\phi_1) & A_{12} &= -A_{11}/2 \\ A_{22} &= (1-S^2)\{(1-S_1^2/2)\mathcal{D}_1+(1-2S_1^2)S_1\phi_1/2 \end{aligned} \quad (3.33)$$

$$H_{11}=Q+2\mathcal{J}_1+\mathcal{J}_2+2\mathcal{H}_1+\mathcal{H}_2+\mathcal{O}_1$$

$$H_{12}=Q+2\mathcal{J}_1-\mathcal{J}_2+2\mathcal{H}_1-\mathcal{H}_2+\mathcal{O}_2$$

$$H_{12}=- (Q/2)-\mathcal{J}_1-\mathcal{J}_2-\mathcal{H}_1-\mathcal{H}_2+\mathcal{O} \quad (3.34)$$

where

$$Q=(s_a s_a \sigma_a \pi_a s_b s_b \pi_b \varpi_b, H s_a s_a \sigma_a \pi_a s_b s_b \pi_b \varpi_b) \quad (3.35)$$

$$\mathcal{J}_1=J_1+(J_3+J_4+J_5)/2-J_{10}-(J_{12}+J_{13}+J_{14}+J_{15})/2$$

$$\mathcal{J}_2=J_2+(J_6+J_7+J_8+J_9)/2-J_{11}-(J_{16}+J_{17}+J_{18}+J_{19})/2 \quad (3.36)$$

$$\mathcal{H}_1=N_1+N_2-(N_5+N_6) \quad \mathcal{H}_2=N_3+N_4-(N_7+N_8) \quad (3.37)$$

$$J_1=(s_a s_b \sigma_b \pi_b s_a \pi_a \varpi_a) \quad J_2=(s_b s_a \sigma_b \pi_a s_a s_b \pi_b \varpi_a) \quad J_3=(s_a s_b s_b \pi_a \sigma_a s_a \pi_b \varpi_b)$$

$$J_4=(\sigma_b s_b s_b \pi_b s_a \pi_a \varpi_a) \quad J_5=(s_b s_b \sigma_a \pi_a s_a s_b \pi_b \varpi_b) \quad J_6=(s_a s_a \sigma_a \pi_b s_b s_b \pi_a \varpi_b)$$

$$J_7=(s_b s_b \sigma_a \pi_b s_a s_a \pi_a \varpi_b) \quad J_8=(s_b s_a s_b \pi_b s_a \sigma_a \pi_a \varpi_b) \quad J_9=(\sigma_b s_b s_b \pi_a s_a s_a \pi_b \varpi_a)$$

$$J_{10}=(s_b s_a \sigma_a \pi_a s_a s_b \pi_b \varpi_b) \quad J_{11}=(s_a s_b \sigma_a \pi_b s_b s_a \pi_a \varpi_b) \quad J_{12}=(s_a s_a \sigma_b \pi_b s_b s_b \pi_a \varpi_a)$$

$$J_{13}=(s_b s_b \sigma_b \pi_b s_a s_a \pi_a \varpi_a) \quad J_{14}=(\sigma_b s_a s_b \pi_b s_a s_b \pi_a \varpi_a) \quad J_{15}=(s_a s_a s_b \pi_a \sigma_a s_b \pi_b \varpi_b)$$

$$J_{16}=(s_a s_a \sigma_b \pi_b s_b s_a \pi_a \varpi_b) \quad J_{17}=(s_a s_a \sigma_b \pi_a s_b s_b \pi_b \varpi_a) \quad J_{18}=(s_b s_b \sigma_b \pi_a s_a s_a \pi_b \varpi_a)$$

$$J_{19}=(\sigma_b s_a s_b \pi_a s_a s_b \pi_b \varpi_a) \quad (3.38)$$

$$N_1=(s_a \sigma_a s_b \pi_a s_b s_a \pi_b \varpi_b) \quad N_2=(s_a \sigma_b s_a \pi_b s_b s_b \pi_a \varpi_a) \quad N_3=(s_a \sigma_a s_b \pi_b s_b s_a \pi_a \varpi_b)$$

$$N_4=(s_a \sigma_b s_a \pi_a s_b s_b \pi_b \varpi_a) \quad N_5=(s_b \sigma_a s_b \pi_a s_a s_a \pi_b \varpi_b) \quad N_6=(s_b \sigma_b s_a \pi_b s_a s_b \pi_a \varpi_a)$$

$$N_7=(s_b \sigma_a s_b \pi_b s_a s_a \pi_a \varpi_b) \quad N_8=(s_b \sigma_b s_a \pi_a s_a s_b \pi_b \varpi_a) \quad (3.39)$$

$$\begin{aligned} \mathcal{O}_1 &= -C_{\pi s \pi} \{ 2\mathcal{D}_1 + (1-S^2)(1+S_1^2/2) - S_1\phi_1 - (1+S^2+S_1^2+SS_1+2S^2S_1^2)(\delta^2/2) \} \\ &\quad + C_{ss\pi\varpi} \{ (\mathcal{D}_1 - 2S_1\phi_1)SS_1 + (1-S^2)SS_0S_1^2 + (S\phi_1/2) \\ &\quad + (4+S_1^2-SS_1-S^2S_1^2)(\delta^2/2) \} - C_{s\pi\pi\sigma}(1-S^2)(2SS_1^2+S+S_1/2)\delta \\ &\quad + C_{s\sigma\pi\varpi}(1-S^2)(1-2SS_1)S_1\delta/2 - C_{\sigma\pi\pi\sigma}(1-S^2)^2(1-2S_1^2)/2 \\ &\quad + C_{\sigma\sigma\pi\varpi}(1-S^2)^2S_1^2/2 + C_{\pi\varpi\varpi\pi}(1-S^2)(S_1\phi_1-\mathcal{D}_1)/2 \\ &\quad - D_{s\sigma s}(1-S^2)(1+2S_1^2)SS_1/2 \\ &\quad - D_{s\pi\pi\pi} \{ (2S_1^2+1/2)S\phi_1 - (\mathcal{D}_1+\delta^2/2)SS_1 + (1-S^2)(\delta^2/2-S_0S)S_1^2 + 2S^2\delta^2 \} \\ &\quad + D_{\pi\sigma\sigma\pi}(1-S^2)(4-S_1^2)S\delta + D_{\pi\sigma\sigma\pi}(1-S^2)^2(4-S_1^2)/2 - D_{\pi\varpi\varpi\pi}(1-S^2)S_1\phi_1/2 \end{aligned}$$

$$\begin{aligned}
& -F_{s\sigma s} \{ (5/2) + (4+S_1) \mathcal{D}_1 + (1+S+S_1^2-SS_1) \delta^2/2 - S_1 \phi_1 \} \\
& -F_{s\pi\pi\sigma} (1-S^2) \{ \phi_1 + (1-S^2)/2 \} \\
& -L_{s\pi\pi s} \{ 4S \mathcal{D}_1 + (1-S^2)(2+S_1^2)S - 2SS_1 \phi_1 - (4S+S_1+6SS_1^2+S_1S^2)(\delta^2/2) \} \\
& -L_{\pi s\pi s} \{ (1+4S_1^2) \phi_1 + 2(1-S^2)S_0S_1^2 - 2S_1 \mathcal{D}_1 \} - L_{s\pi\pi s} (1-S^2)(2-SS_1)\delta/2 \\
& -L_{\pi s\pi s} (1-S^2)(2S_1-S)S_1\delta/2 + L_{s\pi\pi\sigma} (1-S^2)(2-4S_1^2-SS_1)\delta/2 \\
& + L_{\pi s\pi\sigma} (1-S^2)(8+SS_1-2S_1^2)\delta/2 + (L_{\pi\pi\pi\sigma} - L_{\pi\sigma\pi\pi}/2)(1-S^2)^2S_1 \\
& -L_{\pi\pi\pi\pi} (1-S^2)(S_1 \mathcal{D}_1 - \phi_1) \quad (3.40) (a)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_2 = & -C_{s\pi s\pi} \{ 2\mathcal{D}_1 + (1-S^2)(1-S_1^2/2) + (S_1-S/2+S_1^2) \phi_1 - (2-S^2-S_1^2/2 \\
& + SS_1+S^2S_1^2) \delta^2 \} - C_{ss\pi\pi} \{ SS_1 \mathcal{D}_1 - (1-4S_1^2)S \phi_1/2 + (1-S^2)SS_0S_1^2 \\
& - (2+2SS_1+4S^2S_1^2-S^2-S_1^2)\delta^2/2 \} + C_{s\pi\pi\sigma} (1-S^2)(2S+S_1-2SS_1^2)\delta \\
& + C_{s\sigma\pi\pi} (1-S^2)(1+2SS_1)S_1\delta + \{ C_{s\pi\pi\sigma} (1-S_1^2) - C_{s\sigma\pi\pi} S_1^2 \} (1-S^2)^2 \\
& - C_{\pi\pi\pi\pi} (1-S^2)(\mathcal{D}_1 - 2S_1 \phi_1)/2 + D_{s\sigma s} (1-S^2)(1-2S_1^2)SS_1 \\
& + D_{\pi\sigma s\pi} (1-S^2)\{ 2S(1-S_1^2) - S_1 \} - D_{s\pi\pi s} \{ SS_1 \mathcal{D}_1 + 2SS_1^2 \phi_1 \\
& + (S^2-S_1^2-SS_1-S^2S_1^2) \delta^2 \} - D_{\pi\sigma\pi\pi} (1-S^2)^2(1-S_1^2) + D_{\pi\pi\pi\pi} (1-S^2)S_1 \phi_1/2 \\
& - F_{s\sigma s} \{ 2\mathcal{D}_1 + (1-S^2)(2+S_1S_0-S_1^2) + (S^2+S_1^2/2)\delta^2 \} - F_{s\pi\pi\sigma} (1-S^2)(\mathcal{D}_1 - \delta^2) \\
& + L_{s\pi\pi s} \{ 4S \mathcal{D}_1 + (1-S^2)(2-S_1^2)S - 2SS_1 \phi_1 - (2S+S_1/2+SS_1^2+S^2S_1/2)\delta^2 \} \\
& + L_{\pi s\pi s} \{ 2S_1 \mathcal{D}_1 + 2(1-S^2)S_0S_1^2 - (1-4S_1^2) \phi_1 + (2S+S_1+S^2S_1-2SS_1^2)\delta^2 \} \\
& + L_{s\pi\pi s} (1-S^2)(2-SS_1)\delta + L_{\pi\sigma\pi\pi} (1-S^2)(2S_1+S)S_1\delta \\
& + L_{s\pi\pi\sigma} (1-S^2)(2-SS_1-2S_1^2)\delta + L_{\pi s\sigma\pi} (1-S^2)(2+SS_1-2S_1^2)\delta \\
& + (L_{s\pi\pi\sigma} - L_{\pi\sigma\pi\pi})(1-S^2)S_1 + L_{\pi\pi\pi\pi} (1-S^2)(\mathcal{D}_1 - \phi_1)S_1 \quad (3.40) (b)
\end{aligned}$$

$$\begin{aligned}
\mathcal{C} = & C_{\pi s\pi s} \{ \mathcal{D}_1 + (1-S^2)(1+S_1^2)/2 - S_1 \phi_1 - (S+S_1)S_1\delta^2/2 \} \\
& + C_{ss\pi\pi} \{ (1+2S_1^2)S \phi_1 + (1-S^2)SS_0S_1^2 - (2\mathcal{D}_1 + \delta^2)SS_1 \}/2 \\
& - C_{s\sigma\pi\pi} (1-S^2)(1-2SS_1)S_1\delta/2 + (C_{s\pi\pi\sigma} \delta - C_{\pi\pi\pi\pi} \phi_1)(1-S^2)S_1/2 \\
& - C_{s\sigma\pi\pi} (1-S^2)^2S_1^2/2 + D_{s\sigma s} (1-S^2)(1+S_1^2)SS_1/2 \\
& + D_{s\pi\pi s} \{ (1+2S_1^2)S \phi_1 + (1-S^2)(SS_0 + \delta^2)S_1^2 - (2\mathcal{D}_1 + \delta^2)SS_1 \}/2 \\
& + F_{s\sigma s} \{ \mathcal{D}_1 + (1-S^2)(1+S_1^2) - S_1 \phi_1 - (1+S_1^2+SS_1-S^2)\delta^2/2 \} \\
& + F_{s\pi\pi\sigma} (1-S^2)(1+2\mathcal{D}_1-S^2)/2 - L_{s\pi\pi s} \{ 2S \mathcal{D}_1 + (1-S^2)(1+S_1^2)S - 2SS_1 \phi_1 \\
& - (S_1+2SS_1^2+S_1S^2)\delta^2/2 \} - L_{\pi s\pi s} \{ (1+2S_1^2) \phi_1 - 2S_1 \mathcal{D}_1 + (1-S^2)S_0S_1^2 \\
& + (1+S^2)S_1\delta^2/2 \} + L_{s\sigma\pi\pi} (1-S^2)(2-SS_1)\delta/2 + L_{\pi s\pi\pi} (1-S^2)(2S_1-S)S_1\delta/2 \\
& + (L_{s\pi\pi\sigma} - L_{s\sigma\pi\pi})(1-S^2)SS_1\delta/2 + (L_{\pi\sigma\pi\pi} - L_{s\pi\pi\sigma})(1-S^2)^2S_1/2
\end{aligned}$$

$$+L_{\pi\pi\pi\pi}(1-S^2)(S_1\mathcal{D}_1-\mathcal{E}_1)/2 \quad (3.40)(c)$$

$$(iv) \quad (s^2\sigma\pi, s^2\sigma\pi)^1\mathcal{D}_g$$

$$\mathcal{A}_{kj}=\mathcal{A}_{kj}^{(ii)}, \quad H_{kj}=H_{kj}^{(ii)}+E_{kj} \quad (k, j=1, 2) \quad (3.41)$$

where $\mathcal{A}_{kj}^{(ii)}$ and $H_{kj}^{(ii)}$ mean \mathcal{A}_{kj} and H_{kj} in the case (ii) respectively, and E_{kj} represents small deviations as follows:

$$E_{11}=-C_{\pi\pi\pi\pi}(D_1^2S^2+S_1^2)-D_{\pi\pi\pi\pi}\{D_1^2+S_0^2(1-S^2)^2\}$$

$$E_{22}=C_{\pi\pi\pi\pi}\{(1-S^2)S-S_1^2/2\}, \quad E_{12}=-E_{11}/2 \quad (3.42)$$

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Electronic States of C_2 -Molecule, III*Numerical Values and Reduction Formulas of Integrals*

Gentaro ARAKI and Wataro WATARI
Faculty of Engineering, Kyoto University

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Introduction

In the first⁽¹⁾ and second⁽²⁾ parts of this paper we discussed the electronic energy of the C_2 -molecule. The numerical evaluation of various integrals were carried out by making use of Kotani, Amemiya, and Simose's table.⁽³⁾ In order to make use of the table for our calculation we had to prepare some supplementary formulas and tables for the integrals. We think some of these formulas and tables are useful for other calculation. For this reason they will be given in what follows.

§ 1. Definitions

The definition of orbitals and integrals in our calculation are somewhat different from Kotani's. Therefore an account for this point will here be given. The orbitals in our calculation are defined by

$$\begin{aligned} s &= N_s r \exp(-xr/2) & \pi &= N_\pi (x+iy) \exp(-xr/2) \\ \sigma &= N_\sigma z \exp(-xr/2) & \varpi &= N_\pi (x-iy) \exp(-xr/2) \end{aligned} \quad (1.1)$$

where the origin of coordinates is at an atomic nucleus. We shall refer to this origin as the origin of orbitals. Two nuclei are distinguished by a and b . If an origin of a π -orbital, for example, is at the a -nucleus we denote the orbital by π_a . In order to account for the relation between integrals in our definition and Kotani's it is convenient to introduce new orbitals defined by

$$\xi = 2^{-1/2}(\pi + \varpi) \quad \eta = 2^{-1/2}(\pi - \varpi)/i \quad (1.2)$$

The angular part of these orbitals are proportional to x and y respectively. If we put $x=2\delta$ we can represent orbitals in Kotani's definition as follows:

$$\begin{aligned} a_s &= s_a & b_s &= s_b & a_\sigma &= \sigma_a & b_\sigma &= -\sigma_b \\ a_\pi &= \xi_a & b_\pi &= \xi_b & a_{\pi'} &= \eta_a & b_{\pi'} &= \eta_b \end{aligned} \quad (1.3)$$

We define the Hermitian inner product of two functions by

$$(\varphi, \psi) = \int \int_{-\infty}^{+\infty} \varphi^*(q_1, q_2, \dots, q_m) \psi(q_1, q_2, \dots, q_m) dq_1 dq_2 \dots dq_m \quad (1.4)$$

where q_1, q_2, \dots, q_m ($m=3$, or 6) stand for x, y, z or $x_1, y_1, z_1, x_2, y_2, z_2$. Further a product of two orbitals is abbreviated in such a way that

$$\varphi\psi = \varphi(x_1, y_1, z_1)\psi(x_2, y_2, z_2) \quad (1.5)$$

In this way we define various integrals as follows:

$$S = (s_a, s_b) \quad \delta = (s_a, \sigma_b) \quad (1.6)$$

$$S_0 = (\sigma_a, \sigma_b) \quad S_1 = (\pi_a, \pi_b) = (\varpi_a, \varpi_b)$$

$$I_{\varphi\psi} = (\varphi_a, r_b^{-2}\psi_b) \quad J_{\varphi\psi} = (\varphi_a, r_b^{-1}\psi_b) \quad K_{\varphi\psi} = (\varphi_a, r_b^{-1}\psi_a) \quad (1.7)$$

$$C_{\varphi\psi\chi\zeta} = (\varphi_a\psi_b, r_{12}^{-1}\chi_b\zeta_a)$$

$$D_{\varphi\psi\chi\zeta} = (\varphi_a\psi_b, r_{12}^{-1}\chi_a\zeta_b)$$

$$F_{\varphi\psi\chi\zeta} = (\varphi_a\psi_a, r_{12}^{-1}\chi_a\zeta_a) \quad (1.8)$$

$$L_{\varphi\psi\chi\zeta} = (\varphi_a\psi_a, r_{12}^{-1}\chi_a\zeta_b)$$

where r_b is a distance between an electron and the b -nucleus, r_{12} is a distance between the first and second electrons, and φ, ψ, χ , or ζ stands for one of s, σ, π , and ϖ .

The integrals defined by (1.6) and (1.7) have the general form $(\varphi_c, \mathcal{J}\psi_d)$ where c or d is one of a and b , and \mathcal{J} denotes $1, r_b^{-1}$ or r_b^{-2} . This integral will be denoted by $(\varphi\psi)$ for the sake of simplicity. The axial quantum numbers of $\mathcal{J}\psi_d$ and ψ_d are the same. Therefore φ_c and $\mathcal{J}\psi_d$ are orthogonal unless the axial quantum numbers of φ_c and ψ_d are the same. Thus non-vanishing integrals are only $(ss), (s\sigma), (\sigma s), (\sigma\sigma), (\pi\pi) = (\varpi\varpi)$, and the remaining integrals all vanish. For this reason we obtain, by making use of (1.2), the following relations between integrals in our and Kotani's definitions:

$$(\xi\xi) = (\eta\eta) = (\pi\pi) = (\varpi\varpi) \quad (1.9)$$

Our $S, \delta, S_0, S_1, J_{\sigma\sigma}$, and $J_{\pi\pi}$ are equal to Kotani's $S_{ss}, -S_{s\sigma}, -S_{\sigma\sigma}, S_{\pi\pi}, J_{\sigma\sigma}$ and $-J_{\sigma\sigma}$ respectively.

The integrals defined by (1.8) can be written in the form $(\varphi_c\psi_d, r_{12}^{-1}\chi_p\zeta_q)$ where c, d, p , or q is one of a and b . This integral will be denoted by $(\varphi\psi\chi\zeta)$. It vanishes unless the axial quantum number of $\varphi\psi$ and $\chi\zeta$ are equal, for the same reason as in the case of $(\varphi\psi)$. We have therefore

$$\begin{aligned} (\xi\xi\xi\xi) &= \{(\pi\pi\pi\pi) + (\pi\varpi\pi\varpi) + (\pi\varpi\varpi\pi)\} / 2 \\ (\xi\eta\xi\eta) &= \{(\pi\pi\pi\pi) + (\pi\varpi\pi\varpi) - (\pi\varpi\varpi\pi)\} / 2 \\ (\xi\eta\eta\xi) &= \{(\pi\pi\pi\pi) - (\pi\varpi\pi\varpi) + (\pi\varpi\varpi\pi)\} / 2 \end{aligned} \quad (1.10)$$

The axial quantum numbers of $\varpi_c^*(\mathbf{x})\varpi_d(\mathbf{x})$ and $\pi_c^*(\mathbf{x})\pi_d(\mathbf{x})$ both vanish, namely they are axially symmetrical with respect to the molecular axis where \mathbf{x} denotes a position vector of an electron. Therefore they are the same function. For this reason the three equations in (1.10) reduce to

$$\begin{aligned}
 (\xi\xi\xi\xi) &= (\pi\pi\pi\pi) + (\pi\varpi\varpi\pi)/2 \\
 (\xi\eta\xi\eta) &= (\pi\pi\pi\pi) - (\pi\varpi\varpi\pi)/2 \\
 (\xi\eta\eta\xi) &= (\pi\varpi\varpi\pi)/2
 \end{aligned} \tag{1.11}$$

and the inverse transformation is given by

$$\begin{aligned}
 (\pi\pi\pi\pi) &= (\pi\varpi\pi\varpi) = (\xi\xi\xi\xi) - (\xi\eta\eta\xi) \\
 &= (\xi\eta\xi\eta) + (\xi\eta\eta\xi) \\
 &= \{ (\xi\xi\xi\xi) + (\xi\eta\xi\eta) \} / 2
 \end{aligned} \tag{1.12} (a)$$

$$(\pi\varpi\varpi\pi) = (\xi\xi\xi\xi) - (\xi\eta\eta\xi) = 2(\xi\eta\eta\xi) \tag{1.12} (b)$$

Further if α and β represent s or σ we have

$$\begin{aligned}
 (\xi\alpha\xi\beta) &= (\eta\alpha\eta\beta) = (\pi\alpha\pi\beta) \\
 (\alpha\xi\xi\beta) &= (\alpha\eta\eta\beta) = (\alpha\pi\pi\beta)
 \end{aligned} \tag{1.13}$$

§ 2. Reduction Formulas

In this section we give reduction formulas which were not written in the table of Kotani, Amemiya and Simose.³⁾

$$(i) \quad F_{ssss} = F_{soss} = F_{s\pi s\pi} = F_0$$

$$F_{soos} = F_0 + 4F_2 \quad F_{s\pi o\pi} = F_0 - 2F_2$$

$$F_{s\pi s\pi} = F_0 + F_2 \quad F_{\pi\varpi\varpi\pi} = 3F_2$$

$$F_{soos} = F_{s\pi s\pi} = F_1 \quad F_{\pi o\sigma\pi} = 6F_2$$

where F_0 , F_1 , and F_2 are Slater's⁽¹¹⁾ F_k and their numerical values are given by

$$F_0 = 0.181640625 \kappa \quad F_1 = 0.040147569 \kappa$$

$$F_2 = 0.0035156250 \kappa$$

$$(ii) \quad I_{ss} = \frac{\kappa^2}{24} \left(\frac{\kappa R}{2} \right)^3 \left\{ \frac{1}{3} A_0 \left(\frac{\kappa R}{2} \right) + A_2 \left(\frac{\kappa R}{2} \right) \right\}$$

where R is a distance between two nuclei and $A_n(x)$ is a function defined by Eq. (10) of Kotani, Amemiya, and Simose.³⁾

(iii) In the following reduction formulas* for $L_{\varphi\psi\chi\xi}$ the functions $G_\tau^\nu(m, n, \kappa R/2)$ and $W_\tau^\nu(m, n, \kappa R/2)$ are written in the abbreviated form $G_\tau^\nu(m, n)$ and $W_\tau^\nu(m, n)$ respectively. They are defined respectively by Eqs. (19) and (17) of Kotani, Amemiya and Simose.³⁾

* In the private letter E. Ishiguro, K. Hijikata, and T. Nakamura communicated us that they obtained the same result.

$$\begin{aligned}
L_{ssss} = & \frac{4x}{9} \left(\frac{xR}{4} \right)^9 \left[\left\{ \frac{1}{5} W_0(0,4) - \frac{2}{3} W_0(2,4) + W_0(4,4) \right\} G_0(0) \right. \\
& + \left\{ \frac{2}{5} W_0(0,3) - \frac{4}{3} W_0(2,3) + 2 W_0(4,3) \right\} G_0(1) \\
& - \left\{ \frac{2}{5} W_0(0,1) - \frac{4}{3} W_0(2,1) + 2 W_0(4,1) \right\} G_0(3) \\
& - \left\{ \frac{1}{5} W_0(0,0) - \frac{2}{3} W_0(2,0) + W_0(4,0) \right\} G_0(4) \\
& + \left\{ \frac{4}{7} W_2(0,4) - \frac{4}{3} W_2(2,4) \right\} G_2(0) + \left\{ \frac{8}{7} W_2(0,3) - \frac{8}{3} W_2(2,3) \right\} G_2(1) \\
& - \left\{ \frac{8}{7} W_2(0,1) - \frac{8}{3} W_2(2,1) \right\} G_2(3) - \left\{ \frac{4}{7} W_2(0,0) - \frac{4}{3} W_2(2,0) \right\} G_2(4) \\
& + \frac{8}{35} \{ W_4(0,4) G_4(0) + 2 W_4(0,3) G_4(1) - 2 W_4(0,1) G_4(3) \\
& \quad \left. - W_4(0,0) G_4(4) \} \right]
\end{aligned}$$

$$\begin{aligned}
L_{ssss} = & \frac{2x}{3\sqrt{3}} \left(\frac{xR}{4} \right)^9 \left[\left\{ -2 W_0(4,1) + \frac{4}{3} W_0(2,1) - \frac{2}{5} W_0(0,1) \right\} \{ G_0(4) + G_0(2) \} \right. \\
& + \left\{ 2 W_0(4,3) - \frac{4}{3} W_0(2,3) + \frac{2}{5} W_0(0,3) \right\} \{ G_0(2) + G_0(0) \} \\
& + \left\{ -2 W_0(4,2) + \frac{4}{3} W_0(2,2) - 2 W_0(4,0) + \frac{14}{15} W_0(2,0) - \frac{2}{5} W_0(0,0) \right\} G_0(3) \\
& + \left\{ 2 W_0(4,4) + \frac{2}{3} W_0(4,2) + \frac{2}{5} W_0(4,0) - \frac{4}{3} W_0(2,2) + \frac{2}{5} W_0(0,2) \right\} G_0(1) \\
& + \left\{ \frac{8}{3} W_2(2,1) - \frac{8}{7} W_2(0,1) \right\} \{ G_2(4) + G_2(2) \} \\
& + \left\{ -\frac{8}{3} W_2(2,3) + \frac{8}{7} W_2(0,3) \right\} \{ G_2(2) + G_2(0) \} \\
& + \left\{ \frac{8}{3} W_2(2,2) + \frac{32}{21} W_2(2,0) - \frac{8}{7} W_2(0,0) \right\} G_2(3) \\
& + \left\{ -\frac{8}{3} W_2(2,4) + \frac{8}{7} W_2(0,4) - \frac{8}{3} W_2(2,2) + \frac{8}{7} W_2(0,2) \right\} G_2(1) \\
& + \frac{16}{35} \{ -W_4(0,1) (G_4(4) + G_4(2)) + W_4(0,3) (G_4(2) + G_4(0)) \\
& \quad + (-W_4(0,2) - W_4(0,0)) G_4(3) + (W_4(0,4) + W_4(0,2)) G_4(1) \} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{ssss}}{L_{ssss\sigma}} \Big\} = \frac{2x}{3\sqrt{3}} \left(\frac{xR}{4} \right)^9 & \left[\pm \left\{ \frac{4}{3} W_0(4,3) - \frac{4}{15} W_0(4,1) \right\} G_0(0) \pm \left\{ \frac{8}{3} W_0(3,3) - \frac{8}{15} W_0(3,1) \right\} G_0(1) \right. \\
& \mp \left\{ \frac{8}{3} W_0(1,3) - \frac{8}{15} W_0(1,1) \right\} G_0(3) \mp \left\{ \frac{4}{3} W_0(0,3) - \frac{4}{15} W_0(0,1) \right\} G_0(4) \\
& + \left\{ 2W_1(4,4) - \frac{16}{5} W_1(4,2) + \frac{6}{5} W_1(4,0) \right\} G_1(0) \\
& + \left\{ 4W_1(3,4) - \frac{32}{5} W_1(3,2) + \frac{12}{5} W_1(3,0) \right\} G_1(1) \\
& - \left\{ 4W_1(1,4) - \frac{32}{5} W_1(1,2) + \frac{12}{5} W_1(1,0) \right\} G_1(3) \\
& - \left\{ 2W_1(0,4) - \frac{16}{5} W_1(0,2) + \frac{6}{5} W_1(0,0) \right\} G_1(4) \\
& \mp \left\{ \frac{4}{3} W_2(4,3) + \frac{4}{21} W_2(4,1) \right\} G_2(0) \mp \left\{ \frac{8}{3} W_2(3,3) + \frac{8}{21} W_2(3,1) \right\} G_2(1) \\
& \pm \left\{ \frac{8}{3} W_2(1,3) + \frac{8}{21} W_2(1,1) \right\} G_2(3) \pm \left\{ \frac{4}{3} W_2(0,3) + \frac{4}{21} W_2(0,1) \right\} G_2(4) \\
& + \left\{ -\frac{4}{5} W_3(4,2) + \frac{4}{5} W_3(4,0) \right\} G_3(0) + \left\{ -\frac{8}{5} W_3(3,2) + \frac{8}{5} W_3(3,0) \right\} G_3(1) \\
& - \left\{ -\frac{8}{5} W_3(1,2) + \frac{8}{5} W_3(1,0) \right\} G_3(3) - \left\{ -\frac{4}{5} W_3(0,2) + \frac{4}{5} W_3(0,0) \right\} G_3(4) \\
& \pm \frac{16}{35} \{ W_4(4,1) G_4(0) + 2W_4(3,1) G_4(1) - 2W_4(1,1) G_4(3) \\
& \qquad \qquad \qquad - W_4(0,1) G_4(4) \} \Big]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{ss\sigma\sigma}}{L_{ss\sigma\sigma s}} \Big\} = \frac{2x}{3} \left(\frac{xR}{4} \right)^9 & \left[\pm \left\{ \frac{4}{3} W_0(3,1) - \frac{4}{15} W_0(1,1) \right\} \{ G_0(4) + G_0(2) \} \right. \\
& \mp \left\{ \frac{4}{3} W_0(3,3) - \frac{4}{15} W_0(1,3) \right\} \{ G_0(2) + G_0(0) \} \\
& \pm \left\{ \frac{4}{3} W_0(3,2) - \frac{4}{15} W_0(1,2) + \frac{4}{3} W_0(3,0) - \frac{4}{15} W_0(1,0) \right\} G_0(3) \\
& \mp \left\{ \frac{4}{3} W_0(3,4) - \frac{4}{15} W_0(1,4) + \frac{4}{3} W_0(3,2) - \frac{4}{15} W_0(1,2) \right\} G_0(1) \\
& \left. - \left\{ 2W_1(4,1) - \frac{16}{5} W_1(2,1) + \frac{6}{5} W_1(0,1) \right\} \{ G_1(4) + G_1(2) \} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ 2W_1(4,3) - \frac{16}{5}W_1(2,3) + \frac{6}{5}W_1(0,3) \right\} \{ G_1(2) + G_1(0) \} \\
& - \left\{ 2W_1(4,2) - \frac{16}{5}W_1(2,2) - 2W_1(2,0) + 2W_1(4,0) + \frac{6}{5}W_1(0,0) \right\} G_1(3) \\
& + \left\{ 2W_1(4,4) - \frac{6}{5}W_1(4,2) + \frac{6}{5}W_1(0,4) - \frac{16}{5}W_1(2,2) + \frac{6}{5}W_1(0,2) \right\} G_1(1) \\
& \pm \left\{ \frac{4}{3}W_2(3,1) + \frac{4}{21}W_2(1,1) \right\} \{ G_2(4) + G_2(2) \} \\
& \pm \left\{ \frac{4}{3}W_2(3,3) + \frac{4}{21}W_2(1,3) \right\} \{ G_2(2) + G_2(0) \} \\
& \mp \left\{ \frac{4}{3}W_2(3,2) + \frac{4}{21}W_2(1,2) + \frac{4}{3}W_2(3,0) + \frac{4}{21}W_2(1,0) \right\} G_2(3) \\
& \pm \left\{ \frac{4}{3}W_2(3,4) + \frac{4}{21}W_2(1,4) + \frac{4}{3}W_2(3,2) + \frac{4}{21}W_2(1,2) \right\} G_2(1) \\
& + \left\{ \frac{4}{5}W_3(2,1) - \frac{4}{5}W_3(0,1) \right\} \{ G_3(4) + G_3(2) \} \\
& - \left\{ \frac{4}{5}W_3(2,3) - \frac{4}{5}W_3(0,3) \right\} \{ G_3(2) + G_3(0) \} \\
& + \left\{ \frac{4}{5}W_3(2,2) - \frac{4}{5}W_3(0,0) \right\} G_3(3) \\
& - \left\{ \frac{4}{5}W_3(2,4) - \frac{4}{5}W_3(0,4) + \frac{4}{5}W_3(2,2) - \frac{4}{5}W_3(2,0) \right\} G_3(1) \\
& \pm \frac{16}{35}W_4(1,1) \{ G_4(4) + G_4(2) \} \\
& \pm \frac{16}{35}W_4(3,1) \{ G_4(2) + G_4(0) \} \\
& \pm \frac{16}{35} \{ W_4(2,1) + W_4(0,1) \} G_4(3) \\
& \mp \frac{16}{35} \{ W_4(4,1) + W_4(2,1) \} G_4(1) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\sigma\sigma\sigma} &= \frac{2\kappa}{3} \left(\frac{\kappa R}{4} \right)^{\sigma\tau} \Bigg[\left\{ \frac{2}{3}W_0(4,4) - \frac{12}{5}W_0(4,2) + \frac{2}{3}W_0(4,0) \right\} G_0(0) \\
& + \left\{ \frac{4}{3}W_0(3,4) - \frac{24}{5}W_0(3,2) + \frac{4}{3}W_0(3,0) \right\} G_0(1) \\
& + \left\{ -\frac{4}{3}W_0(1,4) + \frac{24}{5}W_0(1,2) - \frac{4}{3}W_0(1,0) \right\} G_0(3)
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -\frac{2}{3}W_0(0,4) + \frac{12}{5}W_0(0,2) - \frac{2}{3}W_0(0,0) \right\} G_0(4) \\
& + \left\{ \frac{4}{3}W_2(4,4) - \frac{8}{7}W_2(4,2) + \frac{4}{3}W_2(4,0) \right\} G_2(0) \\
& + \left\{ \frac{8}{3}W_2(3,4) - \frac{16}{7}W_2(3,2) + \frac{8}{3}W_2(3,0) \right\} G_2(1) \\
& + \left\{ -\frac{8}{3}W_2(1,4) + \frac{16}{7}W_2(1,2) - \frac{8}{3}W_2(1,0) \right\} G_2(3) \\
& + \left\{ -\frac{4}{3}W_2(0,4) + \frac{8}{7}W_2(0,2) - \frac{4}{3}W_2(0,0) \right\} G_2(4) \\
& + \frac{16}{35} \{ -W_4(4,2)G_4(0) - 2W_4(3,2)G_4(1) + 2W_4(1,2)G_4(3) \\
& \qquad \qquad \qquad + W_4(0,2)G_4(4) \} \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\sigma\sigma\sigma\sigma} = \frac{2x}{3} \left(\frac{xR}{4} \right)^9 & \Big[\left\{ 2W_0(4,2) - \frac{4}{3}W_0(2,2) + \frac{2}{5}W_0(0,2) \right\} \{ G_0(0) - G_0(4) \} \\
& + \left\{ 2W_0(4,4) - \frac{4}{3}W_0(2,4) - \frac{8}{5}W_0(4,0) + \frac{4}{3}W_0(2,0) - \frac{2}{5}W_0(0,0) \right\} G_0(2) \\
& - \left\{ 4W_0(4,1) - \frac{8}{3}W_0(2,1) + \frac{4}{5}W_0(0,1) \right\} G_0(3) \\
& + \left\{ 4W_0(4,3) - \frac{8}{3}W_0(2,3) + \frac{4}{5}W_0(0,3) \right\} G_0(1) \\
& + \left\{ \frac{8}{3}W_2(2,2) - \frac{8}{7}W_2(0,2) \right\} \{ G_2(4) - G_2(0) \} \\
& + \left\{ -\frac{8}{3}W_2(2,4) + \frac{8}{7}W_2(0,4) + \frac{8}{3}W_2(2,0) - \frac{8}{7}W_2(0,0) \right\} G_2(2) \\
& - \left\{ -\frac{16}{3}W_2(2,1) + \frac{16}{7}W_2(0,1) \right\} G_2(3) \\
& + \left\{ -\frac{16}{3}W_2(2,3) + \frac{16}{7}W_2(0,3) \right\} G_2(1) \\
& - \frac{16}{35}W_4(0,2) (G_4(4) - G_4(0)) + \frac{16}{35} (W_4(0,4) - W_4(0,0)) G_4(2) \\
& + \frac{32}{35} \{ -W_4(0,1)G_4(3) + W_4(0,3)G_4(1) \} \Big]
\end{aligned}$$

$$\frac{L_{\sigma\sigma\sigma\sigma}}{L_{\sigma\sigma\sigma\sigma}} = \frac{2x}{\sqrt{3}} \left(\frac{xR}{4} \right)^9 \Big[\pm \left\{ \frac{4}{3}W_0(2,3) - \frac{4}{15}W_0(2,1) \right\} \{ G_0(4) - G_0(0) \}$$

$$\begin{aligned}
& \mp \left\{ \frac{4}{3} W_0(4,3) - \frac{4}{15} W_0(4,1) - \frac{4}{3} W_0(0,3) + \frac{4}{15} W_0(0,1) \right\} G_0(2) \\
& \pm \left\{ \frac{8}{3} W_0(1,3) - \frac{8}{15} W_0(1,1) \right\} G_0(3) \mp \left\{ \frac{8}{3} W_0(3,3) - \frac{8}{15} W_0(3,1) \right\} G_0(1) \\
& + \left\{ -2 W_1(2,4) + \frac{16}{5} W_1(2,2) - \frac{6}{5} W_1(2,0) \right\} \{ G_1(4) - G_1(0) \} \\
& - \left\{ -2 W_1(4,4) + \frac{16}{5} W_1(4,2) + \frac{4}{5} W_1(0,4) - \frac{16}{5} W_1(0,2) + \frac{6}{5} W_1(0,0) \right\} G_1(2) \\
& + \left\{ -4 W_1(1,4) - \frac{32}{5} W_1(1,2) - \frac{12}{5} W_1(1,0) \right\} G_1(3) \\
& - \left\{ -4 W_1(3,4) + \frac{32}{5} W_1(3,2) - \frac{12}{5} W_1(3,0) \right\} G_1(1) \\
& \mp \left\{ \frac{4}{3} W_2(2,3) + \frac{4}{21} W_2(2,1) \right\} \{ G_2(4) - G_2(0) \} \\
& \pm \left\{ \frac{4}{3} W_2(4,3) + \frac{4}{21} W_2(4,1) - \frac{4}{3} W_2(0,3) - \frac{4}{21} W_2(0,1) \right\} G_2(2) \\
& \mp \left\{ \frac{8}{3} W_2(1,3) + \frac{8}{21} W_2(1,1) \right\} G_2(3) \pm \left\{ \frac{8}{3} W_2(3,3) + \frac{8}{21} W_2(3,1) \right\} G_2(1) \\
& + \frac{4}{5} \{ W_3(2,2) - W_3(2,0) \} \{ G_3(4) - G_3(0) \} \\
& - \frac{4}{5} \{ W_3(4,2) - W_3(4,0) - W_3(0,2) + W_3(0,0) \} G_3(2) \\
& + \frac{8}{5} \{ W_3(1,2) - W_3(1,0) \} G_3(3) - \frac{8}{5} \{ W_3(3,2) - W_3(3,0) \} G_3(1) \\
& \pm \frac{16}{35} W_4(2,1) \{ G_4(4) - G_4(0) \} \pm \frac{32}{35} W_4(3,1) G_4(1) \\
& \mp \frac{32}{35} W_4(1,1) G_4(3) \mp \frac{16}{35} \{ W_4(4,1) - W_4(0,1) \} G_4(2) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{3000} &= \frac{2x}{\sqrt{3}} \left(\frac{xR}{4} \right)^9 \Big[\left\{ -\frac{2}{3} W_0(1,4) + \frac{12}{5} W_0(1,2) - \frac{2}{3} W_0(1,0) \right\} \{ G_0(4) + G_0(2) \} \\
& - \left\{ -\frac{2}{3} W_0(3,4) + \frac{12}{5} W_0(3,2) - \frac{2}{3} W_0(3,0) \right\} \{ G_0(2) + G_0(0) \} \\
& + \left\{ -\frac{2}{3} W_0(2,4) + \frac{12}{5} W_0(2,2) + \frac{26}{15} W_0(2,0) - \frac{2}{3} W_0(0,4) - \frac{2}{3} W_0(0,0) \right\} G_0(3)
\end{aligned}$$

$$\begin{aligned}
& - \left\{ -\frac{2}{3}W_0(4,4) + \frac{26}{15}W_0(4,2) - \frac{2}{3}W_0(0,4) + \frac{12}{5}W_0(2,2) - \frac{2}{3}W_0(2,0) \right\} G_0(1) \\
& + \left\{ -\frac{4}{3}W_2(1,4) + \frac{8}{7}W_2(1,2) - \frac{4}{3}W_2(1,0) \right\} \{G_2(4) + G_2(2)\} \\
& - \left\{ -\frac{4}{3}W_2(3,4) + \frac{8}{7}W_2(3,2) - \frac{4}{3}W_2(3,0) \right\} \{G_2(2) + G_2(0)\} \\
& + \left\{ -\frac{4}{3}W_2(2,4) + \frac{8}{7}W_2(2,2) - \frac{4}{21}W_2(2,0) - \frac{4}{3}W_2(0,4) - \frac{4}{3}W_2(0,0) \right\} G_2(3) \\
& - \left\{ -\frac{4}{3}W_2(4,4) - \frac{4}{21}W_2(4,2) - \frac{4}{3}W_2(4,0) + \frac{8}{7}W_2(2,2) - \frac{4}{3}W_2(2,0) \right\} G_2(1) \\
& + \frac{16}{35}W_4(2,1) \{G_4(4) + G_4(2)\} - \frac{16}{35}W_4(2,3) \{G_4(2) + G_4(0)\} \\
& + \frac{16}{35} \{W_4(2,2) + W_4(2,0)\} G_4(3) - \frac{16}{35} \{W_4(2,4) + W_4(2,2)\} G_4(1) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\pi s \pi s} = \frac{x}{3} \left(\frac{xR}{4} \right)^9 & \Big[\left\{ 2W_0(4,4) - \frac{4}{3}W_0(2,4) + \frac{2}{5}W_0(0,4) \right\} \{G_0(0) - G_0(2)\} \\
& + \left\{ 2W_0(4,2) - \frac{4}{3}W_0(2,2) + \frac{2}{5}W_0(0,2) \right\} \{G_0(4) - G_0(0)\} \\
& + \left\{ 2W_0(4,0) - \frac{4}{3}W_0(2,0) + \frac{2}{5}W_0(0,0) \right\} \{G_0(2) - G_0(4)\} \\
& + \left\{ -\frac{8}{3}W_2(2,4) + \frac{8}{7}W_2(0,4) \right\} \{G_2(0) - G_2(2)\} \\
& + \left\{ -\frac{8}{3}W_2(2,2) + \frac{8}{7}W_2(0,2) \right\} \{G_2(4) - G_2(0)\} \\
& + \left\{ -\frac{8}{3}W_2(2,0) + \frac{8}{7}W_2(0,0) \right\} \{G_2(2) - G_2(4)\} \\
& + \frac{16}{35} \{W_4(2,0) - W_4(0,0)\} G_4(4) + \frac{16}{35} \{W_4(0,0) - W_4(4,0)\} G_4(2) \\
& + \frac{16}{35} \{W_4(4,0) - W_4(2,0)\} G_4(0) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{s \pi s \pi s} = \frac{4x}{3} \left(\frac{xR}{4} \right)^9 & \Big[\left\{ \frac{1}{3}W_0(4,4) - \frac{2}{5}W_0(4,2) + \frac{1}{15}W_0(4,0) \right\} G_0(0) \\
& + \left\{ \frac{2}{3}W_0(3,4) - \frac{4}{5}W_0(3,2) + \frac{2}{15}W_0(3,0) \right\} G_0(1) \\
& - \left\{ \frac{2}{3}W_0(1,4) - \frac{4}{5}W_0(1,2) + \frac{2}{15}W_0(1,0) \right\} G_0(3)
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{1}{3} W_0(0,4) - \frac{2}{5} W_0(0,2) + \frac{1}{15} W_0(0,0) \right\} G_0(4) \\
& + \left\{ -\frac{1}{3} W_2(4,4) + \frac{2}{7} W_2(4,2) + \frac{1}{21} W_2(4,0) \right\} G_2(0) \\
& + \left\{ -\frac{2}{3} W_2(3,4) + \frac{4}{7} W_2(3,2) + \frac{2}{21} W_2(3,0) \right\} G_2(1) \\
& - \left\{ -\frac{2}{3} W_2(1,4) + \frac{4}{7} W_2(1,2) + \frac{2}{21} W_2(1,0) \right\} G_2(3) \\
& - \left\{ -\frac{1}{3} W_2(0,4) + \frac{2}{7} W_2(0,2) + \frac{1}{21} W_2(0,0) \right\} G_2(4) \\
& + \frac{4}{35} \{ W_4(2,4) - W_4(0,4) \} G_4(0) + \frac{8}{35} \{ W_4(2,3) - W_4(0,3) \} G_4(1) \\
& - \frac{8}{35} \{ W_4(2,1) - W_4(0,1) \} G_4(3) - \frac{4}{35} \{ W_4(2,0) - W_4(0,0) \} G_4(4) \Big]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{\pi\pi\pi\pi}}{L_{\pi\pi\pi\pi}} \Big\} &= \frac{x}{3} \left(\frac{xR}{4} \right)^9 \Big[\left\{ -W_1^1(3,3) + \frac{1}{5} W_1^1(1,3) \right\} G_1^1(0) + \left\{ -W_1^1(2,3) + \frac{1}{5} W_1^1(1,2) \right\} G_1^1(1) \\
&+ \left\{ -\frac{1}{5} W_1^1(1,1) + W_1^1(1,3) \right\} G_1^1(2) + \left\{ -\frac{1}{5} W_1^1(0,1) + W_1^1(0,3) \right\} G_1^1(3) \\
&\pm \left\{ -\frac{1}{21} W_2^1(0,3) + \frac{1}{9} W_2^1(2,3) \right\} G_2^1(0) \pm \left\{ -\frac{1}{21} W_2^1(0,2) + \frac{1}{9} W_2^1(2,2) \right\} G_2^1(1) \\
&\pm \left\{ -\frac{1}{9} W_2^1(1,2) + \frac{1}{21} W_2^1(0,1) \right\} G_2^1(2) \pm \left\{ -\frac{1}{9} W_2^1(0,2) + \frac{1}{21} W_2^1(0,0) \right\} G_2^1(3) \\
&+ \frac{1}{45} \{ W_3^1(1,3) G_3^1(0) + W_3^1(1,2) G_3^1(1) - W_3^1(1,1) G_3^1(2) - W_3^1(0,1) G_3^1(3) \} \\
&\mp \frac{1}{175} \{ W_4^1(0,3) G_4^1(0) + W_4^1(0,2) G_4^1(1) - W_4^1(0,1) G_4^1(2) - W_4^1(0,0) G_4^1(3) \} \Big]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{\pi\sigma\pi\pi}}{L_{\pi\pi\pi\pi}} \Big\} &= \frac{x}{\sqrt{3}} \left(\frac{xR}{4} \right)^9 \Big[\pm \left\{ \frac{4}{3} W_0(3,4) - \frac{4}{15} W_0(1,4) \right\} \{ G_0(0) - G_0(2) \} \\
&\pm \left\{ \frac{4}{3} W_0(3,2) - \frac{4}{15} W_0(1,2) \right\} \{ G_0(4) - G_0(0) \} \\
&\pm \left\{ \frac{4}{3} W_0(3,0) - \frac{4}{15} W_0(1,0) \right\} \{ G_0(2) - G_0(4) \} \\
&+ \left\{ 2W_1(4,4) - \frac{16}{5} W_1(2,4) + \frac{6}{5} W_1(0,4) \right\} \{ G_1(0) - G_1(2) \}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ 2W_1(4,2) - \frac{16}{5}W_1(2,2) + \frac{6}{5}W_1(0,2) \right\} \{ G_1(4) - G_1(0) \} \\
& + \left\{ 2W_1(4,0) - \frac{16}{5}W_1(2,0) + \frac{6}{5}W_1(0,0) \right\} \{ G_1(2) - G_1(4) \} \\
& \pm \left\{ -\frac{4}{3}W_2(3,2) - \frac{4}{21}W_2(1,2) \right\} \{ G_2(4) - G_2(0) \} \\
& \pm \left\{ -\frac{4}{3}W_2(3,0) - \frac{4}{21}W_2(1,0) \right\} \{ G_2(2) - G_2(4) \} \\
& \pm \left\{ -\frac{4}{3}W_2(3,4) - \frac{4}{21}W_2(1,4) \right\} \{ G_2(0) - G_2(2) \} \\
& + \frac{4}{5} \{ W_3(0,2) - W_3(2,2) \} \{ G_3(4) - G_3(0) \} \\
& + \frac{4}{5} \{ W_3(0,0) - W_3(2,0) \} \{ G_3(2) - G_3(4) \} \\
& + \frac{4}{5} \{ W_3(0,4) - W_3(2,4) \} \{ G_3(0) - G_3(2) \} \\
& \pm \frac{16}{35} \{ (W_4(1,2) - W_4(1,0)) G_4(4) \pm \frac{16}{35} \{ W_4(1,0) - W_4(1,4) \} G_4(2) \\
& \pm \frac{16}{35} \{ W_4(1,4) - W_4(1,2) \} G_4(0) \}
\end{aligned}$$

$$\begin{aligned}
L_{\pi\pi\sigma\pi} = \frac{4x}{\sqrt{3}} \left(\frac{xR}{4} \right)^0 & \left[- \left\{ \frac{1}{3}W_0(1,4) - \frac{2}{5}W_0(1,2) + \frac{1}{15}W_0(1,0) \right\} \{ G_0(4) + G_0(2) \} \right. \\
& + \left\{ \frac{1}{3}W_0(3,4) - \frac{2}{5}W_0(3,2) + \frac{1}{15}W_0(3,0) \right\} \{ G_0(2) + G_0(0) \} \\
& - \left\{ \frac{1}{3}W_0(2,4) - \frac{2}{5}W_0(2,2) - \frac{1}{3}W_0(0,2) + \frac{1}{3}W_0(0,4) + \frac{1}{15}W_0(0,0) \right\} G_0(3) \\
& + \left\{ \frac{1}{3}W_0(4,4) - \frac{1}{15}W_0(4,2) + \frac{1}{15}W_0(4,0) - \frac{2}{5}W_0(2,2) + \frac{1}{15}W_0(2,0) \right\} G_0(1) \\
& + \left\{ \frac{1}{3}W_2(1,4) - \frac{2}{7}W_2(1,2) - \frac{1}{21}W_2(1,0) \right\} \{ G_2(4) + G_2(2) \} \\
& - \left\{ \frac{1}{3}W_2(3,4) - \frac{2}{7}W_2(3,2) - \frac{1}{21}W_2(3,0) \right\} \{ G_2(2) + G_2(0) \} \\
& + \left\{ \frac{1}{3}W_2(2,4) - \frac{2}{7}W_2(2,2) - \frac{1}{3}W_2(2,0) + \frac{1}{3}W_2(0,4) - \frac{1}{21}W_2(0,0) \right\} G_2(3) \\
& - \left\{ \frac{1}{3}W_2(4,4) + \frac{1}{21}W_2(4,2) - \frac{1}{21}W_2(4,0) - \frac{2}{7}W_2(2,2) - \frac{1}{21}W_2(2,0) \right\} G_2(1)
\end{aligned}$$

$$\begin{aligned}
& + \frac{4}{35} \{ (W_4(0,1) - W_4(2,1)) \} \{ G_4(4) + G_4(2) \} \\
& + \frac{4}{35} \{ W_4(0,0) - W_4(2,2) \} G_4(3) \\
& + \frac{4}{35} \{ W_4(2,3) - W_4(0,3) \} \{ G_4(2) + G_4(0) \} \\
& + \frac{4}{35} \{ W_4(4,2) + W_4(2,2) - W_4(4,0) - W_4(2,0) \} G_4(1) \Big]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{\sigma\pi\pi s}}{L_{\pi s 0\pi}} \Big\} &= \frac{\kappa}{\sqrt{3}} \left(\frac{\kappa R}{4} \right)^{9\Gamma} \Big[\left\{ -W_1^1(2,3) + \frac{1}{5} W_1^1(2,1) \right\} G_1^1(0) + \left\{ -W_1^1(3,3) + \frac{1}{5} W_1^1(3,1) \right\} G_1^1(1) \\
&+ \left\{ W_1^1(0,3) - \frac{1}{5} W_1^1(0,1) \right\} G_1^1(2) + \left\{ W_1^1(1,3) - \frac{1}{5} W_1^1(1,1) \right\} G_1^1(3) \\
&\pm \left\{ \frac{1}{9} W_2^1(2,2) - \frac{1}{21} W_2^1(2,0) \right\} G_2^1(0) \pm \left\{ \frac{1}{9} W_2^1(2,3) - \frac{1}{21} W_2^1(0,3) \right\} G_2^1(1) \\
&\pm \left\{ -\frac{1}{9} W_2^1(0,2) + \frac{1}{21} W_2^1(0,0) \right\} G_2^1(2) \pm \left\{ -\frac{1}{9} W_2^1(2,1) + \frac{1}{21} W_2^1(0,1) \right\} G_2^1(3) \\
&+ \frac{1}{45} \{ W_3^1(1,2) G_3^1(0) + W_3^1(1,3) G_3^1(1) - W_3^1(1,0) G_3^1(2) - W_3^1(1,1) G_3^1(3) \} \\
&\mp \frac{1}{175} \{ W_4^1(0,2) G_4^1(0) + W_4^1(0,3) G_4^1(1) - W_4^1(0,0) G_4^1(2) - W_4^1(0,1) G_4^1(3) \} \Big]
\end{aligned}$$

$$\begin{aligned}
\frac{L_{s\pi\pi\sigma}}{L_{\pi\sigma s\pi}} \Big\} &= \frac{\kappa}{\sqrt{3}} \left(\frac{\kappa R}{4} \right)^{9\Gamma} \Big[\mp \left\{ -W_1^1(2,3) + \frac{1}{5} W_1^1(3,0) \right\} G_1^1(0) \mp \left\{ -W_1^1(2,2) + \frac{1}{5} W_1^1(2,0) \right\} G_1^1(1) \\
&\mp \left\{ W_1^1(1,2) - \frac{1}{5} W_1^1(1,0) \right\} G_1^1(2) \mp \left\{ W_1^1(0,2) - \frac{1}{5} W_1^1(0,0) \right\} G_1^1(3) \\
&- \left\{ \frac{1}{9} W_2^1(3,3) - \frac{1}{21} W_2^1(1,3) \right\} G_2^1(0) - \left\{ \frac{1}{9} W_2^1(2,3) - \frac{1}{21} W_2^1(2,1) \right\} G_2^1(1) \\
&- \left\{ -\frac{1}{9} W_2^1(1,3) + \frac{1}{21} W_2^1(1,1) \right\} G_2^1(2) - \left\{ -\frac{1}{9} W_2^1(0,3) + \frac{1}{21} W_2^1(0,1) \right\} G_2^1(3) \\
&\mp \frac{1}{45} \{ W_3^1(3,0) G_3^1(0) + W_3^1(2,0) G_3^1(1) - W_3^1(1,0) G_3^1(2) - W_3^1(0,0) G_3^1(3) \} \\
&+ \frac{1}{175} \{ W_4^1(3,1) G_4^1(0) + W_4^1(2,1) G_4^1(1) - W_4^1(1,1) G_4^1(2) - W_4^1(0,1) G_4^1(3) \} \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\sigma\sigma\sigma\sigma} &= 4\kappa \left(\frac{\kappa R}{4} \right)^{9\Gamma} \Big[\left\{ \frac{1}{3} W_0(4,2) - \frac{6}{5} W_0(2,2) + \frac{1}{3} W_0(0,2) \right\} G_0(0) \\
&+ \left\{ \frac{2}{3} W_0(4,3) - \frac{12}{5} W_0(2,3) + \frac{2}{3} W_0(0,3) \right\} G_0(1) \\
&+ \left\{ \frac{1}{3} W_0(4,4) - \frac{6}{5} W_0(4,2) + \frac{6}{5} W_0(2,0) - \frac{1}{3} W_0(0,0) \right\} G_0(2)
\end{aligned}$$

$$\begin{aligned}
& - \left\{ \frac{2}{3} W_0(4,1) - \frac{12}{5} W_0(2,1) + \frac{2}{3} W_0(0,1) \right\} G_0(3) \\
& - \left\{ \frac{1}{3} W_0(4,2) - \frac{6}{5} W_0(2,2) + \frac{1}{3} W_0(0,2) \right\} G_0(4) \\
& + \left\{ \frac{2}{3} W_2(4,2) - \frac{4}{7} W_2(2,2) + \frac{2}{3} W_2(0,2) \right\} G_2(0) \\
& + \left\{ \frac{4}{3} W_2(4,3) - \frac{8}{7} W_2(2,3) + \frac{4}{3} W_2(0,3) \right\} G_2(1) \\
& + \left\{ \frac{2}{3} W_2(4,4) - \frac{4}{7} W_2(2,4) + \frac{4}{7} W_2(2,0) - \frac{2}{3} W_2(0,0) \right\} G_2(2) \\
& - \left\{ \frac{4}{3} W_2(4,1) - \frac{8}{7} W_2(2,1) + \frac{4}{3} W_2(0,1) \right\} G_2(3) \\
& - \left\{ \frac{2}{3} W_2(4,2) - \frac{4}{7} W_2(2,2) + \frac{2}{3} W_2(0,2) \right\} G_2(4) \\
& - \frac{8}{35} W_4(2,2) G_4(0) - \frac{16}{35} W_4(2,3) G_4(1) \\
& + \frac{8}{35} \{ W_4(2,0) - W_4(2,4) \} G_4(2) \\
& + \frac{16}{35} W_4(1,2) G_4(3) + \frac{8}{35} W_4(2,2) G_4(4) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\pi\sigma\pi\sigma} = 2\kappa \left(\frac{\kappa R}{4} \right)^0 & \Big[\left\{ \frac{1}{3} W_0(4,4) - \frac{23}{15} W_0(4,2) + \frac{1}{3} W_0(4,0) + \frac{6}{5} W_0(2,2) - \frac{1}{3} W_0(2,0) \right\} G_0(0) \\
& - \left\{ \frac{1}{3} W_0(4,4) - \frac{6}{5} W_0(4,2) + \frac{6}{5} W_0(2,0) - \frac{1}{3} W_0(0,0) \right\} G_0(2) \\
& + \left\{ \frac{1}{3} W_0(4,2) - \frac{1}{3} W_0(4,0) - \frac{6}{5} W_0(2,2) + \frac{23}{15} W_0(2,0) - \frac{1}{3} W_0(0,0) \right\} G_0(4) \\
& + 2 \left\{ \frac{1}{3} W_2(4,4) - \frac{13}{21} W_2(4,2) + \frac{1}{3} W_2(4,0) + \frac{2}{7} W_2(2,2) - \frac{1}{3} W_2(2,0) \right\} G_2(0) \\
& - 2 \left\{ \frac{1}{3} W_2(4,4) - \frac{2}{7} W_2(4,2) + \frac{2}{7} W_2(2,0) - \frac{1}{3} W_2(0,0) \right\} G_2(2) \\
& + 2 \left\{ \frac{1}{3} W_2(4,2) - \frac{1}{3} W_2(4,0) - \frac{2}{7} W_2(2,2) + \frac{13}{21} W_2(2,0) - \frac{1}{3} W_2(0,0) \right\} G_2(4) \\
& - \frac{8}{35} \{ W_4(4,2) - W_4(2,2) \} G_4(0) \\
& + \frac{8}{35} \{ W_4(4,2) - W_4(2,0) \} G_4(2) \\
& - \frac{8}{35} \{ W_4(2,2) - W_4(2,0) \} G_4(4) \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\sigma\pi\sigma\pi} = & \frac{4}{3}\chi\left(\frac{\kappa R}{4}\right)^9 \left[\left\{ W_0(4,2) - \frac{6}{5}W_0(2,2) + \frac{1}{5}W_0(0,2) \right\} G_0(0) \right. \\
& + \left\{ 2W_0(4,3) - \frac{12}{5}W_0(2,3) + \frac{2}{5}W_0(0,3) \right\} G_0(1) \\
& + \left\{ W_0(4,4) - \frac{6}{5}W_0(4,2) - \frac{4}{5}W_0(4,0) + \frac{6}{5}W_0(2,0) - \frac{1}{5}W_0(0,0) \right\} G_0(2) \\
& + \left\{ -2W_0(4,1) + \frac{12}{5}W_0(2,1) - \frac{2}{5}W_0(0,1) \right\} G_0(3) \\
& + \left\{ -W_0(4,2) + \frac{6}{5}W_0(2,2) - \frac{1}{5}W_0(0,2) \right\} G_0(4) \\
& + \left\{ -W_2(4,2) + \frac{6}{7}W_2(2,2) + \frac{1}{7}W_2(0,2) \right\} G_2(0) \\
& + \left\{ -2W_2(4,3) + \frac{12}{7}W_2(2,3) + \frac{2}{7}W_2(0,3) \right\} G_2(1) \\
& + \left\{ -W_2(4,4) + \frac{6}{7}W_2(4,2) + \frac{8}{7}W_2(4,0) - \frac{6}{7}W_2(0,2) - \frac{1}{7}W_2(0,0) \right\} G_2(2) \\
& + \left\{ 2W_2(4,1) - \frac{12}{7}W_2(2,1) - \frac{2}{7}W_2(1,0) \right\} G_2(3) \\
& + \left\{ W_2(4,2) - \frac{6}{7}W_2(2,2) - \frac{1}{7}W_2(0,2) \right\} G_2(4) \\
& + \frac{12}{35} \{ W_4(2,2) - W_4(2,0) \} G_4(0) + \frac{24}{35} \{ W_4(3,2) - W_4(3,0) \} G_4(1) \\
& + \frac{12}{35} \{ W_4(4,2) - W_4(4,0) - W_4(2,0) + W_4(0,0) \} G_4(2) \\
& + \frac{24}{35} \{ -W_4(1,2) + W_4(1,0) \} G_4(3) + \frac{12}{35} \{ -W_4(2,2) + W_4(2,0) \} G_4(4) \Big] \\
\frac{L_{\sigma\pi\pi\sigma}}{L_{\pi\sigma\sigma\pi}} = & \chi\left(\frac{\kappa R}{4}\right)^9 \left[\pm \left\{ \frac{1}{5}W_1^1(2,0) - W_1^1(2,2) \right\} G_1^1(0) \pm \left\{ \frac{1}{5}W_1^1(3,0) - W_1^1(3,2) \right\} G_1^1(1) \right. \\
& \pm \left\{ \frac{1}{5}W_1^1(0,0) - W_1^1(0,2) \right\} G_1^1(2) \pm \left\{ \frac{1}{5}W_1^1(0,1) - W_1^1(2,1) \right\} G_1^1(3) \\
& + \left\{ \frac{1}{21}W_2^1(1,2) - \frac{1}{9}W_2^1(3,2) \right\} G_2^1(0) + \left\{ \frac{1}{21}W_2^1(3,1) - \frac{1}{9}W_2^1(3,3) \right\} G_2^1(1) \\
& - \left\{ \frac{1}{21}W_2^1(0,1) - \frac{1}{9}W_2^1(0,3) \right\} G_2^1(2) - \left\{ \frac{1}{21}W_2^1(1,1) - \frac{1}{9}W_2^1(1,3) \right\} G_2^1(3) \\
& \mp \frac{1}{45} \{ W_3^1(0,2)G_3^1(0) + W_3^1(0,3)G_3^1(1) - W_3^1(0,0)G_3^1(2) - W_3^1(0,1)G_3^1(3) \} \\
& + \frac{1}{175} \{ W_4^1(1,2)G_4^1(0) + W_4^1(1,3)G_4^1(1) - W_4^1(0,1)G_4^1(2) - W_4^1(1,1)G_4^1(3) \} \Big]
\end{aligned}$$

$$\begin{aligned}
L_{\pi\pi\pi\pi} = & \frac{\alpha}{2} \left(\frac{\alpha R}{4} \right)^9 \left[\frac{4}{3} \left\{ W_0(4,4) - \frac{11}{5} W_0(4,2) + \frac{6}{5} W_0(2,2) + \frac{1}{5} W_0(0,4) - \frac{1}{5} W_0(0,2) \right\} G_0(0) \right. \\
& + \frac{4}{3} \left\{ -W_0(4,4) + \frac{6}{5} W_0(4,2) + \frac{4}{5} W_0(0,4) - \frac{6}{5} W_0(0,2) + \frac{1}{5} W_0(0,0) \right\} G_0(2) \\
& + \frac{4}{3} \left\{ W_0(4,2) - \frac{6}{5} W_0(2,2) - W_0(0,4) + \frac{7}{5} W_0(0,2) - \frac{1}{5} W_0(0,0) \right\} G_0(4) \\
& + 4 \left\{ -\frac{1}{3} W_2(4,4) + \frac{13}{21} W_2(4,2) - \frac{2}{7} W_2(2,2) + \frac{1}{21} W_2(0,4) - \frac{1}{21} W_2(0,2) \right\} G_2(0) \\
& + 4 \left\{ \frac{1}{3} W_2(4,4) - \frac{2}{7} W_2(4,2) - \frac{8}{21} W_2(4,0) + \frac{2}{7} W_2(2,0) + \frac{1}{21} W_2(0,0) \right\} G_2(2) \\
& + 4 \left\{ -\frac{1}{3} W_2(4,2) + \frac{2}{7} W_2(2,2) + \frac{1}{3} W_2(0,4) - \frac{5}{21} W_2(0,2) - \frac{1}{21} W_2(0,0) \right\} G_2(4) \\
& + \frac{16}{35} \{ W_4(4,2) - W_4(2,2) - W_4(0,4) + W_4(0,2) \} G_4(0) \\
& + \frac{16}{35} \{ W_4(4,2) + W_4(0,4) + W_4(0,2) - W_4(0,0) \} G_4(2) \\
& + \frac{16}{35} \{ W_4(2,2) - 2W_4(0,2) + W_4(0,0) \} G_4(4) \Big] \\
L_{\pi\pi\pi\pi} = & \frac{\alpha}{36} \left(\frac{\alpha R}{4} \right)^9 \left[\frac{1}{2} \left\{ W_2^2(2,2) - \frac{1}{7} W_2^2(2,0) \right\} G_2^2(0) + \frac{1}{2} \left\{ -W_2^2(2,0) + \frac{1}{7} W_2^2(0,0) \right\} G_2^2(2) \right. \\
& + \frac{1}{525} \{ -G_4^2(0) W_4^2(2,0) + G_4^2(2) W_4^2(0,0) \} \Big]
\end{aligned}$$

§ 3. Numerical Values of Integrals*

Table I.

αR	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
S	0.948311	0.921317	0.889732	0.854102	0.815019	0.773120	0.729074	0.683572
S_0	0.735759	0.612926	0.482519	0.351484	0.225559	0.109104	0.005130	-0.084588
S_1	0.907436	0.861007	0.808847	0.752839	0.694721	0.636018	0.578015	0.521745
δ	-0.276114	-0.349749	-0.386473	-0.429610	-0.463606	-0.488197	-0.503538	-0.510139

αR	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
S	0.637275	0.590839	0.544853	0.499844	0.456263	0.414481	0.374790	0.337402	0.302459
S_0	-0.159319	-0.219165	-0.264856	-0.297555	-0.318692	-0.329834	-0.332576	-0.328460	-0.318929
S_1	0.467998	0.417347	0.370170	0.326676	0.286945	0.250947	0.218570	0.189643	0.163957
δ	-0.508779	-0.500421	-0.486131	-0.467008	-0.444131	-0.418514	-0.391080	-0.362645	-0.333905

* More precise values for $D_{\pi\pi\pi\pi}$ and $L_{\pi\pi\pi\pi}$ are obtained by E. Ishiguro, K. Hijikata, and T. Nakamura though the result is not yet published. Related tables were published by several authors.⁽⁴⁾⁻⁽¹⁰⁾

Table II

χR	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
K_{ss}/κ	0.240608	0.232923	0.223387	0.212602	0.201159	0.189551	0.178158	0.167239
K_{sa}/κ	0.077067	0.084819	0.087999	0.087583	0.084590	0.079930	0.074322	0.068342
K_{so}/κ	0.274475	0.273745	0.268152	0.258620	0.246331	0.232421	0.217833	0.203277
$K_{\pi\pi}/\kappa$	0.223673	0.212511	0.201004	0.189593	0.178572	0.168116	0.158320	0.149221
I_{ss}/κ	0.235034	0.226443	0.216157	0.204471	0.191725	0.178273	0.164455	0.150580
I_{sa}/κ	0.123897	0.143229	0.157005	0.165516	0.169294	0.169008	0.163378	0.159121
I_{so}/κ	-0.053099	-0.065692	-0.076489	-0.085044	-0.091159	-0.094844	-0.096265	-0.095684
I_{sa}/κ	0.153283	0.114527	0.076701	0.041860	0.011278	-0.014410	-0.035057	-0.050859
$I_{\pi\pi}/\kappa$	0.214596	0.168464	0.181293	0.163818	0.146613	0.130102	0.114577	0.100220
I_{ss}/κ^2	0.081751	0.078590	0.074377	0.069389	0.063908	0.058189	0.052443	0.046836

χR	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
K_{ss}/κ	0.156958	0.147398	0.138586	0.130511	0.123141	0.116425	0.110310	0.104740	0.099659
K_{sa}/κ	0.062339	0.056570	0.051178	0.046239	0.041774	0.037774	0.034215	0.031057	0.028263
K_{so}/κ	0.189241	0.176027	0.163795	0.152607	0.142454	0.133237	0.125034	0.117610	0.110930
$K_{\pi\pi}/\kappa$	0.140817	0.133083	0.125981	0.119463	0.113485	0.107994	0.102949	0.098305	0.094024
I_{ss}/κ	0.136914	0.123677	0.110383	0.099124	0.088017	0.077766	0.068390	0.059882	0.052219
I_{sa}/κ	0.150909	0.141343	0.130940	0.120133	0.109270	0.098622	0.088390	0.078718	0.069699
I_{so}/κ	-0.093420	-0.089808	-0.085175	-0.079824	-0.074022	-0.067996	-0.061933	-0.055980	-0.050248
I_{sa}/κ	-0.062234	-0.069723	-0.073921	-0.075422	-0.074789	-0.072528	-0.069084	-0.064831	-0.060080
$I_{\pi\pi}/\kappa$	0.087127	0.075327	0.064799	0.055487	0.047315	0.040192	0.034021	0.028704	0.024144
I_{ss}/κ^2	0.041489	0.036485	0.031875	0.027682	0.023912	0.020554	0.017589	0.014990	0.012727

Table III

$C_{\varphi\psi\chi\chi}/\chi$

χR	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
$\varphi\psi\chi\chi$								
$ssss$	0.162657	0.153251	0.142638	0.131333	0.119080	0.106802	0.094613	0.082797
$ssss$	0.052522	0.064400	0.068816	0.073251	0.075164	0.074734	0.072280	0.067947
$soss$	0.118372	0.091130	0.063872	0.038438	0.016135	-0.002270	-0.016465	-0.026569
$soos$	0.013873	0.001745	-0.010692	-0.022416	-0.032556	-0.040493	-0.046106	-0.049203
$soos$	0.049642	0.054316	0.059081	0.063337	0.066544	0.068367	0.068434	0.066919
$\pi s \pi s$	0.154831	0.141568	0.127540	0.113057	0.098686	0.084887	0.072008	0.060280
$\pi \pi s s$	0.036312	0.031799	0.028725	0.025503	0.022270	0.019141	0.016208	0.013532
$\pi s \pi \pi$	0.036709	0.032337	0.029381	0.026243	0.023054	0.019934	0.016977	0.014253
$s o s s$	-0.035303	-0.032793	-0.025882	-0.015796	-0.004071	0.007805	0.018607	0.027491
$\pi s \pi o$	0.051897	0.057591	0.061812	0.063224	0.062162	0.059108	0.054602	0.049178
$\pi \pi s o$	0.013854	0.015456	0.016283	0.016352	0.015791	0.014762	0.013415	0.011897
$s o \pi s$	0.016440	0.018356	0.019339	0.019422	0.018760	0.017538	0.015941	0.014140
$s o s o$	0.101655	0.070995	0.046784	0.030544	0.021995	0.019687	0.021636	0.025863
$\pi o \pi \pi$	0.106945	0.079640	0.053649	0.030770	0.012011	-0.002310	-0.012393	-0.018765
$\pi o \pi o$	0.002150	-0.001035	-0.003886	-0.006173	-0.007790	-0.008733	-0.009072	-0.008923
$\pi s o \pi$	0.016047	0.017790	0.019045	0.019657	0.019589	0.018898	0.017703	0.016149
$\pi \pi \pi \pi$	0.149357	0.133066	0.116038	0.099214	0.083308	0.068801	0.055962	0.044888
$\pi \pi \pi \pi$	0.017002	0.015134	0.013181	0.011252	0.009429	0.007772	0.006303	0.005024

χR	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$\varphi\psi\chi\zeta$									
$ssss$	0.071591	0.061180	0.051691	0.043196	0.035717	0.029235	0.023698	0.019032	0.015151
σsss	0.062913	0.056878	0.050478	0.044041	0.037827	0.032022	0.026765	0.022064	0.017991
$\sigma\sigma s$	-0.032975	-0.036245	-0.037017	-0.035926	-0.033552	-0.030391	-0.026841	-0.023202	-0.019687
$\sigma\sigma\sigma$	-0.050043	-0.048970	-0.046408	-0.042791	-0.038528	-0.033969	-0.029395	-0.025041	-0.020962
$\sigma s\sigma$	0.063923	0.059722	0.054648	0.049046	0.043233	0.037479	0.031994	0.026923	0.022358
$\pi s\pi s$	0.049840	0.040716	0.032896	0.026299	0.020816	0.01642	0.012687	0.009779	0.007478
$\pi\sigma^+ss$	0.011149	0.009072	0.007296	0.005804	0.004570	0.003564	0.002754	0.002111	0.001605
$\pi s\pi\pi$	0.011805	0.009654	0.007801	0.006234	0.004929	0.003859	0.002968	0.002302	0.001756
$\sigma\sigma s$	0.034013	0.038080	0.039863	0.039705	0.038030	0.035277	0.031854	0.028104	0.024298
$\pi s\pi\sigma$	0.043309	0.037383	0.031686	0.026432	0.021720	0.017609	0.014099	0.011161	0.008742
$\pi\sigma^+\sigma$	0.010325	0.008789	0.007378	0.006057	0.004876	0.003943	0.003124	0.002448	0.001899
$\sigma\pi\pi s$	0.012273	0.010450	0.008745	0.007205	0.005853	0.004693	0.003719	0.002915	0.002261
$\sigma\sigma\sigma$	0.030705	0.034974	0.037966	0.039386	0.039252	0.037775	0.035264	0.032058	0.028471
$\sigma\pi\pi\pi$	-0.022114	-0.023165	-0.022588	-0.020960	-0.018738	-0.016266	-0.013782	-0.011443	-0.009335
$\pi\sigma^+\sigma\sigma$	-0.008413	-0.007665	-0.006789	-0.005869	-0.004968	-0.004129	-0.003376	-0.002721	-0.002164
$\pi\sigma\sigma\pi$	0.014383	0.012537	0.010718	0.009004	0.007444	0.006065	0.004877	0.003874	0.003043
$\pi\pi\pi\pi$	0.035546	0.027817	0.021533	0.016502	0.012527	0.009430	0.007041	0.005217	0.003840
$\pi\sigma^+\pi\sigma^+\pi$	0.003980	0.003423	0.002640	0.002016	0.001525	0.001129	0.000851	0.000628	0.000460

Table IV $D_{\varphi\psi\chi\zeta}/\alpha$

χR	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
$\varphi\psi\chi\zeta$								
$ssss$	0.175969	0.172033	0.168122	0.163720	0.158907	0.153778	0.148421	0.142921
σsss	0.020342	0.024446	0.027958	0.030820	0.033000	0.034508	0.035378	0.035664
$\sigma\sigma s$	0.177402	0.175122	0.172393	0.169219	0.165596	0.161547	0.157104	0.152316
$\sigma\sigma\sigma$	0.029629	0.024659	0.019353	0.014021	0.008942	0.004306	0.000253	-0.003140
$\pi s\pi\pi$	0.174352	0.170488	0.165987	0.160970	0.155564	0.149896	0.144080	0.138223
$\pi s\pi\pi$	0.036490	0.034628	0.032527	0.030265	0.027912	0.025541	0.023210	0.020963
$\sigma\sigma\sigma$	0.011340	0.014889	0.018622	0.022343	0.025840	0.028960	0.031556	0.033638
$\sigma\pi\pi\pi$	0.024843	0.029225	0.032626	0.035059	0.036580	0.037283	0.037289	0.036727
$\pi\sigma\pi\pi$	0.006592	0.009078	0.009943	0.010445	0.010609	0.010492	0.010146	0.009628
$\sigma\sigma\sigma$	0.187213	0.183599	0.180009	0.176487	0.172939	0.169346	0.165591	0.161584
$\pi\sigma\pi\sigma$	0.172497	0.170883	0.168585	0.165585	0.161923	0.157647	0.152859	0.147681
$\sigma\sigma\sigma\pi$	0.006305	0.004538	0.002766	0.001128	-0.000264	-0.001394	-0.002248	-0.002837
$\pi\pi\pi\pi$	0.175280	0.170291	0.164688	0.158663	0.152385	0.146020	0.139691	0.133495
$\pi\sigma^+\pi\sigma^+$	0.018096	0.016641	0.015081	0.013486	0.011900	0.010389	0.008981	0.007699

χR	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$\varphi\psi\chi\zeta$									
$ssss$	0.137358	0.131808	0.126332	0.120983	0.115805	0.110828	0.106077	0.101560	0.097289
σsss	0.035441	0.034791	0.033799	0.032546	0.031110	0.029555	0.027939	0.026308	0.024697
$\sigma\sigma s$	0.147247	0.141971	0.136567	0.131110	0.125678	0.120329	0.115120	0.110089	0.105270
$\sigma\sigma\sigma$	-0.005859	-0.007932	-0.009407	-0.010365	-0.010890	-0.011061	-0.010959	-0.010653	-0.010203
$\pi s\pi\pi$	0.132415	0.126727	0.121215	0.115920	0.110869	0.106079	0.101555	0.097296	0.093298
$\pi s\pi\pi$	0.018836	0.016854	0.015027	0.013364	0.011861	0.010517	0.009320	0.008259	0.007325
$\sigma\sigma\sigma$	0.034875	0.035578	0.035695	0.035301	0.034485	0.033334	0.031938	0.030377	0.028721
$\sigma\pi\pi\pi$	0.035725	0.034398	0.032850	0.031168	0.029422	0.027666	0.025940	0.024274	0.022685
$\pi\sigma\pi\pi$	0.008992	0.008286	0.007548	0.006810	0.006100	0.005419	0.004793	0.004243	0.003706
$\sigma\sigma\sigma$	0.157269	0.152628	0.147679	0.142476	0.137088	0.131594	0.126075	0.120607	0.115253
$\pi\sigma\pi\sigma$	0.142236	0.136643	0.131010	0.125429	0.119972	0.114996	0.109641	0.104829	0.100279
$\pi\sigma\sigma\pi$	-0.003194	-0.003361	-0.003373	-0.003274	-0.003096	-0.002870	-0.002619	-0.002358	-0.002101
$\pi\pi\pi\pi$	0.127505	0.121769	0.116318	0.111166	0.106318	0.101770	0.097512	0.093529	0.089808
$\pi\sigma^+\pi\sigma^+$	0.006551	0.005541	0.004663	0.003908	0.003264	0.002721	0.002265	0.001883	0.001567

Table V $L_{\varphi\psi\chi\chi}/x$

$\varphi\psi\chi\chi$	xR	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
$SSSS$		0.170373	0.164569	0.157821	0.150272	0.142085	0.133357	0.124305	0.115070
σSSS		0.009994	0.012217	0.013984	0.015414	0.016476	0.017161	0.017495	0.017439
$S\sigma SS$		0.045167	0.054764	0.063056	0.069841	0.074868	0.077983	0.080022	0.079958
$SSS\sigma$		-0.064506	-0.077387	-0.088124	-0.096511	-0.102328	-0.105750	-0.106722	-0.106004
$SS\sigma\sigma$		0.028028	0.022326	0.016298	0.010339	0.004751	-0.000229	-0.004454	-0.007863
$\sigma\sigma\sigma\sigma$		0.124776	0.104314	0.072668	0.046163	0.022866	0.001625	-0.016415	-0.031212
$\sigma\sigma\sigma S$		0.038480	0.037802	0.037066	0.036260	0.035328	0.034236	0.033065	0.031486
$S\sigma\sigma S$		0.170809	0.165276	0.158725	0.151694	0.143869	0.135522	0.126785	0.117800
$\pi S\pi S$		0.170043	0.164207	0.157376	0.149561	0.141496	0.132771	0.123065	0.113706
$S\pi S\pi$		0.161576	0.150643	0.140721	0.129136	0.117325	0.105623	0.094270	0.083520
$\pi\pi\pi S$		0.037044	0.035382	0.033467	0.031333	0.029054	0.026693	0.024309	0.021956
$\pi S\pi\pi$		0.035930	0.033796	0.031379	0.028797	0.026136	0.023478	0.020889	0.018423
$\sigma\sigma S\sigma$		-0.058728	-0.071439	-0.082528	-0.091627	-0.098505	-0.103072	-0.105376	-0.105590
$S\sigma\sigma\sigma$		-0.000252	-0.002734	-0.005333	-0.009884	-0.013670	-0.017189	-0.020112	-0.022594
$\sigma\sigma\sigma S$		0.048664	0.058964	0.067890	0.075205	0.080747	0.084463	0.086392	0.086669
$\pi\sigma\pi S$		0.043433	0.052730	0.060672	0.067143	0.072965	0.075106	0.076628	0.076655
$\pi S\pi\sigma$		-0.067379	-0.077976	-0.090882	-0.098839	-0.104178	-0.107013	-0.107562	-0.106120
$S\pi\pi\sigma$		0.011719	0.013376	0.014424	0.014910	0.014906	0.014530	0.013805	0.012778
$\sigma\pi\pi S$		0.002660	0.003147	0.003533	0.003810	0.004029	0.004046	0.004022	0.003910
$S\pi\pi\sigma$		-0.019820	-0.022923	-0.025093	-0.026365	-0.026824	-0.026585	-0.025779	-0.024537
$\pi\sigma S\pi$		0.011728	0.013386	0.014437	0.014922	0.014920	0.014520	0.013819	0.012980
$\pi S\sigma\pi$		0.005248	0.006037	0.006555	0.006811	0.006785	0.006663	0.006343	0.005928
$\sigma\sigma S\sigma$		0.136516	0.109408	0.081812	0.053185	0.027132	0.003872	-0.016053	-0.032290
$\pi\sigma\pi\sigma$		0.118922	0.094100	0.068546	0.043774	0.020886	0.000777	-0.016269	-0.030046
$\sigma\pi\sigma\pi$		0.156677	0.137686	0.127624	0.126971	0.115764	0.104991	0.093809	0.073995
$\sigma\pi\pi\sigma$		0.006532	0.004806	0.003086	0.001460	-0.000042	-0.000964	-0.002014	-0.002741
$\pi\sigma\sigma\pi$		0.010893	0.010869	0.010706	0.010414	0.010049	0.009346	0.008076	0.007941
$\pi\pi\pi\pi$		0.164010	0.153698	0.142270	0.130276	0.118105	0.106102	0.094532	0.083595
$\pi\pi\pi\pi$		0.018228	0.016852	0.015357	0.013812	0.012275	0.010794	0.009398	0.008087

$\varphi\psi\chi\chi$	xR	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$SSSS$		0.105809	0.097016	0.087761	0.079204	0.071077	0.063441	0.056337	0.049788	0.042589
σSSS		0.017111	0.016516	0.015718	0.014766	0.013712	0.012600	0.011467	0.010364	0.009262
$S\sigma SS$		0.078706	0.076306	0.072987	0.068973	0.064475	0.059838	0.054757	0.049833	0.045060
$SSS\sigma$		-0.103381	-0.099351	-0.094242	-0.088362	-0.081989	-0.075362	-0.068684	-0.062116	-0.055823
$SS\sigma\sigma$		-0.010457	-0.012283	-0.013425	-0.013982	-0.014061	-0.013771	-0.013187	-0.012415	-0.011530
$\sigma\sigma\sigma\sigma$		-0.042397	-0.050603	-0.056027	-0.059051	-0.060076	-0.059500	-0.057685	-0.054960	-0.051606
$\sigma\sigma\sigma S$		0.029836	0.028032	0.026111	0.024118	0.022095	0.020090	0.018130	0.016234	0.014470
$S\sigma\sigma S$		0.108718	0.099665	0.090784	0.082184	0.073958	0.066180	0.058772	0.052160	0.045969
$\pi S\pi S$		0.104712	0.095164	0.086250	0.077715	0.069593	0.062072	0.055054	0.048634	0.042715
$S\pi S\pi$		0.073449	0.064242	0.055830	0.047991	0.041513	0.035546	0.030311	0.025748	0.021794
$\pi\pi\pi S$		0.019678	0.018399	0.015581	0.013779	0.012015	0.010458	0.009057	0.007750	0.006704
$\pi S\pi\pi$		0.016117	0.014891	0.012245	0.010465	0.008994	0.007661	0.006497	0.005548	0.004621
$\sigma\sigma S\sigma$		-0.103935	-0.100732	-0.096314	-0.090971	-0.085013	-0.078699	-0.072243	-0.065820	-0.059639
$S\sigma\sigma\sigma$		-0.024235	-0.024995	-0.025371	-0.025051	-0.024228	-0.023125	-0.021707	-0.020118	-0.018435
$\sigma\sigma\sigma S$		0.085457	0.083005	0.079567	0.075369	0.070653	0.065625	0.060456	0.055345	0.050286
$\pi\sigma\pi S$		0.075397	0.073069	0.069817	0.065941	0.061632	0.056402	0.052300	0.047593	0.042995
$\pi S\pi\sigma$		-0.103015	-0.098621	-0.093184	-0.087091	-0.080567	-0.074556	-0.067206	-0.060676	-0.054401
$S\pi\pi\sigma$		0.011889	0.010739	0.009617	0.008513	0.007482	0.006446	0.005632	0.004837	0.004130
$\sigma\pi\pi S$		0.003749	0.003810	0.003533	0.003246	0.003074	0.002772	0.002357	0.002088	0.001826
$S\pi\pi\sigma$		-0.022995	-0.021238	-0.019387	-0.017506	-0.015658	-0.013784	-0.012222	-0.010684	-0.009279

$\pi\sigma\pi$	0.011852	0.010751	0.009629	0.008532	0.007491	0.006421	0.005639	0.004845	0.004136
$\pi^*\sigma\pi$	0.005410	0.004635	0.004126	0.003628	0.003289	0.002856	0.002342	0.001998	0.001687
$\sigma\sigma\sigma$	-0.044936	-0.054654	-0.060342	-0.063781	-0.065162	-0.064682	-0.062525	-0.059939	-0.056347
$\pi\sigma\pi\sigma$	-0.040666	-0.048329	-0.053368	-0.056136	-0.057063	-0.056863	-0.054696	-0.052144	-0.049032
$\sigma\pi\sigma\pi$	0.073695	0.064578	0.056196	0.048773	0.042050	0.036078	0.030819	0.026221	0.022206
$\sigma\pi\pi\sigma$	-0.003283	-0.003604	-0.003707	-0.003566	-0.003344	-0.003160	-0.002936	-0.002681	-0.002361
$\pi\sigma\sigma\pi$	0.007280	0.006582	0.005887	0.005169	0.004428	0.003719	0.003040	0.002619	0.002342
$\pi\pi\pi\pi$	0.073414	0.064074	0.055588	0.048004	0.041243	0.035279	0.030055	0.025729	0.021589
$\pi^*\sigma^*\sigma^*\pi$	0.006954	0.005911	0.004994	0.004197	0.003508	0.002897	0.002418	0.001925	0.001641

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On the Hydrodynamics of Degenerating Bose-Einstein Gases

Sadao NAKAJIMA

Physical Institute, Nagoya University

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The hydrodynamic equations of the degenerating ideal Bose-Einstein gas are derived, assuming that the distribution function of gas atoms has a singularity of δ -type in momentum space:

$$f = N_s \delta(\mathbf{p} - \mathbf{P}_s) + f_n.$$

For f_n , a particular approximate solution is proposed, in which the mean momentum of the normal atoms is different from the momentum of the condensed atoms, \mathbf{P}_s . The resulting thermohydrodynamic equations are very similar to those of the two fluid model of liquid helium II. In both cases, for instance, the two component fluids exert the frictional forces on each other. Nevertheless the degenerating ideal Bose-Einstein gas does not exhibit such peculiar properties as second sound, fountain pressure, etc., because of the vanishing chemical potential in this case.

§ 1. Introduction

In its recent development, the two fluid theory has accomplished successfully a consistent description of such peculiar properties of liquid helium II as second sound, fountain effect, anomalous heat conductivity, etc. The theory however, should be regarded, at least from the formalistic point of view, as a *phenomenological* model. After having analyzed the most of important phenomena, we are now to attack the problem to justify the two fluid model on the basis of the molecular kinetic theory.

As F. London¹⁾ explained in detail, the theory of the degenerating ideal Bose-Einstein gas played historically the suggestive role in the development of the two fluid model. It is well known that the ideal B. E. gas begins "the condensation in momentum space" at a certain temperature. That is, the number of the atoms existing at the lowest energy level begins to increase rapidly with decreasing temperature. This condensation in momentum space can also be interpreted as a kind of equilibrium between two phases and the degenerating gas as a mixture of two component fluids, separated not in ordinary space but in momentum space. Let N_n be the number of the excited atoms per unit volume at temperature T , then we have

$$N_n = 2.612 [2\pi m k / h^3]^{3/2} T^{3/2}, \quad (1.1)$$

where k is Boltzmann's constant, h Planck's constant, and m the mass of the single atom. The remaining atoms are all condensed at the lowest energy level. Only the excited atoms make contribution to the thermal properties of the gas.

For instance, the entropy per unit volume is given by

$$S = S_0 N_n. \quad (1.2)$$

S_0 is a constant given by

$$S_0 = (5 \times 2.612 / 2 \times 1.341) Nk,$$

where N is the total number of atoms per unit volume. Thus, as for the *thermostatic* properties, one can find certain parallelism between the degenerating ideal B. E. gas and liquid helium II. The parallelism might presumably be due to the feature of symmetry peculiar to Bose-Einstein particles and be retained also with respect to the *thermodynamic* properties. No molecular kinetic theory, however, has been developed which derives rigorously the two fluid model from this feature of symmetry. It will certainly be a very difficult problem. As the first step to the solution, it may therefore have some importance to examine the thermohydrodynamics of the degenerating ideal B. E. gas, which is the purpose of the present paper.

There exist indeed the theories of Landau²⁾ and Green,³⁾ in which B. E. statistics does not necessarily play an essential role. The recent experiments on the properties of the isotope He^3 , however, indicate that B. E. statistics should decidedly be important. As for the quantum effect due to the small mass of helium atoms, the zero-point motion in He^3 is more violent than that in He^4 . In fact, the recent experiment⁴⁾ shows that He^3 is also the quantum liquid. That is, under the ordinary pressure, He^3 remains as the liquid even at the absolute zero of temperature and the pressure necessary to solidify liquid He^3 is higher than that in the case of liquid He^4 . On the other hand, there is no indication of superfluidity or the lambda transition of liquid He^3 . We should therefore conclude that one can not explain the peculiarities of liquid helium II by consideration of the quantum mechanical zero-point motion alone and without taking account of B. E. statistics.

In § 2, we formulate the equations of motion in the two fluid theory in the form suitable to be compared with those derived in the succeeding sections. In § 3, the thermohydrodynamic equations of the degenerating ideal B. E. gas are derived, assuming that the distribution function of gas atoms has a singularity of δ -function type in momentum space. In § 4, a particular approximate distribution function is proposed and the results are compared with the two fluid model formulated in § 2.

§ 2. The two fluid model

We shall formulate the two fluid model by means of the method of Tolman and Fine.⁵⁾ We consider liquid helium II as a sort of fluid mixture consisting of the normal and the superfluids. Let the density and the velocity of the normal fluid be ρ_n and \mathbf{V}_n , those of the superfluid ρ_s and \mathbf{V}_s . The total density is given

by $\rho = \rho_n + \rho_s$ and the velocity of the center of gravity by $\mathbf{V}_1 = (\rho_n/\rho) \mathbf{V}_n + (\rho_s/\rho) \mathbf{V}_s$.

For the entropy s per unit mass of liquid helium II, Tisza⁶⁾ assumed that

$$s = s_\lambda(\rho_n/\rho), \quad (2.1)$$

where s_λ is a constant representing the entropy at the lambda point (2.19°K). The detailed discussion of the assumption was given elsewhere.⁷⁾ Empirically the assumption is justified for the temperature range from the lambda point to 1°K, while below 1°K it is in contradiction to the experimental result of second sound. The general formalism of the two fluid model without the use of the assumption (2.1) has been also developed by Gorter⁸⁾ and Usui.⁹⁾ For the sake of simplicity, however, we assume here "Tisza's relation" (2.1).

The chemical potential g in general should be regarded as a function of pressure p , temperature T , and the normal fluid concentration $\xi = (\rho_n/\rho)$. In thermal equilibrium, ξ is the function of p and T , which is determined by the condition

$$(\partial g / \partial \xi)_{pT} = 0. \quad (2.2)$$

We assume that this condition is always satisfied everywhere, even in non-stationary states.

From the assumption (2.1), the partial entropy of the superfluid vanishes and that of the normal fluid is equal to s_λ . Hence the reversible flow of the entropy is given by

$$s_\lambda \rho_n \mathbf{V}_n = \rho s \mathbf{V}_n. \quad (2.3)$$

From the assumption (2.2), the partial enthalpy of the superfluid is equal to g and that of the normal fluid to $g + T s_\lambda$. Hence the flow of the enthalpy is given by

$$g \rho_s \mathbf{V}_s + (g + T s_\lambda) \rho_n \mathbf{V}_n = (\rho u + p) \mathbf{V}_1 + \rho s T (\mathbf{V}_n - \mathbf{V}_1), \quad (2.4)$$

where u is the internal energy per unit mass. Thus, the first and the second laws of thermodynamics take the following forms:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_n V_n^2 + \frac{1}{2} \rho_s V_s^2 + \rho u \right) + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho_n V_n^2 V_{nj} + \frac{1}{2} \rho_s V_s^2 V_{sj} + \right. \\ \left. + (\rho u + p) V_{1j} + T \rho s (V_{nj} - V_{1j}) + q_j + V_{ni} \pi_{ij}^* \right) = 0, \end{aligned} \quad (2.5)$$

$$\frac{\partial}{\partial t} (\rho s) + \frac{\partial}{\partial x_j} \left(\rho s V_{nj} + \frac{q_j}{T} \right) = \left(\frac{\partial S}{\partial t} \right)_{irr}, \quad (2.6)$$

where π^* is the viscous stress tensor, q the heat current, and $(\partial S / \partial t)_{irr}$ the rate of irreversible production of entropy. The suffix i or j distinguishes the three cartesian components and one should as usual sum over double indices.

The conservation law of mass is decomposed into

$$\frac{\partial \rho_n}{\partial t} + \frac{\partial}{\partial x_j} (\rho_n V_{nj}) = \Gamma, \quad \frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x_j} (\rho_s V_{sj}) = -\Gamma, \quad (2.7)$$

where Γ is the rate of transformation from the superfluid to the normal one. From Eq. (2.7) and the conservation law of momentum,

$$\frac{\partial}{\partial t} (\rho_n V_{ni} + \rho_s V_{si}) + \frac{\partial}{\partial x_j} (\rho_n V_{ni} V_{nj} + \rho_s V_{si} V_{sj} + p \delta_{ij} + \pi_{ij}^*) = 0, \quad (2.8)$$

we have

$$\rho_s \frac{dV_{si}}{dt} + \rho_n \frac{dV_{ni}}{dt} + \frac{\partial p}{\partial x_i} + \frac{\partial \pi_{ij}^*}{\partial x_j} + \Gamma (V_{ni} - V_{si}) = 0. \quad (2.9)$$

From Eqs. (2.5), (2.6), (2.9), and the thermostatic relation

$$du = T ds + (p/\rho^2) d\rho, \quad (2.10)$$

we have

$$\rho_s \frac{dV_{si}}{dt} = -\frac{\rho_s}{\rho} \frac{\partial p}{\partial x_i} + s \rho_s \frac{\partial T}{\partial x_i} - \frac{\Gamma}{2} (V_{ni} - V_{si}) + f_i \quad (2.11)$$

and

$$T \left(\frac{\partial S}{\partial t} \right)_{rr} = -\frac{q_j}{T} \frac{\partial T}{\partial x_j} - V_{ni} \frac{\partial \pi_{ij}^*}{\partial x_j} + f_i (V_{ni} - V_{si}). \quad (2.12)$$

f_i is the separation parameter, which is, from the phenomenological point of view, undetermined except that

$$f_i (V_{ni} - V_{si}) \geq 0. \quad (2.13)$$

In our previous paper,¹⁰⁾ we have assumed that f_i gives the mutual friction between two fluids, which was introduced by Gorter. There is another possibility pointed out by Zilsel.¹¹⁾ That is, we put

$$f_i = (\Gamma/2) (V_{ni} - V_{si}). \quad (2.14)$$

Then, from Eq. (2.11),

$$\frac{dV_{si}}{dt} = -\frac{\partial q}{\partial x_i}, \quad (2.15)$$

and, from Eq. (2.9),

$$\rho_n \frac{dV_{ni}}{dt} = -\frac{\rho_n}{\rho} \frac{\partial p}{\partial x_i} - s \rho_s \frac{\partial T}{\partial x_i} - \Gamma (V_{ni} - V_{si}). \quad (2.16)$$

In this case, the mutual friction arises from the term representing the transformation from one component fluid to the other, since from Eq. (2.13) it must be that

$$\Gamma \geq 0. \quad (2.17)$$

In our phenomenological derivation, however, it is uncertain whether the condition (2.17) is really satisfied.

§ 3. Hydrodynamics of the degenerating ideal B.E. gas

We shall consider the ideal B.E. gas. The number of atoms, which are situate in the volume element $d\mathbf{r}$ and have the momentum lying in the range $d\mathbf{p}$, is equal to $f(\mathbf{r}, \mathbf{p})d\mathbf{r}d\mathbf{p}/h^3$ by definition. The change of the distribution function f is governed by Boltzmann's equation

$$\frac{\partial f}{\partial t} + \frac{p_j}{m} \frac{\partial f}{\partial x_j} = \Delta f, \quad (3.1)$$

where we consider for simplicity the case of no external force. Δf on the right of Eq. (3.1) represents the change of the distribution function arising from collisions and is, in the case of B.E. statistics, given by¹²⁾

$$\begin{aligned} \Delta f = & \int A(p, p_1; p_2 p_3) [f(p_2)f(p_3)(1+f(p))(1+f(p_1)) \\ & - f(p)f(p_1)(1+f(p_2))(1+f(p_3))] d\omega_1 d\omega_2 d\omega_3, \end{aligned} \quad (3.2)$$

where A is the transition probability of the collision $p + p \rightleftharpoons p_2 + p_3$, and $d\omega = d\mathbf{p}/h^3$.

As we have seen in § 2, the distribution function of the degenerating gas in equilibrium has a singularity of δ -function type in momentum space:

$$f_0 = N_s \delta(\mathbf{p} - \mathbf{P}) + f_n(\mathbf{p} - \mathbf{P}), \quad (3.3)$$

where

$$f_n(\mathbf{p} - \mathbf{P}) = [\exp \{ (\mathbf{p} - \mathbf{P})^2 / 2mkT \} - 1]^{-1}. \quad (3.4)$$

In fact, substituting Eq. (3.3) in Eq. (3.1), the right hand side of the latter vanishes (see § 4). If the number N_s of the condensed atoms, temperature T , and the mean momentum \mathbf{P} of the gas are all constant in time and uniform in space, the left side of Eq. (3.1) also vanishes.

We shall assume *a priori* that the distribution function in general has a singularity of δ -function type in momentum space:

$$f = N_s \delta(\mathbf{p} - \mathbf{P}_s) + f_n. \quad (3.5)$$

N_s is the number of the condensed atoms per unit volume and \mathbf{P}_s is their momentum, which is not necessarily equal to the mean momentum \mathbf{P} of the total atoms. We define the mean momentum \mathbf{P}_n of the normal atoms by

$$N_n \mathbf{P}_n = \int \mathbf{p} f_n d\omega, \quad N_n = \int f_n d\omega. \quad (3.6)$$

Now we multiply Eq. (3.1) by 1, p_i , and $(p^2/2m)$ respectively and then

integrate them in momentum space. From the conservation laws,

$$\int \Delta f \cdot d\omega = 0, \quad \int p_i \Delta f \cdot d\omega = 0, \quad \int (p^2/2m) \Delta f \cdot d\omega = 0, \quad (3.7)$$

we obtain the following results:

$$\frac{\partial}{\partial t} (N_n + N_s) + \frac{\partial}{\partial x_j} (N_n P_{nj} + N_s P_{sj}) = 0, \quad (3.8)$$

$$\frac{\partial}{\partial t} (N_n P_{ni} + N_s P_{si}) + \frac{\partial}{\partial x_j} (N_n P_{ni} P_{nj} + N_s P_{si} P_{sj} + \pi_{ij}) = 0, \quad (3.9)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2m} N_n P_n^2 + \frac{1}{2m} N_s P_s^2 + E \right) + \frac{\partial}{\partial x_j} \left(\frac{1}{2m} N_n P_n^2 P_{nj} + \right. \\ \left. + \frac{1}{2m} N_s P_s^2 P_{sj} + E \frac{P_{nj}}{m} + \pi_{ij} \frac{P_{ni}}{m} + q_j \right) = 0. \end{aligned} \quad (3.10)$$

Here

$$\pi_{ij} = \int (1/m) (p_i - P_{ni}) (p_j - P_{nj}) f_n d\omega \quad (3.11)$$

is the stress tensor.

$$E = \int [(p - P_n)^2 / 2m] f_n d\omega \quad (3.12)$$

is the thermal energy per unit volume.

$$q_j = \int [(p - P_n)^2 / 2m] [(p_j - P_{nj}) / m] f_n d\omega \quad (3.13)$$

is the heat current. These quantities are defined in reference to the distribution and the mean momentum of the normal atoms. Similarly we define the entropy per unit volume by

$$S = k \int \sigma f_n d\omega, \quad (3.14)$$

where

$$\sigma = f_n^{-1} \ln(1 + f_n) + \ln(1 + f_n^{-1}). \quad (3.15)$$

Now the equation for the distribution function f_n takes the form

$$\frac{\partial f_n}{\partial t} + \frac{p_j}{m} \frac{\partial f_n}{\partial x_j} = \gamma, \quad (3.16)$$

where

$$\gamma = \Delta f - [(\partial/\partial t) + (p_j/m)(\partial/\partial x_j)] N_s \delta(p - P_s). \quad (3.17)$$

We multiply Eq. (3.16) by σ and integrate it in momentum space. Then we have

$$\frac{\partial}{\partial t} \int \sigma f_n d\omega + \frac{\partial}{\partial x_j} \int \frac{p_j}{m} \sigma f_n d\omega = \int \gamma \cdot \ln(1 + f_n^{-1}) d\omega,$$

which is transcribed as

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x_j} \left(S \frac{P_{nj}}{m} + \Sigma_j \right) = \left(\frac{\partial S}{\partial t} \right)_{irr}, \quad (3.18)$$

where

$$\Sigma_j = k \int (\sigma/m) (p_j - P_{nj}) f_n d\omega, \quad (3.19)$$

$$\left(\frac{\partial S}{\partial t} \right)_{irr} = k \int \gamma \cdot \ln(1 + f_n^{-1}) d\omega. \quad (3.20)$$

If f is equal to f_0 given by (3.3), q_j , Σ_j and $(\partial S/\partial t)_{irr}$ vanish, E is equal to the internal energy per unit volume in thermal equilibrium, and π_{ij} is the unit tensor multiplied by the hydrostatic pressure $p = (2/3)E$.

§ 4. A particular approximate distribution function

In the first place, we shall show that the force acting on the condensed atoms vanishes generally, so long as the distribution function has the form (3.5). Substitution of (3.5) in (3.2) gives the expression having the singularity of δ -function type. On the other hand, substitution of (3.5) in the left hand side of (3.1) gives the expression containing the derivative of δ -function, the coefficient of which therefore necessarily vanishes. Thus we have

$$\frac{\partial P_{sj}}{\partial t} + \frac{P_{sj}}{m} \frac{\partial P_{st}}{\partial x_j} = 0. \quad (4.1)$$

Now the conservation law of mass, (3.8), is decomposed into

$$\frac{\partial N_n}{\partial t} + \frac{\partial}{\partial x_j} (N_n P_{nj}) = \Gamma, \quad \frac{\partial N_s}{\partial t} + \frac{\partial}{\partial x_j} (N_s P_{sj}) = -\Gamma, \quad (4.2)$$

where Γ is given by integration of (3.17). From Eqs. (3.7) and (4.1),

$$\begin{aligned} I &= - \int \left(\frac{\partial N_s}{\partial t} + \frac{p_j}{m} \frac{\partial N_s}{\partial x_j} \right) \delta(\mathbf{p} - \mathbf{P}_s) d\omega \\ &= N_s^2 \int A(P_s, P_s; p_2, p_3) [(1 + f_n(p_2))(1 + f_n(p_3)) \\ &\quad - f_n(p_2)f_n(p_3)] d\omega_2 d\omega_3 \\ &\quad + N_s \int A(P_s, p_1; p_2, p_3) [f_n(p_1)(1 + f_n(p_2))(1 + f_n(p_3)) \\ &\quad - (1 + f_n(p_1))f_n(p_2)f_n(p_3)] d\omega_1 d\omega_2 d\omega_3. \end{aligned} \quad (4.3)$$

From the conservation law of momentum, (3.9), and Eq. (3.1),

$$N_n \left(\frac{\partial}{\partial t} + \frac{P_{nj}}{m} \frac{\partial}{\partial x_j} \right) P_{ni} + \frac{\partial}{\partial x_j} \pi_{ij} + \frac{\Gamma}{m} (P_{ni} - P_{si}) = 0. \quad (4.4)$$

Comparing (4.1) and (4.4) with the corresponding equations in § 2, we see that there is no fountain pressure proportional to temperature gradient in the degenerating ideal B.E. gas.

Now we introduce here a particular approximate distribution function

$$f = f_0^* + f_{n1}, \quad (4.5)$$

where

$$f_0^* = N_s \delta(\mathbf{p} - \mathbf{P}_s) + f_{n1}(\mathbf{p} - \mathbf{P}_n). \quad (4.6)$$

$f_{n0}(\mathbf{p} - \mathbf{P}_n)$ has the same functional form as (3.4) with the mean momentum \mathbf{P}_n , instead of \mathbf{P} . f_{n1} represents a small correction term. The definition of \mathbf{P}_n , (3.6), gives the restriction on f_{n1}

$$\int (\mathbf{p}_i - \mathbf{P}_{ni}) f_{n1} d\omega = 0. \quad (4.7)$$

We shall define the parameter T contained in f_{n0} through the condition

$$\int (1/2m) (\mathbf{p} - \mathbf{P}_n)^2 f_{n1} d\omega = 0. \quad (4.8)$$

Hence the thermal energy E is the same function of temperature T as in thermal equilibrium and the stress tensor takes the form

$$\pi_{ij} = p \delta_{ij} + \pi_{ij}^*,$$

where p is the hydrostatic pressure given by $(2/3)E$ and

$$\pi_{ij}^* = \int (1/m) (\mathbf{p}_i - \mathbf{P}_{ni}) (\mathbf{p}_j - \mathbf{P}_{nj}) f_{n1} d\omega. \quad (4.9)$$

To the first order of f_{n1} , the entropy (3.14) is calculated as

$$\begin{aligned} S = & k \int [\ln(1 + f_{n0}) + f_{n0} \ln(1 + f_{n0}^{-1})] d\omega \\ & + k \int f_{n1} \ln(1 + f_{n0}^{-1}(\mathbf{p} - \mathbf{P}_n)) d\omega. \end{aligned}$$

The first term represents the entropy in thermal equilibrium, while the second term vanishes because of the condition (4.8). Thus, the thermostatic relation (2.10) still remains valid.

The flow of the entropy in (3.10) reduces to

$$\begin{aligned} \Sigma_j &= k \int f_{n1} \ln(1 + f_{n0}^{-1}(\mathbf{p} - \mathbf{P}_n)) (1/m) (\mathbf{p}_j - \mathbf{P}_{nj}) d\omega \\ &= q_j/T. \end{aligned} \quad (4.10)$$

Finally the flow of the enthalpy is, from (3.10), equal to

$$(E+p)(\mathbf{P}_n/m) = (G+TS_0N_n)(\mathbf{P}_n/m). \quad (4.11)$$

This has just the same form as obtained in § 2, since the chemical potential $G=E+p-TS$ in the degenerating ideal B.E. gas vanishes.

We can therefore apply the method of Tolman and Fine to the conservation laws (3.8), (3.9), (3.10) and (3.18), and obtain the same results as derived in § 2. In fact, if we put $\eta=0$ in the equation (2.15), then we have the equation (4.1). The equation (2.16) also agrees with the equation (4.4), since in the degenerating gas

$$(\partial g/\partial x_i) = (1/\rho)(\partial p/\partial x_i) - s(\partial T/\partial x_i) = 0.$$

The lack of second sound or fountain pressure in the ideal B.F. gas may therefore be attributed to the fact that the chemical potential vanishes in this case.

Now we calculate explicitly the rate of irreversible production of entropy, neglecting the terms of higher order in f_{n1} . From (3.20),

$$\left(\frac{\partial S}{\partial t}\right)_{irr} = k \int \gamma \ln(1+f_{n0}^{-1}) d\omega - k \int \frac{1}{1+f_{n0}} \frac{f_{n1}}{f_{n0}} \gamma d\omega. \quad (4.12)$$

To compute the second term, we may put

$$\gamma \simeq [(\partial/\partial t) + (p_j/m)(\partial/\partial x_j)] f_{n0}(\mathbf{p} - \mathbf{P}_n).$$

Then, taking account of the conditions (4.7) and (4.8), the second term reduces to

$$q_j \frac{\partial}{\partial x_j} \left(\frac{1}{T} \right) - \frac{1}{T} \frac{\pi_{ij}^*}{m} \frac{\partial P_{ni}}{\partial x_j}.$$

The first term of (4.12) is calculated by means of (4.7) and (3.17). The results are summarized as

$$T \left(\frac{\partial S}{\partial t} \right)_{irr} = I' \frac{(\mathbf{P}_s - \mathbf{P}_n)^2}{2m} - \frac{q_j}{T} \frac{\partial T}{\partial x_j} - \frac{\pi_{ij}^*}{m} \frac{\partial P_{ni}}{\partial x_j}, \quad (4.13)$$

which agrees with the result obtained by the method of Tolman and Fine in § 2.

Though it is not easy to prove the positive definite character of I' in general, the proof is very simple in the case of the particular distribution f_0^* . We put $f_n = f_{n0}(\mathbf{p} - \mathbf{P}_n)$ in Eq. (4.3), then we have

$$\begin{aligned} I' = & N_s^2 \int A(P_s, P_s; p_2, p_3) f_{n0}(p_2) f_{n0}(p_3) \times \\ & \times [\exp \{ (\mathbf{P}_s - \mathbf{P}_n)^2 / m k T \} - 1] d\omega_2 d\omega_3 \\ & + N_s \int A(P_s, p_1; p_2 p_3) f_{n0}(p_1) f_{n0}(p_2) f_{n0}(p_3) \times \\ & \times \exp \{ (\mathbf{p}_1 - \mathbf{P}_n)^2 / 2 m k T \} [\exp \{ (\mathbf{P}_s - \mathbf{P}_n)^2 / 2 m k T \} - 1] d\omega_1 d\omega_2 d\omega_3, \end{aligned} \quad (4.14)$$

since the probability A does not vanish only when the conservation laws of energy and momentum are satisfied. We can see that $\Gamma=0$ if $\mathbf{P}_s=\mathbf{P}_n$ and $\Gamma'>0$ otherwise.

§ 5. Summary

The distribution function of the degenerating ideal Bose-Einstein gas in thermal equilibrium has a singularity of δ -type in momentum space.

Even if the distribution of the normal atoms with reference to their mean momentum remains unaltered, the deviation of the latter quantity from the momentum of the condensed atoms does cause the irreversible process, in which the mutual friction acts between the two fluids.

The deviation in the functional form from equilibrium distribution gives rise to the ordinary irreversible processes such as viscous flow, conduction of heat, etc.

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On the So-called Pseudo Spinors

Ken-iti GOTO

Department of Physics, Osaka University

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Recently Yang has treated with new interaction terms of Fermi type, considering the so-called pseudo spinors. In this paper we want to give some remarks about the representation by pseudo spinors.

The spinor representation of the rotation or the Lorentz group is on the one hand "two-valued" and on the other hand "up to a sign" representation. These two kinds of representations are quite similar but not of the same idea. Corresponding to an arbitrary group element $a=(a_{ij})$, there exists a spinor transformation S , satisfying the relation

$$S P_i S^{-1} = \sum_j a_{ji} P_j \quad (1)$$

where P_1, \dots, P_4 are appropriate matrices obeying to the commutation relation

$$P_i P_j + P_j P_i = 2g_{ij}. \quad (2)$$

The correspondence $a \rightarrow S(a)$ constructs the so-called spinor representation of the group. From the relation (1), if $S(a)$ corresponds to a , all $\rho S(a)$ also correspond to a , where ρ is an arbitrary number. The correspondence

$$a \rightarrow \rho S(a), \text{ or all } S(a) \text{ of (1)} \quad (3)$$

forms a many valued representation, or the so-called ray representation. But this can be reduced, at most to the two valued representation by the condition

$$(\det S(a))^2 = 1. \quad (4)$$

In this case the correspondence is

$$a \rightarrow \pm S(a). \quad (5)$$

This is a two-valued continuous representation of the group and a one-valued continuous representation of its covering group. On the other hand correspondence

$$a \rightarrow \text{an arbitrary } S(a) \text{ of (1)} \quad (6)$$

constructs a representation up to a factor, in the sense that

$$S(a) S(b) = \omega S(ab) \quad (7)$$

where ω is a number. This representation is one valued but "im Grossen" not continuous. Also in this case we can confine ω , at most, to ± 1 , by the condition (4), so the correspondence

$$a \rightarrow S(a) \quad (8)$$

constructs a representation up to a sign, in the sense that

$$S(a) S(b) = \pm S(ab). \quad (9)$$

This representation is also one valued but not continuous. We can not construct one-valued continuous spinor representation of the rotation or the Lorentz group because of the topological nature of the group.

First let us consider the case of the rotation group. This group consists of the two not connected branches, proper rotation a^+ and improper rotation a^- . Let us consider the two sorts of correspondence

$$a^+ \rightarrow S^+ \text{ (or } \pm S^+), a^- \rightarrow S^- \text{ (or } \pm S^-) \quad (10a)$$

and

$$a^+ \rightarrow S^+ \text{ (or } \pm S^+), a^- \rightarrow -S^- \text{ (or } \mp S^-). \quad (10b)$$

To distinct between these two sorts is meaningless in the sense of two valued representation. But it has meaning in the sense of the one-valued continuous representation of the covering group or of the one-valued not continuous representation up to a sign of the group. Thus in the latter two cases we can admit the so-called pseudospinors. Next the Lorentz group consists of four not connected branches,

a_+^+ : orthochronous proper Lorentz transformation,

a_-^+ : antichronous proper " " "

a_+^- : orthochronous improper " " "

a_-^- : antichronous improper " " "

Now let us take the following matrices as P_1, \dots, P_4

$$P_1 = \begin{pmatrix} 0 & iP \\ -iP & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & iQ \\ -iQ & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & iR \\ -iR & 0 \end{pmatrix}, P_4 = \begin{pmatrix} 0 & iE \\ -iE & 0 \end{pmatrix} \quad (12)$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case of the orthochronous proper transformation a_+^+ , spinor

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\psi_1, \psi_2; \varphi^{\dot{1}}, \varphi^{\dot{2}}) \quad (13)$$

transforms as

$$S_+^+ = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad (14)$$

where

$$S_1 = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_2 = \begin{pmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{a} \end{pmatrix}, \quad (15)$$

$$\det \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix} = \det \begin{pmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{a} \end{pmatrix} = 1.$$

In the case of typical transformations in other three branches a_+^+ , a_+^- , a_-^- , spinor transforms as follows.

(a_+^+) space and time inversion: $x_1 \rightarrow -x_1$, $x_2 \rightarrow -x_2$, $x_3 \rightarrow -x_3$, $x_4 \rightarrow -x_4$,

$$S_+^+ = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad (16)$$

$$(\Psi_1 \rightarrow \Psi_1, \Psi_2 \rightarrow \Psi_2, \Psi_3 \rightarrow -\Psi_3, \Psi_4 \rightarrow -\Psi_4),$$

$$(\phi_1 \rightarrow \phi_1, \phi_2 \rightarrow \phi_2, \phi^{\dot{1}} \rightarrow -\phi^{\dot{1}}, \phi^{\dot{2}} \rightarrow -\phi^{\dot{2}}).$$

(a_+^-) space inversion: $x_1 \rightarrow -x_1$, $x_2 \rightarrow -x_2$, $x_3 \rightarrow -x_3$, $x_4 \rightarrow x_4$,

$$S_+^- = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}, \quad (17)$$

$$(\Psi_1 \rightarrow \Psi_3, \Psi_2 \rightarrow \Psi_4, \Psi_3 \rightarrow \Psi_1, \Psi_4 \rightarrow \Psi_2),$$

$$(\phi_1 \rightarrow \phi^{\dot{1}}, \phi_2 \rightarrow \phi^{\dot{2}}, \phi^{\dot{1}} \rightarrow \phi_1, \phi^{\dot{2}} \rightarrow \phi_2).$$

(a_-^-) time inversion: $x_1 \rightarrow x_1$, $x_2 \rightarrow x_2$, $x_3 \rightarrow x_3$, $x_4 \rightarrow -x_4$,

$$S_-^- = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad (18)$$

$$(\Psi_1 \rightarrow \Psi_3, \Psi_2 \rightarrow \Psi_4, \Psi_3 \rightarrow -\Psi_1, \Psi_4 \rightarrow -\Psi_2),$$

$$(\phi_1 \rightarrow \phi^{\dot{1}}, \phi_2 \rightarrow \phi^{\dot{2}}, \phi^{\dot{1}} \rightarrow -\phi_1, \phi^{\dot{2}} \rightarrow -\phi_2).$$

Such as there are two sorts of spinor transformations (10a) (10b) in the case of the rotation group, there are four sorts of spinors, in the sense of one-valued continuous representation of the covering group of the Lorentz group, transforming as follows according to a_+^+ , a_+^- , a_-^+ , a_-^- respectively.

Spinor of the first kind Ψ^I :

$$\begin{array}{cccc} a_+^+ : & a_+^- : & a_-^+ : & a_-^- : \\ \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, & \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix}, & \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & S_1 \\ -S_2 & 0 \end{pmatrix}. \end{array} \quad (19a)$$

Spinor of the second kind Ψ^{II} :

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix}, \begin{pmatrix} 0 & -S_1 \\ -S_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -S_1 \\ S_2 & 0 \end{pmatrix}. \quad (19b)$$

Spinor of the third kind Ψ^{III} :

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} -S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -S_1 \\ S_2 & 0 \end{pmatrix}. \quad (19c)$$

Spinor of the fourth kind Ψ^{IV} :

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} -S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \begin{pmatrix} 0 & -S_1 \\ -S_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & S_1 \\ -S_2 & 0 \end{pmatrix}. \quad (19d)$$

But (19a) and (19b) as well as (19c) and (19d) have the same trace of transformation matrices, so the representations by Ψ^I and Ψ^{II} as well as those by Ψ^{III} and Ψ^{IV} are respectively equivalent.

Indeed, if we indicate with $A \sim B$ that A transforms as B , the following relations hold between these four sorts of spinors.

$$(\Psi_1^{II}, \Psi_2^{II}, \Psi_3^{III}, \Psi_4^{II}) \sim (\Psi_1^I, \Psi_2^I, -\Psi_3^I, -\Psi_4^I), \quad (20a)$$

$$(\Psi_1^{IV}, \Psi_2^{IV}, \Psi_3^{IV}, \Psi_4^{IV}) \sim (\Psi_1^{III}, \Psi_2^{III}, -\Psi_3^{III}, -\Psi_4^{III}) \quad (20b)$$

or

$$\begin{aligned} \Psi^{II} &\sim P_5 \Psi^I, \\ \Psi^{IV} &\sim P_5 \Psi^{III}, \end{aligned} \quad (21)$$

where

$$P_5 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} = -iP_1P_2P_3P_4. \quad (22)$$

As $P_5\Psi^{II}$ is a spinor of the first kind, we can put

$$P_5\Psi^{II} = \Phi^I, \quad (23)$$

and then

$$\tilde{\Psi}^I P_5 \Psi^{II} = \tilde{\Psi}^I \Phi^I, \quad (24)$$

($\tilde{\Psi}$ is contragradient to Ψ).

Therefore $\tilde{\Psi}^I P_5 \Psi^{II}$ is not a new form of invariant at all. But if spinors must be solutions of a certain wave equation, another circumstance will arise. When Ψ^{II} is a solution of the given wave equation, the corresponding spinor $\Phi^I = P_5 \Psi^{II}$ is not always another solution of it and may be different from the solution Ψ^I . So only when Φ^I is not a solution, or when the wave equation is not invariant under the transformation P_5 of representation bases, $\tilde{\Psi}^I P_5 \Psi^{II}$ becomes a new form of invariant.

If we look upon the spinor transformation as a one valued but not continuous up to a sign representation of the Lorentz group, we can take at random the signs of S in (16), (17), (18), and there are many more sorts of spinors.

Finally we wish to mention that the same discussion as above made about the first rank spinor is also possible about any higher and odd rank spinor and expinor.

In concluding the author should express his thanks to Prof. K. Husimi, Prof. G. Araki and Prof. S. Watanabe for their kind interests and useful discussions to this problem about the so called pseudo spinors.

Note on the Bloch-Nordsieck's Method

Gyo TAKEDA, Yasutaka TANIKAWA, Tosiya TANIUTI

Physical Department, Kobe University

and

Keiiti SAEKI

Faculty of Education, Kobe University

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The Bloch-Nordsieck's method has been applied to mesonic systems by several authors. Here we examine the convergence character of this method and show that it is about the same with that of the current perturbation method. So the $B-N$'s method will be unsuccessful if the current perturbational treatment is incorrect for mesonic systems.

§ 1. The Bloch-Nordsieck's method applied to mesonic systems

Since the Bloch-Nordsieck's method¹⁾ was successfully applied to radiative processes in order to overcome the infrared catastrophe of electromagnetic systems, this same method has been applied also to mesonic systems.²⁾ But, in latter cases, careful attention was never made about the applicability of it. So we want to inquire how is the case compared with the current perturbation method.

Here, as an illustration, we take up the symmetrical pseudoscalar meson theory with pseudovector coupling. Employing the same notations with those of the previous paper,³⁾ the Hamiltonian becomes

$$H = \alpha \cdot \mathbf{p} + m\beta + 1/2 \sum_{\alpha, \mathbf{k}} \omega_{\mathbf{k}} (P_{\alpha, \mathbf{k}}^2 + Q_{\alpha, \mathbf{k}}^2) + (g/\mu) \sum_{\alpha, \mathbf{k}} (2\Omega\omega_{\mathbf{k}})^{-1/2} \tau_{\alpha} (\boldsymbol{\sigma} \cdot \mathbf{k} + \gamma_5 \omega_{\mathbf{k}}) (Q_{\alpha, \mathbf{k}} \cos \mathbf{k} \cdot \mathbf{r} - P_{\alpha, \mathbf{k}} \sin \mathbf{k} \cdot \mathbf{r}). \quad (1)$$

The $B-N$'s method consists essentially in replacing uncommuting quantities α , β , $\vec{\tau}$, $\boldsymbol{\sigma}$ by their classical representatives. To do this, we introduce three operators A , T and S , given by

$$A = \beta, \quad T = \mathbf{t} \cdot \vec{\tau}, \quad S = \mathbf{s} \cdot \boldsymbol{\sigma} \quad (2)$$

where \mathbf{s} and \mathbf{t} are classical unit vectors. All of them have eigenvalues 1 and -1 , so we can divide the wave function Ψ into eight parts according to signs of their eigenvalues

$$\Psi = (\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}, \chi^{++}, \chi^{+-}, \chi^{-+}, \chi^{--}). \quad (3)$$

ϕ and χ belongs to $A=1$ and -1 respectively. The first suffix on right shoulder of the wave function ϕ and χ corresponds to the sign of the eigenvalue of T , and the second to that of S .

From the Schrödinger equation

$$(H-E)\Psi=0, \quad (4)$$

we can make eight equations

$$\frac{1 \pm A}{2} \cdot \frac{1 \pm T}{2} \cdot \frac{1 \pm S}{2} (H-E)\Psi=0. \quad (5)$$

For example, preferring plus signs one and all, we have

$$\begin{aligned} & \{m + (1/2) \sum_{\alpha, k} \omega_k (P_{\alpha, k}^2 + Q_{\alpha, k}^2) - E \\ & + (g/\mu) \sum_{\alpha, k} (2Q\omega_k)^{-1/2} t_{\alpha}(\mathbf{s} \cdot \mathbf{k}) (Q_{\alpha, k} \cos \mathbf{k} \cdot \mathbf{r} - P_{\alpha, k} \sin \mathbf{k} \cdot \mathbf{r})\} \phi^{++} \\ & = - (g/\mu) \sum_{\alpha, k} (2Q\omega_k)^{-1/2} (Q_{\alpha, k} \cos \mathbf{k} \cdot \mathbf{r} - P_{\alpha, k} \sin \mathbf{k} \cdot \mathbf{r}) \\ & \times \{t_{\alpha}(\boldsymbol{\sigma} + \mathbf{s} \cdot \mathbf{k}) \phi^{+-} + (t_{\alpha} + \tau_{\alpha})(\mathbf{s} \cdot \mathbf{k}) \phi^{-+} + (t_{\alpha} + \tau_{\alpha})(\boldsymbol{\sigma} + \mathbf{s} \cdot \mathbf{k}) \phi^{--}\} \\ & - \gamma_5(\mathbf{s} \cdot \mathbf{p}) \chi^{++} - \{\gamma_5(\mathbf{s} \cdot \mathbf{p}) + (\boldsymbol{\alpha} \cdot \mathbf{p})\} \chi^{+-} \\ & - (g/\mu) \sum_{\alpha, k} (\omega_k/2Q)^{1/2} (Q_{\alpha, k} \cos \mathbf{k} \cdot \mathbf{r} - P_{\alpha, k} \sin \mathbf{k} \cdot \mathbf{r}) \\ & \times \{\gamma_5 t_{\alpha} \chi^{++} + \gamma_5 (t_{\alpha} + \tau_{\alpha}) \chi^{+-}\}. \end{aligned} \quad (6)$$

Other equations can be deduced from Eq. (6) by each combination of the following transformations:

- (i) $m \rightarrow -m$, $\phi \rightarrow \chi$ and $\chi \rightarrow \phi$.
- (ii) $\mathbf{t} \rightarrow -\mathbf{t}$ and change of the sign of first suffixes.
- (iii) $\mathbf{s} \rightarrow -\mathbf{s}$ and change of the sign of second suffixes.

When we attend to ϕ^{++} , the $B-N$'s approximation is to omit other parts of Ψ and solve Eq. (6) without the right side.

In this approximation, a wave function and an proper energy corresponding to a single nucleon are as follows:

$$\begin{aligned} \phi_0^{++} = & u^{++} Q^{-1/2} \exp i \{ \sum_{\alpha, k} A_{\alpha, k} \sin \mathbf{k} \cdot \mathbf{r} (Q_{\alpha, k} + (1/2) A_{\alpha, k} \cos \mathbf{k} \cdot \mathbf{r}) \} \\ & \times \Pi_{\alpha, k} h_0 (Q_{\alpha, k} + A_{\alpha, k} \cos \mathbf{k} \cdot \mathbf{r}), \end{aligned} \quad (7)$$

$$E_0 = m + \sum_{\alpha, k} (\omega_k/2) - (1/2) \sum_{\alpha, k} \omega_k A_{\alpha, k}^2, \quad (8)$$

where

$$A_{\alpha, k} = (g/\mu) (2Q\omega_k^3)^{-1/2} t_{\alpha}(\mathbf{s} \cdot \mathbf{k}) \quad (9)$$

and u^{++} is a unit spinor.

As an appraisal of errors introduced by this approximation, we shall adopt norms of other parts of Ψ and calculate them in the first approximation.

Then, solving Eq. (5), we obtain

$$\begin{aligned}
(1/12\pi)(g^2/4\pi)(K_c/\mu)^4 &> |\phi^{+-}|^2, |\phi^{-+}|^2 \\
&> (1/6\pi)(g^2/4\pi)(K_c/\mu)^2 \exp\{-(1/3\pi)(g^2/4\pi)(K_c/\mu)^2\}, \\
|\phi^{--}|^2 &\sim (1/3\pi)(g^2/4\pi)(K_c/\mu)^2, \\
|\chi^{++}|^2 &\sim (1/24\pi)(g^2/4\pi)(K_c/\mu)^2(K_c/m)^2, |\chi^{+-}|^2 \sim (1/240\pi)(g^2/4\pi)(K_c/m)^2, \\
|\chi^{-+}|^2 &\sim (1/16\pi)(g^2/4\pi)(K_c/\mu)^2(K_c/m)^2, |\chi^{--}|^2 = 0.
\end{aligned} \tag{10}$$

K_c is a cut-off meson momentum, and we have used a relation

$$m \gg K_c \gg \mu,$$

which is not satisfied in practical cases but sufficient for rough estimation. Exact expressions for $|\phi^{+-}|^2$ and $|\phi^{-+}|^2$ are

$$|\phi^{+-}|^2 = \sum_{n, n', \dots} |c^{+-}(n_{\alpha k}, n'_{\alpha' k'}, \dots)|^2, |\phi^{-+}|^2 = \sum_{n, n', \dots} |c^{-+}(n_{\alpha k}, n'_{\alpha' k'}, \dots)|^2 \tag{11}$$

with

$$\begin{aligned}
C^{+-}(n_{\alpha k}, n'_{\alpha' k'}, \dots) &= -(\sum_{\alpha, k} n_{\alpha, k} \omega_k)^{-1} \sum_{\alpha, k} (g/\mu) (2\Omega \omega_k)^{-1/2} \\
&\times t_{\alpha}(\sigma - \mathbf{s}, \mathbf{k}) u^{++} K(-A_{\alpha k}, n_{\alpha k}; A_{\alpha k}, I_{\alpha k}) \Pi_{\alpha k} K(-A_{\alpha k}, n_{\alpha k}, O_{\alpha k}), \\
C^{-+}(n_{\alpha k}, n'_{\alpha' k'}, \dots) &= -(\sum_{\alpha, k} n_{\alpha, k} \omega_k)^{-1} \sum_{\alpha, k} (g/\mu) (2\Omega \omega_k)^{-1/2} \\
&\times (\tau_{\alpha} - t_{\alpha})(\mathbf{s} \cdot \mathbf{k}) u^{++} K(A_{\alpha k}, n_{\alpha k}; A_{\alpha k}, I_{\alpha k}) \Pi'_{\alpha k} K(-A_{\alpha k}, n_{\alpha k}; A_{\alpha k}, O_{\alpha k}).
\end{aligned} \tag{12}$$

$K(-A_{\alpha k}, n_{\alpha k}; A_{\alpha k}, m_{\alpha k})$ is the same quantity used in B - N 's paper.

If we replace $\sum_{\alpha, k} n_{\alpha k} \omega_k$ by μ in Eqs. (12) and make summations of Eqs. (11), we obtain the upper limit of $|\phi^{+-}|^2$ and $|\phi^{-+}|^2$. And the lower one is obtained by limiting the summations to those terms which satisfy a condition

$$n_{\alpha k} + n'_{\alpha' k'} + \dots = 1.$$

We have already employed these limits in the first equation of (10).

§ 2. Comparison with perturbation method

Next we consider a nucleon state at rest, with positive energy, σ - and τ -spin parallel to \mathbf{s} and \mathbf{t} respectively, by the usual perturbation method. Then, a norm of the first approximation is

$$\sim (3/4\pi)(g^2/4\pi)(K_c/\mu)^2 + (3/32\pi)(g^2/4\pi)(K_c/\mu)^2(K_c/m)^2,$$

of which the second term comes from a negative energy part. Comparing with Eqs. (10), we shall find each of them has the same order of magnitude, irrespective of the magnitude of $g^2/4\pi$.

Now, the wave function of the $B-N$'s method including higher approximations can be written as follows:

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots \quad (15)$$

and if it is admitted to develop ψ_i in g ,

$$\begin{aligned} \psi_0 &= \psi_{0,0} + g\psi_{0,1} + g^2\psi_{0,2} + \cdots, \\ \psi_1 &= g\psi_{1,1} + g^2\psi_{1,2} + \cdots, \\ \psi_2 &= g^2\psi_{2,2} + \cdots, \\ &\dots\dots\dots \end{aligned} \quad (16)$$

$\psi_{0,0}$ is just the same with the corresponding perturbational wave function of 0-th approximation. So both wave functions, Bloch-Nordsieck's and perturbational, which are solutions of Eq. (4), have about the same region of convergence including $g=0$ and coincide with each other in the limit of $g \rightarrow 0$. This means that the $B-N$'s method cannot improve the perturbational one and is rather injurious in its complexity.

§ 3. Discussion of results

Before discussing the above results, we make similar estimation for an electron wave function. The results are

$$|\psi_1|^2 \sim f(v) (e^2/4\pi) (K_e/m)^2 + g(v) (e^2/4\pi)^2 (K_e/m)^2 \quad (17)$$

in $B-N$'s method, where

$$\begin{aligned} f(v) &= (1-v^2)/4\pi, \\ g(v) &= \begin{cases} 16v^2/9\pi^2 & \text{for } v \ll 1, \\ (1-v^2)/\pi^2 |\log(1-v^2)|^2 & \text{for } 1-v \ll 1, \end{cases} \end{aligned} \quad (18)$$

and, in the perturbation methods,

$$|\psi_1|^2 \sim (1/4\pi) (1-v^2/3) (e^2/4\pi) (K_e/m)^2 + (2v^2/3\pi) (e^2/4\pi) \log(K_e/K_{\min}) \quad (19)$$

K_{\min} is a cut-off momentum of the low frequency side, and the last term of (19) is related to the well known infrared catastrophe. Here we have considered a electron with an average velocity v , because radiation fields around it are produced principally by its mass motion.

But, in cases of mesonic fields, it is no matter whether a nucleon is at rest or in motion.

This infrared catastrophe destroys the convergency of perturbation method and is due to a strong coupling between an electron and low frequency photons.

That is, the Fourier decompose of interaction Hamiltonian contains a factor $\omega^{-1/2}$ and the energy difference between an initial and final state combined by this is ω , if both of them belong to the positive or negative energy state. So transitions between positive energy states or those between negative energy states are produced very oftenly by low frequency photons and we cannot neglect them in 0-approximation.

In B - N 's method, by replacing α by its average ν , we have succeeded in introducing these transitions in 0-approximation. So remaining transitions are only positive to negative, or negative to positive transitions, and energy differences as large as $2m$ eliminate the infrared catastrophe.

Now we are in a position to answer the question "Why the B - N 's method doesn't improve the perturbation method when applied to mesonic systems?" In the first B - N 's approximation we have neglected states ϕ^{+-} , ϕ^{-+} etc., each of which contains many different states characterized by numbers of unbound mesons. We shall denote them by $\phi^{+-}(n_{\alpha k})$, $\phi^{-+}(n_{\alpha k})$ and so on. And in the next approximation transitions to these neglected states can occur, probabilities of which depend on (I) the magnitude of the energy differences between initial and final states and (II) the aspect of mesonic bound fields around a nucleon.

As for the latter point, the origin of the bound fields $Q_{\alpha k}$ of ϕ^{++} are shifted by $A_{\alpha k} \cos \mathbf{k} \cdot \mathbf{r}$ as seen in Eq. (7) and those of ϕ^{+-} , ϕ^{-+} , χ^{+-} and χ^{-+} by $-A_{\alpha k} \cos \mathbf{k} \cdot \mathbf{r}$. This difference brings a damping factor $\exp \{-(I/3\pi)(g^2/4\pi)(K_c/\mu)^2\}$ to the corresponding probabilities between the former and the latters, for example, $\phi^{++}(\alpha k)$ to $\phi^{+-}(n_{\alpha k})$, while many states with different $n_{\alpha k}$ can be excited.

These two competing effects are just compensated in electro-magnetic cases, but this time circumstances will not be so simple.

In Table I we tabulate transition schemes appearing in the next approximation with the differences of energy and those of the aspect of bound field between initial and final states. Even if the damping factor reduces transition probabilities from ϕ^{++} to ϕ^{+-} and ϕ^{-+} , we can easily see that the presence of ϕ^{--} transition

Table I. Transition Schemes

Initial state	Final state	Energy difference	Aspect of bound field
$\phi^{++}(\alpha k)$	$\phi^{+-}(n_{\alpha k})$	$\sum \alpha k n_{\alpha k} \omega_k$	\times
	$\phi^{-+}(n_{\alpha k})$	$\sum \alpha n_{\alpha k} \omega_k$	\times
	$\phi^{--}(1_{\alpha k})$	ω_k	\bigcirc
	$\chi^{++}(1_{\alpha k})$	$\sim 2m$	\bigcirc
	$\chi^{+-}(n_{\alpha k})$	$\sim 2m$	\times
	$\chi^{-+}(n_{\alpha k})$	$\sim 2m$	\times
	χ^{--} (forbidden)	$\sim 2m$	\bigcirc

The origin of bound field of a final state is shifted to the same or opposite direction with that of a initial state ϕ^{++} , according as its corresponding transition scheme has a sign \bigcirc or \times .

makes $B-N$'s method as worse as the perturbation method.

Until now, we only consider the symmetrical pseudoscalar theory with pseudo-vector coupling. But $B-N$'s method is unsuccessful for any of other meson theories except the neutral scalar and pseudoscalar ones in which cases mesonic radiations are only produced by mass motions of a nucleon.

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Meson-Nucleon Scattering

Yoichiro NAMBU and Yoshio YAMAGUCHI

Osaka City University

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Using the symmetry properties in the charge space, the relationships between various modes of meson-nucleon scattering are discussed. The connection between our formalism and meson theories is also given.

§ 1. Preliminary notes about the charge space

Let τ be the isotopic spin of a nucleon, the components of which are denoted by τ_1, τ_2, τ_3 . For the sake of simplicity, we describe the proton and neutron state by charge wave function p and n , respectively:

$$\tau_3 p = p, \quad \tau_3 n = -n,$$

suppressing the other coordinates specifying the momenta and ordinary spin. The isotopic spin of a meson is denoted by ω , whose components ω_1, ω_2 and ω_3 obey the same commutation relations as the components of angular momentum operator. The charge wave functions for π -meson are denoted by

$$\begin{array}{ll} m^+ & \pi^+ \\ m^0 & \text{for } \pi^0\text{-meson,} \\ m^- & \pi^- \end{array}$$

with

$$\omega_3 m^+ = m^+, \quad \omega_3 m^0 = 0, \quad \omega_3 m^- = -m^-.$$

Explicit representation of the ω 's is as follows:

$$\omega_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and that of τ 's:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us consider a system composed of a nucleon and a meson, and put

$$\mathbf{T} = \omega + \frac{1}{2} \boldsymbol{\tau}, \quad \mathbf{T}^2 = T(T+1), \quad T_3 = \omega_3 + \frac{1}{2} \tau_3.$$

The eigenvalues of T and T_3 are

$$T = 3/2 \text{ and } 1/2; \quad T_3 = 3/2, 1/2, -1/2 \text{ and } -3/2, \quad |T_3| \leq T;$$

and the total charge number of this system is given by $T_3 + 1/2$. The projection operators for the state corresponding to $T = 3/2$ and $1/2$ are given by

$$P_{3/2} = \frac{(\omega \boldsymbol{\tau}) + 2}{3} \quad \text{and} \quad P_{1/2} = \frac{1 - (\omega \boldsymbol{\tau})}{3},$$

respectively.

§ 2. The scattering of "symmetrical" mesons by nucleon— The charge independent case

The scattering of a meson and a nucleon is characterized by the scattering matrix R , which involves the momenta, spin and isotopic spin of the partaking particles. We assume first of all that R is, apart from other variables, charge independent*, i.e., R is commutable with each of the components of \mathbf{T} . (More general cases are discussed in the subsequent sections.) Then R is expressed in terms of two parameter matrices a and b , which describe the scattering amplitudes for the charge states corresponding to $T = 3/2$ and $T = 1/2$, respectively:

$$R = a.P_{3/2} + b.P_{1/2} = \frac{2a+b}{3} + \frac{a-b}{3}(\omega \boldsymbol{\tau}). \quad (2.1)$$

From this expression we readily get the scattering amplitudes for various scattering modes, at the same incident energy (and if necessary, the same angle). The results are listed in Table I**. The differential cross section per unit solid angle is given by $|R|^2$. We see from the Table that there are very close connections between the various cross sections, some of which, such as those for $\pi^+ + p \rightarrow p + \pi^+$, $\pi^- + p \rightarrow p + \pi^-$ and $\pi^- + p \rightarrow n + \pi^0$, are indeed observable at laboratories, and hence may serve to check our working hypothesis of charge independence. The charge independence hypothesis, though fairly well established in other phenomena like nuclear forces and offering a very important and fruitful guiding principle to reduce ambiguous possibilities or to predict some new results to be observed, is itself a mere hypothesis at the present stage, and remains to be checked by further experiments. The meson-nucleon scattering considered here thus gives us

* In this paper, we do not take into account the Coulomb forces and the fine structure in the mass values of nucleon and meson. In practical comparison with experiments, especially at low energies, these effects must not be neglected.

** Mr. Kinoshita has kindly informed us that the same results were derived by W. Heitler, Proc. Roy. Irish Acad. 51 A (1946), 33.
Also see, K. A. Brueckner and K. M. Watson, Phys. Rev. 83 (1951), 1.

a very effective means for this purpose.

Table I

Scattering Mode	Scattering Amplitude
$\pi^+ + p \rightarrow p + \pi^+$, $\pi^- + n \rightarrow n + \pi^-$	a
$\pi^- + p \rightarrow p + \pi^-$, $\pi^+ + n \rightarrow n + \pi^+$	$(1/3)(a+2b)$
$\pi^- + p \rightarrow n + \pi^0$, $\pi^+ + n \rightarrow p + \pi^0$	$(\sqrt{2}/3)(a-b)$
$\pi^0 + n \rightarrow p + \pi^-$, $\pi^0 + p \rightarrow n + \pi^+$	$(\sqrt{2}/3)(a-b)$
$\pi^0 + n \rightarrow n + \pi^0$, $\pi^0 + p \rightarrow p + \pi^0$	$(1/3)(2a+b)$

§ 3. The scattering of mesons by a nucleon—More general case

Next we drop the restriction that was used in the preceding section, i.e., the charge independence, and assume only the conservation law of total charge. Then the scattering matrix R is only required to be commutable with the operator T_3 , so that R in its most general form can be written, for example, as follows:

$$R = A + B\tau_3 + C\omega_3 + D\omega_3^2 + E\omega_3\tau_3 + F(\omega\tau) + G\frac{[\omega \times \tau]_3}{\sqrt{2}i} + H\omega_3^2\tau$$

$$+ K\frac{\omega_3(\omega\tau) + (\omega\tau)\omega_3}{\sqrt{2}} + L\frac{\omega_3 \cdot [\omega \times \tau]_3 + [\omega \times \tau]_3 \cdot \omega_3}{\sqrt{2}i}, \quad (3.1)$$

where the ten parameters A, B, \dots, L (really matrices) are some functions of spin and momenta of nucleon and meson. That there are ten and only ten linearly independent parameters can be shown as follows. The total charge degree of freedom for a system composed of a meson and a nucleon is $3 \times 2 = 6$. Of these six states, the states $T_3 = 1/2$ and $-1/2$ are doubly degenerate, corresponding to $T = 3/2$ and $1/2$. Hence the most general matrix commutable with T has $1 + 4 + 4 + 1 = 10$ non-zero matrix elements. Since no other restrictions are considered, they can be chosen independently. The eq. (3.1) corresponds to a special decomposition of the general matrix R in terms of ω and τ . The matrix elements are listed in Table II.

Table II

$\pi^+ + p \rightarrow p + \pi^+$	$A + B + C + D + E + F + H + \sqrt{2}K$
$\pi^+ + n \rightarrow n + \pi^+$	$A - B + C + D - E - F - H - \sqrt{2}K$
$\pi^0 + p \rightarrow p + \pi^0$	$A + B$
$\pi^+ + n \rightarrow p + \pi^0$	$\sqrt{2}F - G + K - L$
$\pi^0 + p \rightarrow n + \pi^+$	$\sqrt{2}F + G + K + L$
$\pi^0 + n \rightarrow p + \pi^-$	$\sqrt{2}F - G - K - L$
$\pi^- + p \rightarrow n + \pi^0$	$\sqrt{2}F + G - K - L$
$\pi^0 + n \rightarrow n + \pi^0$	$A - B$
$\pi^- + p \rightarrow p + \pi^-$	$A + B - C + D - E - F + H + \sqrt{2}K$
$\pi^- + n \rightarrow n + \pi^-$	$A - B - C + D + E + F - H - \sqrt{2}K$

Under some conditions it may happen that R acquires certain extra symmetry properties in the charge space. They are, for example, as follows:

I. R is invariant against the inversion or interchange of the states p and n , and m^+ and m^- .

$$B=C=G=H=K=0. \quad (3.I)$$

Physically this means that, e.g., the scattering amplitude for the process $\pi^+ + p \rightarrow p + \pi^+$ is equal to that of $\pi^- + n \rightarrow n + \pi^-$.

II. R is antisymmetrical with respect to the above mentioned procedure. In this case,

$$A=D=E=F=L=0. \quad (3.II)$$

The scattering amplitude for $\pi^+ + p \rightarrow p + \pi^+$ is equal to that for $\pi^- + n \rightarrow n + \pi^-$ except for a sign.

III. R is invariant with respect to all rotations in charge space. This is the charge independent case considered in the previous section and is characterized by

$$A, B \neq 0, \quad \text{and others} = 0.$$

A and F are related to the constants in section 2 by

$$A = \frac{2a+b}{3}, \quad F = \frac{a-b}{3}.$$

§ 4. The generalized charge space—Introduction of neutral meson of neutral theory type*

In the preceding discussions the neutral meson π^0 was treated on an equal basis as the charged π^\pm ; that is, the transformation properties of charge wave function of π^\pm and π^0 are essentially those of the spherical harmonics of the first degree in the ordinary spaces, or what amounts to the same thing, π^\pm and π^0 together form a vector in the charge space. In this respect we can say that these mesons are of "symmetrical type". However, the charge state of a neutral meson can also be described by a "scalar" wave function (or spherical harmonics of degree zero) in the charge space. Such a neutral meson may be regarded to represent the neutral meson which appeared in Bethe's neutral theory, and will hereafter be denoted by π' , and the corresponding charge wave function by m' . Both π^0 and π' can coexist, and are supposed to be distinguishable, if equal in other respects, at least by their "transformation properties".

The inclusion of π' -meson is achieved by the generalization of charge space to a four-dimensional one. The 1, 2, 3-axes bear the same meaning as before, while the new fourth dimension concern the π' -meson. Then the operators A , ω_j , etc. appearing in the preceding section are now expressed in the form

* We are indebted to Messrs. Aizu, Kinoshita and Utiyama for their valuable discussions on the formulation in this section.

$$A \rightarrow \left(\begin{array}{ccc|c} & & & 0 \\ & A & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), \quad \omega_j \rightarrow \left(\begin{array}{ccc|c} & & & 0 \\ & \omega_j & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), \quad \text{etc.}$$

The generalized space thus obtained and the transformations in this space have, as will readily be noticed, similar structures to those of the Minkowski space which is a generalization of the three-dimensional Galilei space. But since here only those transformations which preserve the third axis β have a definite physical meaning, it is immaterial whether the generalized four dimensional space is really Minkowskian or Euclidean, and either of them may be adopted without loss of generality.

In the four dimensional charge space, the operators $\omega(\omega_1, \omega_2, \omega_3)$ may be regarded as the space-space part of a six-component antisymmetric tensor \mathcal{Q} , the space-time part of which, denoted by $\omega'(\omega'_1, \omega'_2, \omega'_3)$, now describes transitions between π' and the original π^\pm , π^0 -mesons:

$$-\pi^0 + p \rightarrow p + \pi', \quad \pi^+ + p \rightarrow n + \pi', \quad \text{etc.}$$

The tensor \mathcal{Q} plays the role of infinitesimal rotation operators of the four-dimensional charge space for meson. A different analogy is obtained by comparing \mathcal{Q} to the electromagnetic field quantities, ω to the magnetic and ω' to the electric field.

The concrete representation of ω' is easily found in standard text books on group theory. We adopt here the following representation:

$$\omega'_1 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \\ \hline -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{array} \right), \quad \omega'_2 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & i/\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i/\sqrt{2} \\ \hline -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \end{array} \right),$$

$$\omega'_3 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \end{array} \right).$$

Further we define an operator

$$\mathbf{1} = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right).$$

Assuming the commutability with T_3 (charge conservation), that part of the scattering matrix which describes processes involving π' turns out, in its most general form, as

$$\begin{aligned}
 R' = & A' \mathbf{1} + B' \mathbf{1} \tau_3 + C' \omega_3 + D' \frac{[\omega \times \omega']_3 - [\omega' \times \omega]_3}{2i} + E' \omega'_3 \tau_3 + F' (\omega' \tau) \\
 & + G' \frac{[\omega' \times \tau]_3}{\sqrt{2} i} + H' \frac{[\omega \times \omega']_3 - [\omega' \times \omega]_3}{2i} \tau_3 + K' \frac{\omega_3 (\omega' \tau) + (\omega' \tau) \omega_3}{\sqrt{2}} \\
 & + L' \frac{\omega' \cdot [\omega' \times \tau]_3 + [\omega' \times \tau]_3 \cdot \omega_3}{\sqrt{2} i}, \quad (4.1)
 \end{aligned}$$

where A', B', \dots, L' are, as before, some functions of spin and momenta. The matrix elements corresponding to the various scattering modes represented by R' are listed in Table III.

Extra requirements on the symmetry properties of R' can also be invoked as in the previous discussions. Thus,

I R' is symmetrical with respect to the inversion;

$$B' = E' = F' = H' = L' = 0.$$

II R' is antisymmetrical with respect to the inversion;

$$A' = C' = D' = G' = K' = 0.$$

III R' is invariant with respect to all rotations in the (three-dimensional) charge space;

$$A', F' \neq 0, \text{ all others} = 0.$$

Table III

$\pi' + p \rightarrow p + \pi'$	$A' + B'$
$\pi' + n \rightarrow n + \pi'$	$A' - B'$
$\pi' + p \rightarrow n + \pi^+$	$-\sqrt{2} F' - G' + K' - L'$
$\pi' + n \rightarrow p + \pi^-$	$\sqrt{2} F' - G' + K' + L'$
$\pi' + p \rightarrow p + \pi^0$	$C' + D' + E' + F' + H'$
$\pi' + n \rightarrow n + \pi^0$	$C' + D' - E' - F' - H'$
$\pi^0 + p \rightarrow p + \pi'$	$C' - D' + E' + F' - H'$
$\pi^0 + n \rightarrow n + \pi'$	$C' - D' - E' - F' + H'$
$\pi^- + p \rightarrow n + \pi'$	$\sqrt{2} F' + G' + K' - L'$
$\pi^+ + n \rightarrow p + \pi'$	$-\sqrt{2} F' + G' + K' + L'$

§ 5. The relation to meson theories

Interaction Hamiltonians appearing in meson theories can be expressed in the typical form:

$$\sum_{j=1}^4 f_j \tau_j O_j \varphi_j, \quad \begin{cases} f_1=f_2; & O_1=O_2; \\ \tau_4=1; \end{cases}$$

where φ_j 's describe meson field and O_j 's mean the source density of nucleon (O_j may contain the derivatives operating to φ_j).

The so-called symmetrical theory is specified by the well-known relations:

$$f_1=f_2=f_3, \quad O_1=O_2=O_3; \quad f_4=0. \quad (5.1)$$

Assuming a single type of neutral meson, "un" symmetrical meson theories consisting of charged meson and some type of neutral meson can now be classified in two kinds:

(i) (charged meson) + (π^0 -meson)

$$f_1=f_2, \quad O_1=O_2; \quad f_4=0; \quad (5.i)$$

(ii) (charged meson) + (π' -meson)

$$f_1=f_2, \quad O_1=O_2; \quad f_3=0. \quad (5.ii)$$

The symmetrical theory is the special case of (i). In the case (i),

$$R'=0 \quad (5.i')$$

corresponding to the requirement that π' must be completely eliminated. While in the case (ii),

$$\begin{cases} A=B=F=G=K=L=0, \\ C'=D'=E'+F'=H'=0 \end{cases} \quad (5.ii')$$

in order to eliminate the π^0 -meson.

As is well-known, the symmetry properties of Hamiltonian are preserved in any results derived from this Hamiltonian. Thus a charge independent Hamiltonian in the symmetrical meson theory must lead to a charge independent transition matrix for, e.g., meson-nucleon scattering. The cases of less symmetrical theories are already stated. These correspondences may give some definite criteria for deciding among the existing meson theories. Unfortunately we have not yet sufficient experimental results to derive any definite conclusions. We must, therefore, leave further detailed, undoubtedly interesting discussions to future program.

A remark may be added on the nature of the scattering matrix R . As is well known, it is related to the S -matrix by

$$S=1+R.$$

R is neither hermitic nor unitary, and so are the coefficients A, B, \dots . Since the only observable quantity is the cross section or the square of modulus of R , the number of cross sections necessary to determine the magnitude of the coefficients is in general larger than that of these coefficients. In the symmetrical theory, for instance, the three cross sections

$$\pi^+ + p \rightarrow p + \pi^+,$$

$$\pi^- + p \rightarrow \begin{cases} p + \pi^-, \\ n + \pi^0, \end{cases}$$

are only just enough to determine $|A|, |B|$ and the relative phase of A and B .

On the Mesonic Correction to the β -Decay

Tsuneyuki KOTANI,

Kobayashi Institute of Physical Research, Tokyo,

Shigeru MACHIDA, Seitaro NAKAMURA, Hisao TAKEBE, Minoru UMEZAWA,

Department of Physics, University of Tokyo; and

Tets YOSHIMURA,

Department of Physics, Tokyo Bunrika University

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General types of the mesonic correction to the β -decay are obtained from the invariance requirements. Explicit calculations are performed, using Feynman-Dyson method, up to the second order of meson-nucleon coupling constants for the symmetrical pseudoscalar meson theory with both pseudoscalar and pseudovector couplings. It is shown that the mesonic correction is small and does not affect the selection rules and spectra in the Konopinski's forbidden theory.

§ 1. Introduction

Of late years, experiments on the β -decay have made the remarkable progress and the following facts have been confirmed: ft -values of β -decay fall into several groups, that is, $\sim 10^3$ (mirror nuclei), $10^5 \sim 10^7$ (other allowed transitions), $10^6 \sim 10^9$ (first forbidden, $\Delta I=1$), $10^8 \sim 10^9$ (first forbidden, $\Delta I=2$), $10^{12} \sim 10^{13}$ (second forbidden), $\sim 10^{16}$ (third forbidden), and this classification by the ft value is consistent with the shape of spectra predicted by Konopinski's forbidden theory and the results are in good agreement with Mayer's shell model.¹⁾⁻³⁾

But, it seems rather difficult to classify the transitions of the elements of $ft \sim 10^6$ to be the first forbidden ones in contrast to the transitions of the α shape spectra, ft -values of which are known to be the distinct values $\sim 10^8$. There remains, moreover, only one transition, RaE-decay, the shape of which has not yet been explained by the Konopinski's forbidden theory. The ft -value ($\sim 10^9$) of C^{14} decay cannot be explained sufficiently, which is expected to be the allowed transition.⁴⁾

And there are several problems which remain to be investigated, that is;

Does any of these five types of interaction really exist?

Don't the higher order radiative and mesonic corrections to the β -decay affect the Konopinski's forbidden theory?

Recently it has been shown that either of the tensor interaction or vector and pseudovector interactions are necessary to explain the shapes of all the experimental β -spectra. But does it remain unchanged, if we take into account the

higher order radiative and mesonic corrections?

The spectrum of RaE cannot be explained even if more than two interactions may be introduced and no new suitable interaction may explain the spectrum since the RaE β -decay is probably the first forbidden transition. Recently, radiative correction was investigated by Hanawa, Miyazima, Nakano and Watanabe⁵⁾ and Merzbacher⁶⁾. According to them, radiative correction does not affect the Konopinski's forbidden theory and so give no transition of new shape of the spectrum.

Does the higher order mesonic correction affect the Konopinski's forbidden theory? This is one of the purpose of our present work. But our calculation is accompanied by the divergence difficulties and moreover the conclusions obtained are somewhat ambiguous since the results can be obtained only in the interaction representation.

Robson⁷⁾ has experimented recently on a neutron decay and concluded that the half-life of neutron is 12.8 ± 2.5 minutes and its shape of energy spectrum is of allowed type.

As this is the elementary process, the decisive conclusion may be expected, in contrast to the case of nuclear β -decay in which we encounter the ambiguities relating to the nuclear structure. Other purpose of this work is, therefore, to estimate the order of magnitude of the mesonic correction to the β -decay of the neutron.

§ 2. Evaluation of the correction terms

We shall start from the equation of the interaction representation of the form

$$\{H'_{int} + H''_{int} - i\partial/\partial\sigma\} \Psi[\sigma] = 0,$$

where H'_{int} is the interaction energy density between nucleon and lepton and H''_{int} is the interaction energy density between nucleon and meson. We assume the five types of Fermi interaction between nucleon and lepton. Then H'_{int} is

$$H'_{int} = G(\bar{\psi}(x)B\tau_{PN}\psi(x))(\bar{\phi}(x)\beta\varphi(x)) + \text{conj}, \quad (1)$$

where $\psi(x)$ describe the nucleon field, $\varphi(x)$ and $\phi(x)$ the neutrino field and electron field respectively, G the coupling constant, β and B the sedenions characterizing the type of interaction, given in the following table.

Type of coupling	Scalar	Vector	Tensor	Pseudovector	Pseudoscalar
G	G_s	G_v	G_t	$G_{p.v.}$	$G_{p.s.}$
β	1	$i\gamma_\mu$	$\sigma_{\mu\nu}$	$i\gamma_5\gamma_\mu$	$i\gamma_5$
B	1	$i\Gamma_\mu$	$\Sigma_{\mu\nu}$	$i\Gamma_5\Gamma_\mu$	$i\Gamma_5$

(2)

We assume the pseudoscalar π meson with the pseudoscalar and pseudovector couplings with nucleon. Then H''_{int} is

$$H_{int}'' = \sum_{i=1}^3 f_i \bar{\psi}(x) i \Gamma_5 \tau_i \psi(x) \mathcal{X}_i(x) + \sum_{i=1}^3 \frac{g_i}{\mu} \bar{\psi}(x) i \Gamma_5 \Gamma_\mu \tau_i \frac{\partial \mathcal{X}_i(x)}{\partial x_\mu}. \quad (3)$$

The field quantities, of course, satisfy the usual field equations and the commutation relations for free fields, of which the mass parameters are taken to be the observed ones, since they are assumed to have already been renormalized.

We evaluate the β -decay probabilities, in accordance with the Dyson's formula,

$$U = \sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} (-i)^n \frac{1}{n!} \int dx_1 \cdots \int dx_n P(H(x_1) \cdots H(x_n)).$$

The first order transition matrix element, U_1 , for the neutron decay can be written immediately,

$$U_1 = c(\bar{\psi}(F) B \tau_{FN} \psi(I)) (\bar{\phi}(p) \beta \varphi(q)) \quad (4)$$

$$\text{where} \quad C = -iG\delta(I - q - p - F)/4(2\pi)^3. \quad (5)$$

I and F are the initial and final nucleon four-momenta and p and q are the electron and neutrino four momenta, respectively.

Here, we shall consider the general properties of matrix elements U_{2n+1} 's. It is easily shown that the Gf^{2n} (or Gg^{2n}) order matrix element U_{2n+1} can be written, in the momentum space, as follows;

$$N^{(2n)}(f, v) \cdot C(\bar{\psi}(F) B' \tau_{FN} \psi(I)) (\bar{\phi}(p) \beta \varphi(q)) \quad (6)$$

where $N^{(2n)}(f, v)$ includes the f^{2n} (or g^{2n}) and the powers of $v(=F-I)$. B' 's are the functions consisting of I , F and I' and have the same rotational properties as that of B and β . Any matrix elements obtained from the scalar, vector, tensor, pseudovector and pseudoscalar interactions can be reduced to the linear combinations of the fundamental several types of the component matrix elements, given in the Table (I).*

Thus it is shown that U_{2n+1} 's can be rewritten by the suitable linear combinations of the component matrix elements, and that the possible types of matrix elements in any higher order mesonic corrections are limited only to the above types.

We restrict ourselves to the terms U_1 and U_3 . To find U_3 we must consider the Feynman-Dyson graphs shown in Fig. 1.

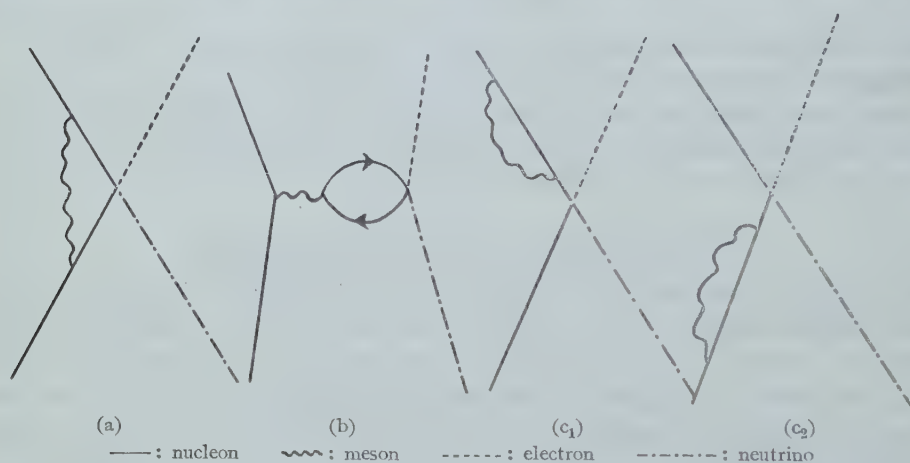
Here the neutral meson can contribute to the graphs (a) and (c), and the charged meson can contribute to the graphs (b) and (c). The contribution from the graph (c) corresponds only to the mass and charge renormalizations. Thus U_3 may split into two components as $U_3 = U_a + U_b$.

Writing the matrix elements for the pseudoscalar coupling of the pseudoscalar

* The definition of the "component" matrix element and the analogous discussion for the γ - π process have been given by Z. Koba, T. Kotani and S. Nakai^[8].

Table (I).

Type of interaction	Component matrix element (B')	Coefficients (pseudoscalar coupling)	
		$N_a^{(2)} \left/ \left(\frac{f^2}{(4\pi)^2} \right) \right.$	$N_b^{(2)} \left/ \left(\frac{f^2}{(4\pi)^2} \right) \right.$
Scalar	1	$-0.16 v^2/M^2$	0
Vector	$i\Gamma_\mu$	-0.003	0
	v_μ/M	0	0
	$i\sum_{\mu\nu} v_\nu/M$	0.47	0
Tensor	$\sum_{\mu\nu} (v_\mu \Gamma_\nu - v_\nu \Gamma_\mu)/M$	$-0.20 \cdot v^2/M^2$	0
	$\frac{1}{2M} \{ \Gamma_\mu (\Gamma_\nu) \Gamma_\nu - \Gamma_\nu (\Gamma_\nu) \Gamma_\mu \}$	-0.16	0
	$i(F_\mu I_\nu - F_\nu I_\mu)/M^2$	0	0
		-0.32	0
Pseudovector	$i\Gamma_5 \Gamma_\mu$	$-0.01 v^2/M^2$	0
	$\Gamma_5 v_\mu/M$	0.37	$-(2/3)(v^2/(b^2 + \mu^2))$
	$i\Gamma_5 \sum_{\mu\nu} v_\nu/M$	0	0
Pseudoscalar	$i\Gamma_5$	$0.16 v^2/M^2$	$(1/6) \{ v^4/(M^2(v^2 + \mu^2)) \}$

Fig. 1. The mesonic correction of the β -decay of the neutron.

π -meson with nucleon arising from the graphs (a) and (b), and performing the integration over coordinates, we obtain the following expressions.

$$U_a = N_a^{(2)}(f, v) c(\bar{\psi}(F) A_R^{(2)} \psi(I)) (\bar{\psi}(p) \beta \varphi(q), \quad (7)$$

$$U_b = N_b^{(2)}(f, v) c(\bar{\psi}(F) A_R^{(2)} \psi(I)) (\bar{\psi}(p) \beta \varphi(q), \quad (8)$$

where

$$A_R^{(2)}(F, I) = \frac{-if^2}{(2\pi)^4} \int dt i\Gamma_5 \tau_3 \frac{i\Gamma(F-t) - M}{(F-t)^2 + M^2} B \tau_{PN} \frac{i\Gamma(I-t) - M}{(I-t)^2 + M^2} i\Gamma_5 \tau_3 \frac{1}{t^2 + \mu^2}, \quad (9)$$

$$A_R^{(2)}(F, I) = \frac{if}{(2\pi)^4} i\Gamma_5 \tau_{PN} \frac{1}{v^2 + \mu^2} \int dt S \rho \left[\frac{i\Gamma t - M}{t^2 + M^2} B \tau_{PN} \frac{i\Gamma(t+v) - M}{(t+v)^2 + M^2} i\Gamma_5 \tau_{NP} \right] \quad (10)$$

and M and μ are the nucleon and meson mass respectively. The mass difference between neutron and proton is ignored when calculating the mesonic corrections. The finite parts renormalized in company with the divergent term and contracted out of only the constants, M and μ , are consistently defined for the graph (a) by the terms which are remained if the initial four-momentum I is equal to the final momentum F . For the graph (b), the regulator method is used, because it is difficult by using the above method to distinguish the required finite correction from the divergent part. The results obtained are given in Table (I). Each component matrix element is small as compared with the first order matrix element, (4), e.g. 1/1800, and the selection rule is not changed as will be shown in §3. The numerical factors of these corrections are, as a result, sufficient to know the order of magnitudes in spite of some ambiguities.

In the case of the pseudovector coupling we adopted the method which is progressed by Koba, Mugibayashi and Nakai⁹⁾. It would then seem that the second order mesonic correction due to the pseudovector coupling of the π -meson with the nucleon is split into two terms, one of which is equivalent to that due to the pseudoscalar coupling. The other contains both the divergent terms, which can be renormalized for the graph (a) or dropped by the regulator method for the graph (b).

§ 3. The selection rules and the energy spectra

We have calculated in interaction representation, then, when we want to apply this results to the β -decay of nuclei, we must transform it into the Schrödinger representation. But it may be permitted to apply it directly in the approximation in which the effects of nuclear binding is ignored. In the following we will proceed on this assumption.

As shown above, the matrix elements of Gf^{2n} order correction are written in the form (6).

Since B 's have the same rotational properties as that of B 's, then the same selection rule is deduced from this transition matrix elements of Gf^{2n} order correction as that deduced from the matrix elements of the first order perturbation. On account of this facts, the selection rule of the former forbidden theory is preserved even if we take into account the higher order mesonic correction.

Moreover, as shown in §2, possible types of matrix element which appear in the higher order mesonic corrections are limited. Among these, the terms including the factor v_μ or $F_\mu I_\nu - I_\mu F_\nu$ are negligibly small. The resultant terms are all the same as that in the first order perturbation. Then the higher order mesonic corrections do not affect the shape of energy spectra which are predicted by the former forbidden theory.

Above considerations seem to be legitimate in the case of allowed transitions. But in the case of forbidden transitions it is mentioned that in the above consi-

deration the results obtained in the interaction representation are, as it stands, regarded as that to be obtained in the Schrödinger representation.

§ 4. Discussion

As shown above, the selection rule and spectra predicted in the forbidden theory are not affected by the higher order mesonic and radiative corrections, and this fact confirms the recent success in the Konopinski's forbidden theory.

But, on the other hand, the more confirmative the Konopinski's forbidden theory is, the more difficult to be explained the remaining problems become.

In the case of higher order correction which has been considered here, we must consider the nuclear binding effect; e.g. the effect of Pauli exclusion principle for the virtual states of nucleon in the nuclei and the fact that the nucleon in the nuclei exchange the meson with other nucleon. We can treat this effect as the shift from the free nucleon decay and show that these effects do not affect the selection rules and the shapes of the energy spectra.

Here, as already mentioned, we must notice that in our consideration the matrix elements in the interaction representation are, as it stands, regarded as that in Schrödinger representation. In spite of this approximation, our conclusions about nuclear β -decay can be regarded as plausible.

The above calculation has confirmed the smallness of the mesonic corrections, and so we shall discuss the β -decay of the neutron with confidence by using the five Fermi interactions without the higher order correction. This discussion will be published in the near future.

Two of us (T. K. and S. M.) wish to express our thanks to the "Yukawa Yomiuri Fellowship" for the financial aid. We wish to acknowledge our appreciation to Mr. H. Umezawa for many helpful discussions.

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Letters to the Editor

Wave Equations in de Sitter Space*

K. Goto

*Department of Physics,
Osaka University*

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In this paper we attempted to give wave equations in de Sitter space, applying 5-dimensional spinor analysis which we obtained formerly.¹⁾ As it was shown by Robertson, de Sitter space can be described, by a suitable variable-transformation, as a 4-dimensional hyper sphere of radius R

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 = R^2 \quad (1)$$

in a 5-dimensional flat space, whose metric is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 + dx_5^2. \quad (2)$$

According to the postulation of the relativistic invariance, our wave equations must be invariant under linear transformations in de Sitter space namely under those which leave

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 \quad (3)$$

invariant. One of such operators along de Sitter space, which is invariant under the transformation in the space, is the rotation operator

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad (4)$$

where

$$p_\mu = \frac{\partial}{\partial x_\mu} \quad (\mu=1, 2, 3, 5), \quad p_4 = -\frac{\partial}{\partial x_4}.$$

Near the point $(0, 0, 0, 0, R)$ de Sitter space approaches to Minkowski space. So near such point our equation must approach to the well known wave equation in Minkowski space. The operator approaches near such point to

the followings:

$$\begin{aligned} m_{\mu\nu} &\rightarrow 0 \quad (\mu, \nu \neq 5), \\ m_{5\mu} &= -m_{\mu 5} \rightarrow R p_\mu. \end{aligned} \quad (5)$$

Now we propose the following equation as the wave equation in de Sitter space which approaches to the wave equation of Bhabha's type in Minkowski space near the point $(0, 0, 0, 0, R)$.

$$\{\alpha^{\mu\nu} m_{\mu\nu} - 2R\alpha\} \psi = 0, \quad (6)$$

$$\alpha^{\mu\nu} = I^{\mu\nu} \quad (7)$$

where $I^{\mu\nu}$ is the representation matrices of infinitesimal transformation of the group of linear transformation which leave (3) invariant. This equation is relativistic invariant because of the tensor character of $I^{\mu\nu}$ and $m_{\mu\nu}$. Near the point $(0, 0, 0, 0, R)$ this equation approaches to Bhabha's wave equation

$$\{\alpha^\mu p_\mu + \alpha\} \psi = 0, \quad \alpha^\mu = I^{\mu 5} \quad (8)$$

on account of (5).

For the field of the first rank spinor, (6) becomes

$$\begin{aligned} \{\alpha^\mu \alpha^\nu m_{\mu\nu} - 2R\alpha\} \psi &= 0, \\ \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu &= 2g^{\mu\nu} \end{aligned} \quad (9)$$

which was proposed by Dirac formerly,²⁾ because we can put in the case of spinor representation $D_{(1/2, 1/2)}$

$$I^{\mu\nu} = \alpha^\mu \alpha^\nu. \quad (10)$$

For the case of the representation $D_{(1,1)}$ by the symmetric spinor of the second rank, we can calculate by means of our spinor analysis in five dimensions, and (6) becomes

$$m_{\lambda\mu} F_{\mu\nu} - m_{\nu\mu} F_{\mu\lambda} = R\alpha F_{\lambda\nu} \quad (11)$$

where $F_{\mu\nu}$ are antisymmetric tensors such as

$$\begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} & F_{15} \\ & 0 & F_{23} & F_{24} & F_{25} \\ & & 0 & F_{34} & F_{35} \\ & & & 0 & F_{45} \\ & & & & 0 \end{pmatrix} \\
 = \begin{pmatrix} 0 & -\phi_{44} & \phi_{34} & \phi_{11} & -i\phi_{13} \\ & 0 & -\phi_{33} & \phi_{12} & -i\phi_{14} \\ & & 0 & \phi_{22} & -i\phi_{23} \\ & & & 0 & -i\phi_{24} \\ & & & & 0 \end{pmatrix}. \quad (12)$$

These tensors become near the point $(0, 0, 0, 0, R)$ as

$$(F_{\mu\nu}) \rightarrow \begin{pmatrix} 0 & -G_3 & G_2 & F_1 & iU_1 \\ & 0 & -G_1 & F_2 & iU_2 \\ & & 0 & F_3 & iU_3 \\ & & & 0 & iU_0 \\ & & & & 0 \end{pmatrix} \quad (13)$$

and the equation (11) becomes Yukawa's equations for the vector meson.

For the case of the representation $D_{(1,0)}$ also we can calculate in the same way, and (6) becomes

$$m_{\mu\nu} U_\nu = R x U_\mu, \quad (14)$$

where U_μ is a five dimensional vector and has the following relation with the anti-symmetric spinor of the second rank.

$$(U_1, U_2, U_3, U_4, U_5) \\
 = (\phi_{13}, \phi_{14}, \phi_{23}, \phi_{24}, -i(\phi_{12} + \phi_{34})). \quad (15)$$

This vector splits in a 4-vector and a 4-scalar near the point $(0, 0, 0, 0, R)$ as

$$(U_\mu) \rightarrow (U_1, U_2, U_3, U_0, iU), \quad (16)$$

and the equation (14) becomes Yukawa's equations for the scalar meson. It is a quite interesting fact that the wave equations for mesons have very symmetrical forms such as (11), (14), and these splits into several equations when the space becomes flat.

In concluding, we wish to express our

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V-Particles and Nuclear Phenomena*

S. Ôneda

*Institute for Theoretical Physics,
Kanazawa University*

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In the preceding note,¹⁾ a possible theory of V -particles was discussed. Thereafter, there appear many experiments²⁾ on these particles. According to Manchester group³⁾ their decay schemes are; (A) $V^0 \rightarrow P + \pi^-$, $m_{V^0} = (2203 \pm 12)m$, (B) $\tau^0 \rightarrow \pi^+ + \pi^-$, $m_{\tau^0} = (796 \pm 27)m$, or $\tau^0 \rightarrow \mu^+ + \mu^-$, $m_{\tau^0} = (705 \pm 32)m$. Pasadena group⁴⁾ also proposed another possible scheme based on the following symmetrical three-particle processes; $V^0 \rightarrow \text{nucleon} + \text{meson} + \text{meson}$, $m_{V^0} \sim 2600m$. But as yet there is no evidence for the existence of neutral secondaries and the momentum spectra of primary V -particle provide some evidences against these interpretations (see reference (3)). At present, V^0 -particle and τ -meson being in subsidiary position with hardly any apparent reason for existing at all, we proceed the discussion and try to make them play some role in nuclear phenomena. Here we shall discuss the phenomena associated with the fairly strong interactions responsible for V and τ production. As regards the interactions such as (2) in the reference (1) which lead to V and τ -decay, there may be many alternative ones which were discussed by Nambu et al.⁵⁾ in detail. Moreover, the attempt to regard V

as nucleon- τ -meson system⁶⁾ is very interesting, though relativistic treatment of bound state has not yet been established. The main results of such a theory, however, will be the same as those obtained from our theory which treats V -particle as elementary particle.

(A) *The strength of the coupling V - N - τ*

The possible interactions are as follows:

$$G_1\{\phi\phi\varphi.U\}+C.C. \text{ such as } V^0\rightarrow p+\tau^-,$$

$$G_2\{\phi\phi\varphi.U^*\}+C.C. \text{ such as } V^0\rightarrow n+\tau^0,$$

$$G_3\{\phi\phi\varphi.U^*\}+C.C. \text{ such as } V^+\rightarrow p+\tau^0,$$

$$G_4\{\phi\phi\varphi.U^*\}+C.C. \text{ such as } V^+\rightarrow n+\tau^+.$$

According to the coupling G and g ($NN\pi$ interaction), the production processes of V and τ are

$$(a) \quad N+N\rightarrow N+V+\tau \text{ through } Gg^2 \text{ or } G^3,$$

$$(b) \quad N+N\rightarrow V+V \text{ through } G^2,$$

$$(c) \quad \tau+N\rightarrow V+\tau \text{ through } G_1.$$

Namely, V and τ appear as V - τ pair or V - V pair. Such pair-wise productions should be examined but present experiments seem not so certain as to reject them. The threshold energies of the incident nucleon or photon in the laboratory system are given as follows in each case:

$$(a) \quad E_N = \{m_\tau(1+m_V/m_N) + m_\tau^2/2m_N \\ + 1/2m_N$$

$$\times (m_V + 3m_N)(m_V - m_N)\}$$

$$\times c^2 \sim 1.5 \text{ Bev.},$$

$$(b) \quad E_N = \{2(m_V + m_N)(m_V - m_N)/m_N\}$$

$$\times c^2 \sim 0.9 \text{ Bev.},$$

$$(c) \quad E_\tau = \{(m_V + m_\tau)^2 - m_N^2\}/2m_N$$

$$\times c^2 \sim 0.9 \text{ Bev.}$$

These threshold energies are much larger than those of π -meson production in the corresponding reactions. So the intensities of parent nucleon or photon for V and τ production will be smaller than one-tenth of those for π production, considering the power energy spectra of the ancestor nucleon or

photon components in cosmic rays. Moreover, the largeness of the mass of V and τ -meson will increase the energy denominators and decrease the cross-section for these processes. These arguments tempt us to assume that at least G must be greater than $1/10 g$ or rather comparable with g in order to explain the abundance of V and τ -mesons. As regards the charged heavy unstable particles, details have not yet appeared, but Manchester Group suggested that they are probably the charged counter part of τ^0 -meson. If it were true, the coupling G_3 and G_4 are unnecessary. Then G_1 is responsible for V^0 and charged τ -meson and G_2 for V^0 and τ^0 -meson. It should be noted that in our theory negative proton will only appear as $V_{\text{anti.}}^0 \rightarrow p^- + \pi^+$, and the production of $V_{\text{anti.}}^0$ will be very small compared with V^0 , so negative proton will not appear in the decay products of V^0 particle.

(B) *Contributions to the nucleon moments*

When we take $G \sim g$, V particle and τ -meson will have some effects on nucleon moments. Case's calculations⁷⁾ show that the anomalous magnetic moment of nucleon will not be so sensitive to the mass of meson, but the possibilities of the change of the meson clouds around the nucleon due to V -particle should be investigated. Here we checked the tendency by the lowest order calculation. Taking V -particle wave function different in reflection property from that of nucleon, the V - N - τ interactions become $iG\phi\gamma_5\varphi.U^* + C.C.$, and τ -meson behaves as pseudo-scalar coupling even when τ -meson is scalar meson. Using renormalization technique the anomalous magnetic moments due to τ -meson as a function of $\lambda = m_V/m_N$ and $\delta = m_\tau/m_N$ are given by

$$\mu_p' = 1/8\pi^2\hbar c \{-G_3^2 B_1 + G_1^2 B_2\},$$

$$\mu_n' = 1/8\pi^2\hbar c \{-G_4^2 B_1 - G_2^2 B_2\}$$

where

$$B_1 = \frac{1}{2} + (-\lambda^2 + \lambda + \delta^2)$$

$$\begin{aligned}
& -\{(-\lambda^2 + \lambda + \delta^2)(\lambda^2 - 1 - \delta^2) \\
& + \delta^2\} \ln(\lambda/\delta) \\
& + [1/2(\lambda^2 - 1 - \delta^2)\{(-\lambda^2 + \lambda + \delta^2) \\
& \times (\lambda^2 - 1 - \delta^2) + \delta^2\} \\
& - \delta^2(-\lambda^2 + \lambda + \delta^2)] \\
& \times \frac{1}{\sqrt{\delta^2 - (\lambda^2 - 1 - \delta^2)^2/4}} \\
& \times \cos^{-1}\left\{\frac{1}{\lambda\delta}\left(\delta^2 + \frac{\lambda^2 - 1 - \delta^2}{2}\right)\right\}, \\
B_2 = & 1 + (-\lambda^2 + \lambda + \delta^2) \ln(\lambda/\delta) \\
& - 1/2(-\lambda^2 + \lambda + \delta^2)(\lambda^2 - 1 - \delta^2) + \delta^2\} \\
& \times \frac{1}{\sqrt{\delta^2 - (\lambda^2 - 1 - \delta^2)^2/4}} \\
& \times \cos^{-1}\left\{\frac{1}{\lambda\delta}\left(\delta^2 + \frac{\lambda^2 - 1 - \delta^2}{2}\right)\right\}.
\end{aligned}$$

If we put $\lambda=1$ and $m_\tau=m_\pi$, these results (case (a)) agree with Case's ones. Changing the sign of λ in the above expressions, the values when both V and nucleon are the same spinor (case (b)) are obtained. The value of B_1 and B_2 are for example for the choice $m_V=2200m$, $m_N=1800m$, $m_\tau=900m$

$$(a) \quad B_1=0.32, \quad B_2=0.20.$$

$$(b) \quad B_1=-0.60, \quad B_2=-0.53.$$

Case's values for the symmetrical pseudo-scalar meson theory are $\mu_p^{(2)}=g^2/8\pi^2\hbar c \times 0.11$ and $\mu_n^{(2)}=-g^2/8\pi^2\hbar c \times 0.82$, where meson current contributions are not so great as desired. This situation is fairly improved in the fourth order calculation⁸⁾ where

$$\mu_p^{(4)}=\{g^2/8\pi^2\hbar c\}^2 \times 0.33$$

and

$$\mu_n^{(4)}=-\{g^2/8\pi^2\hbar c\}^2 \times 0.18.$$

The fourth order corrections of our theory (case (a)) will show approximately the same tendency as the above result obtained for π -meson. Taking G_1 and G_2 nearly as large as g and omitting G_3 and G_4 , the coupling G_1 will contribute to increase the anomalous magnetic moment of proton and quantitative agreement with experiment may have some

hope. The interaction G_3 , if it exists, will give the same tendency when we consider the fourth order correction. At present there is no reason for the necessity of the coupling G_4 , (G_4 predicts the pairwise production of charged V and charged τ -meson). Of course, the doubt for the convergency of the perturbation theory prevents us from decisive conclusion. On the other hand, G interaction may contribute to neutron-electron interaction but we may not expect any quantitative results from this problem at present stage.

(C) Other nuclear phenomena

As regards the nuclear forces G interaction appears firstly as the fourth order effect. The force range due to τ -meson will be about one sixth of that due to π -meson, and τ -meson will not contribute to the phenomena outside the nuclear force range. But from the high energy nuclear events such as high energy nuclear scattering the criticism of $V-N-\tau$ interaction and the choice of G may be offered. Moreover there appear the view points which treat the $V-N-\tau$ interaction as modified cohesive interaction introduced by Pais and Sakata.⁹⁾ This theory was already discussed by Sawada.¹⁰⁾ The experimental evidences are not so clear and we cannot go into more details. In meson theory, we have accumulated fairly correct knowledge about π -meson, but it cannot be said at present confidently that the nuclear phenomena can be solved by the single existence of π -meson. The appearance of V and τ -meson may shed some light on the future theory of the elementary particles. The author wishes to express his sincere thanks to Prof. S. Ozaki for his encouragement throughout this work.

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Nucleonic Components at High Altitudes

K. Sakihama and S. Takagi

*Department of Physics,
Kyoto University*

November 1, 1951

Recent many experiments have clarified some aspects on the nucleonic components at high altitudes. Especially, Lord¹⁾ shew larger neutron-proton ratio in high energy nucleon components than that expected from the nucleonic cascade theory²⁾ which assumed the primary cosmic radiation consists of only proton component. According to Bradt and Peter's experiments,³⁾ almost equal number of nucleons as the number of proton component come into the top of the atmosphere in the forms of α particles and heavier nuclei, so it is of some interest to inquire whether or not we can explain the high energy neutron-proton ratio in terms of the contribution of primary α particles.

Following the current treatment of nucleonic cascade theory, we postulate as the number

(or probability) of emerging nucleons of energy (E' , $E'+dE'$), when a nucleon of energy E collide with air nuclei passing through the unit atmospheric depth,

$$V(E, E')dE' = V(E'/E)dE'/E. \quad (1)$$

As for the corresponding quantity for α particle, we assume

$$\begin{aligned} W(E, E')dE' &= W(E'/E)dE'/E \\ &= 4(l_n/l_\alpha)V(E'/E/4)dE'/(E/4), \end{aligned} \quad (2)$$

where l_n and l_α are collision mean free path of nucleon and α particle respectively. This means the neglect of correlation between the constituents of α particle and that we consider α -air-nucleus collision as four independent nucleon-nucleus collisions.

Let $p(E, x)$, $n(E, x)$ and $\alpha(E, x)$ be the numbers of protons, neutrons and α particles with energy E found at depth x respectively. Then the diffusion equations are, neglecting the ionization loss,

$$\left. \begin{aligned} \frac{\partial p(E, x)}{\partial x} &= -p(E, x) + \int_E^\infty \{kp(E', x) \\ &\quad + k'n(E', x)\}V(E/E')dE'/E' \\ &\quad + \frac{1}{2} \int_{4E}^\infty \alpha(E', x)W(E/E')dE'/E', \\ \frac{\partial n(E, x)}{\partial x} &= -n(E, x) + \int_E^\infty \{k'p(E', x) \\ &\quad + kn(E', x)\}V(E/E')dE'/E' \\ &\quad + \frac{1}{2} \int_{4E}^\infty \alpha(E', x)W(E/E')dE'/E', \\ \frac{\partial \alpha(E, x)}{\partial x} &= -\sigma\alpha(E, x), \end{aligned} \right\} \quad (3)$$

where we take collision length as unit (we assume collision length being equal to 65 g/cm²) and assume that α components decrease with depth with constant absorption coefficient σ . k and k' are the probabilities

that emerging nucleons from air nucleus are protons (neutrons) and neutrons (protons) when colliding nucleon is proton (neutron) respectively.

Experimentally,³⁾ proton- α ratio in primary flux is 4:1 and primary α spectrum has a resemblance to that of protons. Primary spectrum of protons has often been assumed to be power spectrum, which is not correct strictly. But, for simplicity, we employ this assumption and put

$$p(E, 0) = \begin{cases} (4/5) \gamma E_0^\gamma / E^{1+\gamma} & E > E_0, \\ 0 & E < E_0, \end{cases}$$

$$\alpha(E, 0) = \begin{cases} (1/5) \gamma \epsilon_0^\gamma / E^{1+\gamma} & E > \epsilon_0, \\ 0 & E < \epsilon_0, \end{cases} \quad (4)$$

where E_0 and ϵ_0 are latitude cut-off energies for protons and α 's respectively. Under the initial conditions (4) we can solve (3) and get for $E > \epsilon_0 > E_0$

$$p(>E, x) = \int_E^\infty p(E, x) dE$$

$$= \int_E^\infty dE \int_0^{\pi/2} 2\pi p(E, x/\cos\theta) \sin\theta d\theta$$

$$= \pi (E_0/E)^\gamma [(4/5) (f(\lambda x) + f(\mu x))$$

$$+ (1/5) (\epsilon_0/E_0)^\gamma W(\gamma+1) (\sigma + V(\gamma+1) - 1)^{-1} (f(\lambda x) - f(\mu x))], \quad (5)$$

$$n(>E, x) = \pi (E_0/E)^\gamma [(4/5) (f(\lambda x)$$

$$- f(\mu x)) + (1/5) (\epsilon_0/E_0)^\gamma$$

$$\times W(\gamma+1) (\sigma + V(\gamma+1) - 1)^{-1}$$

$$\times (f(\lambda x) - f(\mu x))],$$

where

$$V(\gamma+1) = \int_0^1 v^\gamma V(v) dv,$$

$$W(\gamma+1) = \int_0^{1/4} w^\gamma W(w) dw$$

$$= \frac{4}{4^\gamma} \frac{l_n}{l_\alpha} V(\gamma+1), \quad (6)$$

$$\lambda = 1 - V(\gamma+1),$$

$$\mu = 1 - (k - k') V(\gamma+1),$$

and

$$f(x) = e^{-x} + x E_i(-x).$$

Thus,

$$\frac{p(>E, x) - n(>E, x)}{p(>E, x) + n(>E, x)}$$

$$= \frac{f(\mu x)}{f(\lambda x) \{1 + \alpha(1 - f(\sigma x)/f(\lambda x))\}}, \quad (7)$$

where

$$\alpha = \frac{1}{4} \left(\frac{\epsilon_0}{E_0} \right)^\gamma \frac{W(\gamma+1)}{\sigma + V(\gamma+1) - 1}$$

$$= \frac{l_n}{l_\alpha} \left(\frac{\epsilon_0/4}{E_0} \right)^\gamma \frac{V(\gamma+1)}{\sigma + V(\gamma+1) - 1}$$

$$= \frac{l_n}{l_\alpha} \left(\frac{\epsilon_0/4}{E_0} \right)^\gamma \frac{1 - \lambda}{\sigma - \lambda}. \quad (8)$$

To explain the altitude variation of nucleonic components, we must take

$$\lambda = 1 - V(\gamma+1) \simeq \frac{1}{2} \quad (9)$$

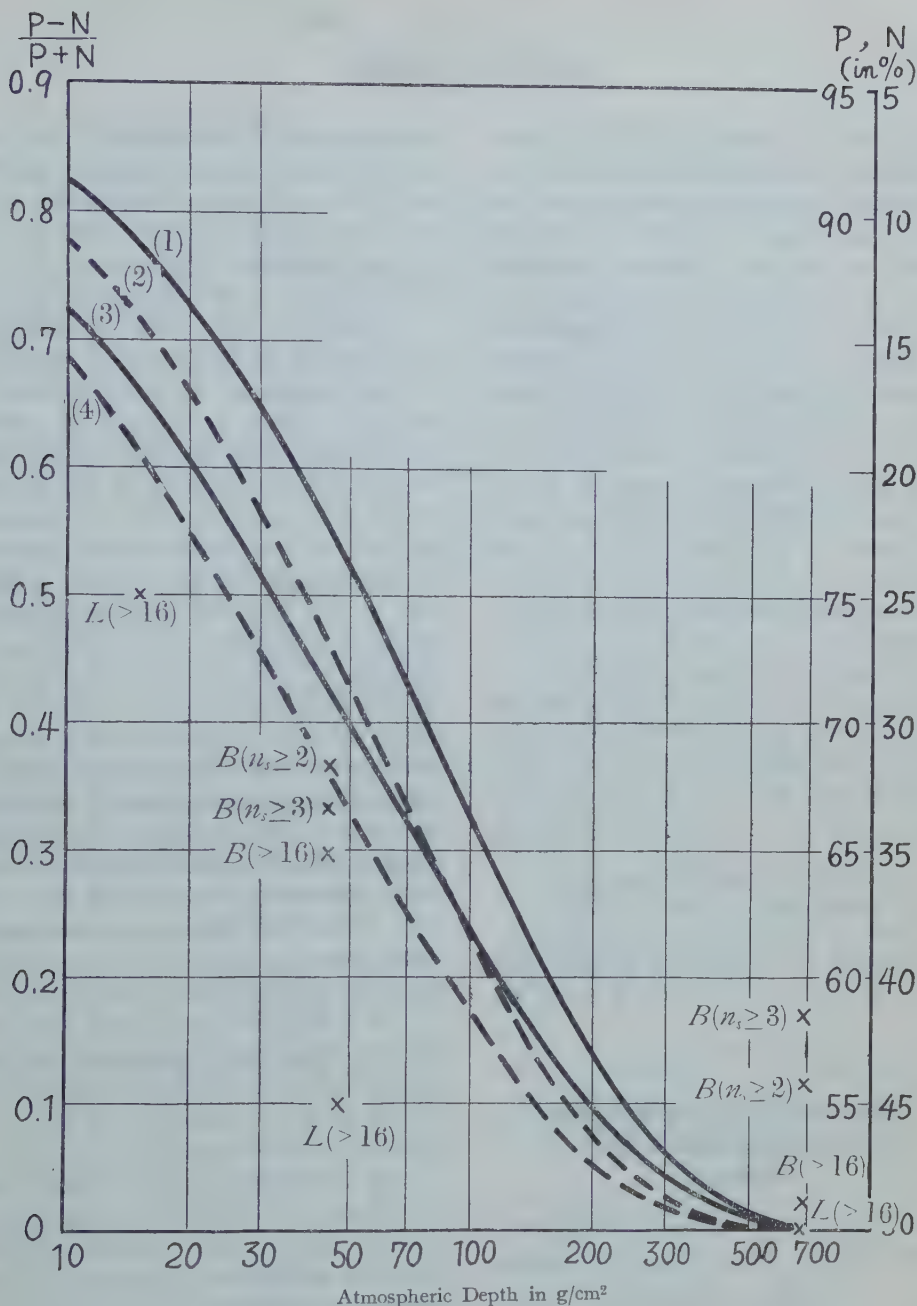
corresponding to the absorption mean free path $\simeq 130$ g/cm².

$$\text{If } l_n/l_\alpha = 3, \quad \sigma = 65/40^{33}, \quad (10)$$

we have at geomagnetic latitude $54^\circ N$ ($E_0 = 1$ Bev, $\epsilon_0/4 = 0.35$ Bev)

$$\alpha = \frac{4}{3} (0.35)^\gamma = \begin{cases} 0.4201 & \text{for } \gamma = 1.1, \\ 0.2237 & \text{for } \gamma = 1.7. \end{cases}$$

In Fig. 1 we have plotted $f(\mu x)/f(\lambda x)$ (without taking into account α particles in primaries) and (7) for $\mu = 1$ ($k = k' = 1/2$) and $\mu = 7/6$ ($k = 1/3$, $k' = 2/3$) taking $\gamma = 1.1$ which brings larger effect than that for higher γ . The experimental points have been taken from the Lord's data and Bristol's. $L(>16)$ mean $(p-n)/(p+n)$ of the stars with more than 16 prongs in Lord's, and $B(n_s \geq 2)$, $B(n_s \geq 3)$ and $B(>16)$ those with two or more than two shower particles, three or more than three shower particles and more



$$(1) : f\left(\frac{x}{65}\right) / f\left(\frac{x}{130}\right)$$

$$(2) : f\left(\frac{7}{6} \cdot \frac{x}{65}\right) / f\left(\frac{x}{130}\right)$$

$$(3) : \frac{f\left(\frac{x}{65}\right)}{f\left(\frac{x}{130}\right)} \cdot \frac{1}{\left\{1 + 0.42 \left(1 - f\left(\frac{x}{40}\right) / f\left(\frac{x}{130}\right)\right)\right\}}$$

$$(4) : \frac{f\left(\frac{7}{6} \cdot \frac{x}{65}\right)}{f\left(\frac{x}{130}\right)} \cdot \frac{1}{\left\{1 + 0.42 \left(1 - f\left(\frac{x}{40}\right) / f\left(\frac{x}{130}\right)\right)\right\}}$$

than 16 prongs in Bristol's, where p and n are the numbers of stars induced by charged and neutral particles respectively. If we consider the charged particles contain π^\pm mesons, the empirical values of $(p-n)/(p+n)$ would be reduced. Although, as is easily seen, Lord's data are in disagreement with Bristol's and we cannot draw a decisive conclusion on high energy neutron-proton ratio at high altitudes on account of low accuracy of experiments, it seems difficult to explain high neutron-proton ratio of Lord's experiments at least, as the effect on neutron-proton ratio of α particles in primaries is rather small in our treatment.

Our conclusion essentially depends on the assumption (1) and (2). Assumption of power spectrum is rather insensitive for neutron-proton ratio, since (7) for $\alpha=0.2237$ ($\gamma=1.7$) is not appreciably different from that for $\alpha=0.4201$ ($\gamma=1.1$). Numerical values (10) would not be different from true values so that our conclusion is not altered.

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"Infra-red Catastrophe" -like Divergency in Meson-Decay Process*

D. Itô

Physics Institute, Tokyo Bunrika University

November 12, 1951

It is well-known that no "*Infra-red Catastrophe*" is able to appear in the process of meson-bremsstrahlung by deflection of a heavy particle in an external field. The reason is readily understood if we take into account the fact that the "*Infra-red Catastrophe*" originates from the poles due to the

vanishing energy denominators in the perturbation formulae of the transition probability, but in the corresponding meson-process the energy conservation in intermediate states can no longer be satisfied on account of the non-vanishing mass of the emitted meson.

Instead of the ordinary emission of meson by the heavy particle, if we consider a decay process of the heavy particle into more lighter one and a meson, the conservation of energy in the intermediates states becomes possible, because of the possibility of real meson-decay processes. Consequently, the integrals of the total transition probability of this process will be divergent as in the case of the infra-red catastrophe appeared in the bremsstrahlung in quantum electrodynamics.

The problem of the infra-red catastrophe has been discussed by many authors¹⁾, and clarified that the infra-red divergence appeared in the bremsstrahlung is cancelled out by that of the radiative correction of the elastic scattering process. Recently, T. Kinoshita²⁾ analysed this problem in more detail and showed that this cancellation can be accomplished for each one of the Feynman diagrams of corresponding processes.

It is the aim of this note to demonstrate that the "*Infra-red Catastrophe*"-like divergency in the meson-decay process can be treated in same manner as in the case of ordinary infra-red divergence.

In accordance with Kinoshita's results, we confine ourselves to the discussion of cancellation only upon a special term of the transition probability, represented by Feynman's diagram in Fig. 1.

Using the following simplest model:

- $U(x)$: heavy charged scalar field, (mass M)
- $\phi(x)$: light charged scalar field, (mass m)
- $\phi(x)$: neutral scalar field, (mass μ)
- $V(x)$: external field,

$$H(x) = g(\bar{U}\phi + \bar{\psi}U)\phi,$$

$$K(x) = \bar{U}UV, \quad (1)$$

$$K'(x) = \bar{\psi}\phi V,$$

the transition probability w_A of the decay process is calculated as

$$w_A = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int_{\epsilon_k = \mu}^{\epsilon_{\max.}} \frac{d\mathbf{K}}{\epsilon_k} \int \frac{d\mathbf{P}'}{e_{p'-k}} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{[E_p^2 - E_{p'}^2][e_{p'-k}^2 - e_{p-k}^2]} \delta(E_p - e_{p'-k} - \epsilon_k), \quad (2)$$

$$\text{where } \begin{pmatrix} E_p \\ e_p \end{pmatrix} \equiv \sqrt{\mathbf{P}^2 + \begin{pmatrix} M^2 \\ \mu^2 \end{pmatrix}} \text{ respectively,}$$

and $\delta(E_p - e_{p'-k} - \epsilon_k) = \delta$ -function, which guarantees the energy-conservation in process "A".

The integrand has two poles. One of which corresponds to the energy conservation in the intermediate state B , and the other corresponds to the conservation in the intermediate state C . In the vicinity of these two poles, w_A takes following asymptotic forms

$$w_A(B) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2E_p' e_{p'-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{e_{p'-k}^2 - e_{p-k}^2} \frac{\delta(E_p - e_{p'-k} - \epsilon_k)}{e_{p'-k} + \epsilon_k - E_{p'}}, \quad (3)$$

$$w_A(C) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2e_{p-k} e_{p'-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{E_p^2 - E_{p'}^2} \frac{\delta(E_p - e_{p'-k} - \epsilon_k)}{e_{p'-k} - e_{p-k}}, \quad (4)$$

In analogy with the theory of ordinary infra-red divergence, the transition probability of the elastic scattering w_B must be considered, and is calculated as follows

$$w_B = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int_{\epsilon_k = \mu}^{\infty} \frac{d\mathbf{K}}{\epsilon_k} \int \frac{d\mathbf{P}'}{E_{p'}} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{[e_{p-k}^2 - (E_p - \epsilon_k)^2][e_{p'-k}^2 - (E_{p'} - \epsilon_k)^2]} \times$$

$$\times \delta(E_p - E_{p'}), \quad (5)$$

Two poles of this integrand correspond to the conservation of energy in two intermediate states A and B respectively, and the asymptotic forms are

$$w_B(A) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2E_p' e_{p'-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{e_{p-k}^2 - e_{p'-k}^2} \frac{\delta(E_p - E_{p'})}{e_{p'-k} + \epsilon_k - E_p}, \quad (6)$$

$$w_B(C) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2E_{p'} e_{p-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{e_{p'-k}^2 - e_{p-k}^2} \frac{\delta(E_p - E_{p'})}{e_{p-k} + \epsilon_k - E_p}. \quad (7)$$

In this case, we must calculate the transition probability w_C of the process of meson-decay accompanied by double scattering. This process corresponds to cutting of the diagram in Fig. 1, at the vertical line C . w_C is calculated as

$$w_C = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int_{\epsilon_k = \mu}^{\epsilon_{\max.}} \frac{d\mathbf{K}}{\epsilon_k} \int \frac{d\mathbf{P}'}{e_{p-k}} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{[E_{p'}^2 - E_p^2][e_{p'-k}^2 - e_{p-k}^2]} \delta(E_p - e_{p'-k} - \epsilon_k), \quad (8)$$

and two asymptotic forms at A and B are

$$w_C(A) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2e_{p-k} e_{p'-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{E_p^2 - E_{p'}^2} \frac{\delta(E_p - e_{p'-k} - \epsilon_k)}{e_{p'-k} - e_{p-k}}, \quad (9)$$

$$w_C(B) = \frac{g^2}{4(2\pi)^8} \frac{1}{E_p} \int \frac{d\mathbf{K}d\mathbf{P}'}{2E_{p'} e_{p-k} \epsilon_k} \times \frac{|V(\mathbf{P}-\mathbf{P}')|^2}{e_{p'-k}^2 - e_{p-k}^2} \frac{\delta(E_p - e_{p'-k} - \epsilon_k)}{E_{p'} - e_{p-k} \epsilon_k}. \quad (10)$$

Comparing the asymptotic forms of divergent integrals of these transition probabilities, we can easily find out a kind of "reciprocity relations";

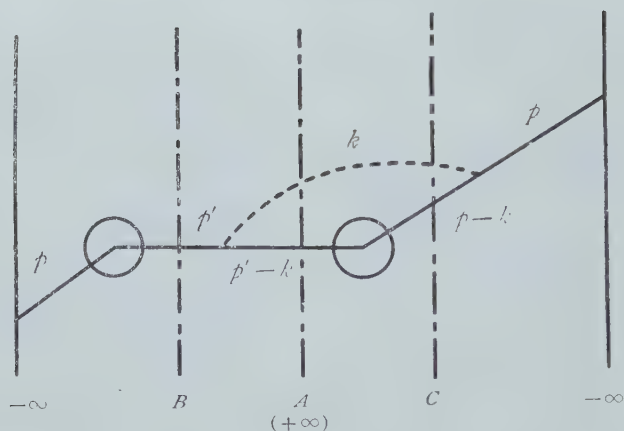


Fig. 1

$$\begin{aligned}
 w_A(B) &= -w_B(A) \\
 w_R(C) &= -w_C(B) \\
 w_C(A) &= -w_A(C),
 \end{aligned} \quad (11)$$

holding between them.

From view-point of the asymptotic forms of divergence, we can summarize the above results as follows

$$\begin{aligned}
 w_A &\sim w_A(B) + w_A(C), \\
 w_B &\sim w_B(A) + w_B(C), \\
 w_C &\sim w_C(A) + w_C(B).
 \end{aligned} \quad (12)$$

From these asymptotic forms and the above reciprocity relations, the "Infra-red Catastrophe"-like divergences in the meson decay process can be cancelled out by simultaneous consideration of the processes of elastic scattering and of decay with double scattering.

The detailed account will be given in the later issue of this journal. Author expresses his cordial thanks to Dr. T. Miyazima for the interest he has taken in this work, and to Miss. M. Yamazaki and Messrs. H. Tanaka and M. Ishida for their kind discussion.

*). The contents of this letter are read at the annual meeting of the Physical Society of Japan held on October 9th in 1951.

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On the Divergence of the Transition Probability due to Energy Conservation in Intermediate States

D. Itô

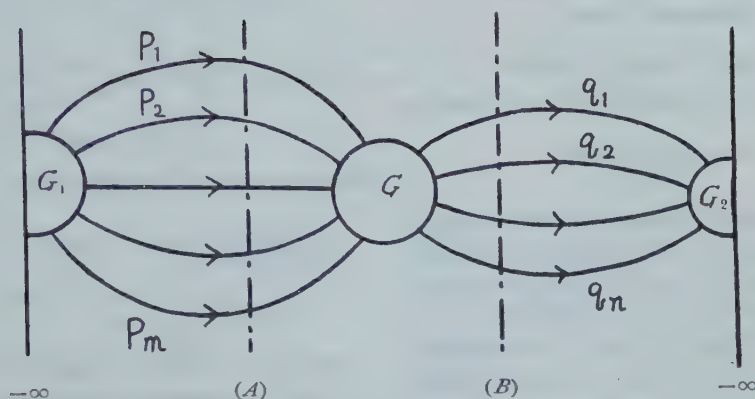
Physics Institute, Tokyo Bun-rika University

November 12, 1951

In the previous letter¹⁾ we have observed that the "Infra-red Catastrophe"-like divergence in the transition probability of meson-decay process can be avoided just in the same manner as in the case of ordinary infra-red catastrophe, and that the reason why this method is successful consists in the "reciprocity relations" holding between the asymptotic forms of divergent integrals. This method will be successfully extended to the more general cases if we are able to give a proof of validity of the reciprocity relations in more general form.

In this letter, we shall give a brief account of the validity of the reciprocity relations in sufficient generality, and by which,

Fig. 1



$$E_i(\mathbf{p}_i) \equiv \sqrt{\mathbf{p}_i^2 + M_i^2}, \quad e_j(\mathbf{q}_j) \equiv \sqrt{\mathbf{q}_j^2 + m_j^2}$$

we show that the divergence of transition probability due to the energy conservation in intermediate states will be eliminated just in the same manner as in the case of infra-red catastrophe.

The most general form of Feynman's diagram²⁾ for transition probability is illustrated in Fig. 1, there, G_1 , G_2 and G represents an arbitrary graph, respectively. Referring to this graph, we can formulate the "reciprocity relation" as follows: The asymptotic form of divergence $w_A(B)$ due to the energy conservation in intermediate state "B" of the process "A", which is obtained from the diagram by cutting at the vertical line A, has just the same form and opposite sign as the asymptotic form $w_B(A)$ of similar divergence due to conservation in "A" of the process "B".

Using same model as in the previous letter, we can obtain following asymptotic forms after some elementary calculations.

$$w_A(B) = \prod_{i=1}^m \int \frac{d\mathbf{p}_i}{E_i(\mathbf{p}_i)} \prod_{j=1}^n \int \frac{d\mathbf{q}_j}{e_j(\mathbf{q}_j)} \times \\ \times \frac{f(\mathbf{p}_1, E_1(\mathbf{p}_1); \dots; \mathbf{q}_1, e_1(\mathbf{q}_1); \dots)}{\sum_{j=1}^n e_j(\mathbf{q}_j) - \sum_{i=1}^m E_i(\mathbf{p}_i)} \times \\ \times \delta(E_{\text{initial}} - \sum_{j=1}^n e_j(\mathbf{q}_j)),$$

$$w_B(A) = \prod_{i=1}^m \int \frac{d\mathbf{p}_i}{E_i(\mathbf{p}_i)} \prod_{j=1}^n \int \frac{d\mathbf{q}_j}{e_j(\mathbf{q}_j)} \times \\ \times \frac{f(\mathbf{p}_1, E_1(\mathbf{p}_1); \dots; \mathbf{q}_1, e_1(\mathbf{q}_1); \dots)}{\sum_{i=1}^m E_i(\mathbf{p}_i) - \sum_{j=1}^n e_j(\mathbf{q}_j)} \times \\ \times \delta_{\text{initial}} - \sum_{i=1}^m E_i(\mathbf{p}_i)).$$

These results show that the reciprocity relation;

$$w_A(B) = -w_B(A),$$

holds for the most general cases.

The possibility of the cancellation of divergence due to energy conservations in intermediate states by simultaneous consideration of related processes is an immediate consequence of this reciprocity relation.

The detailed accounts will be given in the later issue of this journal.

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Divergences arising from Nuclear Forces

T. Nakano and K. Nishijima

Department of Physics, Osaka City University

November 24, 1951

To our regret the renormalization method which has provided us with a powerful means so far in the quantum electrodynamics has lost its validity once applied to the meson field theory. Thus because of the absence of any standard procedure to manage divergences together with the lack of any satisfactory approximation method, we are confronted with hard difficulties in the meson theory much more seriously than in the case of electrodynamics. Under such circumstances, it will be of some importance to analyse the forms of divergences not explored yet.

Here, we shall call our attention to the divergences arising from the fourth order nuclear forces based on the symmetrical pseudoscalar meson theory. The interaction Lagrangian between nucleon and P_8 -meson fields including both pseudoscalar and pseudo-vector couplings is given by

$$L = -i\bar{\psi}\gamma_5\tau_j\psi\phi_j - i(g/x) \times \bar{\psi}\gamma_\mu\tau_j\psi\phi_j \cdot (\partial\phi_j/\partial x_\mu), \quad (1)$$

where j denotes a dummy index with regard to charge coordinates, and runs from 1 to 3 in the symmetrical theory.

In order to evaluate the fourth order S -matrix, it is convenient to utilize Dyson transformation, by which the Lagrangian (1) is transformed into the following equivalent form¹⁾:

$$\bar{L} = \bar{L}_1 + \bar{L}_2 + \bar{L}_3 + \dots, \quad (2)$$

where

$$\bar{L}_1 = -iF\bar{\psi}\gamma_5\tau_j\psi\phi_j, \quad \left(F = f - \frac{2M}{x}g\right)$$

$$\bar{L}_2 = \frac{1}{2} \left(\frac{g}{x}\right)^2 \phi_k \frac{\partial\phi_j}{\partial x_\mu} \bar{\psi}\gamma_\mu[\tau_k\tau_j]\psi,$$

$$\bar{L}_3 = \frac{i}{6} \left(\frac{g}{x}\right)^3 \phi_k \left\{ \phi_l \frac{\partial\phi_j}{\partial x_\mu} \right\} \bar{\psi}\gamma_5\tau_\mu \times \{ \tau_l(\tau_k\tau_j) \} \psi.$$

Now the desired S -matrix can be written as²⁾

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int (dx_1) \dots \int (dx_n) \times P^*(\bar{L}(x_1) \dots \bar{L}(x_n)). \quad (3)$$

Since the second order S -matrix is well known, we shall be concerned only about the fourth order terms. They are given by

$$S_f = \frac{1}{4!} \int (dx_1) \dots \int (dx_4) \times P^*(\bar{L}_1(x_1), \bar{L}_1(x_2), \bar{L}_1(x_3), \bar{L}_1(x_4)),$$

$$S_a = \frac{i^3}{2!} \int (dx_1) \dots \int (dx_3) \times P^*(\bar{L}_1(x_1), \bar{L}_1(x_2), \bar{L}_2(x_3)),$$

$$S_b = -\frac{1}{2!} \int (dx_1) \int (dx_2) \times$$

$$P^*(\bar{L}_2(x_1), \bar{L}_2(x_2)),$$

$$S_c = -\int (dx_1) \int (dx_2) P^*(\bar{L}_1(x_1), \bar{L}_3(x_2)).$$

The first type S_f has already been computed by Watson and Lepore³⁾, and it is known that all divergences occurring in S_f can be eliminated by means of the mass and coupling constant renormalizations. The last one S_c vanishes identically as easily be verified. The other two, S_a and S_b , bear an essential divergent character outside the scope of the renormalization technique.

We shall write down these divergent terms:

$$(a) \quad G^2 \left(\frac{g}{x}\right)^2 \bar{\psi}(x) \gamma_\mu \tau_l \psi(x) \cdot \bar{\psi}(x) \gamma_\mu \tau_l \psi(x) \times (\text{logarithmic divergence}), \quad (5a)$$

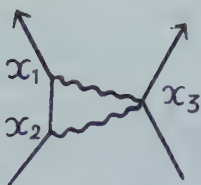


Fig. (a)

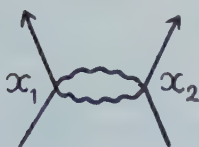


Fig. (b)

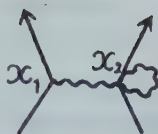


Fig. (c)

$$(b) \quad \left(\frac{g}{x}\right)^4 \bar{\psi}(x) \gamma_\mu \tau_L \psi(x) \cdot \bar{\psi}(x) \gamma_\mu \tau_L \psi(x) \times (\text{quadratic divergence}), \quad (5b)$$

$$\left(\frac{g}{x}\right)^4 \partial_\lambda (\bar{\psi}(x) \gamma_\mu \tau_L \psi(x)) \times \partial_\lambda (\bar{\psi}(x) \gamma_\mu \tau_L \psi(x)) \times (\text{logarithmic divergence}). \quad (5c)$$

The first two terms are of the same Fermi type interaction, and the last one is of a more singular type one that is no more allowed to exist in the canonical formalism.

Therefore it is clear that these divergences cannot be removed by the renormalization technique in the frame of the present field theory. Moreover, the occurrence of such interactions indicates that singular terms in higher orders will be larger than in lower orders as the energy increases even if the divergences were removed by a suitable technique.

Thus the perturbation method breaks down at high energies, and the convergence radius will be determined by an unknown parameter such as the universal length.⁴⁾ On the other hand, the strong coupling meson theory cannot be applied at high energies because of the failure of its relativistic treatment.

The delta-function type interactions (5a) and (5b) expressed in the momentum space correspond to r^{-3} singularities of the nuclear potential⁵⁾, so that (5b') corresponds to r^{-5} singularity in conformity with the previous calculations.⁶⁾

And this is an example of the statement that the higher the order of approximation,

the more singular the nuclear potential. To conclude, the applicability of the P_8 -meson theory is severely limited below some critical energy determined by an unknown parameter, which fact means, if translated into the language of the coordinate space, that our knowledge about the nuclear forces is restricted only to the outside of some characteristic "critical range."

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- 3) K. M. Watson and J. V. Lepore, Phys. Rev. **76** (1949), 1157.
- 4) W. Heisenberg, ZS. f. Phys. **101** (1936), 533; **113** (1939), 61; J. R. Oppenheimer, H. Snyder and R. Serber, Phys. Rev. **57** (1949), 75.
- 5) L. Rosenfeld, *Nuclear Forces*, (1948) p. 323.
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On the Plasma-like Oscillation

T. Nishiyama

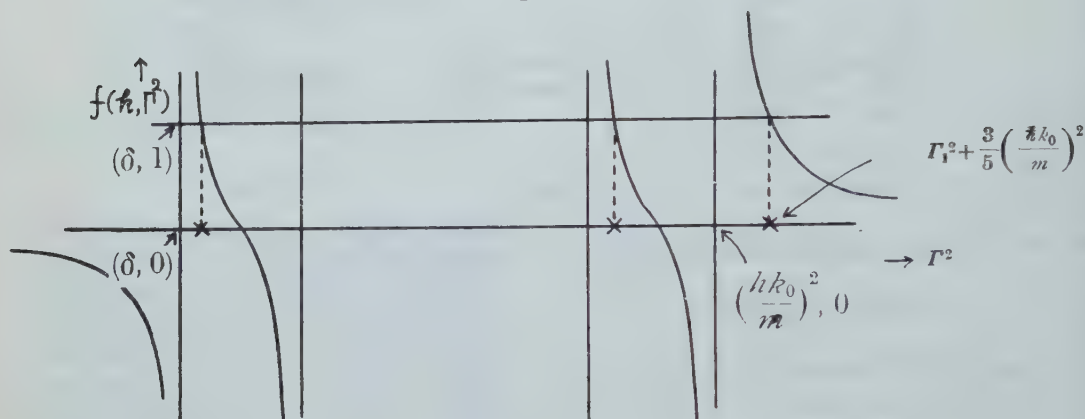
Department of Physics, Osaka University

November 29, 1951

Recently Pines and Bohm¹⁾ have concluded that the lowest unit of the energy loss in the Cerenkov-like radiation is given by the plasma frequency, evaluating the density fluctuation classically.

Such a plasma-like frequency is also obtained²⁾ as a normal mode of the coupled

Fig. 1



oscillators, which were given in our paper³⁾ and derived by selecting out the linear terms of the exact equations to result in the neglect of the direction interaction that diverts electrons excited in a direction to another direction. The normal modes are given as the characteristic values of the matrix of the form

$$\gamma_{st}^2(k) = \frac{\hbar^2}{m^2} (k_0^2 - s^2) \delta_{st} + \frac{\sqrt{s N_s} \sqrt{t N_t}}{m} J G(k)$$

which is positive definite. The characteristic values are demonstrated in the following graphic curve:⁵⁾

$$f(k, I^2) = \frac{G(k) L}{m} \sum_s \frac{s N_s}{I^2 - \frac{\hbar^2}{m^2} (k_0^2 - s^2)}$$

which shows that the almost continuous frequency distribution is shifted slightly and the largest value is shifted to a value larger than $I_1^2 = NG(k)/m$.

This formulation fails to give the angular dependence of the electron excitations as has been pointed out by both Professor Tomonaga and Professor Wentzel. The author would like to express his thanks to them for the valuable discussions and the same gratitude to Professor Husimi for his interest in this problem.

- 1) D. Pines and D. Bohm, Submitted to the Physical Review. The writer thanks to Professor Tomonaga for bringing his notice to a copy of their manuscript.
- 2) G. Wentzel, private communication.
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- 4) S. Tomonaga, Prog. Theor. Phys. **5** (1950), 544.
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A Commentary to Yosida's Theory on FeS_n

S. Miyahara

Department of Physics, Hokkaido University

November 29, 1951

Recently K. Yosida proposed a theory¹⁾ to explain elegantly the unordinary magnetic properties of FeS_n. I think, however, the theory of FeS_n must clarify the fact, that the ferro-para transition due to the sulphur content is almost independent on temperature. An experiment²⁾ shows, that the paramagnetic (at room temperature) FeS_n containing the utmost sulphur does not become ferromagnetic at the liquid nitrogen temperature. This fact cannot be explained by the gradual

lowering of the Curie point according to the decrease of sulphur content, as Yosida's theory shows.

On the other hand, from the pure theoretical standpoint, the Madelung energy (M.E.) must increase according to the number of Fe^{+++} ions and vacancies without electric charge. The M.E. would be smaller, if all Fe ions were divalent and each vacancy took two positive holes making the M.E. same to that of perfectly stoichiometric FeS. Generation of Fe^{+++} ions caused by the jump of positive holes from vacancies to Fe^{++} ions, must increase the energy. Besides, the increase of energy depends on the distribution of Fe^{+++} ions and vacancies.

For convenience it is assumed, that the increase of energy is given as a linear function of the number x of Fe^{+++} ions and simply dependent on the inequality ρ of the distribution on two sub-lattices. Then we must add a positive energy term of a type

$$bx(1-2\rho)^2 \quad (1)$$

to Yosida's expression (6), where b is a certain constant.

The energy expression containing both (1) and Yosida's negative term formally coincides to that of the theory of ferromagnetism of semi-conductor based on the band structure, where we take the positive energy due to the excitation of electrons to a higher band and the negative one proportional to the square of total spins.

Above mentioned improvement of Yosida's theory alters the expression of the Curie point as

$$\Theta = -\frac{2a}{k} \left(x - \frac{b}{a} \right) (1-x).$$

If we take $b/a=0.05$, we can easily show that the pyrrhotite containing less sulphur than $\text{FeS}_{1.1}$ is always paramagnetic.

- 1) K. Yosida, Prog. Theor. Phys. **6** (1951), 356.
- 2) S. Miyahara, Proc. Phys.-Math. Soc. **22** (1940), 358.

Generalized Furry's Theorem for Closed Loops, II*

K. Nishijima

Department of Physics, Osaka City University

November 30, 1951

Previously we have discussed the generalization of Furry's theorem, especially concerned with the parities of Dirac matrices, but not touched upon the isotopic matrices. So it seems instructive to examine the rôle of isotopic matrices in full detail.

We shall employ the same notations with (I) throughout this letter. First let us remember the following relation:

$$C^{-1} \gamma_\mu C = -\gamma_\mu^T. \quad (1)$$

Since this relation is very useful in the theory of closed loops, we shall define an analogous matrix c in the isotopic space by

$$c^{-1} \tau_L c = -\tau_L^T. \quad (L=1, 2, 3) \quad (2)$$

For the most usual representation of the isotopic matrices, i.e.

$$\begin{aligned} \tau_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \tau_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (3)$$

we can put as

$$c = \tau_2. \quad (4)$$

What is necessary for us is only the existence of such a matrix c that satisfies the equation (2).

In (1), we defined two quantities α and $\bar{\alpha}$ for closed loops C and \bar{C} by

$$\alpha = S_p(T_1 T_2 \cdots T_n),$$

$$\bar{\alpha} = S_p(T_n \cdots T_2 T_1).$$

In the meson-nucleon system, each T is equal to one of isotopic matrices τ_1, τ_2, τ_3 and I , so that we have with the aid of equation (2)

$$\begin{aligned}
 \bar{a} &= S_p(T'_n \cdots T'_2 T'_1)^T \\
 &= S_p(T_1^T T_2^T \cdots T_n^T) \\
 &= (-1)^{n-m} S_p(e^{-1} T_1 e \cdot e^{-1} T_2 e \cdots e^{-1} T_n e) \\
 &= (-1)^{n-m} a,
 \end{aligned} \quad (5)$$

where m is the number of the isotopic matrix I , namely the number of pure neutral mesons.

Thus the factor which is responsible for the selection rule becomes

$$\begin{aligned}
 a + (-1)^N \bar{a} \\
 = a(1 + (-1)^{N+n-m}).
 \end{aligned} \quad (6)$$

So the case

$$N + n - m = \text{odd} \quad (7)$$

is forbidden. Especially in the symmetrical theory

$$N + n = \text{odd} \quad (8)$$

is forbidden since $m=0$.

And we see in symmetrical meson theories with even couplings such as (S), (Ps) and (Pv) that the contributions from graphs containing odd order closed loops identically vanish, because in the above case we know

$$N=0.$$

On the other hand, if the coupling is odd, i.e. (V) or (T), we have no selection rule from the above argument since $N=n$.

* K. Nishijima, *Prog. Theor. Phys.* **6** (1951), 614.
This paper is cited as (I) in the text.

On the Short μ -Meson Tracks from π -Meson Decays

T. Nakano, J. Nishimura*
and Y. Yamaguchi

*Osaka City University and Kobe University**

November 30, 1951

Recently Fry has found the following anomalous mode of π - μ decay events with

photographic emulsion (200 micron Ilford C2-plates)¹⁾: (i) Among 3018 π - μ events there were four cases in which μ -meson tracks are unusually short (about 200 microns). Two of them might be interpreted as due to the decay in flight of the π -mesons. Thus the relative frequency of anomalous decay events with short μ -meson track among the normal π - μ decay is estimated as $\sim 2 \times 10^{-4}$, taking into account the finite thickness of emulsions.

(ii) Furthermore 20 cases of short μ -meson tracks with ranges between 480 and 520 microns were also found, and they were regarded as the examples of extreme straggling.

Of course there remains the possibility that these events are quite different from π - μ decay, e.g., at least either parent or daughter particle may not be an usual (π or μ) meson. However, according to Fry, both of parent and daughter particles seem to be really usual mesons. Let us, therefore, try to interpret here these anomalous events as decay but different decay mode from usual one:

$$\pi^\pm \rightarrow \mu^\pm + \nu^0 \quad (1)$$

where ν^0 is the neutral counter particle and $P(E) \times 10^3$

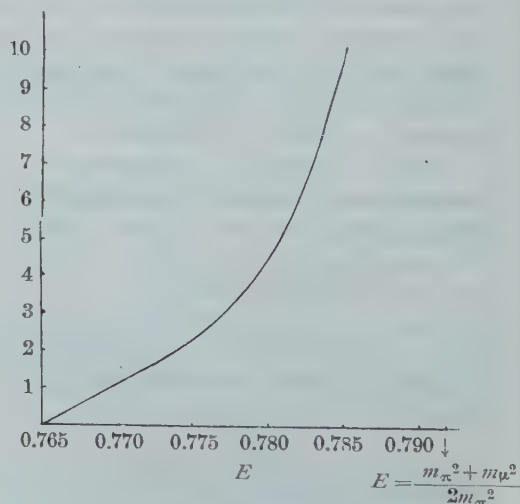


Fig. 1.

probably neutrino.

We first tentatively assume the following two particle decay mode:

$$\pi^\pm \rightarrow \mu^\pm + \mu^0.$$

The mass of neutral particle μ^0 can now be determined from the range of corresponding μ^\pm and it should be as large as $\sim 40 m_e$ (m_e electron mass). However, as there is no other positive evidence for existence of such neutral particles, this interpretation does not seem to be reasonable. Moreover the range of the μ^\pm -mesons are somewhat widely distributed and are not favorable to the two particle decay mode.

We then next assume the three particle decay mode:

$$\pi^\pm \rightarrow \mu^\pm + \mu_1^0 + \mu_2^0,$$

where μ_1^0 and μ_2^0 are neutral particles, and one must be a fermion while the other a boson. If we do not demand new unknown particles, it is easily concluded that μ_1^0 and μ_2^0 should be neutrino ν^0 and photon γ considering masses and energy balance:

$$\pi^\pm \rightarrow \mu^\pm + \nu^0 + \gamma. \quad (2)$$

In other words, this decay mode is a so-called radiative correction for normal π - μ decay (1). If we assume the direct coupling between pseudoscalar π -field and spin 1/2 (μ^\pm, ν^0) field*, the probability of (2) relative to ordinary one (1) is readily found by the well-known Feynman-Dyson method:

$$\begin{aligned} \frac{\delta w}{w} &= \int_{\pi}^{\frac{1-x^2}{2}} P(E) dE, \\ P(E) &= \frac{1}{\pi} \frac{e^2}{4\pi\hbar c} \frac{1-x^2}{(2-x^2)(1+x^2)} \times \\ &\times p \left[2 + \frac{6(1-x^2)}{1+x^2-2E} \right. \\ &- \frac{1}{p} \left\{ (2E-1) \ln \frac{E-x^2+p}{E-x^2-p} \right. \\ &\left. \left. + (3-x^2) \ln \frac{1-E+p}{1-E-p} \right\} \right]; \quad (3) \\ x &= m_\mu/m_\pi; \quad p = \sqrt{E^2 - x^2}; \end{aligned}$$

where m_μ and m_π mean μ - and π -meson mass, respectively, and $m_\pi E$ denotes the total energy of μ -meson (including rest energy). The energy distribution $P(E)$ of μ -meson is shown in Fig. 1. The probability that the kinetic energy of μ -meson is smaller than $\frac{3}{4} m_\pi \left(\frac{1+x^2}{2} - x \right)$ ($m_\pi \frac{1+x^2}{2}$ is the total μ -meson energy in the case of (1)) is $\sim 10^{-4}$, which is in excellent agreement with Fry's result. $P(E)$ shows catastrophic raise at $E = \frac{1+x^2}{2}$, which is due to the same origin as the infrared catastrophe in quantum electro-dynamics and can be removed by the same device as there²⁾. But we do not enter into its details. Anyhow there should be a few number of μ -tracks with ranges larger than 500 microns. Nevertheless, theoretical investigation for straggling shows that the number of such tracks are negligibly small compared with ones caused by straggling.

O'Ceallaigh has found an event which may be interpreted as π - μ decay accompanying electron-pair produced by photon, and seems to be favorable to our assumption (2) (though O'Ceallaigh's opinion concerning this event is different from ours).³⁾

There remain other possibilities to explain these anomalous decays, e.g., the four particle decay, etc. However, the decay modes involving more than four daughter particles seem to be excluded because they lead too low energy for μ -mesons to agree with Fry's results. We believe that our explanation (2) is the most reasonable among many possibilities.

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- 2) D. Ito, Soryushiron-Kenkyu, III-5 (1951), 28. Also see T. Kinoshita, Prog. Theor. Phys. **5** (1950), 1045 (L).
- 3) C. O'Ceallaigh, Phil. Mag. **41** (1950), 838.

* If we adopt other models for coupling between π - and μ -mesons and other elementary particles (cf. S. Nakamura et al., Prog. Theor. Phys. **5** (1950), 140), we get slightly different results.

On the Z-Dependence of the Positive-Negative Ratio of the Mesons produced by Photons on Nuclei

S. Machida and T. Tamura

Department of Physics, University of Tokyo

November 30, 1951

Recently, Littauer and Walker¹⁾ published a remarkable experimental evidence, i.e., minus to plus ratios of 50 Mev π -mesons, produced from nuclei bombarded by 300 Mev X-ray, reduced with increasing mass number A and the results for symmetrical (D, C, O, S, and Ca) and asymmetrical (Be, F, Al, and Bi) nuclei were as were as was shown in Fig. 1. Theoretical values by Hayakawa²⁾ are also shown in Fig. 1 with dotted lines. He intended to explain the experiments by the differences between binding energies of the final nuclei after producing π^- or π^+ .

It was necessary to take into account the effects of nuclear binding to explain the ratio between the photo-meson production cross section from carbon and the one from hydrogen target in energy region not much higher than the threshold³⁾. Afterwards, we pointed out that the interactions between produced mesons and producing nuclei were important to explain the $A^{2/3}$ -dependence of the photo-meson production cross section⁴⁾⁵⁾. Since mesons are too fast to attribute Littauer and Walker's result to the difference between interactions of π^+ or π^- with nuclei, we would again have to attribute it to the behavior of nucleons assuming a suitable nuclear model.

In the following we will treat symmetrical nuclei (excepting deuteron, of course) as a group of α -clusters and asymmetrical nuclei as a group of α -clusters plus zero, one, or two neutrons or protons. We will assume, further, that the binding of α -cluster is so tight that the recoil energy is divided uniformly among all nucleons constituting

the α -cluster which produces π -meson. In the final state, therefore, four nucleons will be all in excited states or some of them will go out from nucleus, and the probability for protons being in nucleus will be reduced compared to the one for neutrons by the effect of Coulomb potential barrier. This reduction of the probability for the final state is more effective in the case of π^- -production than in the case of π^+ -production, since in the final state there are three protons and one neutron for the former case compared to one proton and three neutrons for the latter. Moreover this reduction effect increases with Z , so we can expect to obtain qualitative agreement with the experimental results.

To obtain quantitative results, we will adopt the formula given by Menon et al⁶⁾ for the probability, $P(E)$, for one of four nucleons (which constituted the α -cluster just before the π -production) having kinetic energy E , when the total kinetic energy available for four nucleons is E_0 (in rest system of the α -cluster) ;

$$P(E)dE = E_{\max}^{(5-3n)/2} \frac{\Gamma\{(3n-3)/2\}}{\Gamma\{(3n-6)/2\}} \frac{2}{\pi^{1/2}} \times \\ \times E^{1/2} (E_{\max} - E)^{3n/2-4} dE, \quad (1)$$

where, in our case, $n=4$ and $E_{\max}=(3/4)E_0$. From equation (1) we can obtain the probability, $P(q')$, for one of the four nucleons having the momentum $q'=q/p_{\max}$ in the laboratory system (the rest system of bombarded nucleus) as follows :

$$P(q') \propto [1 - 2F' + F'^2 + (1/3)H'^2] q'^2 d\varepsilon',$$

$$\text{for } 0 < q' < 1 - P'/4,$$

$$P(q') \propto [(1 - 2F' + F'^2)(1 + \mu_{\min}) \\ + H'(1 - F')(1 - \mu_{\min}^2) \\ + (1/3)H'^2(1 + \mu_{\min}^3)] q'^2 d\varepsilon', \quad (2)$$

$$\text{for } 1 - P'/4 < q' < 1 + P'/4,$$

where

$$F' = q'^2 + P'^2/16, \quad H' = q' P'/2,$$

$$\mu_{\min} = (1 - F')/H', \quad (2')$$

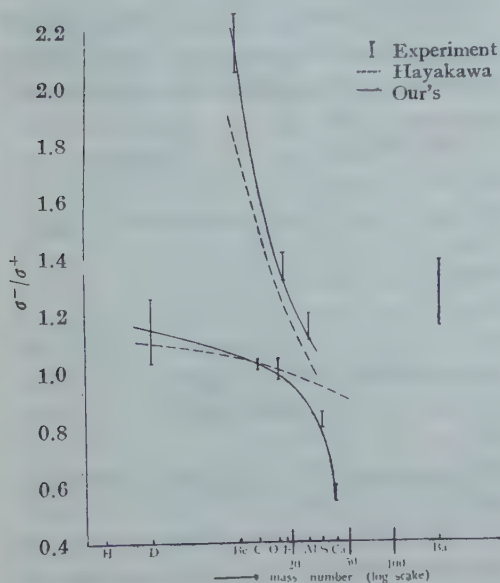
$$P' = P/p_{\max}, \quad p_{\max} = (2ME_{\max})^{1/2},$$

$$P^2 = (\nu - \mu)^2, \quad (2'')$$

and ν and μ are momenta of photon and meson respectively.

Probability, (2), reduces for proton by the effect of Coulomb potential barrier as mentioned above. Therefore, if we write "r.f." (reduction factor) for the ratio between the equation (2) integrated over the interval $0 < q' < 1 + P'/4$ to the cases of proton and neutron, r.f. is smaller than one evidently. If we designate the π^-/π^+ ratio from bare proton and neutron by " k ", then $k(\text{r.f.})^2$ are the ones for symmetrical nuclei. The results are shown in Fig. 1 with full line normalized at carbon, as the value of k is not yet known exactly. The agreement with experimental results is excellent.

For the case of asymmetrical nuclei, we may treat α -clusters as above and extra neutrons as has been done by Hayakawa²⁾, and, afterwards, average them with the weight of their numbers and efficiencies.



Since it is almost impossible to calculate theoretically the efficiency of α -particle for photo-meson production because of the lack of detailed knowledge of the internal wave function of α -particle, we estimate this efficiency, a_e , as follows; we normalize π^-/π^+ -ratio for Be^8 derived as mentioned above to the experimental value of 2.25, assuming the absorption mean free path of π -meson produced from α -cluster to be about equal to the radius of Be^9 nucleus. Then we obtain $a_e \approx 1/3.1$. According to our model, this a_e must be equal to the efficiency of α -nuclei. Since experimental value of efficiency of carbon for this energy is about $1/3^{(7)}$, above value for a_e is quite consistent.

The results obtained for asymmetric nuclei are shown in Fig. 1 with full line, and are in good agreement with experiments.

The fact, that our results are better than Hayakawa's, seems to suggest that we must assume suitable sub-unit (e.g. α -cluster) to account for high energy nuclear reactions. Such features were also pointed out by the discussions of Tamor about π^- -star⁸⁾, of Chew and Steinberger about π^-/π^+ ratio from nucleon-nucleus collision⁹⁾, and of Yoshida¹⁰⁾ and Levinger¹¹⁾ about high energy (γ, p) reaction.

As equation (1) underestimates the probability for low E value⁶⁾, our results might have been reduced than the ones given above, if we had used the correct equation for $P(E)$. However, this effect would be cancelled out and our results would not be altered at any rate, if we correct the assumption that the nuclei are composed completely of α -clusters, which would be an overestimation of the clustering effect.

We have not calculated anew for Bi, because Hayakawa's calculation agrees rather well with experiment and it will not be suitable to treat such a heavy nucleus with large neutron excess from the standpoint of α -model as above.

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- 2) S. Hayakawa, to be published. We are indebted to Mr. Y. Yamaguchi for communicating us Hayakawa's calculation.
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- 5) R. F. Mozley, Phys. Rev. **80** (1950), 493; see also ref. 1)
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- 9) G. F. Chew and J. Steinberger, Phys. Rev. **78** (1950), 497.
- 10) S. Yoshida, Prog. Theor. Phys. **6** (1951).
- 11) J. S. Levinger, Phys. Rev. **80** (1951), 43.

On the High Energy Nuclear Photoelectric Reaction

S. Yoshida

Department of Physics, University of Tokyo

December 1, 1951

Recent experiments¹⁾ have revealed that the high energy protons produced from nuclei irradiated by high energy photon have a marked forward asymmetry. The purpose of this note is to explain these angular distributions, and we calculated it by a similar method as was used by Chew and Goldberger²⁾ for the neutron pick-up process.

We assume that the nuclear photoelectric reaction takes place in a nuclear subunit which is consisted of n nucleons in the nucleus and this subunit emits one proton by absorbing γ -ray. Since we treat high energy nuclear phenomena, we use the plane wave for photon wave instead of decomposing into

multipole radiations. Summing over all possible final states of remaining nucleus, the differential cross section that a proton is emitted in a given solid angle is

$$\sigma(\theta)d\Omega = \frac{1}{8\pi} \frac{e^2}{\hbar c} \frac{n}{n-1} \frac{k}{E_r} \frac{\hbar^2 k^2}{2M} \times \\ \times \rho(\alpha-k) \sin^2 \theta d\Omega, \quad (1)$$

and the energy conservation is given as follow,

$$E_r - \epsilon = \frac{\alpha^2}{2nM} + \frac{n}{n-1} \frac{\hbar^2}{2M} k^2, \quad (2)$$

where ϵ is binding energy of emitted proton, α and E_r are the wave number and energy of photon, r and k are relative distance and relative wave number between one proton and remaining nucleus, θ is the angle between α and k , $\rho(q)$ is the probability that the emitted nucleons has momentum q in the ground state of the subunit composed of n nucleons. In order to compare the angular distribution of protons with experiments we must transform the cross section to laboratory system and average over X-ray spectrum. If we write E_p for the proton energy, Θ for angle between the directions for the emitted proton and incident photon, and q for $\alpha-k$ all in the laboratory system, then we get the cross section $\bar{\sigma}(\Theta)$ that a given energy proton is emitted into a given solid angle as follow,

$$\bar{\sigma}(\Theta)dE_p d\Omega = \frac{1}{E_r} \sigma(\Theta) \frac{\sin^3 \theta}{\sin^3 \Theta} \times \\ \times \frac{d\theta}{d\Theta} \frac{dE_r}{dE_p} dE_p d\Omega \quad (3)$$

where

$$E_r = \frac{\frac{n}{n-1} E_p + \epsilon}{1 + \frac{1}{n-1} \sqrt{\frac{2E_p}{Mc^2}} \cos \Theta}, \quad (4)$$

$$\frac{\hbar^2}{2M} q^2 = E_p - \frac{n+1}{n} \sqrt{\frac{2E_p}{Mc^2}} E_r \cos \Theta. \quad (5)$$

We must now determine $\rho(q)$ by the empirical method. For this purpose it is

convenient to choose $\rho(q)$ so that the differential cross section for $\theta=90^\circ$ may agree with the experimental curve. For $\theta=90^\circ$ we get from (3) and (5)

$$\bar{\sigma}(90^\circ) = \frac{1}{8\pi} \frac{e^2}{\hbar c} \left(\frac{n}{n-1} \right)^2 \frac{E_p}{\left(\frac{n}{n-1} E_p + \epsilon \right)^2} \times \left(\frac{2M}{\hbar^2} E_p \right)^{\frac{1}{2}} \rho(q), \quad (6)$$

where

$$E_p = \frac{\hbar^2}{2M} q^2. \quad (7)$$

On the other hand the experimental cross section for $\theta=90^\circ$ is described as follow by Levinthal and Silverman¹⁾ for carbon irradiated by the 300 Mev X-ray

$$\bar{\sigma}(90^\circ) \sim E_p^{-S}, \quad S = 1.7 \pm 0.1.$$

Substituting the expression into (6) we get

$$\rho(q) = C(q^2 + \alpha^2)^2 q^{-3-2S} \quad (8)$$

where

$$\alpha^2 = \frac{n-1}{n} \frac{2M}{\hbar^2} \epsilon, \quad S = 1.7,$$

and C is a constant. This momentum distribution is shown in Fig. 1 with the Chew-Goldberger's distribution for comparison. These curves are normalized to 1 at 40 Mev. We see in Fig. 1 that both curves agree fairly well.

By using this $\rho(q)$ we calculated the angular distribution for $n=2, 4, 12$ corresponding to the deuteron model, α -particle model and the one particle model for carbon including the recoil of the remaining nucleus, respectively. The results are shown in Fig. 2 for $E_p=40$ Mev and 90 Mev with experimental values. These curves are all normalized to 1 at $\theta=90^\circ$. From these figures we see that at the lower energy (40 Mev) the curve for $n=4$ and 12 are in better agreement with experimental data than the one for $n=2$, but at the higher energy (90

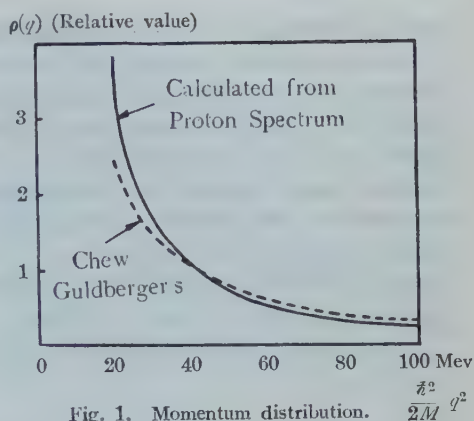


Fig. 1. Momentum distribution.

Relative Cross Section

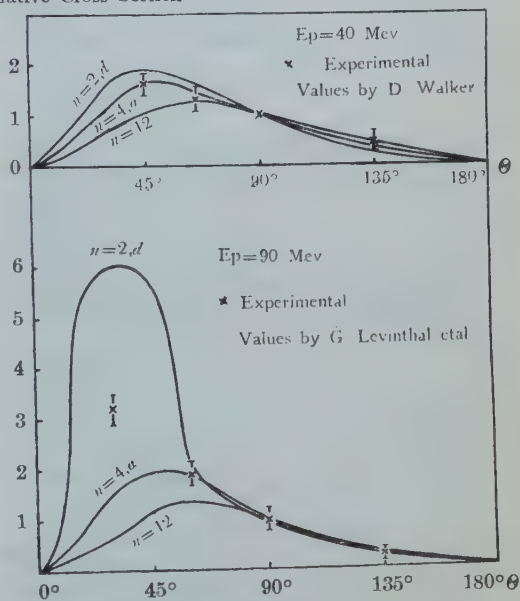


Fig. 2. Angular distribution of proton.

Mev) experimental values lie between the curves for $n=2$ and $n=4$. This would show that at the lower energy one particle model is a good approximation but at the higher energy α -particle model and deuteron model become better approximation.

Recently Levinger published his detailed investigation concerning the nuclear photo-effect³⁾. He calculated the proton energy spectrum and angular distribution using the deuteron model similar to the one which was

used by Heidmann⁴). The energy spectrum of proton calculated by him does not seem to show very good agreement with the experimental data. This is because the α -particle model are better approximation than the deuteron model especially at lower energy. Therefore we can say that the high energy nuclear photoeffect can be explained if we assume the nuclear subunit, and the subunit varies as the energy of reaction changes.

In conclusion the author wishes to thank

Messrs. Y. Fujimoto, H. Horie, T. Tamura and H. Miyazawa for their valuable discussions.

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